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A NOTE ON THE QUADRATIC EXPENDITURE MODEL

Jan van Daal

Arnold H.Q.M. Merkies

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by

Jan van Daal
Erasmus University Rotterdam
and
Arnold H.Q.M. Merkies
Vrije Universiteit, Amsterdam

Summary

In this note we derive the general form of a utility based system of demand equations that are quadratic in income with coefficients that are functions of prices. Our result turns out to be slightly more general than known till now.

Introduction

In this note we shall show that any theoretically plausible (i.e. utility based) quadratic demand system

$$(1) \qquad q^k(p,y) = A^k(p)y^2 + B^k(p)y + C^k, \text{ 1)}$$

where q^k = quantity consumed of good $k(=1,\dots,K)$,
$p = (p^1,\dots,p^K)$ is the price vector of the goods,
y = total amount spent on the goods
and where the A^k, B^k and C^k are sufficiently differentiable functions of the prices, can be written in the form

$$(2) \qquad q^k = \frac{1}{G} H_k y^2 + \left(\frac{1}{G} G_k - \frac{2F}{G} H_k\right)y + \frac{F^2}{G} H_k - \frac{F}{G} G_k + F_k + \chi(H)GH_k,$$

1) Typografical note: All superscripts in letters denote goods: superscripts in arabic numbers denote powers. Subscripts always denote differentiation with respect to prices p or income y.

where F and G are linear–homogeneous functions of prices, H is a zero–homogeneous function of prices and χ is a function of one variable. This means that we claim that the system devised by Howe, Pollak and Wales (1979), further to be called HPW, is not the most general quadratic system because their system results as a special case of (2), namely if $\chi(H) = 0$ identically.

We shall first show that (2) is correct; then we shall conjecture why HPW missed the most general form. We end with presenting the indirect utility function of (2) for the case $\chi(H) = \lambda$, where λ is a non–zero constant.

<u>Proof of statement</u>

Necessary and sufficient conditions for (1) to be based utility are

(<u>i</u>) for all $k = 1,\dots,K$

(3) $$\sum_k p^k A^k = \sum_k p^k C^k = 0; \sum_k p^k B^k = 1,$$

(<u>ii</u>) for all $k = 1,\dots,K$ the function q^k is homogeneous of degree zero in p and y,

(<u>iii</u>) for all $k,j = 1,\dots,K$ the Slutsky element

(4) $$s^{kj} = q^k_j + q^j q^k_y$$

is symmetrical in k and j:

(5) $$s^{kj} = s^{jk},$$

(<u>iv</u>) $S = [s^{kj}]$ is negative semi–definite of rank K–1;

see, e.g., Hurwicz and Uzawa (1971, pp. 123–130). We shall show that (i), (ii) and (iii) lead to (2). In order to satisfy (iv) further specification of F, G, H and χ is needed. Conditions (i), (ii) and (iii), and in particular (iii), are sometimes called the mathematical integrability conditions; together with sufficient smoothness of all functions they guarantee the existence of a function $u = u(q^1,\dots,q^K)$ such that (1) can be derived from the K equations

$u_k = \lambda p^k$ and $\sum_k p^k q^k = y$. Additional conditions on convexity must ensure that u is a meaningful utility function; see Hurwicz (1971, pp. 176–177).

All Slutsky elements s^{kj} are polynomials of degree 3; the four coeficients are symmetrical to the corresponding coefficients of s^{jk}. This yields

$$(6) \qquad A^k_j + A^k B^j = A^j_k + A^j B^k,$$

$$(7) \qquad B^k_j + 2A^k C^j = B^j_k + 2A^j C^k,$$

$$(8) \qquad C^k_j + B^k C^j = C^j_k + B^j C^k.$$

This system of partial differential equations is solved in two steps. First we show that the functions A^k are of the form H_k/G, where G is linear–homogeneous and H is zero–homogeneous in the prices. Second, this result is used to find B^k and C^k.

Rewriting (6) as

$$(9) \qquad A^i(A^k_j - A^j_k) = A^i A^j B^k - A^i A^k B^j,$$

cycling i, k and j and adding the three equations yields

$$(10) \qquad A^i(A^k_j - A^j_k) + A^k(A^j_i - A^i_j) + A^j(A^i_k - A^k_i) = 0.$$

Corresponding identities hold for all triples of prices. This is sufficient for the existence of an integrating factor $G = G(p^1,\dots,p^K)$ such that

$$(11) \qquad GA^1dp^1 + GA^2dp^2 + \dots + GA^Kdp^K = 0$$

is an exact differential equation. In other words there exists a function $H=H(p^1,\dots,p^K)$ such that $H(p) = c$ (constant) is a solution of (11); see, e.g., Guillemine and Pollack (1974, p. 181) or Ince (1967, p. 318). Consequently, for all k

$$(12) \qquad GA^k = H_k,$$

or

$$(13) \qquad A^k = \frac{1}{G} H_k .$$

Substituting this into (6) yields for all k and j

$$\text{(14)} \qquad \frac{GB^k - G_k}{H_k} = \frac{GB^j - G_j}{H_j}.$$

Consequently, there exists a function $F = F(p)$ such that

$$\text{(15)} \qquad GB^k - G_k = -2FH_k$$

for all k, or

$$\text{(16)} \qquad B^k = \frac{1}{G} G_k - \frac{2F}{G} H_k.$$

Substituting (13) and (16) into (7) and rearranging leads to

$$\text{(17)} \qquad \frac{GC^k + FG_k - GF_k}{H_k} = \frac{GC^j + FG_j - GF_j}{H_j}.$$

Hence there is a function $\bar{D}$ of prices such that for all k

$$\text{(18)} \qquad GC^k + FG_k - GF_k = \bar{D}\, H_k,$$

or, with $D = \bar{D}/G$,

$$\text{(19)} \qquad C^k = -\frac{F}{G} G_k + F_k + DH_k.$$

The combination of (8), (16) and (19) requires that

$$\text{(20)} \qquad -\frac{F}{G} G_{kj} + \frac{F}{G^2} G_k G_j - \frac{1}{G} F_j G_k + F_{kj} + DH_{kj} + D_j H_k +$$

$$+ \left(\frac{1}{G} G_k - \frac{2F}{G} H_k\right)\left(-\frac{F}{G} G_j + F_j + DH_j\right)$$

be symmetrical in k and j. Accordingly, deleting the terms that are always symmetrical (such as G_{kj} and $G_k G_j$), also

$$\text{(21)} \qquad D_j H_k + \frac{D}{G} G_k H_j + \frac{2F^2}{G^2} G_j H_k - \frac{2F}{G} F_j H_k$$

must be symmetrical. Collecting all terms with the same index into one side of the symmetry relation results in

$$(22)\qquad \frac{-\frac{D}{G^2}G_k + \frac{1}{G}D_k + \frac{2F^2}{G^3}G_k - \frac{2F}{G^2}F_k}{H_k} = \frac{-\frac{D}{G^2}G_j + \frac{1}{G}D_j + \frac{2F^2}{G^3}G_j - \frac{2F}{G^2}F_j}{H_j},$$

or, for all k:

$$(23)\qquad \frac{(\frac{D}{G} - \frac{F^2}{G^2})_k}{H_k} = \frac{(\frac{D}{G} - \frac{F^2}{G^2})_j}{H_j}.$$

This means that H and $D/G - F^2/G^2$ are functionally dependent: there is a function χ with

$$(24)\qquad \frac{D}{G} - \frac{F^2}{G^2} = \chi(H);$$

see, e.g., Burkil and Burkil (1970, p. 230). Hence

$$(25)\qquad D = \frac{F^2}{G} + \chi(H)G.$$

Substitution into (19) renders for all k

$$(26)\qquad C^k = \frac{F^2}{G}H_k - \frac{F}{G}G_k + F_k + \chi(H)\,GH_k.$$

The homogeneity properties of F, G and H, and of the q^k, follow from the additivity conditions (3). This proves our point.

Remark Instead of starting with (6) and then using the results together with (7) and (8), one could as well start with (8) in deriving the A^k, B^k and C^k because (8) has the same mathematical form as (6). We would have got, then, a system that at first sight is completely different from (2). It can be argued, however, that both can be transformed into each other. We prefer the form (2) because of its easier economic interpretation. It can be written as

$$(27)\qquad q_k = \frac{1}{G}H_k(y-F)^2 + \frac{1}{G}G_k(y-F) + F_k + \chi(H)GH_k.$$

With suitably chosen specifications for the functions of prices one can build in bliss and/or minimum subsistence levels for income y; see HPW.

Howe, Pollak and Wales's solution

HPW found (2) with $\chi(H)=0$. Their f and g are our F and G, respectively. Their linear-homogeneous function α equals our HG. They use a theorem mentioned in Hurwicz and Uzawa (1971, Appendix) stating that a system of differential equations

$$(28) \qquad \frac{\partial z}{\partial p^k} = \phi^k(p,z)$$

with $k = 1,\ldots,K$, has a solution $z = z(p)$ if and only if for all $k,j=1,\ldots,K$

$$(29) \qquad \phi^k_j + \phi^j\phi^k_z = \phi^j_k + \phi^k\phi^j_z.$$

The theorem is used in proving that the system of K partial differential equations

$$(30) \qquad f_k = C^k + B^k f + A^k f^2$$

has a solution $f=f(p)$ provided that the Slutsky requirements (6), (7) and (8) are met. Repeated use of the theorem leads to the existence of a function g such that $g_k/g = B^k + 2A^k f$ and a function α such that $A^k=(\alpha_k-\alpha g_k/g)/g^2$. With these functions they obtain their result.

Our approach described above, indicates that HPW, in adopting (30), did not use the most general system of differentiable equations. Consequently, they did not find the most general quadratic expenditure system, although they found a very large subset. Their discovery of (30), however brillant, appears to be arbitrary. It is, after all, only a choice and does not inevitably follow from (6), (7) and (8). Our attention to this problem was drawn when we found that Lewbel (1987) introduced a so-called extended PIGL demand system that for the case k=2 (Lewbel's k is an exponent) is not a special case of the general quadratic expenditure system that he takes the same as HPW do. This indicated the direction into we had to search.

The indirect utility function

We did not succeed in finding the indirect utility function for the most general case (2). Lewbel (1987), however, inspired us to the following indirect utility function

$$(31) \qquad \nu(p,y) = \psi\left(-\frac{G}{y-F}\right) - H,$$

where ψ is such that

$$(32) \qquad \psi(z) = \int \frac{dz}{1+\lambda z^2}$$

for the case $\chi(H)=\lambda$, where λ is a constant $\neq 0$. By using Roy's theorem it can be checked that (31) indeed generates our system with $\chi(H)=\lambda$.

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