SERIE RESEARCH MEMORANDA

NONLINEAIR REGRESSION WITH DISCRETE EXPLANATORY VARIABLES, WITH AN APPLICATION TO THE EARNINGS FUNCTION

H.J. Bierens

J. Hartog

Researchmemorandum 1988-3 Febr.'88

VRIJE UNIVERSITEIT
FACULTEIT DER ECONOMISCHE WETENSCHAPPEN
EN ECONOMETRIE
AMSTERDAM

NONLINEAR REGRESSION WITH DISCRETE EXPLANATORY VARIABLES, WITH AN APPLICATION TO THE EARNINGS FUNCTION: *)

MATHEMATICAL APPENDIX **)

by Herman J. Bierens [1] and Joop Hartog [2]

1) Department of Econometrics
Free University
P.O. Box 7161
1007 MC Amsterdam

2) Department of Economics
University of Amsterdam
Jodenbreestraat 23
1011 NH Amsterdam

*) Forthcoming in the Journal of Econometrics

**) This separate appendix contains the proofs of the theorems in the published article.

APPENDIX : Mathematical Proofs

Proof of Theorem 1. Let θ be a random drawing from F. Then by hypothesis of the theorem the distribution of the random variable $\theta'(x_1-x_2)$, where $x_1 \in X$ and $x_2 \in X$ are non-random, is atomless for $x_1 \neq x_2$. Consequently we have

$$\text{(A1)} \qquad P\{\theta'(x_1-x_2) = 0\} = \begin{cases} 1 & \text{if } x_1 = x_2 , \\ 0 & \text{if } x_1 \neq x_2 . \end{cases}$$

From (A1) and the countability of X we now conclude

$$\text{(A2)} \qquad P(\theta' x_1 = \theta' x_2 \text{ for some } (x_1,x_2) \in X \times X \text{ with } x_1 \neq x_2)$$

$$\leq \sum_{x_1 \in X,\ x_2 \in X,\ x_1 \neq x_2} P\{\theta'(x_1-x_2) = 0\} = 0$$

This proves the theorem

Q.E.D.

Proof of Theorem 4. The strong consistency results in Theorem 4 follow straightforwardly from Kolmogorov's strong law or large numbers and Theorem 2.2.5 of Bierens (1981). For proving asymptotic normality, observe that by the central limit theorem

$$\text{(A3)} \qquad \left(\frac{1}{n} \Sigma_{j=1}^{n} x_j x_j'\right) \sqrt{n}(\hat{\theta}-\theta_0) = \frac{1}{\sqrt{n}} \Sigma_{j=1}^{n} (y_j-\theta_0' x_j)x_j$$

$$\rightarrow N_k[0,\ E(y_j - \theta_0' x_j)^2 x_j x_j'] \qquad \text{in distr.}$$

and that by Kolmogorov's strong law of large numbers,

$$\text{(A4)} \qquad \frac{1}{n} \Sigma_{j=1}^{n} x_j x_j' \rightarrow E\, x_j x_j' \qquad \text{a.s.} ,$$

The asymptotic normality result follows now from Theorem 2.2.14 in Bierens (1981). A similar proof can be found in White (1980).

Q.E.D.

Proof of Theorem 5. Since θ_0 is a linear separator its components are non-zero, possibly except the components corresponding with nonvarying components of X. Therefore the functions $M_1(\theta)$ are continuously differentiable in a neighborhood of θ_0, and so is $z(x,\theta)$ for each $x \in X$. Using

Theorem 2.3.3. of Bierens (1981) it is now not hard to verify that for some compact neighborhood S_0 of θ_0 ,

(A5) $$\sup_{|z|\leq 1} \sup_{\theta \in S_0} | \psi_\ell(z|\theta) - \bar{\psi}_\ell(z|\theta)| \to 0 \quad \text{a.s.}$$

for $\ell=0,1,2,\ldots$, where similarly to (12).

(A6) $$E\bar{\psi}_{r_1} (z(x_j,\theta)|\theta)\, \bar{\psi}_{r_2}(x_j,\theta)|\theta) = \begin{cases} 1 & \text{if } r_1 = r_2 , \\ 0 & \text{if } r_1 \neq 2_2 . \end{cases}$$

Defining

(A7) $$\bar{\gamma}_\ell(\theta) = E\, y_j\, \bar{\psi}_\ell(z(x_j,\theta)|\theta)$$

it follows from (A5) that

(A8) $$\sup_{\theta \in S_0} | \hat{\gamma}_\ell(\theta) - \bar{\gamma}_\ell(\theta)| \to 0 \quad \text{a.s.}$$

Since by Theorem 4, $\hat{\theta} \to \theta_0$ a.s., the theorem under review now easily follows from (A6) and (A8) and Theorem 2.2.5 of Bierens (1981).

Q.E.D.

For proving Theorems 6 and 7 we need the following lemma's.

Lemma A1. Let u be a random variable in $\mathbb{R}$, satisfying $E|u| < \infty$ and let z be a random variable in a bounded subset Z of $\mathbb{R}$. Then $P(E(u|z) = 0) < 1$ if and only if for some $\delta > 0$, $Eue^{\tau z} \neq 0$ for all $\tau \in (-\delta,0) \cup (0,\delta)$.

Proof. Lemma A1 follows straightforwardly from the Proof of Theorem 2 of Bierens (1982).

Q.E.D.

Lemma A2. Let the conditions of Lemma A1 be satisfied. Let

(A9) $$T = \{\tau \in \mathbb{R} : Eue^{\tau z} = 0\}.$$

<u>If</u> $P(E(u|z) = 0) < 1$ <u>then</u> T <u>is countable and any bounded subset of</u> T <u>is finite</u>.

<u>Proof</u>: Let $\tau_0 \in T$. From lemma A1 it follows that there exists a $\delta > 0$ such that

(A10) $$Eue^{\tau_0 z} e^{\tau z} \neq 0 \text{ for all } \tau \in (-\delta,0) \cup (0,\delta) ,$$

hence

(A11) $$Eue^{\tau z} \neq 0 \text{ for all } \tau \in (-\delta+\tau_0,\tau_0) \cup (\tau_0,\tau_0+\delta) .$$

Obviously (A11) implies that for every $\tau_0 \in T$

(A12) $$\inf_{\tau \in T, \tau \neq \tau_0} |\tau-\tau_0| > 0 ,$$

which in its turn implies that Lemma A2 holds.

Q.E.D.

<u>Proof of Theorem 6</u>. First we note that we may replace $\hat{z}(x_j,\theta^*)$ in (14), (15) and (17) by $\theta^{*\prime}x_j$. However, using $\hat{z}$ has the advantage that τ then becomes independent of the scale of $\theta^{*\prime}x_j$.

Now let

(A13) $$\bar{z}(x_j,\theta^*) = \frac{2\,\theta^{*\prime}x_j - \max_{x \in X} \theta^{*\prime}x - \min_{x \in X} \theta^{*\prime}x}{\max_{x \in X} \theta^{*\prime}x - \min_{x \in X} \theta^{*\prime}x} .$$

Since X is finite and contains only points with positive probability mass, we have

(A14) $$\lim_{n\to\infty} P(\{x_1,\ldots,x_n\} \supset X) = 1 ,$$

hence

(A15) $$\lim_{n\to\infty} P\{\hat{z}(x_j,\theta^*) = \bar{z}(x_j,\theta^*) \text{ for } j=1,\ldots,n\} = 1 .$$

Therefore we may replace $\hat{z}$ by $\bar{z}$ without loss of generality.

Assume that (18) holds. Then

(A16) $$E(y_j|x_j) = g(x_j) = \Sigma_{\ell=0}^{m-1} \gamma_\ell(\theta_0)\ \psi_\ell(z(x_j,\theta_0)|\theta_0) \quad \text{a.s.}$$

so that

(A17) $$(1/\sqrt{n})\ \Sigma_{j=1}^n\ (y - \hat{g}_m(x\ |\hat{\theta}))e^{\tau\bar{z}(x_j,\theta^*)}$$

$$= (1/\sqrt{n})\ \Sigma_{j=1}^n(u_j+g(x_j)-\hat{g}_m(x_j|\theta_0)+\hat{g}_m(x_j|\theta_0)-\hat{g}_m(x_j|\hat{\theta}))e^{\tau\bar{z}(x_j,\theta^*)}$$

$$= (1/\sqrt{n})\ \Sigma_{j=1}^n\ u_j\ e^{\tau\bar{z}(x_j,\theta^*)}$$

$$- (1/\sqrt{n})\ \Sigma_{j=1}^n\Sigma_{\ell=0}^{m-1}\ (\hat{\gamma}_\ell(\theta_0) - \gamma_\ell(\theta_0))\psi_\ell(z(x_j,\theta_0)|\theta_0)e^{\tau\bar{z}(x_j,\theta^*)}$$

$$- (1/\sqrt{n})\ \Sigma_{j=1}^n\ (\hat{g}_m(x_j|\hat{\theta})-\hat{g}_m(x_j|\theta_0))e^{\tau\bar{z}(x_j,\theta^*)}$$

$$= \tilde{c}_1(\tau,\theta^*) - \tilde{c}_2(\tau,\theta^*) - \tilde{c}_3(\tau,\theta^*)\ ,\ \text{say.}$$

Observe from (9) that for $\ell=0,1,2,\ldots.$

(A18) $$\hat{\gamma}_\ell(\theta_0) - \gamma_\ell(\theta_0) = \frac{1}{n}\ \Sigma_{j=1}^n\ u_j\ \psi_\ell(z(x_j,\theta_0)|\theta_0)\ ,$$

hence

(A19) $$\tilde{c}_2(\tau,\theta^*) = \Sigma_{\ell=0}^{m-1}\ (\frac{1}{n}\ \Sigma_{j=1}^n\ u_j\ \psi_\ell(z(x_j,\theta_0)|\theta_0)$$

$$.\ (\frac{1}{n}\ \Sigma_{j=1}^n\psi_\ell(z(x_j,\theta_0)|\theta_0)e^{\tau\bar{z}(x_j,\theta^*)}\ .$$

Denoting

(A20) $$\tilde{\tilde{c}}_2(\tau,\theta^*) =$$

$$\Sigma_{\ell=0}^{m-1}\ (\frac{1}{\sqrt{n}}\ \Sigma_{j=1}^n\ u_j\ \bar{\psi}_\ell(z(x_j,\theta_0)|\theta_0)E(\bar{\psi}_\ell(z(x_j,\theta_0)|\theta_0)e^{\tau\bar{z}(x_j,\theta^*)}),$$

where $\bar{\psi}_\ell(x|\theta_0)$ is the probability limit of $\psi_\ell(z|\theta_0)$, it is easy to verify that

(A21) $$\text{plim}_{n\to\infty}\ \{\tilde{\tilde{c}}_2(\tau,\theta^*) - \tilde{c}_2(\tau,\theta^*)\} = 0.$$

Next, observe that by the mean value theorem there exists a random vector $\tilde{\theta}(\tau,\theta^*)$ satisfying

(A22) $$||\tilde{\theta}(\tau,\theta^*) - \theta_0\ || \le ||\ \hat{\theta} - \theta_0\ || \qquad \text{a.s.}$$

and

(A23) $$\tilde{c}_3(\tau,\theta^*) = \frac{1}{\sqrt{n}}\,\Sigma_{j=1}^{n}(\hat{g}_m(x_j|\hat{\theta}) - \hat{g}_m(x_j|\theta_0))e^{\tau\bar{z}(x_j,\theta^*)}$$

$$= \sqrt{n}(\hat{\theta}-\theta_0)'\,\frac{1}{n}\,\Sigma_{j=1}^{n}(\partial/\partial\theta')\hat{g}_m(x_j|\tilde{\theta}(\tau,\theta^*))e^{\tau\bar{z}(x_j,\theta^*)} .$$

Moreover, from (A22) it follows

(A24) $$\underset{n\to\infty}{\text{plim}}\ \frac{1}{n}\,\Sigma_{j=1}^{n}\ (\partial/\partial\theta')\hat{g}_m(x_j|\tilde{\theta}(\tau,\theta^*))e^{\tau\bar{z}(x_j,\theta^*)}$$

$$= \underset{n\to\infty}{\text{plim}}\ \frac{1}{n}\ \Sigma_{j=1}^{n}\ (\partial/\partial\theta')\hat{g}_m(x_j|\hat{\theta})e^{\tau\bar{z}(x_j,\theta^*)}$$

$$= \underset{n\to\infty}{\text{plim}}\ \frac{1}{n}\,\Sigma_{j=1}^{n}\ (\partial/\partial\theta')\hat{g}_m(x_j|\theta_0)e^{\tau\bar{z}(x_j,\theta^*)}$$

$$= \bar{\xi}_m(\tau|\theta^*),\ \text{say.}$$

Denoting

(A25) $$\tilde{\tilde{c}}_3(\tau,\theta^*) = \sqrt{n}(\hat{\theta}-\theta_0)'\ \bar{\xi}_m(\tau|\theta^*)$$

we thus have

(A26) $$\underset{n\to\infty}{\text{plim}}\ \{\tilde{c}_3(\tau,\theta^*) - \bar{\tilde{c}}_3(\tau,\theta^*)\} = 0.$$

Furthermore, observe that

(A27) $$\sqrt{n}(\hat{\theta}-\theta_0) = (\frac{1}{n}\ \ \Sigma_{j=1}^{n}\ x_jx_j')^{-1}\ \ \frac{1}{\sqrt{n}}\,\Sigma_{j=1}^{n}\ x_j(y_j-x_j'\theta_0)$$

and that by (A4),

(A28) $$\underset{n\to\infty}{\text{plim}}\ \{\sqrt{n}(\hat{\theta}-\theta_0) - (Ex_jx_j')^{-1}\ \ \frac{1}{\sqrt{n}}\,\Sigma_{j=1}^{n}\ x_j(y_j-x_j'\theta_0)\} = 0 .$$

Thus denoting

(A29) $$\tilde{\bar{\tilde{c}}}_3(\tau,\theta^*) = \bar{\xi}_m(\tau,\theta^*)'(Ex_jx_j')^{-1}\ \ \frac{1}{\sqrt{n}}\,\Sigma_{j=1}^{n}\ x_j(y_j-x_j'\theta_0)$$

we have

(A30) $$\underset{n\to\infty}{\text{plim}}\ \{\bar{\tilde{c}}_3(\tau,\theta^*) - \tilde{\bar{\tilde{c}}}_3(\tau,\theta^*)\} = 0 .$$

From (A15), (A17), (A21), (A26) and (A30) we now obtain

(A31) $$\text{plim}\ \{\frac{1}{\sqrt{n}}\ \Sigma_{j=1}^{n}(y_j-\hat{g}_m(x_j|\hat{\theta}))e^{\tau\bar{z}(x_j,\theta^*)} - \hat{d}(\tau|\theta^*)\} = 0\ ,$$

where

(A32) $$\hat{d}(\tau|\theta^*) = \tilde{c}_1(\tau,\theta^*) - \tilde{\tilde{c}}_2(\tau,\theta^*) - \tilde{\tilde{\tilde{c}}}_3(\tau,\theta^*)$$

$$= \frac{1}{\sqrt{n}}\ \Sigma_{j=1}^{n}\ u_j\ e^{\tau\bar{z}(x_j,\theta^*)}$$

$$- \frac{1}{\sqrt{n}}\ \Sigma_{j=1}^{n}\ u_j\Sigma_{\ell=0}^{m-1}\ \bar{\psi}_\ell(z(x_j,\theta_0)|\theta_0).E(\bar{\psi}_\ell(z(x_j,\theta_0)|\theta_0)e^{\tau\bar{z}(x_j,\theta^*)}\ )$$

$$- \bar{\xi}_m(\tau|\theta^*)'(E\ x_jx_j')^{-1}\ \ \frac{1}{\sqrt{n}}\ \Sigma_{j=1}^{n}x\ (y_j-x_j'\theta_0)$$

$$= \ \frac{1}{\sqrt{n}}\ \Sigma_{j=1}^{n}\ u_j\ \bar{\rho}_{j,n}(\tau|\theta^*) - \bar{\xi}_m(\tau|\theta^*)'(Ex_j\ x_j')\frac{1}{\sqrt{n}}\ \ \Sigma_{j=1}^{n}\ x_j(y_j-x_j'\theta)$$

with

(A33) $$\bar{\rho}_{j,m}(\tau|\theta^*) = e^{\tau\bar{z}(x_j,\theta^*)} - \Sigma_{\ell=0}^{m-1}\ \bar{\psi}_\ell(z(x_j,\theta_0)|\theta_0)E(\bar{\psi}_\ell(z(x_j,\theta_0)|\theta_0)e^{\tau\bar{z}(x_j,\theta^*)}\ )$$

Realizing that the terms in (A32) are i.i.d. with zero mean and variance

(A34) $$\bar{s}_j^2(\tau|\theta^*) = E\{u_j\bar{\rho}_{j,m}(\tau|\theta^*) - \bar{\xi}_m(\tau|\theta^*)'(Ex_jx_j')^{-1}x_j(y_j-x_j'\theta_0)\}^2$$

$$= E\{u_j^2\ \bar{\rho}_{j,m}^2(\tau|\theta^*)\} - 2\ E\{u_j\bar{\rho}_{j,m}(\tau|\theta^*)\bar{\xi}_m(\tau|\theta^*)'(Ex_jx_j')^{-1}x_j(y_j-x_j'\theta_0)\}$$

$$+ \bar{\xi}_m(\tau|\theta^*)'(Ex_jx_j')^{-1}(E(y_j-\theta_0'x_j)^2x_jx_j')(Ex_jx_j')^{-1}\ \bar{\xi}_m(\tau|\theta^*)$$

$$= E\ u_j^2\ \bar{\rho}_{j,m}^2(\tau|\theta^*)$$

$$- 2\ E\ u_j(y_j-x_j'\theta_0)\ \bar{\rho}_{j,m}(\tau|\theta^*)\ x_j'(Ex_jx_j')^{-1}\ \bar{\xi}_m(\tau|\theta^*)$$

$$+ \bar{\xi}_m(\tau|\theta^*)'\ \Omega\ \bar{\xi}_m(\tau|\theta^*)\ ,$$

we have by the central limit theorem

(A35) $$\hat{d}_m(\tau|\theta^*) \to N(0,\bar{s}_m^2(\tau|\theta^*))\ \ \text{in distr.}$$

We leave it to the reader to verify that

(A36) $$\hat{s}_m^2(\tau|\theta^*) \to \bar{s}_m^2(\tau|\theta^*)\ \ \text{a.s.}$$

and that $\bar{s}_m^2(\tau|\theta^*) > 0$ for $\tau \neq 0$. Combining (A31), (A35) and (A36), part

(19) of Theorem 6 follows.

Next, assume that (18) fails to hold. Then by (A15) ,

(A37) $$\frac{1}{n}\Sigma_{j=1}^{n}(y_j - \hat{g}_m(x_j|\hat{\theta}))e^{\tau\hat{z}(x_j,\theta^*)}$$
$$\to E(y_j - \Sigma_{\ell=0}^{m-1}\gamma_\ell(\theta_0)\bar{\psi}_\ell(z(x_j,\theta_0)|\theta_0))e^{\tau\bar{z}(x_j,\theta^*)} \quad \text{a.s.}$$

Moreover, it is not hard to verify that also now $\hat{s}_m^2(\tau|\theta^*)$ converges a.s. to a limit $\bar{s}_*^2(\tau|\theta^*)$, say, which is positive for $\tau\neq 0$. Thus part (20a) follows straightforwardly from (A37) and Lemma's A1 and A2.

Finally, the conclusion that we may substitute $\hat{\theta}$ for θ^* follows from the fact that by Theorem 4

(A38) $$\text{plim}\ \{\frac{1}{\sqrt{n}}\Sigma_{j=1}^{n}(y_j-\hat{g}_m(x_j,\hat{\theta}))e^{\tau\bar{z}(x_j,\hat{\theta})}$$
$$-\frac{1}{\sqrt{n}}\Sigma_{j=1}^{n}(y_j-\hat{g}_m(x_j,\hat{\theta}))e^{\tau\bar{z}(x_j,\theta_0)}\ \} = 0$$

provided (18) is satisfied. Proving (A38) is not too hard and therefore left to the reader.

Q.E.D.

Proof of Theorem 7. The result (19) is equivalent with

(A39) $$\lim_{n\to\infty} E\, e^{i\,t\hat{\eta}_m(\tau|\theta^*)} = e^{-\frac{1}{2}t^2} \qquad \text{for every } t \in \mathbb{R}$$

If τ and θ^* are random and independent from the data-generating process then we have similarly,

(A40) $$E(e^{i\,t\hat{\eta}_m(\tau|\theta^*)}|\tau,\theta^*) \to e^{-\frac{1}{2}t^2} \quad \text{a.s.}$$

Hence by bounded convergence,

(A41) $$E\, e^{i\,t\hat{\eta}_m(\tau|\theta^*)} = E[E(e^{i\,t\,\hat{\eta}_m(\tau|\theta^*)}|\tau,\theta^*)] \to e^{-\frac{1}{2}t^2}$$

which proves that (19) carries over if θ^* and τ are random.

Now suppose that (18) fails to hold. Lemma A2 implies that (20a) hold for $\tau \in R\backslash T$, where T is a countable subset of $\mathbb{R}$. But since τ is now continuously distributed we have

(A42) $$P(\tau \in \mathbb{R}\backslash T) = 1.$$

Moreover, Theorem 1 implies that θ^* is a.s. a linear separator. Therefore (20a) also holds for the random τ and θ^* involved.

Q.E.D.

Proof of Theorem 8. From the mean value theorem it follows

(A43) $$\hat{\gamma}_\ell(\hat{\theta}) - \hat{\gamma}_\ell(\theta_0) = [(\partial/\partial\theta)\hat{\gamma}_\ell(\tilde{\theta})]\,(\hat{\theta}-\theta_0),$$

where $\tilde{\theta}$ is a mean value satisfying $||\tilde{\theta}-\theta_0|| \le ||\hat{\theta}-\theta_0||$. From this result we see

(A44) $$\text{plim}\left\{\sqrt{n}\left[\begin{bmatrix}\hat{\gamma}_0(\hat{\theta})\\ \vdots \\ \hat{\gamma}_{m-1}(\hat{\theta})\end{bmatrix} - \begin{bmatrix}\hat{\gamma}_0(\theta_0)\\ \vdots \\ \hat{\gamma}_{m-1}(\theta_0)\end{bmatrix}\right] - \bar{\Gamma}_m\sqrt{n}(\hat{\theta}-\theta_0)\right\} = 0,$$

where

(A45) $$\bar{\Gamma}_m = \text{plim}\,\hat{\Gamma}_m .$$

Moreover, from (9) and (10) it follows that

(A46) $$\hat{\gamma}_\ell(\theta_0) - \gamma_\ell(\theta_0) = \frac{1}{n}\Sigma_{j=1}^n u_j\psi_\ell(z(x_j,\theta_0)|\theta_0) .$$

Combining (A44) and (A46) and using (A28) we see that

(A47) $$\text{plim}\left[\sqrt{n}\left[\begin{bmatrix}\hat{\gamma}_0(\hat{\theta})\\ \vdots \\ \hat{\gamma}_{m-1}(\hat{\theta})\end{bmatrix} - \begin{bmatrix}\gamma_0(\theta_0)\\ \vdots \\ \gamma_{m-1}(\theta_0)\end{bmatrix}\right] - \{\bar{\Gamma}_m(Ex_jx_j')^{-1}\frac{1}{\sqrt{n}}\Sigma_{j=1}^n x_j(y_j-x_j'\theta_0) + \frac{1}{\sqrt{n}}\Sigma_{j=1}^n u_j\Psi_{j,m}\}\right] = 0$$

where

(A48) $$\Psi_{j,m} = \begin{bmatrix}\psi_0(z(x_j,\theta_0)|\theta_0)\\ \vdots \\ \psi_{m-1}(z(x_j,\theta_0)|\theta_0)\end{bmatrix}$$

But the random vectors

(A49) $$d_j = \bar{\Gamma}_m (Ex_jx_j')^{-1} x_j(y_j - x_j'\theta_0) + u_j \Psi_{j,m}$$

are indepenent with zero mean vector and variance matrix

(A50) $$E\, d_jd_j' = \bar{\Gamma}_m \Omega \bar{\Gamma}_m' + \bar{\Gamma}_m (Ex_jx_j')^{-1} E(u_j^2 x_j \Psi_{j,m}') + E(u_j^2 \Psi_{j,m} x_j')(Ex_jx_j')^{-1} \bar{\Gamma}_m' + E(u_j^2 \Psi_{j,m} \Psi_{j,m}') .$$

Denoting

(A51) $$\bar{\Delta}_m = \lim_{n\to\infty} \frac{1}{n} \Sigma_{j=1}^n E\, u_j^2 \Psi_{j,m} \Psi_{j,m}' ,$$

(A52) $$\bar{\Sigma}_m = \lim_{n\to\infty} \frac{1}{n} \Sigma_{j=1}^n (u_j^2 \Psi_{j,m} x_j')(Ex_jx_j')^{-1}$$

(A53) $$\Lambda_m = \bar{\Gamma}_m \Omega \bar{\Gamma}_m + \bar{\Sigma}_m \bar{\Gamma}_m + \bar{\Gamma}_m \bar{\Sigma}_m' + \bar{\Delta}_m ,$$

we thus have by the central limit theorem

(A54) $$\frac{1}{\sqrt{n}} \Sigma_{j=1}^n d_j \to N_m (0, \Lambda_m) \text{ in distr.}$$

Combining (A47) and (A54), the first part of Theorem 8 follows.

We leave it to the reader to verify the second part.

Q.E.D.

<u>Proof of Theorem 9</u>: Similarly to the proof of Theorem 7.