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CLOSED QUEUEING NETWORKS WITH

BATCH SERVICES

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Abstract

In this paper we study queueing networks which allow multiple changes at a given time. The model has a natural application to discrete-time queueing networks but also describes queueing networks in continuous time.

It is shown that product-form results which are known to hold when there are single changes at a given instant remain valid when multiple changes are allowed.

Keywords: Batch Services, Discrete-time networks of queues, Continuous-time networks of queues, Product form.

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1. Introduction

A general model for queueing networks with product-form solutions was introduced by Jackson (1957). He considered an open network of N queues in which customers arrive from outside the system to queue i in a Poisson stream, undergo an exponentially distributed service time and then are routed to another queue j with probability $p_{i,j}$ or leave the system with probability $1 - \sum_{j=1}^{N} p_{i,j}$. He showed that the steady-state occupancy distribution of the queueing network factorizes into a product form over the queues. For brevity we refer to a network as having *product form* in such a situation. Jackson (1963) extended his model, whilst retaining product form, by allowing the arrival streams to be dependent on the total number of customers in the queue and the service rates to be arbitrary functions of the queue size.

Since Jackson's papers many other authors have discussed stochastic networks with product form. Gordon and Newell (1967) showed that a closed network of N queues and routeing rules similar to Jackson's also possesses product form. Whittle (1968) showed that product form exists in a migration process in which transition of single individuals from colony j to colony k takes place with intensity dependent on the number of individuals in colony j. Baskett, Chandy, Muntz and Palacios (1975) looked at open, closed and mixed networks of queues and showed that provided the service discipline at queue i is one of three types (processor sharing, infinite server or last-come first-served preemptive resume) the service time distribution at that queue can be generally distributed and the network will still retain product form.

This work was put in a more general context by Kelly (1975a), (1975b), (1976) and (1979). Kelly (1975a) introduced the concept of the *symmetric* queue as a queue in which the proportion of service given to a particular position when there are n customers in the queue is equal to the probability that a customer arriving to find $n-1$ customers already in the queue will be allocated to that position. In a further paper (1975b) he analysed a network in which the rate of service at node i is of the form

$$\frac{\phi(\mathbf{n}-\mathbf{e}_i)}{\phi(\mathbf{n})}, \tag{1.1}$$

where the state of the network is given by $\mathbf{n}$ and $\mathbf{e}_i$ is a vector with a one in the ith position and zeros everywhere else, and showed that it has product form. Generally distributed service times were introduced by Kelly (1976) where he showed that the service time at a symmetric queue can be generally distributed without affecting the product form of the steady state distribution. Queues with any of the three service disciplines of Baskett, Chandy, Muntz and Palacios mentioned above are examples of Kelly's symmetric queues as are queues with the WEIRDP discipline of Chandy and Martin (1983). Queues with the Last Batch Processor Sharing disciplines of Noetzel (1979) can also be represented as symmetric queues.

Chandy, Howard and Towsley (1977) looked at a network of queues each of which is a generalization of a symmetric queue. They set up global balance equations with the state space supplemented by residual service times and separate these into partial balance equations which are satisfied by product form solutions.

Chandy and Martin (1983) considered a queue partitioned into distinguished classes where the rate at which customers at distinguished class i are served when the occupancy of the whole queue is given by $\mathbf{n}$ can be written in the form (1.1) and where the proportion of service allocated to position j of distinguished class i when the state is $\mathbf{n}$ is equal to the probability that a customer arriving to distinguished class i to find a state of $\mathbf{n}-\mathbf{e}_i$ is then allocated to position j.

By considering each distinguished class as a node in a network of queues the above conditions can be interpreted as a generalization of the notion of a symmetric queue so that the arrival rates and service allocations are functions of the complete state of the network rather than just the state of the relevant queue. Chandy and Martin's results then can be interpreted as showing that these conditions are necessary and sufficient for a network of queues to have product form.

Hordijk and van Dijk (1981), (1983a) and (1983b) discussed product form models of queues in which the routeing of customers can be state dependent in some way. Using an associated adjoint process as the key to their analysis they considered networks with blocking and showed that there is a trade-off between the generality of the blocking function and the degree of balance required from the routeing matrix for a network to have product form.

In all of the above models only one transition is assumed to take place at a time. This may be (although not always) a natural assumption to make if we are thinking in continuous time. However in models of discrete-time queueing networks it is natural to assume that multiple transitions can occur simultaneously.

Models of discrete time queues have been discussed by Daduna and Schassberger (1983), Walrand (1983) and Pujolle, Claude, and Seret (1986). Each of these authors adopted rules for handling the simultaneous occurrence of multiple transitions. Daduna and Schassberger considered a network of queues with (discrete) phase type occupation times in which the only simultaneous transitions allowed were the incrementation of phases. Pujolle *et al.* looked first at a single queue and then at a tandem system of queues in which only one simultaneous arrival and departure were allowed. Walrand analysed a network of queues in which the probability that a queue with i customers present will serve j of them in a time slot is of the form

$$S(i,j) = \frac{c(i)}{j!}\alpha(i)\dots\alpha(i-j+1) \tag{1.2}$$

where $\alpha(0) = 0$, $\alpha(j) > 0$ for $j > 0$ and $c(i)$ is a normalizing constant.

In this paper we discuss closed networks of queues which are generalisations of the product form networks in the literature in two ways;

(a) different size batches of customers can depart from several queues simultaneously, and

(b) the rate at which batches are worked on has a form similar to (1.1), but the arbitrary function in the numerator can be different from the arbitrary function in the denominator.

The net effect of this structure is to allow multiple changes of a form more general than the discrete-time changes discussed by the authors above.

In Section 2 we discuss closed networks in which the individual customers are labelled. Section 3 discusses how the results of section 2 can be applied to derive results for closed networks in which the individual customers are not distinguished. In section 4 we look at networks in which customers occupy labelled positions at each queue. The probability that a given pattern of customers leave their current queue can now depend on the type of customer in each position.

2. A closed network with multiple transitions

Consider a closed queueing network which has nodes $1, \ldots, N$ and customers labelled $1, \ldots, S$. The states of the network are defined by vectors $\mathbf{s} = (s_1, \ldots, s_S)$ where $s_j \in \{1, \ldots, N\}$ is the node occupied by customer j.

Changes in the network are assumed to happen according to the following mechanism. There is a function $q(\mathbf{s}, \mathcal{T})$ which describes the propensity for a set of customers with labels given by $\mathcal{T} \subseteq \{1, \ldots, S\}$ to change nodes when the state is $\mathbf{s}$. In continuous time $q(\mathbf{s}, \mathcal{T})$ is the rate that a pattern of customers given by $\mathcal{T}$ simultaneously make a transfer, while in discrete time $q(\mathbf{s}, \mathcal{T})$ denotes the probability that the customers in $\mathcal{T}$ make a transfer in the same time interval.

Conditional on it changing nodes, the probability that customer j is routed from s_j to another node a_j is given by $p_j(s_j, a_j)$ independently of the state of the network or which other customers are in $\mathcal{T}$. Assume that $\sum_{k=1}^{N} p_j(i,k) = 1$ for all $j = 1, \dots, S$ and $i = 1, \dots N$ and describe by $\mathbf{s}|\mathcal{T}^c$ the restriction of $\mathbf{s}$ to positions in $\mathcal{T}^c = \{1, \dots, S\} \backslash \mathcal{T}$. Then the probability conditional on $\mathcal{T}$ that the state changes from $\mathbf{s}$ to $\mathbf{a}$ is

$$p(\mathbf{s}, \mathbf{a}|\mathcal{T}) = \prod_{j \in \mathcal{T}} p_j(s_j, a_j) \tag{2.1}$$

for any vectors $\mathbf{s}$ and $\mathbf{a}$ with $\mathbf{s}|\mathcal{T}^c = \mathbf{a}|\mathcal{T}^c$.

Theorem 1 (labelled customers)

Assume that the set of states $\mathbf{s}$ which can be reached according to the routeing rules of the network is irreducible. Suppose furthermore that for all subsets $\mathcal{T}$ of $\{1, \dots, S\}$

$$q(\mathbf{s}, \mathcal{T}) = \frac{\Psi(\mathbf{s}|\mathcal{T}^c)}{\Phi(\mathbf{s})}, \tag{2.2}$$

where $\Phi(.)$ and $\Psi(.)$ are arbitrary given functions. Then the equilibrium distribution of the queueing network is

$$\pi(\mathbf{s}) = C\Phi(\mathbf{s}) \prod_{j=1}^{S} y_j(s_j), \tag{2.3}$$

where C is a normalizing constant and for $j = 1, \dots, S$ the $y_j(i)$ satisfy

$$y_j(i) = \sum_{k=1}^{S} y_j(k) p_j(k, i) \quad (1 \leq j \leq S). \tag{2.4}$$

Proof

Firstly consider a family of Markov Chains on N states, the one step transition matrix for chain j being $[p_j(k,i)]$. Each chain is finite and irreducible and so positive recurrent. Relation (2.4) represents the global balance equations for chain j and so possesses a positive solution, unique to a scale factor.

Returning to the original sysytem we show that, for an initial distribution given by (2.3), the probability of moving into state $\mathbf{s}$ due to changes in the positions given by $\mathcal{T}$ is balanced by the probability of moving out of state $\mathbf{s}$ due to changes in the positions given by $\mathcal{T}$. This balance is embodied in equation (2.5) below. On summation over all $\mathcal{T}$, any distribution which satisfies (2.5) for all $\mathcal{T}$ will also satisfy the global balance equations for the network and hence be its unique stationary distribution.

The balance equations are

$$\pi(\mathbf{s})q(\mathbf{s},\mathcal{T}) = \sum_{\mathbf{a}} \pi(\mathbf{a})q(\mathbf{a},\mathcal{T})p((\mathbf{a},\mathbf{s}|\mathcal{T}) \quad \forall \mathcal{T}, \tag{2.5}$$

where the summation is over all states $\mathbf{a}$ derivable from $\mathbf{s}$ by changing the entries in positions given by $\mathcal{T}$. Substitution from (2.1), (2.2), (2.3) and use of the fact that $\mathbf{s}|\mathcal{T}^c = \mathbf{a}|\mathcal{T}^c$ reduces the right hand side of equation (2.5) to

$$C\sum_{\mathbf{a}} \Psi(\mathbf{s}|\mathcal{T}^c)\left[\prod_{j\in\mathcal{T}^c} y_j(s_j)\right]\prod_{j\in\mathcal{T}} y_j(a_j)p_j(a_j,s_j). \tag{2.6}$$

From (2.4), (2.2) and (2.3), this is equal to the left hand side of (2.5). □

Remark

Since the function $\Psi(.)$ does not appear in the equilibrium solution (2.3), varying it does not vary the equilibrium distribution except possibly for the normalizing constant C.

We may also choose $\Psi(.)$ to permit only certain sorts of transition sets $\mathcal{T}$. Thus if the function $\Psi(\mathbf{s}|\mathcal{T}^c)$ is put equal to 0 for all $\mathcal{T}$ containing more than (say) l elements, the rate of transfers of more than l customers will be zero. The standard Jackson network results follow from Theorem 1 by setting $l = 1$.

Consequently, by varying the choice of $\Psi(.)$, classes of networks with considerably different structures can be shown to have the same form of equilibrium distribution.

We have thus two generalizations to the queueing network literature. The first is the use of batch services and the second the use of an arbitrary function $\Psi(.)$ in the numerator of (2.2) which is different from the arbitrary function $\Phi(.)$ in the denominator. There is a wide class of closed queueing network models for which the probability of a given pattern of transfers can be written in this form. The problem of analysing such networks reduces to the problem of finding suitable functions $\Psi(.)$ and $\Phi(.)$ to reflect the physical situation. The equilibrium distribution is then given by (2.3) with the appropriate function $\Phi(.)$ inserted.

Example 1 (A discrete-time network in which customers move independently)

Consider a discrete-time queueing network in which changes to the network can happen only at fixed time points. Assume that the decision of a particular customer j to change nodes at any time point is made independently of the current state and of which other customers make a transfer. Denote the probability that customer j decides to change nodes by $q_j > 0$ for $j = 1, \ldots, S$. Then the probability that the set of customers changing nodes is given by $\mathcal{T}$ is

$$q(\mathbf{s}, \mathcal{T}) = \prod_{j \in \mathcal{T}} q_j \prod_{j \in \mathcal{T}^c} [1 - q_j]$$

To analyse this network we need only to find functions $\Phi(.)$ and $\Psi(.)$ for which (2.2) holds.

This can be achieved with the choice

$$\Phi(\mathbf{s}) = \prod_{j=1}^{S} \frac{1}{q_j}$$

and

$$\Psi(\mathbf{s}|\mathcal{T}^c) = \prod_{j\in\mathcal{T}^c} \frac{1-q_j}{q_j}.$$

3. Networks where individual customers are not labelled

In our analysis so far we have labelled and distinguished between individual customers. However in practice it is more likely that sets of customers will be essentially identical. For example the set of customers $\{1,\ldots,S\}$ may be partitioned into sets $\mathcal{K}_l$, $(1 \le l \le L)$ of customers of identical type. In example 1 we considered each customer to be in its own set in the partition. The other extreme has all customers in just one set.

In most models with customer types it is impossible to distinguish between customers of identical type. Thus many states $\mathbf{s}$ are physically indistinguishable. In such models it is often reasonable to postulate that service and routeing characteristics depend only on the number of customers of each type present at each node and the number of customers of each type undergoing transfer, rather than on the actual labels of the customers. Making these assumptions leads us to the following result.

Let the incidence matrix $\mathbf{n}(\mathbf{s}) = (n_{i,l}(\mathbf{s}))$ for $1 \le i \le N$ and $1 \le l \le L$ give the number of $\mathcal{K}_l$-type customers at node i when the process has state $\mathbf{s}$. It is conventional to denote the state by $\mathbf{n}$ rather than $\mathbf{s}$. Theorem 1 specializes as follows.

Theorem 2 (unlabelled customers)

In the situation of Theorem 1 let $\Phi(\mathbf{s})$ depend on $\mathbf{s}$ only through $\mathbf{n}(\mathbf{s})$ and let $p_j(i,k)$ depend on j only through the class $\mathcal{K}_l$ in which j resides. Then the equilibrium probabilities may be written as

$$\pi(\mathbf{n}) = C\Phi(\mathbf{n}) \prod_{i=1}^{N} \prod_{l=1}^{L} \frac{y_l(i)^{n_{i,l}}}{n_{i,l}!}, \tag{3.1}$$

where $y_l(i)$ satisfies

$$y_l(i) = \sum_{k=1}^{N} y_l(k) p_l(k,i).$$

Proof

Replacing $y_j(i)$ by $y_l(i)$ for all $j \in \mathcal{K}_l$ in equation (2.3) gives

$$\pi(\mathbf{s}) = C\Phi(\mathbf{s}) \prod_{l=1}^{L} \prod_{i=1}^{N} y_l(i)^{n_{i,l}}. \tag{3.2}$$

The function $\Phi(\mathbf{s})$ depends on s only through $\mathbf{n}(\mathbf{s})$ and there are $\prod_{i=1}^{N} \prod_{l=1}^{L} |\mathcal{K}_l|! \,/ n_{i,l}!$ ways of choosing $\mathbf{s}$ so that $\mathbf{n}(\mathbf{s}) = \mathbf{n}$. Summing (3.2) over such $\mathbf{s}$ and changing the normalizing constant appropriately gives equation (3.1). □

Corollary

Represent by $d_{i,l}$ $(1 \le i \le N; 1 \le l \le L)$ the number of $\mathcal{K}_l$-type customers completing service together at node i. If, for each $\mathbf{d}$, the probability of a batch service pattern $\mathbf{d}$ when the state is $\mathbf{n}$ is of the form

$$q(\mathbf{n},\mathbf{d}) = \frac{\Psi^*(\mathbf{n}-\mathbf{d})}{\Phi^*(\mathbf{n}) \prod_{i=1}^{N} \prod_{l=1}^{L} d_{i,l}!} \tag{3.3}$$

where $\Psi^*(.)$ and $\Phi^*(.)$ are arbitrary given functions then

$$\pi(\mathbf{n}) = C\Phi^*(\mathbf{n}) \prod_{i=1}^{N} \prod_{l=1}^{L} y_l(i)^{n_{i,l}}. \tag{3.4}$$

Proof

For given values $n_{i,l}$ the number of possibilities $\mathcal{T}$ which result in the same service pattern $\mathbf{d}$ is $\prod_{i=1}^{N}\prod_{l=1}^{L}\binom{n_{i,l}}{d_{i,l}}$. For each of these $\Psi(.)$ and $\Phi(.)$ are the same. Consequently, summing (2.2) over these $\mathcal{T}$, we get

$$\begin{aligned} q(\mathbf{n},\mathbf{d}) &= \frac{\Psi(\mathbf{n}|\mathcal{T}^c)}{\Phi(\mathbf{n})}\prod_{i=1}^{N}\prod_{l=1}^{L}\binom{n_{i,l}}{d_{i,l}} \\ &= \left[\frac{\prod_{i=1}^{N}\prod_{l=1}^{L} n_{i,l}!}{\Phi(\mathbf{n})}\right]\left[\frac{\Psi(\mathbf{n}|\mathcal{T}^c)}{\prod_{i=1}^{N}\prod_{l=1}^{L}(n_{i,l}-d_{i,l})!}\right]\left[\frac{1}{\prod_{i=1}^{N}\prod_{l=1}^{L} d_{i,l}!}\right]. \end{aligned}$$

From the argument dependence of the functions involved, comparison with (3.3) provides

$$\Psi^*(\mathbf{n}-\mathbf{d}) = \frac{A\Psi(\mathbf{n}|\mathcal{T}^c)}{\prod_{i=1}^{N}\prod_{l=1}^{L}(n_{i,l}-d_{i,l})!}$$

and

$$\Phi^*(\mathbf{n}) = \frac{A\Phi(\mathbf{n})}{\prod_{i=1}^{N}\prod_{l=1}^{L} n_{i,l}!}$$

for some constant A. Substitution for $\Phi(\mathbf{n})$ in (3.1) gives (3.4) for a suitable constant C.
□

Example 2

In many discrete time queueing networks the probability of a given number of customers of type l departing from node i depends on the total number of type l customers that are actually present at node i. One natural form that $q(\mathbf{n},\mathbf{d})$ may take is the following.

$$q(\mathbf{n},\mathbf{d}) = \prod_{l=1}^{L}\prod_{i\in\mathcal{K}_l}\binom{n_{i,l}}{d_{i,l}}[1-q_{i,l}(n_{i,l}-d_{i,l})]\prod_{j=n_{i,l}-d_{i,l}+1}^{n_{i,l}} q_{i,l}(j). \tag{3.5}$$

Note the similarity between (3.5) and the form (1.2) used by Walrand (1983). This form reflects the situation where a batch of departures is essentially made up of a set of customers each making its own decision whether to transfer or not, dependent on the current number of customers of the same type at the same queue.

On taking

$$\Phi^*(\mathbf{n}) = \frac{1}{\prod_{i=1}^{N}\prod_{l=1}^{L} n_{i,l}!}\prod_{i=1}^{N}\prod_{l=1}^{L}\prod_{j=1}^{n_{i,l}}\frac{1}{q_{i,l}(j)}$$

and

$$\Psi^*(\mathbf{n}-\mathbf{d}) = \frac{1}{\prod_{i=1}^{N}\prod_{l=1}^{L}(n_{i,l}-d_{i,l})!}\prod_{i=1}^{N}\prod_{l=1}^{L}[1-q_{i,l}(n_{i,l}-d_{i,l})]\prod_{j=1}^{n_{i,l}-d_{i,l}}\frac{1}{q_{i,l}(j)},$$

we see that $q(\mathbf{n},\mathbf{d})$ given by (3.5) has the form (3.3).

4. Queues where positions are defined

In many queueing networks the rate at which customers are served is a function of the whole state at any given node rather than just the numbers of each type of customer present. For example in a network of two nodes with two customer types we may wish to distinguish between the state $((1,2):\phi)$ which has a type one customer in positon 1 and a type two customer in position 2 of queue 1 and the state $((2,1):\phi)$ in which the customers are the other way round. To model such networks the structure defined in section 2 needs to be modified slightly.

The state vector $\mathbf{r}$ now needs to include information on both the queue and position of the customers. Thus let $r_j = (i_j, m_j)$, where i_j is the queue occupied by customer j and m_j is the position in that queue occupied by customer j. Assume as in section 3 that the customers are partitioned into classes $\mathcal{K}_l$.

Clearly the set of possible $\mathbf{r}$ has to be restricted so that the same position in a given queue is not occupied by two different customers, so we assume that $r_j \neq r_k$ if $j \neq k$. It is also desirable that positions $1,\ldots,n_i$ are occupied when there are n_i customers at node i. To ensure that this is the case we adopt the following rules for relabelling customers when transitions occur.

We deem that all departures occur before any arrivals take place at a transition instant. If the state is $\mathbf{r}$ and customers given by $\mathcal{T}$ leave their current queues, all customers in positions higher than the positions of customers in $\mathcal{T}$ move down to fill all the gaps, but preserve their relative order. The customers in $\mathcal{T}$ are routed to new nodes according to their routeing matrices to produce a new state $\mathbf{r}'$ and the number of customers at node i is now (say) n_i'. It remains to assign positions to these customers. Let d_i' be the number of customers at node i in $\mathbf{r}'|\mathcal{T}$. Then a convenient way to allocate positions to the d_i' customers newly arrived to node i is to assign them to each of the $n_i'! \,/(n_i' - d_i')!$ possible patterns with equal probability. All customers that were originally in the queue are moved up in an order preserving way to make room.

With the customer positions at each queue included the state space carries much more information than the previous description detailed in section 2. However, in practice, what usually matters in determining transition probabilities is not the positions occupied by each labelled customer, but rather the occupancy or distribution of types of customer over the positions in the queues. There will in general be many states $\mathbf{r}$ which have the same occupancy. Thus we define the following function.

Let $\mathbf{c}(\mathbf{r}) = (c_{i,m}(\mathbf{r}))$ for $1 \leq i \leq N$ and $1 \leq m \leq n_i$ be defined so that its entries give the type of customer in position m of queue i. We denote the state of the network by $\mathbf{c}$. By representing the state in this form we assign a type of customer to each queue and occupied position.

Theorem 3

Consider a network of queues for which the states $\mathbf{r}$ contain information on the position of customers as described above. Assume that

$$q(\mathbf{r}, \mathcal{T}) = \frac{\Psi(\mathbf{r}|\mathcal{T}^c)}{\Phi(\mathbf{r})}$$

where the arbitrary functions $\Psi(\mathbf{r}|\mathcal{T}^c)$ and $\Phi(\mathbf{r})$ depend on $\mathbf{r}$ only through $\mathbf{c}(\mathbf{r})$ and that the routeing rules $p_j(i,k)$ depend only on the class l in which customer j resides. Thus we write $\Phi(\mathbf{c})$ for $\Phi(\mathbf{c}(\mathbf{r}))$ where $\mathbf{r}$ is such that $\mathbf{c}(\mathbf{r}) = \mathbf{c}$.

Then the equilibrium distribution of the queueing network is

$$\pi(\mathbf{c}) = C\Phi(\mathbf{c}) \prod_{i=1}^{N} \prod_{l=1}^{L} \frac{y_l(i)^{n_{i,l}}}{n_i!}, \tag{4.1}$$

where for $l = 1, \ldots, L$ the $y_l(i)$ satisfy

$$y_l(i) = \sum_{k=1}^{N} y_l(k) p_l(k,i). \tag{4.2}$$

Proof

Equation (2.3) gives the equilibrium distribution for a network of queues where customer positions are not taken into account. When we include position there are $\prod_{i=1}^{N} n_i!$ different states $\mathbf{r}$ corresponding to the same state $\mathbf{s}$. Since the allocation rules are random each of these states $\mathbf{r}$ will have equilibrium probability

$$\pi(\mathbf{r}) = \frac{\pi(\mathbf{s})}{\prod_{i=1}^{N} n_i!}. \tag{4.3}$$

Now sum over all $\mathbf{r}$ corresponding to a given $\mathbf{c}(\mathbf{r})$. There are $\prod_{l=1}^{L} |\mathcal{K}_l|!$ such $\mathbf{r}$, over which the values of n_i for $1 \le i \le N$ remain constant. Thus summing we get

$$\pi(\mathbf{c}) = \frac{\pi(\mathbf{s})}{\prod_{i=1}^{N} n_i!} \prod_{l=1}^{L} |\mathcal{K}_l|! \, . \tag{4.4}$$

The assumption that customers' routeing probabilities depend only on their class means that for a given state $\mathbf{r}$

$$\prod_{i=1}^{N} \prod_{l=1}^{L} y_l(i)^{n_i} = \prod_{j=1}^{S} y_j(i_j). \tag{4.5}$$

Using this, the fact that $\Phi(\mathbf{r})$ depends only on $\mathbf{c}(\mathbf{r})$, (2.3) and (4.4) gives (4.1). □

Remark

It is natural to describe a pattern of departures from $\mathbf{c}$ by giving the set $\mathcal{D}$ of nodes and positions (i, m) $(1 \leq i \leq N;\ 1 \leq m \leq n_i)$ from which departures have taken place. The state $\mathbf{c}|\mathcal{D}^c$ then gives the type of customer in the remaining positions after customers given by $\mathcal{D}$ have left, assuming as before that all customers have been packed downwards to the lowest possible positions. We can translate $q(\mathbf{r}, \mathcal{T})$ into $q(\mathbf{c}, \mathcal{D})$ in the following way.

If the state is $\mathbf{c}$ then the process must be in some state $\mathbf{r}$ with $\mathbf{c}(\mathbf{r}) = \mathbf{c}$. For this $\mathbf{r}$, $\mathcal{D}$ is completely determined by $\mathcal{T}$ and vice versa. Thus

$$q(\mathbf{r}, \mathcal{T}) = \frac{\Psi(\mathbf{r}|\mathcal{T}^c)}{\Phi(\mathbf{r})} = \frac{\Psi(\mathbf{c}(\mathbf{r})|\mathcal{D}^c)}{\Phi(\mathbf{c}(\mathbf{r}))}, \tag{4.6}$$

since $\Phi(\mathbf{r})$ and $\Psi(\mathbf{r}|\mathcal{T}^c)$ depend on $\mathbf{r}$ only through $\mathbf{c}(\mathbf{r})$. We can therefore define the probability that customers at nodes and positions given by $\mathcal{D}$ depart when the occupancy is given by $\mathbf{c}$ as

$$q(\mathbf{c}, \mathcal{D}) = \frac{\Psi(\mathbf{c}|\mathcal{D}^c)}{\Phi(\mathbf{c})}. \tag{4.7}$$

Example 3

As a simple illustrative example of how Theorem 3 works consider a network of just two nodes with three customers, two of type 1 and and one of type 2. If both of the type 1 customers are at the same node then they can be served only if they are in adjacent positions, in which case they are served with probability 1/2 as is the type 2 customer alone. If the two type 1 customers are at opposite nodes then they are both served simultaneously with probability 1/4 or the type 2 customer is served alone with probability 3/4.

Denoting a state by $\mathbf{c} = (\mathbf{u_1} : \mathbf{u_2})$ where $\mathbf{u_{j_i}}$ is the type of customer in position i of queue j this network has the following states

$$\mathbf{c_1} = ((1,1,2) : \phi)$$

$$\mathbf{c_2} = ((1,2,1) : \phi)$$

$$\mathbf{c_3} = ((2,1,1) : \phi)$$

$$\mathbf{c_4} = (\phi : (1,1,2))$$

$$\mathbf{c_5} = (\phi : (1,2,1))$$

$$\mathbf{c_6} = (\phi : (2,1,1))$$

$$\mathbf{c_7} = ((1) : (1,2))$$

$$\mathbf{c_8} = ((1) : (2,1))$$

$$\mathbf{c_9} = ((2) : (1,1))$$

$$\mathbf{c_{10}} = ((1,1) : (2))$$

$$\mathbf{c_{11}} = ((1,2) : (1))$$

$$\mathbf{c_{12}} = ((2,1) : (1))$$

Because of the small number of customers involved in this example we are able to unambiguously define the transfer patterns just by giving the nodes and types of customer which transfer. Hence the following transfer patterns are possible

$$\mathbf{d_1} = ((1,1) : \phi)$$

$$\mathbf{d_2} = (\phi : (1,1))$$

$$\mathbf{d_3} = ((1) : (1))$$

$$\mathbf{d_4} = (\phi : (2))$$

$$\mathbf{d_5} = ((2) : \phi)$$

The above assumptions about transfer probabilities imply

$$q(\mathbf{c_1}, \mathbf{d_1}) = q(\mathbf{c_1}, \mathbf{d_5}) = q(\mathbf{c_3}, \mathbf{d_1}) = q(\mathbf{c_3}, \mathbf{d_5}) = 1/2$$

$$q(\mathbf{c_4}, \mathbf{d_2}) = q(\mathbf{c_4}, \mathbf{d_4}) = q(\mathbf{c_6}, \mathbf{d_2}) = q(\mathbf{c_6}, \mathbf{d_4}) = 1/2$$

$$q(\mathbf{c}_2, \mathbf{d}_5) = q(\mathbf{c}_5, \mathbf{d}_4) = 1$$

$$q(\mathbf{c}_7, \mathbf{d}_3) = q(\mathbf{c}_8, \mathbf{d}_3) = q(\mathbf{c}_{11}, \mathbf{d}_3) = q(\mathbf{c}_{12}, \mathbf{d}_3) = 1/4$$

$$q(\mathbf{c}_7, \mathbf{d}_4) = q(\mathbf{c}_8, \mathbf{d}_4) = q(\mathbf{c}_{11}, \mathbf{d}_5) = q(\mathbf{c}_{12}, \mathbf{d}_5) = 3/4$$

$$q(\mathbf{c}_9, \mathbf{d}_2) = q(\mathbf{c}_9, \mathbf{d}_5) = q(\mathbf{c}_{10}, \mathbf{d}_1) = q(\mathbf{c}_{10}, \mathbf{d}_4) = 1/2$$

which can be achieved by putting

$$\Psi((2):\phi) = \Psi((1,1):\phi) = \Psi(\phi:(2)) = \Psi(\phi:(1,1)) = 1$$

$$\Psi((1):(1)) = 3$$

$$\Psi(.) = 0 \quad \text{otherwise}$$

and

$$\Phi(\mathbf{c}_1) = \Phi(\mathbf{c}_3) = \Phi(\mathbf{c}_4) = \Phi(\mathbf{c}_6) = 2$$

$$\Phi(\mathbf{c}_2) = \Phi(\mathbf{c}_5) = 1$$

$$\Phi(\mathbf{c}_7) = \Phi(\mathbf{c}_8) = \Phi(\mathbf{c}_{11}) = \Phi(\mathbf{c}_{12}) = 4$$

$$\Phi(\mathbf{c}_9) = \Phi(\mathbf{c}_{10}) = 2$$

The equilibrium distribution is then given by (4.1) where the $y_l(i)$ satisfy the routeing equations for the network.

References

Baskett F., Chandy K., Muntz R. and Palacios J. (1975) - *Open, Closed and Mixed Networks of Queues with Different Classes of Customers*, J. Assoc. Comp. Mach., 22, 248–260.

Chandy K., Howard J. and Towsley D. (1977) - *Product Form and Local Balance in Queueing Networks*, J. Assoc. Comp. Mach., **24**, 250–263.

Chandy K. and Martin J. (1983) - *A Characterisation of Product Form Queuing Networks*, J. Assoc. Comp. Mach., **30**, 286–299.

Daduna H. and Schassberger R. (1983) - *Networks of Queues in Discrete Time*, Zeitschrift für Operations Research, **27**, 159–175.

Gordon W. and Newell G. (1967) - *Closed Queueing Systems with Exponential Servers*, Operations Research, **15**, 254–265.

Hordijk A. and Van Dijk N. (1981) - *Networks of Queues with Blocking*, Performance 81, (ed K. Kylstra), North Holland, 51–65.

Hordijk A. and Van Dijk N. (1983a) - *Networks of Queues*, Proc. International Seminar on Modelling and Performance Evaluation Methodology, INRIA, **1**, 79–135.

Hordijk A. and Van Dijk N. (1983b) - *Adjoint Processes, Job Local Balance and Insensitivity for Stochastic Networks*, Bull. 44th Session Int. Stat. Inst., **50**, 776–788.

Jackson J. (1957) - *Networks of Waiting Lines*, Operations Research, **5**, 518–521.

Jackson J. (1963) - *Jobshop-like Queueing Systems*, Management Sc., **10**, 131–142.

Jansen U. and König D. (1980) - *Insensitivity and Steady-State Probabilities in Product Form for Queueing Networks*, Elektron. Informationsverarb. Kybernetik., **16**, 385–397.

Kelly F. (1975a) - *Networks of Queues with Customers of Different Types*, J. Appl. Prob., **12**, 542–544.

Kelly F. (1975b) - *Markov Processes and Markov Random Fields*, Bull. Inst. Int. Statist., **46**, 397–404.

Kelly F. (1976) - *Networks of Queues*, Adv. Appl. Prob., **8**, 416–432.

Kelly F. (1979) - *Reversibility and Stochastic Networks*, Wiley, London.

König D. and Jansen U. (1974) - *Stochastic Processes and Properties of Invariance for Queueing Systems with Speeds and Temporary Interruptions*, Trans 7th Prague Conference Inf Th, Stat Dec Fns and Rand Proc, 335–343.

Noetzel A. (1979) - *A Generalized Queueing Discipline for Product Form Queueing Networks*, J. Assoc. Comp. Mach. **26**, 779–793.

Pujolle G., Claude J.P. and Seret D. (1986) - *A Discrete Queueing System with a Product Form Solution*,

Walrand J. (1983) - *A Discrete-time Queueing Network*, J. Appl. Prob., **20**, 903–909.

Whittle P. (1968) - *Equilibrium Distributions for an Open Migration Process*, J. Appl. Prob., **5**, 567–571.

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