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SPATIAL BIRTH-DEATH PROCESSES WITH MULTIPLE CHANGES AND APPLICATIONS TO BATCH SERVICE NETWORKS AND CLUSTERING PROCESSES

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Spatial birth-death processes with multiple changes and applications to batch service networks and clustering processes

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Abstract

Reversible spatial birth-death processes are studied with simultaneous jumps of multi-components. A relationship is established between (i) a product form solution, (ii) a partial symmetry condition on the jumprates and (iii) a solution of a deterministic concentration equation. Applications studied are: 1. Reversible networks of queues with batch services and blocking 2. Clustering processes such as in polymerization chemistry. As illustrated by examples, known results are hereby unified and extended. An expectation interpretation of the transition rates is included.

Keywords: Queueing network, Clustering process, Batch service, Product form, Concentration equation, Symmetry property.

1 Introduction

A spatial birth-death process is introduced in which more components can change at the same time. With this process a wide variety of stochastic processes such as queueing networks with simultaneous changes such as due to batch servicing or a discrete-time structure and clustering processes such as from polymer chemistry can be modelled.

It is shown that any two of the following set of three properties implies the third:

- A partial symmetry property of the transition rates.
- A solution of the deterministic traffic or concentration equations.
- A product form distribution which solves the detailed balance equation.

The stationary product form distribution turns out to be totally independent of the symmetric part of the transition rates. Blocking phenomena and state dependent activation of transitions, such as state dependent delay of transitions in queueing networks, can so be modelled.
As applications queueing networks and clustering processes are considered. General references on both are [10], [19]. Results from [2], [6], [10], [15], [17], [19] are hereby extended to non-convex state spaces, blocking and state dependent delay of transitions in queueing networks and blocking in clustering processes. This is illustrated by various examples.
The outline of this paper is as follows. In section 2 we present our model and main result. In section 3 some important applications are given. In section 4 we give an expectation interpretation of the factors in the transition rates and some general remarks.

2 Model and main result

We consider a continuous time Markov chain with state space $S \subset N_o^n = \{0, 1, 2, \cdots\}^n$ for fixed $n \in N \cup \{\infty\}$. A state $E \in S$ is a vector with components $E_i \in N_o$, $i = 1, \cdots, n$. If $E, M \in N_o^n$ then $E + M$ denotes the vector with components $E_i + M_i$, $i = 1, \cdots, n$.
Let $q(E, E')$ denote the transition rate from state E to state E'. We assume that the transition rates have the form

$$q(E + M, E + M') = \lambda(M, M')\frac{\psi(M, M'; E)}{\phi(E + M)} \tag{2.1}$$

for all $E, M, M' \in N_o^n$ such that both $E + M, E + M' \in S$, while $q(E + M, E + M')$ is assumed to be zero whenever $E + M$ or $E + M'$ is not contained in S. Here the functions $\lambda(\cdot)$, $\psi(\cdot)$, $\phi(\cdot)$ are arbitrary nonnegative functions upon which regularity conditions will be imposed shortly.

Remark 2.1 (Transition rates) Note that (2.1) is no restriction since $\psi(\cdot)$ may contain full information on M, M', $E + M$ and $E + M'$, i.e. for any given $\lambda(\cdot)$ and $\phi(\cdot)$ we may set

$$\psi(M, M'; E) = \frac{q(E + M, E + M')\phi(E + M)}{\lambda(M, M')}$$

if $\lambda(M, M') \neq 0$ and an arbitrary function $\psi(\cdot)$ if $\lambda(M, M') = 0$.

First let us briefly illustrate our notation.

Examples (Notation)

A. Queueing networks

1. Occupation numbers
Consider a queueing network of n nodes. The state of the network is given by

$\bar{m} = (m_1, \cdots, m_n)$ where m_i is the number of customers at node i (occupation number). A transition $\bar{m} + \bar{t} \to \bar{m} + \bar{t}'$ represents that t_i customers leave and t_i' customers enter node i, while m_i customers at node i do not move at all.

2. Positions

Now consider a closed queueing network with n customers present. A state $\bar{p}$ represents the position of the customers, i.e. p_i is the number of the node at which customer i is present. In a transition $\bar{p} + \bar{s} \to \bar{p} + \bar{s}'$ we assume that either $p_i = 0$ and $s_i \neq 0$ or $p_i \neq 0$ and $s_i = 0$. In the first case this indicates that customer i moves from node s_i to node s_i', in the second case that customer i does not move at all and that $p_i = p_i'$ and $s_i = s_i' = 0$.

B. Clustering processes

Consider a process in which basic units are grouped in clusters. Two clusters of size i and j can associate to form a cluster of size $i + j$ and conversely a cluster of size $i + j$ can dissociate into a cluster of size i and a cluster of size j. Symbolically

$$A_i + A_j \rightleftarrows A_{i+j}$$

where A_k denotes a cluster of size k.

A state $\bar{m} = (m_1, m_2, \cdots)$ represents for all i the number of clusters of size i. In a transition $\bar{m} + \bar{v} \to \bar{m} + \bar{v}'$ a number v_i of clusters of size i, $i = 1, 2, \cdots$ interacts to form a number v_i' of new clusters of size i, $i = 1, 2, \cdots$.

For simplicity we make the following regularity assumptions.

(A1) The Markov chain is irreducible for a set $V \subset S$ and there exists a unique stationary distribution π at V.

(A2) The function $\phi(\cdot)$ is strictly positive, i.e. for all $E \in V$

$$\phi(E) > 0.$$

(A3) The total transition rate out of each state is uniformly bounded, i.e. for all $E \in V$ and some constant N

$$q(E) = \sum_{E'} q(E, E') \leq N < \infty.$$

(A4) If $\lambda(M, M') > 0$ then also $\lambda(M', M) > 0$ for all $M, M' \in N_o^n$.

Remark 2.2 (Assumption (A4)) Note that (A4) is no restriction since we may set $\psi(M', M; E) = 0$ for all $E \in N_o^n$.

The following property for $\psi(\cdot)$ plays a keyrole throughout this paper.

Definition 2.1 (Symmetry property) *The function $\psi(\cdot)$ is said to satisfy the symmetry property when $\psi(\cdot)$ is a symmetrical function in the first two arguments at V whenever $\lambda(\cdot)$ is positive, i.e. for all $E, M, M' \in N_o^n$ such that $E+M, E+M' \in V$ and both $\lambda(M, M')$ and $\lambda(M', M) > 0$:*

$$\psi(M, M'; E) = \psi(M', M; E). \tag{2.2}$$

Remark 2.3 The function $\psi(\cdot)$ is allowed to be zero and may depend on both state $E+M$ and $E+M'$. Blocking can thus be included as will be illustrated in section 3.

Remark 2.4 The function $\lambda(\cdot)$ is essential in (2.1). Without $\lambda(\cdot)$ all transition rates are symmetric in M, M' whenever the symmetry property is valid. The explicit appearance of $\lambda(\cdot)$ in (2.1) is also very natural. In queueing networks $\lambda(\cdot)$ may represent the routing probability and in chemical reactions $\lambda(\cdot)$ represents the reaction speed, but we do not make this restriction.

In accordance with [10, pp.6-7] we give the following definition of reversibility.

Definition 2.2 (Reversibility) *A stationary Markov chain is reversible at V when for all $E, E' \in V$*

$$\pi(E)q(E, E') = \pi(E')q(E', E). \tag{2.3}$$

As summing (2.3) over all E yields the standard global balance equations, a distribution $\pi(\cdot)$ satisfying the detailed balance equations (2.3) is necessarily stationary. For verifying stationarity we may thus restrict our attention to (2.3).
A vector $c = (c_1, c_2, \cdots)$ is called positive when $c_i \geq 0$ for all i and $c_i > 0$ for at least one i.

We are now able to give our main theorem.

Theorem 2.1 *Suppose that for some positive solution $c = (c_1, c_2, \cdots)$ and all $M, M' \in N_o^n$ for which there exists at least one $E \in N_o^n$ such that $E+M, E+M' \in V$:*

$$\lambda(M, M') \prod_k c_k^{M_k} = \lambda(M', M) \prod_k c_k^{M'_k}. \tag{2.4}$$

Then under the symmetry property and with B a normalizing constant, the process is reversible and the stationary distribution is given by

$$\pi(E) = B\phi(E) \prod_k c_k^{E_k} \qquad E \in V. \tag{2.5}$$

Proof We will show that (2.5) is a solution of the detailed balance equation (2.3) for all $E, E' \in V$.
Note that $q(E, E') = 0$ for $E \in V$, $E' \notin V$ by definition of V and that by assumption

$\pi(E) = 0$ for $E \notin V$. Therefore it is sufficient to show that (2.5) satisfies (2.3) for all $E, E' \in V$. To this end, let $E + M, E + M' \in V$, then from (2.1) and (2.5)

$$
\begin{aligned}
\pi(E+M)q(E+M,E+M') &= B\phi(E+M)\left(\prod_k c_k^{E_k+M_k}\right)\lambda(M,M')\frac{\psi(M,M';E)}{\phi(E+M)} \\
&= B\psi(M,M';E)\left(\prod_k c_k^{E_k}\right)\lambda(M,M')\left(\prod_k c_k^{M_k}\right), \\
\pi(E+M')q(E+M',E+M) &= B\phi(E+M')\left(\prod_k c_k^{E_k+M'_k}\right)\lambda(M',M)\frac{\psi(M',M;E)}{\phi(E+M')} \\
&= B\psi(M',M;E)\left(\prod_k c_k^{E_k}\right)\lambda(M',M)\left(\prod_k c_k^{M'_k}\right).
\end{aligned}
$$

By the symmetry property (2.2) and concentration equation (2.4), we have hereby proven that (2.5) satisfies (2.3) for E and E' of the form $E + M$ and $E + M'$ respectively. Since any transition is of this form, the proof is thus completed. □

Remark 2.5 (Concentration equation) Relation (2.4) is natural in various applications but is not necessarily fulfilled in general. For networks of queues it extends the standard reversible routing or traffic equation to more jobs moving from one node to another at the same time. In chemical reactions such as polymerization processes it is the detailed balance equation for concentrations (cf. [10], [19]).

Remark 2.6 (Blocking) Note that $\psi(\cdot)$ does not appear in the stationary distribution. For different $\psi(\cdot)$ we always find the same stationary distribution. This combined with the fact that $\psi(\cdot)$ is allowed to be zero enables us to model blocking phenomena.

Remark 2.7 From (2.5) it follows that whenever $c_k = 0$, then for all $E \in V$: $E_k = 0$. From this observation we see that assumption (A4) is very natural. To this end, consider the case where $\lambda(M, M') > 0$ and $\lambda(M', M) = 0$ then from (2.4) it follows that $c_k = 0$ for all k such that $M_k > 0$. Then from (2.5) we find that also $E_k = 0$ for all $E \in V$. The reversed implication stating that $c_k = 0$ whenever $E_k = 0$ for all $E \in V$ is not necessary. If for all $E \in V$ we find $E_k = 0$ the term for c_k drops out in (2.4) and (2.5) and we may choose an arbitrary value for c_k.

When n is finite we can also proof the reversed implication of Theorem 2.1. This is stated in the following theorem.

Theorem 2.2 *Suppose $n \in N$, then under the symmetry property the process with transition rates (2.1) is reversible iff (2.4) possesses a positive solution for c.*

Proof We only need to prove the "only if" implication, since the "if" implication is a direct consequence of Theorem 2.1. Let

$$\pi(E) = \phi(E)Q(E).$$

Then, by virtue of (2.2), the detailed balance equation becomes

$$\lambda(M,M')Q(E+M) = \lambda(M',M)Q(E+M'). \tag{2.6}$$

The state space is a countable set, which implies that also the number of combinations M, M' in the transition rates must be countable. From this it follows that we may denumerate the combinations M, M' for which $\lambda(M,M') > 0$. Let $(M,M')_i$ denote the i-th combination. With $(M,M')_i = (K,K')$ let $\lambda_i = \lambda(K,K')$ and $\lambda_i' = \lambda(K',K)$. We now are able to write (2.6) as

$$\lambda_i Q(E+M_i) = \lambda_i' Q(E+M_i').$$

From here the proof is verbally identical to the one given in [19, pp.162-163] provided that we delete all remarks about more than one ergodic class. □

In the remaining part of this section we will show that the symmetry property is essential in reversible processes.

Theorem 2.3 *Suppose there exists a positive solution c of (2.4). Then a product form distribution of the form (2.5) satisfies the detailed balance equation (2.3) iff the transition rates are of the form (2.1) with $\psi(M,M';E)$ satisfying the symmetry property.*

Proof Without loss of generality we may state that the transition rates have the form (see Remark 2.1)

$$q(E+M,E+M') = \lambda(M,M')\frac{\hat{\psi}(M,M';E)}{\phi(E+M)}.$$

Then by (2.4) we find, by substituting (2.5) in (2.3):

$$\hat{\psi}(M,M';E) = \hat{\psi}(M',M;E).$$

Since any transition must be of the stated form the proof of the "only if" implication is thus completed. The reversed implication is a direct consequence of Theorem 2.1. □

Theorem 2.4 *Suppose there exists a positive c such that (2.5) is a solution of (2.3). Then (2.4) has a positive solution c iff the transition rates in (2.3) are of the form (2.1) with $\psi(M,M';E)$ satisfying the symmetry property.*

Proof The "if" implication is proven by direct substitution of (2.5) in (2.3). As in the proof of Theorem 2.1, from (2.1) and (2.5) we find

$$\begin{aligned}
\pi(E+M)q(E+M,E+M') &= B\psi(M,M';E)\left(\prod_k c_k^{E_k}\right)\lambda(M,M')\left(\prod_k c_k^{M_k}\right),\\
\pi(E+M')q(E+M',E+M) &= B\psi(M',M;E)\left(\prod_k c_k^{E_k}\right)\lambda(M',M)\left(\prod_k c_k^{M_k'}\right).
\end{aligned}$$

By equalizing these terms and using the symmetry property (2.2) we then find (2.4). Since any transition is of the stated form, the "if" proof is completed. The "only if" implication is a direct consequence of Theorem 2.3. □

Remark 2.8 The "only if" implication of Theorem 2.2 and the "if" implication of Theorem 2.4 are not equivalent. For $n = \infty$ this is obvious. For $n < \infty$ note that in Theorem 2.4 we assume the stationary distribution to be of product form and to be a solution of (2.3) while in Theorem 2.2 we only assume that there exists a solution of (2.3).

3 Applications

This section contains two application categories of the preceding results: Queueing networks and Clustering processes. In either category both known results for the purpose of unification and illustration and novel results showing some of the possible extensions are presented. We will give simple examples still showing the key features we wish to illustrate. In each example we will show that the symmetry property is valid and therefore that the stationary distribution is of product form.

3.1 Queueing networks

Consider a closed or open queueing network with nodes $1, 2, \cdots, n$ in which all customers are of the same type. The states of the network are defined by vectors $\bar{m} = (m_1, \cdots, m_n)$ where m_i denotes the number of customers at node i.
First let us consider the standard case allowing only one change at a time. Example 3.1.1a is reported in the literature (cf. [7]) but is included so as to illustrate how the given framework applies as well as to illustrate that blocking phenomena are covered. Example 3.1.1b introduces service rates at one node to be influenced by the state at other nodes more general than has been reported in the literature (also see Remark 3.1). Example 3.1.1c extends the standard convexity condition for a product form to hold under blocking to the inclusion of holes. The examples 3.1.2 are all new as both multiple changes and blocking are possible and are direct generalizations of the examples 3.1.1. Detailed contrasting with related literature will be given later on. Most notably, discrete-time results are hereby also included.

3.1.1 Single changes and blocking

Suppose the transition rates of the network can be written as

$$q(\bar{m} + e_j, \bar{m} + e_k) = \lambda_{jk} \frac{\psi(j, k; \bar{m})}{\phi(\bar{m} + e_j)} \quad j, k = 0, 1, \cdots, n \tag{3.1}$$

with e_j, $j = 1, \cdots, n$, the n-vector with a unit in the j-th place, zeros elsewhere. By allowing $e_0 = \bar{0}$ where 0 represents the exterior, open models can be modelled also.

Let V be the set at which the queueing network is irreducible, then for all j,k such that $\bar{m}+e_j, \bar{m}+e_k \in V$, the function $\psi(\cdot)$ must be symmetric, i.e.

$$\psi(j,k;\bar{m}) = \psi(k,j;\bar{m}). \tag{3.2}$$

In this special case, (2.4) simplifies to

$$c_j \lambda_{jk} = c_k \lambda_{kj}$$

which is the standard reversible traffic equation with c_j the throughput at node j and (λ_{jk}) the routing matrix. The stationary distribution at V is given by

$$\pi(\bar{m}) = B\phi(\bar{m}) \prod_k c_k^{m_k}. \tag{3.3}$$

Remark 3.1 To the best of our knowledge, the form (3.1) has not been explicitly reported in the literature on queueing networks. In contrast, in the literature (cf. [1], [10], [18], [19]) one finds $\psi(j,k;\bar{m}) = \phi(\bar{m})$, where $\phi(\cdot)$ is strictly positive. The fact that different functions ψ and ϕ are allowed with the explicit dependence on j,k in ψ enables us to model blocking and network dependent servicing. This will be illustrated in the examples a, b and c below.

a. 1-0 blocking

Consider a closed queueing network of two nodes in which M customers are present. Let the service capacity of queue i be $f_i(x_i)$, $i=1,2$ when x_i customers are present at node i. When no more then N customers are allowed at node 2 the transition rates are given by

$$\begin{aligned} q(\bar{m}+e_1, \bar{m}+e_2) &= \begin{cases} f_1(m_1) & \text{if } m_2 < N \\ 0 & \text{if } m_2 \geq N \end{cases} \\ q(\bar{m}+e_2, \bar{m}+e_1) &= f_2(m_2). \end{aligned}$$

These transition rates have the form (3.1) which is easily seen by setting

$$\phi(\bar{v}) = \prod_{k=1}^{2} \prod_{l=1}^{v_k} \frac{1}{f_k(l)}$$

$$\psi(i,j;\bar{v}) = b(i,j;\bar{v})\phi(\bar{v})$$

$$b(i,j;\bar{v}) = \begin{cases} 1 & \text{if } i=2,\ j=1 \\ 1 & \text{if } i=1,\ j=2,\ v_2 < N \\ 0 & \text{elsewhere} \end{cases}$$

$$\lambda_{jk} = 1 \quad j,k = 1,2.$$

It is easy to see that $b(\cdot)$ and therefore $\psi(\cdot)$ is symmetric at $V = \{(m_1,m_2) : m_1 \geq 0,\ 0 \leq m_2 \leq N,\ m_1+m_2 = M\}$. This leads to the stationary distribution at V:

$$\pi(m_1,m_2) = B\prod_{k=1}^{2}\prod_{l=1}^{m_k}\frac{1}{f_k(l)}.$$

Remark 3.2 Among various others, the above product form can be found for instance in [7]. This example illustrates however that the present framework allows strict blocking, which is non-reversible or non-symmetric by nature, despite the underlying reversibility structure of the total network and the symmetry condition (3.2). More specific, observe that here we have essentially used that (3.2) is required only restricted to V but not at the boundary of V.

b. state dependent delays

Consider a closed queueing network consisting of three nodes. Let the service capacity at node i be $f_i(x_i)$ when x_i customers are present at node i. At node 3 customers are accepted with probability $b(m_3)$, while the service and routing between nodes 1 and 2 is influenced by the number of customers at node 3. This influence is reflected by the appearance of $\tilde{b}(m_3)$ in the transition rates for transitions between nodes 1 and 2. The transition rates for this network are

$$\begin{cases} q(\bar{m}+e_1,\bar{m}+e_2) &= f_1(m_1+1)\tilde{b}(m_3) \\ q(\bar{m}+e_2,\bar{m}+e_1) &= f_2(m_2+1)\tilde{b}(m_3)/2 \\ q(\bar{m}+e_2,\bar{m}+e_3) &= f_2(m_2+1)b(m_3)/2 \\ q(\bar{m}+e_3,\bar{m}+e_2) &= f_3(m_3+1). \end{cases}$$

These transition rates can be written in the form (3.1) by setting

$$\phi(\bar{v}) = \beta(\bar{v})\prod_{k=1}^{3}\prod_{l=1}^{v_k}\frac{1}{f_k(l)}$$

$$\psi(i,j;\bar{v}) = \alpha(i,j;\bar{v})\prod_{k=1}^{3}\prod_{l=1}^{v_k}\frac{1}{f_k(l)}$$

$$\beta(\bar{v}) = \gamma(v_3-1)$$

$$\alpha(1,2;\bar{v}) = \alpha(2,1;\bar{v}) = \gamma(v_3-1)\tilde{b}(v_3)$$

$$\alpha(2,3;\bar{v}) = \alpha(3,2;\bar{v}) = \gamma(v_3)$$

$$\lambda_{12} = 1,\ \lambda_{21} = \lambda_{23} = \frac{1}{2},\ \lambda_{32} = 1$$

with

$$\gamma(x) = \prod_{k=0}^{x} b(k) \ \text{ and the convention } \ \prod_{k=a}^{b} b(k) = 1 \ \text{ if } a > b.$$

It is easy to see that $\psi(\cdot)$ satisfies the symmetry property (3.2). When M customers are present in the network, the stationary distribution at $V = \{(m_1,m_2,m_3) : m_1+m_2+m_3 = M,\ m_1 \geq 0,\ m_2 \geq 0,\ m_3 \geq 0\}$ is given by

$$\pi(\bar{m}) = B\left(\prod_{k=0}^{m_3-1} b(k)\right)\left(\prod_{k=1}^{3}\prod_{l=1}^{m_k}\frac{1}{f_k(l)}\right)2^{m_2}.$$

Below we give two direct examples that can not be found in the literature.

Example 1. Suppose the service speed at node 1 and 2 is decreased when the number of customers at node 3 exceeds a certain level. Then we set

$$b(k) = \tilde{b}(k) = \begin{cases} 1 & \text{if } k \le N_3 \\ 1/2 & \text{elsewhere.} \end{cases}$$

Example 2. Note that from (A2) we must have $b(k) \neq 0$ for all k but that $\tilde{b}(k)$ may have an arbitrary value, including 0, as is illustrated here. Let node 2 produce jobs that can be further served at node 1 and 3. After completing service at node 1 or 3 the jobs return to node 2. Assume a job prefers service at node 3 and will never route to node 1 when at node 3 less than N_1 customers are present. When at node 3 more than N_1 but less than N_2 customers are present a job will not distinguish between node 1 and 3. When more than N_2 customers are present at node 3 a customer will prefer service at node 1, but in this case still some customers are being served at node 3. Now we may choose the following values for $b(\cdot)$ and $\tilde{b}(\cdot)$ for arbitrary $\alpha > 0$

$$\begin{array}{lll} b(k) = 1 & \tilde{b}(k) = 0 & \text{if } k \le N_1 \\ b(k) = \frac{1}{2} & \tilde{b}(k) = \frac{1}{2} & \text{if } N_1 < k \le N_2 \\ b(k) = \dfrac{1}{2(k-N_2)^\alpha} & \tilde{b}(k) = (k-N_2)^\alpha & \text{if } k > N_2. \end{array}$$

c. transition gaps

Now let us investigate the possibility of blocking also in the interior of V. In this case the symmetry property (3.2) will play a restrictive role. To this end, consider an open queueing network of two parallel queues. The transition rates of the network depend on the state of both queues.

Suppose the number of customers in the queues is restricted by

$$\hat{V} = \{(m_1, m_2) : 0 \le m_1 \le 9;\ 0 \le m_2 \le 10;\ m_1 + m_2 \le 15\}.$$

Let all transitions due to the arrival or departure of customers be possible except for some transitions to and from states with $m_1 = m_2 = even$.

Note that $q(\bar{m}, \bar{m}') = 0$ for $\bar{m}, \bar{m}' \in V$ implies that also $q(\bar{m}', \bar{m}) = 0$. For example, suppose that state (2,2) can only be reached from states (2,3) and (3,2), so that

$$\begin{array}{lllll} q((1,2),(2,2)) & = & q((2,2),(1,2)) & = & 0 \\ q((2,1),(2,2)) & = & q((2,2),(2,1)) & = & 0 \end{array}$$

and similarly (4,4) only from (5,4), so that

$$q((\cdot,\cdot),(4,4)) = q((4,4),(\cdot,\cdot)) = 0 \quad \text{for } (\cdot,\cdot) = (4,5),\ (3,4),\ (4,3)$$

while (6,6) cannot be reached at all, that is

$$q((\cdot,\cdot),(6,6)) = ((6,6),(\cdot,\cdot)) = 0 \quad \text{for } (\cdot,\cdot) = (7,6),\ (6,7),\ (5,6),\ (6,5).$$

Assuming all other transitions within $\hat{V}$ to be positive we thus find

$$V = \hat{V} \setminus \{(6,6)\}.$$

When both queues are single server queues with mean service time μ_i^{-1}, $i = 1,2$ and the arrival stream is Poisson λ_i, $i = 1,2$ we can write the transition rates of this network in the form (3.1) with

$$\begin{aligned}
\lambda_{i0} &= \mu_i \\
\lambda_{0i} &= \lambda_i \\
\phi(\bar{m}) &= 1 \\
\psi(i,j;\bar{m}) &= 1
\end{aligned}$$

except for

$$\begin{array}{rcccl}
\psi(1,0;(1,2)) &=& \psi(0,1;(1,2)) &=& 0 \\
\psi(0,2;(2,1)) &=& \psi(2,0;(2,1)) &=& 0 \\
\psi(0,2;(4,4)) &=& \psi(2,0;(4,4)) &=& 0 \\
\psi(1,0;(3,4)) &=& \psi(0,1;(3,4)) &=& 0 \\
\psi(0,2;(4,3)) &=& \psi(2,0;(4,3)) &=& 0 \\
\psi(1,0;(6,6)) &=& \psi(0,1;(6,6)) &=& 0 \\
\psi(0,2;(6,6)) &=& \psi(2,0;(6,6)) &=& 0 \\
\psi(1,0;(5,6)) &=& \psi(0,1;(5,6)) &=& 0 \\
\psi(0,2;(6,5)) &=& \psi(2,0;(6,5)) &=& 0.
\end{array}$$

Despite these exclusions, however, the following standard type product form distribution can now be concluded from (3.3):

$$\pi(\bar{m}) = B\left(\frac{\lambda_1}{\mu_1}\right)^{m_1}\left(\frac{\lambda_2}{\mu_2}\right)^{m_2} \qquad \bar{m} \in V.$$

Remark 3.3 (Convex state space) Note that V is not convex. Our example thus relaxes the "standard" condition of coordinate or graphical convex blocking protocols for a product form to hold (cf. [6], [7], [9], [11], [13]).

3.1.2 Multiple changes

Now consider a closed queueing network in which groups of customers can change nodes simultaneously. Suppose that the transition rate for a transition in which g_i customers leave node i, g_i' customers arrive at node i and m_i customers stay at node i, $i = 1,\cdots,n$ is given by

$$q(\bar{m}+\bar{g},\bar{m}+\bar{g}') = \lambda(\bar{g},\bar{g}')\frac{\psi(\bar{g},\bar{g}';\bar{m})}{\phi(\bar{m}+\bar{g})}$$

with $\bar{g} = (g_1,\cdots,g_n)$, $\bar{g}' = (g_1',\cdots,g_n')$, $\bar{m} = (m_1,\cdots,m_n)$ and symmetric function $\psi(\cdot)$:

$$\psi(\bar{g}, \bar{g}'; \bar{m}) = \psi(\bar{g}', \bar{g}; \bar{m}).$$

Then (2.4) becomes

$$\lambda(\bar{g}, \bar{g}') \prod_k c_k^{g_k} = \lambda(\bar{g}', \bar{g}) \prod_k c_k^{g'_k} \tag{3.4}$$

and the stationary distribution at V is given by

$$\pi(\bar{m}) = B\phi(\bar{m}) \prod_k c_k^{m_k}.$$

Analogous to the case of single changes one can interpret c_j as the throughput at node j with $\lambda(\bar{g}, \bar{g}')$ representing a routing probability. Below we will extend our preceding examples to batch service and routing.

Remark 3.4 (Fixed number of jobs) Note that in closed queueing networks $\sum_k g_k = \sum_k g'_k$. This is not assumed in our general framework as will be seen in the clustering examples.

Remark 3.5 (Reference [17]) Most recently a similar framework of queueing networks with batch services has been investigated and shown to have a product form in [17]. On the one hand, this reference is more general as it allows arbitrary rather than reversible routing as imposed by our general framework description, on the other hand it is essentially more restrictive as the explicit dependence on $\bar{g}$ and $\bar{g}'$ in $\psi(\cdot)$ is not included. Particularly, blocking is thereby excluded while in our framework blocking of particular type transitions is possible as will be illustrated hereafter.

a. 1-0 blocking

Reconsider the cyclic two-node example 3.1.1a with a finite capacity constraint of no more than N customers allowed at node 2, but now with the transition rates for multiple departures.

Assume that customers, which will be assumed identical, decide to change nodes independently. Then the probability that a group $\bar{g}$ decides to change nodes in state $\bar{m} + \bar{g}$ is given by

$$p(\bar{g}, \bar{m} + \bar{g}) = \prod_{k=1}^{2} \binom{m_k + g_k}{g_k} p_k^{g_k} (1 - p_k)^{m_k}$$

where p_k is the probability that a customer at node k decides to leave, as naturally corresponding to a discrete-time formulation.

After leaving their nodes a group $\bar{g}$ is routed to $\bar{g}'$ according to the routing probability $\lambda(\bar{g}, \bar{g}')$. Depending on the number of customers at node 2 customers entering node 2 can be blocked.

The transition rates for this process are given by

$$q(\bar{m}+\bar{g},\bar{m}+\bar{g}') = \begin{cases} \lambda(\bar{g},\bar{g}')p(\bar{g},\bar{m}+\bar{g})\dfrac{1}{g_1'!g_2'!} & \text{if } m_2+g_2' \leq N \\ 0 & \text{otherwise.} \end{cases}$$

By setting

$$\phi(\bar{v}) = \prod_{k=1}^{2} \frac{1}{v_k!}\frac{1}{p_k^{v_k}}$$

$$\psi(\bar{g},\bar{g}';\bar{v}) = b(\bar{g},\bar{g}';\bar{v}) \prod_{k=1}^{2} \frac{1}{v_k!g_k!g_k'!}\left(\frac{1-p_k}{p_k}\right)^{v_k}$$

$$b(\bar{g},\bar{g}';\bar{v}) = \begin{cases} 1 & \text{if } \; v_2+g_2' \leq N \\ 0 & \text{if } \; v_2+g_2' > N \end{cases}$$

$$\lambda(\bar{g},\bar{g}') = 1,$$

it is easily seen that $\psi(\cdot)$ satisfies the symmetry property at $V = \{(m_1,m_2) : m_1 \geq 0,\ 0 \leq m_2 \leq N,\ m_1+m_2 = M\}$. The results of section 2 then apply with c a solution of (3.4). The stationary distribution at V is given by

$$\pi(\bar{m}) = B\prod_{k=1}^{2}\frac{1}{m_k!}\left(\frac{c_k}{p_k}\right)^{m_k}.$$

Remark 3.6 (References [2] and [15]) The above system is applicable as a central server computer system with blocking and multiple departures and arrivals. As such it is closely related to a discrete-time model analyzed in [2], [15]. As most essential contrast, however, in these references the restrictive underlying assumption is made that "no more than one service request is granted" at a time so that no more than one departure is allowed at one of the nodes. No such condition is imposed in our formulation.

b. state dependent delay

Consider a closed queueing network with three nodes as in example 3.1.1b. Assume that jobs are served indepently but released at exponential times such as naturally arising from discrete-time formulations (cf. [2], [14], [15], [18]). After leaving their nodes a group $\bar{g}$ is routed to $\bar{g}'$ according to the routing probability $\lambda(\bar{g},\bar{g}')$. We assume customers can only route between node 1 and node 2 and between node 2 and node 3. Then the vectors $\bar{g},\bar{g}'$ must satisfy $g_1'+g_3' = g_2$ and $g_1+g_3 = g_2'$ (see Remark 3.4). Depending on the number of customers at node 3 customers are delayed while routing between the nodes. Customers routing between node 1 and 2 are influenced by only the number m_3 of customers that remain at station 3 with delay factor $\tilde{b}(m_3)$. Customers routing from node 2 to node 3 have full information about the total number of customers at node 3. These customers are assumed to route one by one with delay factor $b(m_3+h)$ where m_3+h is the number of customers that still resides at node 3. The routing intensity from node 3 to node 2 is not influenced. The transition rates of this process are given by

$$q(\bar{m}+\bar{g},\bar{m}+\bar{g}') = \lambda(\bar{g},\bar{g}') \left(\prod_{k=1}^{3} \frac{1}{g_k! g_k'!} \prod_{l=1}^{g_k} f_k(m_k+l)\right) \left(\tilde{b}(m_3)^{g_1+g_1'} \prod_{k=m_3+g_3}^{m_3+g_3+g_3'-1} b(k)\right).$$

By setting

$$\phi(\bar{v}) = \beta(\bar{v}) \prod_{k=1}^{3} \prod_{l=1}^{v_k} \frac{1}{f_k(l)}$$

$$\psi(\bar{g},\bar{g}';\bar{v}) = \alpha(\bar{g},\bar{g}';\bar{v}) \prod_{k=1}^{3} \frac{1}{g_k! g_k'!} \prod_{l=1}^{v_k} \frac{1}{f_k(l)}$$

$$\beta(\bar{v}) = \prod_{k=0}^{v_3-1} b(k)$$

$$\alpha(\bar{g},\bar{g}';\bar{v}) = \left(\prod_{k=0}^{v_3+g_3+g_3'-1} b(k)\right) \left(\prod_{k=0}^{g_1-1} \tilde{b}(v_3)\right) \left(\prod_{k=0}^{g_1'-1} \tilde{b}(v_3)\right)$$

we find that the stationary distribution at $V = \{(m_1, m_2, m_3) : m_1 + m_2 + m_3 = M, m_1 \geq 0, m_2 \geq 0, m_3 \geq 0\}$ is given by

$$\pi(\bar{m}) = B \left(\prod_{k=0}^{m_3-1} b(k)\right) \left(\prod_{k=1}^{3} \prod_{l=1}^{m_k} \frac{c_k}{f_k(l)}\right).$$

where c is a solution of (3.4).

Remark 3.7 (Discrete-time) The above description is in fact equivalent to a discrete-time formulation in which in a fixed time quantum more jobs can leave and enter a node.

Remark 3.8 (References [14], [17], [18]) This example can not be given in the description of [14], [17], [18]. In [14], [18] the service rate at a node depends only on the number of customers at that node while blocking is excluded. In [17] the transition rates are of the form

$$q(\bar{m}+\bar{g},\bar{m}+\bar{g}') = \lambda(\bar{g},\bar{g}') \frac{\psi(\bar{m})}{\phi(\bar{m}+\bar{g})}$$

from which it can be directly seen that the factors $\tilde{b}(m_3)^{g_1'}$ can not be modelled.

3.2 Clustering processes

In this section we discuss clustering processes in a more general way than illustrated in section 2. Suppose there exists a countable number of possible cluster types, labelled $r = 1, 2, \cdots$. The type of a clusters may represent the size of the cluster as in section 2, but it may also represent the total structure, or some particular characteristic of a cluster.

We consider only the following three types of transitions:

- two clusters may associate to form a single cluster;
- one cluster may dissociate to form two clusters;
- a cluster may change type.

For each of this reactiontypes we introduce a different function $\psi(\cdot)$.

In 3.2.1 below we reformulate the general model of section 2 in terms of clustering processes. In 3.2.2 we discuss a very special type of clustering processes, viz. polymerization processes. From these processes the relation of the concentratrion equation (2.4) becomes clear. In 3.2.3, finally, we generalize polymerization processes and investigate blocking phenomena.

3.2.1 Basic model

Let the state of the process be denoted by vectors $\bar{m} = (m_1, m_2, \cdots)$, where m_i denotes the number of clusters of type i, $i = 1, 2, \cdots$.
Suppose that the transition rates of the process have the form

$$
\begin{aligned}
q(\bar{m} + e_r + e_s, \bar{m} + e_u) &= \lambda_{rsu} \frac{\psi_1(r, s, u; \bar{m})}{\phi(\bar{m} + e_r + e_s)} \\
q(\bar{m} + e_u, \bar{m} + e_r + e_s) &= \mu_{rsu} \frac{\psi_2(r, s, u; \bar{m})}{\phi(\bar{m} + e_u)} \\
q(\bar{m} + e_r, \bar{m} + e_s) &= \kappa_{rs} \frac{\psi_3(r, s; \bar{m})}{\phi(\bar{m} + e_r)}
\end{aligned}
\tag{3.5}
$$

where $\psi_1(r, s, u; \bar{m}) = \psi_2(r, s, u; \bar{m})$ at V, and if not stated differently $V = \{\bar{m} : m_k \geq 0,\ k = 1, 2, 3, \cdots\}$. When (2.4) for this process, i.e.

$$
\begin{aligned}
c_r c_s \lambda_{rsu} &= c_u \mu_{rsu} \\
c_r \kappa_{rs} &= c_s \kappa_{sr}
\end{aligned}
\tag{3.6}
$$

possesses a positive solution c the stationary distribution at V is given by

$$\pi(\bar{m}) = B\phi(\bar{m}) \prod_k c_k^{m_k}.$$

3.2.2 Polymerization processes

The best developed applications of clustering processes are those of polymer chemistry. In these processes one considers clusters which are built up by the formation of bonds between basic chemical building blocks. The process is confined to a region of volume R and supposed to be aggregated over space, which deletes the notion of distance between clusters.
First let us give a deterministic description of the model. The concentration c_r of clusters of type r is defined as

$$c_r = \frac{m_r}{R}.$$

The reaction rate Λ_α of reaction α, defined as the expected number of occurrences of reaction α per unit time and unit volume, is a function of the concentrations. In polymerization processes it is assumed that the reaction rates are proportional to the concentrations ([3], [4], [5], [19]). For association of an r-type and an s-type to a u-type $\Lambda_{rsu} = \lambda_{rsu} c_r c_s$, for dissociation of a u-type in an r-type and an s-type $\Lambda'_{rsu} = \mu_{rsu} c_u$ and for type changing of a u-type in a v-type $\Lambda_{uv} = \kappa_{uv} c_u$.
In the deterministic description the reaction rates represent the actual number of transitions, so detailed balance in the deterministic description is stated by

$$\Lambda_{rsu} = \Lambda'_{rsu}$$

$$\Lambda_{rs} = \Lambda_{sr}$$

i.e. by (3.6).

Remark 3.9 (Concentration equation (3.6)) In contrast with queueing networks the detailed balance equations for concentrations are assumed to have a solution. The parameters λ, μ, κ are chosen such that a solution for c is possible. For clustering processes where type changing is not allowed, i.e. $\kappa_{rs} = 0$ we give an illustration of the physical way to solve the problem. First an assumption about the aggregation coefficients λ_{rsu} is imposed, based on the type of clusters considered. Then the corresponding fragmentation coefficients μ_{rsu} and the concentrations are obtained by solving the concentration equation (3.6). For polymerization processes where the type represents the number of basic units in the cluster, and therefore $\lambda_{rsu} = K_{rs}$ and $\mu_{rsu} = F_{rs}$ this can be seen in [4], [5], [16].

There are many ways to stochasticize the deterministic model. The most natural way is to assume that the transition rates are proportional to the concentrations so that

$$q(\bar{m} + e_r + e_s, \bar{m} + e_u) = \lambda_{rsu} \frac{(m_r + 1 + \delta_{rs})(m_s + 1)}{R}$$

$$q(\bar{m} + e_u, \bar{m} + e_r + e_s) = \mu_{rsu}(m_u + 1)$$

$$q(\bar{m} + e_r, \bar{m} + e_s) = \kappa_{rs}(m_r + 1)$$

where

$$\delta_{rs} = \begin{cases} 1 & \text{if } r = s \\ 0 & \text{otherwise.} \end{cases}$$

By setting

$$\phi(\bar{v}) = \prod_k \prod_{l=1}^{v_k} \frac{R}{l}$$

$$\psi_1(r,s,u;\bar{v}) = \psi_2(r,s,u;\bar{v}) = R\phi(\bar{v})$$
$$\psi_3(r,s;\bar{v}) = R\phi(\bar{v})$$

we then find the stationary distribution

$$\pi(\bar{m}) = B\prod_k \frac{(Rc_k)^{m_k}}{m_k!}.$$

The results above for transition rates and stationary distribution are wellknown and can for example be found in [10], [19].

3.2.3 Generalized polymerization processes

In applications from polymer chemistry it seems natural to assume that the concentrations can be measured. Then from the equilibrium value for the concentrations in very large systems the coefficients λ, μ, κ can be determined such that the deterministic detailed balance condition is fulfilled. From this observation it is natural to assume that the deterministic model always has a solution.
In stochasticizing the deterministic model we have much freedom. For chemical reactions we want the stochastic process to obey the laws of statistical mechanics. Therefore we want the stationary distribution to be of the form ([8], [19])

$$\pi(\bar{m}) = B\Phi(\bar{m})\prod_k \frac{(Rc_k)^{m_k}}{m_k!}. \tag{3.7}$$

The binomial factors $(m_k!)^{-1}$ represent the fact that clusters of the same type are indistinguishable. $\Phi(\bar{m})$ represents the configuration, for example the potential energy of the configuration. The binomial factors might be absorbed in $\Phi(\cdot)$, but this is not natural since $\prod_k \frac{(Rc_k)^{m_k}}{m_k!}$ on its own represents a kind of zeroth-order model, such as the model above.
Below we will investigate what underlying clustering processes indeed exhibit this form. It so turns out that different configuration factors and blocking functions can be included in the transition rates without affecting (3.7). This observation appears to be new in the literature.

From (3.7) it is clear that

$$\phi(\bar{v}) = \Phi(\bar{v})\prod_k\prod_{l=1}^{v_k}\frac{R}{l} \tag{3.8}$$

is the appropriate form for $\phi(\cdot)$. From Theorem 2.3 it follows that the transition rates for this process must be of the form (3.5). When we want the transition rates to be proportional to the number of clusters we must take

$$\begin{aligned}\psi_i(r,s,u;\bar{v}) &= b_i(r,s,u;\bar{v})R\prod_k\prod_{l=1}^{v_k}\frac{R}{l} \qquad i=1,2\\ \psi_3(r,s;\bar{v}) &= b(r,s;\bar{v})R\prod_k\prod_{l=1}^{v_k}\frac{R}{l}\end{aligned} \tag{3.9}$$

for arbitrary functions $b(\cdot)$, where $b_1(r,s,u;\bar{v}) = b_2(r,s,u;\bar{v})$ at V and $b(r,s;\bar{v})$ must be a symmetrical function of r,s at V. These functions $b(\cdot)$ can represent different types of blocking. Two examples are 1-0 blocking and configuration dependence.

Remark 3.10 The explicit form for $\psi_i(\cdot),\ i = 1,2,3$ is stated merely for elegance. With this form the transition rates remain proportional to the concentrations, so it can directly be seen that the process is a stochasticized version of a polymerization process. The explicit form is by no means necessary.

a. 1-0 blocking

We consider three cases, all for closed clustering processes. A clustering process is called closed when clusters are not allowed to leave the system. This is the same definition as in the theory of queueing networks. Note however that the total number of clusters in a closed clustering process is not constant (also see Remark 3.4).

First we assume that only N clusters are allowed, i.e. $\sum_k m_k \leq N$. This might appear in clustering processes confined to a fixed volume R when all clusters have size R/N. In type changing transitions the total number of clusters remains the same, therefore we set $b(r,s;\bar{v}) = 1$. In association transitions the total number is reduced, so also in this case we set $b_1(r,s,u;\bar{v}) = 1$. In dissociation transitions the total number of clusters is increased by 1 in each transition, therefore we set

$$b_2(r,s,u;\bar{v}) = \begin{cases} 1 & \text{if } \sum_k v_k + 1 \leq N \\ 0 & \text{elsewhere.} \end{cases}$$

At $V = \{\bar{m} : \sum_k m_k \leq N\}$ we find $b_1(\cdot) = b_2(\cdot)$ so the symmetry property is valid. The stationary distribution at V is given by (3.7).

Now assume only the number of clusters of type k to be bounded, say $m_k \leq N_k$. This might appear when the total energy of the system is bounded and the formation of clusters of type k absorbs extreme amounts of energy. In this case also type changing transitions are blocked. By setting

$$b(r,s;\bar{v}) = \begin{cases} 1 & \text{if } v_k + \delta_{rk} \leq N_k \\ 1 & \text{if } v_k + \delta_{sk} \leq N_k \\ 0 & \text{elsewhere} \end{cases}$$

$$b_1(r,s,u;\bar{v}) = \begin{cases} 1 & \text{if } v_k + \delta_{uk} \leq N_k \\ 0 & \text{elsewhere} \end{cases}$$

$$b_2(r,s,u;\bar{v}) = \begin{cases} 1 & \text{if } v_k + \delta_{kr} + \delta_{ks} \leq N_k \\ 0 & \text{elsewhere} \end{cases}$$

we find $b_1(\cdot) = b_2(\cdot)$ and $b(r,s,\bar{v}) = b(s,r;\bar{v})$ at $V = \{\bar{m} : m_k \leq N_k\}$, so the symmetry property is valid. The stationary distribution at V is given by (3.7).

In the examples above the total number of basic chemical building blocks is unbounded. In the case often considered in the physical literature on clustering processes, however, the number of building blocks is fixed (cf. [3], [12]). A total number of M identical building blocks is present and a cluster type represents the size of a cluster. In this case M different types of cluster are possible since a cluster of type M contains all building blocks. The following relation for $\bar{m}$ must be valid

$$\sum_{k=1}^{M} km_k = M.$$

Note that km_k is the total number of building blocks contained in clusters of size k. By setting

$$b(r,s;\bar{v}) = 0$$

$$b_i(r,s,u;\bar{v}) = \begin{cases} 1 & \text{if } u = r+s, \ \sum_{k=1}^{M} v_k + u = M \\ 0 & \text{otherwise} \end{cases} \qquad i = 1,2$$

we find that the stationary distribution at $V = \{\bar{m} = (m_1, \cdots, m_M) : \sum_{k=1}^{M} km_k = M\}$ is given by (3.7).

b. Configuration factors; Potentials

Besides blocking phenomena the functions $b(\cdot)$ allow the transition rates to depend on various configurations. We consider the following four possibilities

(T1) $$b_1(r,s,u;\bar{v}) = b_2(r,s,u;\bar{v}) = 1$$
$$b(r,s;\bar{v}) = 1$$

(T2) $$b_1(r,s,u;\bar{v}) = b_2(r,s,u;\bar{v}) = \Phi(\bar{v})$$
$$b(r,s;\bar{v}) = \Phi(\bar{v})$$

(T3) $$b_1(r,s,u;\bar{v}) = b_2(r,s,u;\bar{v}) = \Phi(\bar{v}+e_r+e_s)\Phi(\bar{v}+e_u)$$
$$b(r,s;\bar{v}) = \Phi(\bar{v}+e_r)\Phi(\bar{v}+e_s)$$

(T4) $$b_1(r,s,u;\bar{v}) = b_2(r,s,u;\bar{v}) = \frac{\Phi(\bar{v}+e_r+e_s)\Phi(\bar{v}+e_u)}{\Phi(\bar{v})}$$
$$b(r,s;\bar{v}) = \frac{\Phi(\bar{v}+e_r)\Phi(\bar{v}+e_s)}{\Phi(\bar{v})}.$$

From (3.8) and (3.9) we then conclude the following transition rates:

(T1) $$q(\bar{m}+\bar{g}, \bar{m}+\bar{g}') \propto \frac{1}{\Phi(\bar{m}+\bar{g})}$$

(T2) $$q(\bar{m}+\bar{g}, \bar{m}+\bar{g}') \propto \frac{\Phi(\bar{m})}{\Phi(\bar{m}+\bar{g})}$$

(T3) $$q(\bar{m}+\bar{g}, \bar{m}+\bar{g}') \propto \Phi(\bar{m}+\bar{g}')$$

(T4) $$q(\bar{m}+\bar{g}, \bar{m}+\bar{g}') \propto \frac{\Phi(\bar{m}+\bar{g}')}{\Phi(\bar{m})}.$$

If we rewrite $\pi(\bar{m})$ as

$$\pi(\bar{m}) = B \exp\left[-U(\bar{m}) - \sum_k \varsigma_k m_k\right]$$

then $U(\bar{m}) = -\log \Phi(\bar{m})$ could be regarded as configuration potential and $\varsigma_k = -\log c_k$ as site potentials (cf. [19]). This observation is valid since the process is reversible in the sense of Definition 2.2 and routing reversible in the sense of (3.6). Then with the transition rate written in terms of the configuration potential we find

(T1) $q(\bar{m}+\bar{g}, \bar{m}+\bar{g}') \propto \exp\left[U(\bar{m}+\bar{g})\right]$

(T2) $q(\bar{m}+\bar{g}, \bar{m}+\bar{g}') \propto \exp\left[U(\bar{m}+\bar{g}) - U(\bar{m})\right]$

(T3) $q(\bar{m}+\bar{g}, \bar{m}+\bar{g}') \propto \exp\left[-U(\bar{m}+\bar{g}')\right]$

(T4) $q(\bar{m}+\bar{g}, \bar{m}+\bar{g}') \propto \exp\left[U(\bar{m}) - U(\bar{m}+\bar{g}')\right].$

From this observation it seems justified to state that the (T1) type process is dual to the (T3) type process and that the (T2) type process is dual to the (T4) type process in the sense that the transitions occur due to an inverted potential law. Most importantly it leads to the following conclusion.

Conclusion 3.11 For totally different forms of the transition rates with totally different physical interpretation we find the same stationary distribution (3.7).

This observation has not been made in the literature. In contrast, one usually takes proportional transition rates as in 3.2.1. or (T2) type transition rates (cf. [3], [10], [12], [19]).

4 Expected transition rates and general remarks

In this section we derive an interpretation of the expected transition rates. Further we give some general remarks.

4.1 Expected transition rates

In the following theorem we compute the expected value of the transition rates when the system is in equilibrium.

Theorem 4.1 *Suppose that the symmetry property (2.2) is valid and that there exists a positive solution c of (2.4). Then under the stationary distribution (2.5) we have for any* $M, M' \in N_o^n$

$$\boldsymbol{E}\left[\frac{\psi(M, M'; E)}{\phi(E+M)}\right] = f(M, M', c) \prod_k c_k^{M_k}$$

with

$$f(M,M';c) = B \sum_{\{E \in N_o^n \,:\, E+M \in V\}} \psi(M,M';E) \prod_k c_k^{E_k}$$

where $f(\cdot)$ is a symmetric function of M, M', i.e.

$$f(M,M',c) = f(M',M,c)$$

whenever $\lambda(M,M') > 0$.
The expected transition rate for a transition $M \to M'$ is thus given by

$$\lambda(M,M')f(M,M',c) \prod_k c_k^{M_k}. \tag{4.1}$$

Proof Direct computation of the expectation value gives for any M, M'

$$\begin{aligned} \boldsymbol{E}\left[\frac{\psi(M,M';E)}{\phi(E+M)}\right] &= \sum_{\{E \in N_o^n \,:\, E+M \in V\}} B\psi(M,M';E) \prod_k c_k^{E_k+M_k} \\ &= \left[B \sum_{\{E \in N_o^n \,:\, E+M \in V\}} \psi(M,M';E) \prod_k c_k^{E_k}\right] \prod_k c_k^{M_k} \end{aligned}$$

It remains to prove that $f(\cdot)$ is a symmetric function of M, M' whenever $\lambda(M,M') > 0$. First note that terms with $\psi(\cdot) = 0$ do not add to the sum, and that $\psi(\cdot)$ is a symmetric function of M, M', so we only need to prove

$$\begin{aligned} &\{E \in N_o^n \,:\, E + M \in V,\ \psi(M,M';E) > 0\} = \\ &\quad = \{E \in N_o^n \,:\, E + M' \in V,\ \psi(M,M';E) > 0\} \end{aligned}$$

whenever $\lambda(M,M') > 0$ so that from (A4) it follows that also $\lambda(M',M) > 0$. To this end, note that $q(E+M, E+M') > 0$ whenever both $\lambda(M,M') > 0$ and $\psi(M,M';E) > 0$ so that the two sets are the same. This completes the proof. □

If $f(M,M',c) = 1$, the expected equilibrium rate (4.1) reduces to the basic deterministic transition rate. The explicit appearance of $f(\cdot)$ in (4.1) is a consequence of possible blocking in the stochastic model. In the basic deterministic model the notion of blocking was deleted.
From the symmetry of $f(\cdot)$ we find that the concentration equation (2.4) remains valid since $f(\cdot)$ drops out. So although the left and right hand side in (2.4) do not reflect the transition rates, the solution c remains valid.

Remark 4.1 For open queueing networks with single changes and $\psi(M,M';E) = \phi(E)$ the result stated in Theorem 4.1 is already obtained in [19]. In this case $f(M,M',c) = 1$ and we find that the equilibrium migration stream from node j to node k is given by $c_j\lambda_{jk}$. In our more general setting this result seems to be new.

4.2 General remarks

Remark 4.2 (More types of customers) The results for queueing networks readily extend to networks with more types of customers. In this case we add an extra letter to the node numbers as follows. Let $m_{i\alpha}$ denote the number of α-type customers at node i, where $\alpha \in A$, a set of types. From section 2 it immediately follows that these queueing networks can be modelled also.

Remark 4.3 (Normalizing constant) The normalizing constant B defined as

$$B^{-1} = \sum_{E \in V} \phi(E) \prod_k c_k^{E_k}$$

can be computed. In case of clustering processes a computational technique is illustrated in [10]. In general in the computation of B many problems arise depending on the structure of $\phi(E)$.

Remark 4.4 Queueing networks can be regarded as a special case of clustering processes. By setting $\lambda_{rsu} = \mu_{rsu} = 0$ we find a queueing network with single changes from the clustering process of section 3.2. This observation remains valid in queueing networks with multiple changes which can be seen as a special case of general chemical reactions. Note, however, that the interpretation in queueing networks and clustering processes is different. In queueing networks the number of customers at the nodes is described, and in clustering processes the number of clusters of different types.

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