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ASYMPTOTICALLY EXPONENTIAL EXPANSION FOR THE WAITING TIME PROBABILITY IN THE SINGLE SERVER QUEUE WITH BATCH ARRIVALS

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Asymptotically exponential expansion for the waiting time probability in the single server queue with batch arrivals.

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Abstract. This paper deals with the single server queue with batch arrivals. We show that under suitable conditions the waiting time distribution of an individual customer has an asymptotically exponential expansion. For the practically important case of phase-type distributions for the interarrival time and the service time a computationally useful characterization of the amplitude factor and the decay parameter is given.

Introduction

Queueing models with batch arrivals are important in many practical applications but have received relatively little attention in the literature. An exact analysis for the waiting time distribution of an arbitrary customer in the single server $GI^X/G/1$ queue with batch arrivals is given in Cohen [1976]. For the particular case of Poissonian arrivals and Erlangian services a simple computational method for the waiting time probabilities was given in Eikeboom and Tijms [1987], cf. also Chaudhry and Templeton [1983].

For computational purposes an asymptotic expansion for the waiting time distribution may be very useful. In this paper we show for the batch arrival $GI^X/G/1$ queues that under suitable conditions the waiting time distribution has an asymptotically exponential expansion. For the important case of phase-type distributions for the interarrival time and the service time we are able to give a computationally useful characterization of the coefficients of the amplitude factor and the decay parameter of the asymptotically exponential expansion. Our results generalize the basic results for the single arrival GI/G/1 queue that were obtained in Feller [1971] and Iglehart [1972] by random walk arguments, cf. also the alternative approach in Takahashi [1981]. Our analysis heavily relies on the basic results in Iglehart [1972].

The organization of this paper is as follows. The model is defined and described in section 1. In section 2 we derive the asymptotically exponential expansion for the waiting time distribution in the $GI^X/G/1$ system when the batch size distribution is assumed to be non-arithmetic. In section 3 we give a computationally useful characterization of the coefficients of this expansion for the case of phase-type distributions. In appendix A the results derived for the non-arithmetic case are extended to the case of arithmetic batch size distributions.

1. The model.

In this paper we consider a single server $GI^X/G/1$ queueing system with batch arrival and individual services. The inter-arrival times of the successive batches are independent and identically distributed variables having distribution function A(.) with A(0)=0, average arrival rate $\lambda>0$, and finite variance. The service times of the customers are independent and identically distributed variables having distribution function B(.) with B(0)=0 and mean $\mu<\infty$. Define

$$(1.1)\quad \hat{B}(s) = {}_0\int^\infty e^{-st}dB(t),$$

as the Laplace-Stieltjes transform of B(.). The sizes of the successive batches are independent random variables having a common probability distribution $\{g_k,\ k=0,1,\ldots\}$ with $g_0=0$. It is assumed that the batch sizes have a finite mean γ. The generating function of the batch size distribution is denoted by

$$(1.2)\quad G(s) = \sum_{k=1}^{\infty} g_k s^k$$

The three families of interarrival times, service times and batch sizes are assumed to be independent. Customers belonging to different batches are served in order of arrival while customers belonging to the same batch are served in random order independently of their service times. Also it is assumed for the offered load that

$$(1.3)\quad \rho = \lambda\mu\gamma < 1.$$

Let us number the customer in order of the commencement of their services. Denote the waiting time of the k-th customer by W_k. We are interested in the waiting time distribution of an arbitrary customer. Under the assumption that the batch size distribution is non-arithmetic, we can define the limiting distribution function

$$(1.4)\quad W(x) = \lim_{n\to\infty} \Pr(W_n\leq x), \qquad x\geq 0.$$

It is shown in Cohen [1976] that this limit exists and represents a proper distribution function. For the case of an arithmetic batch size distribution, it is shown in appendix A that the results, obtained for the non-arithmetic case, remain valid with a proper definition of the waiting time distribution of an arbitrary customer.

2. The individual waiting time.

In this section we analyse the waiting time distribution W(.) of an arbitrary customer belonging to a batch arriving when the system has reached statistical equilibrium. The anaysis uses basic results obtained in Cohen [1976], Burke [1975] and Iglehart [1972]. Throughout this section it is assumed that the batch size distribution is non-arithmetic, that is, there exists no integer $d>1$ with $\Sigma_{n=0}^{\infty} g_{nd}=1$.

The waiting time of an arbitrary customer, called a tagged customer, can be considered as the sum of two independent components, namely the waiting time of the customer who is served as first from the batch of the tagged customer and the delay caused by the customers from the batch of the tagged customer who are served before him. To analyse the waiting time distribution of the customer who is served as first from an arbitrary batch we consider a batch as one entity, called a super customer. In this way we create a standard GI/G/1 queueing system with single arrivals.In this queueing system the interarrival time distribution is A(.) and the service time distribution of the super customer is given by

$$(2.1)\quad B^{SC}(x) = \sum_{n=1}^{\infty} g_n B^{n*}(x),$$

where $B^{n*}(.)$ denotes the n-fold convolution of B(.) with itself. It is readily seen that the Laplace-Stieltjes transform of $B^{SC}(.)$ is given by $G(\hat{B}(.))$.Let W_n^{SC} denote the waiting time of the n-th super customer. By its very definition W_n^{SC} is the waiting time of the customer who is the first one served from the n-th batch. The waiting times of the super customers have a limiting distribution function to be denoted by

$$(2.2)\quad W^{SC}(x) = \lim_{n\to\infty} \Pr\{W_n^{SC}\leq x\}.$$

It is important to note that this result does not require the assumption that the batch size distribution is non-arithmetic.

Let U_n denote the (waiting) time elapsed between the beginnings of the services of respectively the customer who is served as first from the batch to which the n-th customer belongs, and the n-th customer. Because the batch size distribution is non-arithmetic we may define

(2.3) $W_B(x) = \lim_{n\to\infty} \Pr(U_n \le x)$.

Then the waiting time distribution W(.) of an arbitrary customer is given by (cf. also Cohen [1976])

(2.4) $W(x) = {}_0\int^x W^{SC}(x-y)dW_B(y), \quad x\geq 0.$

Since the batch size distribution is non-arithmetic it is known from renewal theory (cf. Burke [1975]) that, for j=0,1,...,

$$\lim_{n\to\infty} \Pr\{\text{n-th customer belongs to a batch with size j}\} = \frac{jg_j}{\gamma}$$

Hence

(2.5) $W_B(x) = \sum_{k=1}^{\infty} B^{(k-1)*}(x) \cdot \frac{1}{\gamma} \sum_{j=k}^{\infty} g_j$

with $B^{0*}(x)$ the degenerate distribution with mass 1 at x=0. We now make some mild assumptions to prove that the tail of 1-W(x) decreases exponentially fast to zero.

Assumption 2.1. For some $\alpha>0$,

(2.6) $1-B(x) = o(e^{-\alpha x}), \quad x\to\infty,$

and for some $0<\beta<1$,

(2.7) $g_k = o(\beta^k), \quad k\to\infty.$

Let $\alpha_0 = \sup\{\alpha>0 | \lim_{x\to\infty}(1-B(x))e^{\alpha x}=0\}$. Then the Laplace-Stieltjes transform of B(.) is analytic for $s\in C$ with $Re(s)>-\alpha_0$. Formula (2.7) implies that the generating function of $\{g_k, k=0,1,...\}$ has a radius of convergence R>1. In addition to assumption 2.1. we need the following technical assumption.

Assumption 2.2. Let R be the radius of convergence of the generating function of $\{g_k, k=0,1,...\}$, that is (see (1.2))

$$R = \sup\{s\in R | G(s)<\infty\},$$

and let (see (1.11)),

$$\Psi = \sup\{s<\alpha_0 | \hat{B}(-s)<R\}.$$

Then,

(2.8) $\lim_{s\uparrow\Psi} \; G(\hat{B}(-s)) \; [{}_0\int^\infty e^{-sy} dA(y)] \geq 1,$

where the left hand side may be infinite. If in (2.8) the equality sign holds we also have

(2.9) $\lim_{s\uparrow\Psi} \; \frac{\hat{B}(-\Psi)-\hat{B}(-s)}{\Psi-s} \sum_{k=1}^{\infty} \; kg_k \; [\hat{B}(-s)]^{k-1} < \infty.$

The assumptions 2.1. and 2.2. are satisfied in most cases of practical interest for example for the class of phase-type service time distributions and geometric batch size distributions (cf. theorem 3.3.). In appendix B we prove that the assumptions 2.1. and 2.2. are equivalent to the conditions under which Iglehart [1972] has given the asymptotically exponential expansion of the waiting time distribution in the standard GI/G/1 queue with single arrivals. Thus by the results in Iglehart [1972] we have the following lemma.

Lemma 2.1. The waiting time distribution $W^{sc}(x)$ of an arbitrary super customer satisfies

(2.10) $\lim_{x\to\infty} \; (1-W^{sc}(x))e^{\Theta x} = \gamma_\Theta,$

where Θ is the unique positive solution to

(2.11) $G(\hat{B}(-\Theta))\,{}_0\int^\infty e^{-\Theta x} dA(x) = 1,$

and γ_Θ is a finite, positive constant given by

$$\gamma_\Theta = \frac{1-E(e^{-\Theta I})}{\Theta\mu_\Theta E(N^{sc})} .$$

Here I denotes the length of an idle period in the standard GI/G/1 queue with interarrival time distribution A(.) and service time distribution $B^{sc}(.)$, and N^{sc} denotes the number of super customers arriving during one busy cycle in this GI/G/1 system. Further,

$$\mu_\Theta = -G(\hat{B}(-\Theta))\,{}_0\int^\infty xe^{-\Theta x}dA(x)$$

$$+ \lim_{s\uparrow\Theta} \frac{\hat{B}(-\Theta)-\hat{B}(-\sigma)}{\Theta-s} \quad [\sum_{k=1}^{\infty} kg_k(\hat{B}(-\Theta))^k] \quad . \; {}_0\int^\infty \; e^{-\Theta x}dA(x).$$

With this lemma we are able to prove the following theorem for systems with batch arrivals, where the batch size distribution is non-arithmetic.

Theorem 2.2. Under the assumptions 2.1. and 2.2. the limiting waiting time distribution W(.) of an individual customer satisfies

$$\lim_{x\to\infty} (1-W(x))e^{\Theta x} = \delta_\Theta,$$

where δ_Θ is a finite positive constant defined by

$$\delta_\Theta = \gamma_\Theta \cdot \frac{1-G(\hat{B}(-\Theta))}{\gamma(1-\hat{B}(-\Theta))},$$

with Θ and γ_Θ as defined in lemma 2.1.

Proof. By the definition of $W_B(x)$ (see (2.3)), (2.5) and the assumptions 2.1. and 2.2. we find

$$(2.12) \quad {}_0\int^\infty e^{\Theta x}dW_B(x) = \frac{1-G(\hat{B}(-\Theta))}{\gamma(1-\hat{B}(-\Theta))} < \infty,$$

hence $\delta_\Theta = {}_0\int^\infty \gamma_\Theta e^{\Theta x} dW_B(x)$. Next we find from (2.4) that

$$1-W(x) = {}_0\int^\infty (1-W^{sc}(x-y))dW_B(y) \quad ,$$

and so,

$$(2.13) \quad e^{\Theta x}(1-W(x)) - \delta_\Theta = {}_0\int^\infty [e^{\Theta(x-y)}(1-W^{sc}(x-y))-\gamma_\Theta]e^{\Theta x}dW_B(x).$$

We have to show that for every $\varepsilon>0$ a $K>0$ exists with

$$(2.14) \quad |e^{\Theta x}(1-W(x))-\delta_\Theta| < \varepsilon \quad \text{for } x>K.$$

Let such a $\varepsilon>0$ be given. Then by (2.10) and (2.12) there exists a K_1 which satisfies

$$|(1-W^{SC}(x))e^{\Theta x}-\gamma_\Theta| < \frac{\varepsilon}{2} \quad \{{}_0\int^\infty e^{\Theta y}dW_B(y)\}^{-1}, \quad x\geq K_1.$$

Let $M=\gamma_\theta+\sup\{e^{\theta x}(1-W^{SC}(x))|x\in(-\infty,K_1)\}$. By (2.12) it follows that a K_2 exists with

$$_{K_2}\int^\infty e^{\Theta x}dW_B(x) < \frac{\varepsilon}{2M} .$$

Now, by splitting the integration interval in (2.13) into two disjoint intervals $[0,K_2)$ and $[K_2,\infty)$, we easily obtain that (2.14) holds for $K=K_1+K_2$. This proves the theorem. Q.E.D.

In the next section we discuss how to compute the constants as mentioned in Lemma 2.1. and Theorem 2.2. for some special models.

3. Special models.

The main difficulty in computing the constant γ_θ (see lemma 2.1.) is to find the Laplace-Stieltjes transform of the distribution of the idle period. This difficulty vanishes for the case of a Poissonian arrival stream of the batches, whatever the batch size distribution and service time distributions are. For $K_n^X/K_m/1$ queueing systems with batch arrivals the constant γ_θ can be computed by using results from Cohen (1982) when a weak assumption is made for the batch size distribution. In these systems both the interarrival time distribution and the service time distribution have a rational Laplace-Stieltjes transform. The class of distribution functions with a rational Laplace-Stieltjes transform contains phase-type distributions such as mixtures of Erlangian distributions. If we make the extra assumption that the generating function of the batch size distribution is rational we are able to express the coefficients of the exponential expansions of the waiting time distribution in easily computable formulae of the (complex) roots of a polynomial with real coefficients. This extra assumption is satisfied e.g. for geometric batch size distributions and batch size distributions with finite support, where the latter includes the constant batch size distribution.

In the following we first consider the two cases: (i) the $M^X/G/1$ queue and (ii) the $K_m^X/K_n/1$ queue. Next we give elaborated solutions to the $M^X/D/1$ queue, the $M^X/E_{k,n}/1$ queue and the $M^X/C_2/1$ queue. For each of these three special cases we consider constant and geometrically distributed batch sizes.

Theorem 3.1. For the batch arrival $M^X/G/1$ queue satisfying the assumptions 2.1. and 2.2. the constant θ is the smallest positive solution to

$$(3.1)\quad \lambda(G(\hat{B}(-\theta))-1) = \theta.$$

The constants γ_θ and δ_θ are given by

$$(3.2)\quad \gamma_\theta = \frac{1-\rho}{\lambda\varphi_\theta-1}, \qquad \text{and} \qquad \delta_\theta = \gamma_\theta \frac{\theta}{\lambda(\hat{B}(-\theta)-1)} .$$

where

$$(3.3)\quad \varphi_\theta = \lim_{s\to\theta} \frac{G(\hat{B}(-\theta))-G(\hat{B}(-s))}{\theta-s} .$$

Here $G(.)$ is the generating function of the batch size distribution and $\hat{B}(.)$ the Laplace-Stieltjes transform of the service time distribution.

Proof. Since the arrival process is Poissonian, we have $_0\int^\infty e^{-st}dA(t)=E(e^{-sI})=\lambda/(\lambda+s)$ and the well-known fact from queueing theory that $E(N^{sc})=1/(1-\rho)$. It is now a matter of simple algebra to derive (3.1), (3.2) and (3.3) from lemma 2.1. and theorem 2.2. Q.E.D.

Let us next consider the $K_m^X/K_n/1$ queue. Under the assumption that the generating function of the batch size distribution is rational, we will show that the batch-arrival $K_m^X/K_n/1$ queueing model can be reduced to a single arrival $K_m/K_\ell/1$ queueing model. Next we apply known results for the latter queueing model to solve our batch arrival queueing model.

Recall that the Laplace-Stieltjes transform of a K_n-type distribution is the quotient of two polynomials where the denominator is a polynomial of degree n and the numerator is a polynomial of degree at most (n-1). Here it is no restriction to assume that the polynomials have no common zeros. Also it is no restriction to assume that in the denominator the coefficient of s^n is equal to 1. Let $B(.)$ be a K_n-type distribution function with a Laplace-Stieltjes transform denoted by

(3.4) $\hat{B}(s) = {}_0\int^{\infty} e^{-st} dB(t), \quad Re(s)>z_0,$

where z_0 is the real part of the axis of convergence of the integral. By applying Widder [1946, chapter II, theorem 5.b] we find that z_0 is a singularity of $\hat{B}(.)$. Now, write $\hat{B}(s)=\hat{B}_1(s)/\hat{B}_2(s)$ for s with $Re(s)>z_0$. Because $\hat{B}_1(.)$ is a polynomial we may conclude that $\hat{B}_2(z_0)=0$. Furthermore, all the zeros z of $\hat{B}_2(.)$ satisfy $Re(z)\le z_0$. Because $\hat{B}_1(.)$ and $\hat{B}_2(.)$ have no common zeros and $\hat{B}(s)$ analytic for s with $Re(s)>0$ we have $z_0\le 0$. Since $\hat{B}_2(.)$ has a finite number of zeros we find that $z_0<0$. So $1-B(t)=o(e^{tz})$, as $t\to\infty$, for any $z\in(z_0,0)$ implying that (2.6) in assumption 2.1. holds.

The following lemma states some known results for the $K_m/K_n/1$ queueing system with single arrivals.

Lemma 3.2. Consider a $K_m/K_n/1$ system with an interarrival time distribution F(.) and a service time distribution H(.) with respectively Laplace-Stieltjes transform $\hat{F}(s)=\hat{F}_1(s)/\hat{F}_2(s)$ and $\hat{H}(s)=\hat{H}_1(s)/\hat{H}_2(s)$, where $\hat{F}_i(.)$ and $\hat{H}_i(.)$ are polynomials. Denote the steady state distribution of the waiting time of an arbitrary customer by V(.). This distribution has a tail satisfying

(3.5) $\lim_{x\to\infty} e^{\kappa x}(1-V(x)) = \Gamma_\kappa,$

with κ the positive solution to $\hat{F}(\kappa)\hat{H}(-\kappa)=1$, and

(3.6) $\Gamma_\kappa = \dfrac{1-E(e^{-\kappa I})}{\kappa\mu_\kappa E(N)}$,

where I denotes the length of an idle period, N the number of arriving customers during a wet cycle (see lemma 2.1.) and

$$\mu_\kappa = \frac{d}{dx}\,(\hat{F}(x)\hat{H}(-x))_{x=\kappa},$$

For a $K_m/K_n/1$ system the equation $\hat{F}(x)\hat{H}(-x)=1$ is equivalent to

$$\hat{F}_2(x)\hat{H}_2(-x)-\hat{F}_1(x)\hat{H}_1(-x) = (-1)^n \prod_{i=1}^{n+m} (x-\xi_i) = 0,$$

where the ξ_i are the (complex) roots of the polynomial. It is no restriction to assume that $Re(\xi_i)\ge Re(\xi_{i-1})$ for $i=2,\dots,n+m$. Now the constants in (3.6) can be further specified by

$$\kappa=\xi_{m+1}, \quad E(e^{-\kappa I}) = 1 - \frac{1}{\hat{F}_2(\kappa)} \cdot \prod_{i=1}^{m} (\kappa-\xi_i), \quad E(N) = \frac{\hat{H}_2(0)}{\prod_{i=1}^{n} \xi_{i+m}},$$

and

$$\mu_\kappa = \frac{(-1)^{n-1} \prod_{i=1}^{m} (\kappa-\xi_i) \prod_{i=m+2}^{n+m} (\kappa-\xi_i)}{\hat{F}_2(\kappa)\hat{H}_2(-\kappa)} .$$

Proof. A combination of the results in Iglehart [1972] and Cohen [1982] and the discussion below (3.4) yields, after some rewriting, the desired result.

Next assume that the generating function of the batch size distribution G(s) can be written as

$$(3.7)\ G(s) = \sum_{k=0}^{\infty} g_k s^k = \frac{G_1(s)}{G_2(s)} ,$$

where $G_i(s) = \Sigma_{j=0}^{D_i} g_{ij} s^j$ with $g_{iD_i} \neq 0$ $(i=1,2)$, such that $G_1(.)$ and $G_2(.)$ have no common zeros. Note $g_{10}=0$, since it is assumed that $g_0=0$. To find R_1 the radius of convergence of G(.), we consider the two cases that either $D_2=0$ or $D_2 \geq 1$. For the first case, $D_2=0$, it is easily found that $R=\infty$ because $G_1(.)$ is a polynomial. For the second case, $D_2 \geq 1$, we can define $R_0=\inf\{|r|;r\in C, G_2(r)=0\}$. Because $G_1(.)$ is a polynomial, $R=R_0$. Next, since $|G(s)| \leq 1$ for s with $|s| \leq 1$ and $G_1(.)$ and $G_2(.)$ have no common zeros, each zero r of $G_2(.)$ must satisfy $|r|>1$. So, because $G_2(.)$ is a polynomial of degree D_2 and has at most D_2 distinct roots, $R=R_0>1$. Hence, for all values of D_2, we find that $g_k=o(\beta^k)$ for any $\beta \in (1/R,1]$ implying that (2.7) in assumption 2.1. holds.

In the following theorem we show that for a certain class of queueing systems with batch arrivals the queueing system with super customers, as defined in section 2, is of a $K_m/K_n/1$ type. Once we have this result we can use lemma 3.2. to compute the constants as defined in section 2.

Theorem 3.3. Consider a $G^X/K_n/1$ queueing system with batch arrivals where the service time distribution B(.) has the Laplace-Stieltjes transform $\hat{B}(s)=\hat{B}_1(s)/\hat{B}_2(s)$. The generating function of the batch size distribution can be written as in (3.7). Then this system fulfils the conditions of the assumptions 2.1. and 2.2. Furthermore, letting $D=\max(D_1,D_2)$, it holds that the service time distribution of the super customers $B^{SC}(.)$ (see (2.1)), is a K_{nD}-distribution with a Laplace-Stieltjes transform given by

$$(3.8)\quad \int_0^\infty e^{-st}dB^{SC}(t) = \frac{\sum_{j=1}^{D_1} g_{1j}\hat{B}_1^j(s)\hat{B}_2^{D_1-j}(s)}{\sum_{j=0}^{D_2} g_{2j}\hat{B}_1^j(s)\hat{B}_2^{D_2-j}(s)}\ \hat{B}_2^{D_2-D_1}(s).$$

Proof. In the discussion below (3.4) and (3.7) we have already shown that assumption 2.1. is satisfied. To verify that the condition of assumption 2.2. are satisfied, define R and Ψ as in assumption 2.2. Let z_0 be as in (3.4). Now consider the case that $R=\infty$. For this case $\Psi=-z_0$ and it is easily seen that

$$(3.9)\quad \lim_{s\uparrow\Psi} G(\hat{B}(-s))\ .\ \int_0^\infty e^{-st}dA(t) = \infty.$$

Next take $R<\infty$. For this case $\Psi\in(0,-z_0)$ with $\hat{B}(-\Psi)=R$. Hence (3.9) also holds for $R<\infty$, so assumption 2.2. is satisfied. To prove the second part of the theorem take the Laplace-Stieltjes transform of (2.1) and insert (3.7). Then we find (3.8). Because $G_1(0)=0$ and $G_1(.)$ and $G_2(.)$ have no common zeros, $g_{20}\neq0$ and so the degree of the denominator of (3.8) is nD. To prove that the denominator and numerator of (3.8) have no common zero, suppose s_0 is such a common zero. Now take $\hat{B}_2(-s_0)\neq0$, then $\hat{B}_1(-s_0)/\hat{B}_2(-s_0)$ is a common zero of $G_1(.)$ and $G_2(.)$ and this contradicts the assumption below (3.8). Next take $\hat{B}_2(-s_0)=0$. Then either $g_{1D_1}(\hat{B}_1(-s_0))^{D_1}=0$ or $g_{2D_2}(\hat{B}_1(-s_0))^{D_2}=0$. Both equalities imply that $\hat{B}_1(-s_0)=0$ and so s_0 is a common zero of $\hat{B}_1(.)$ and $\hat{B}_2(.)$ which contradicts the assumption below (3.4). Hence the numerator and denominator have no common zero. This concludes the proof. Q.E.D.

Corollary 3.4. For a $K_m^X/K_n/1$ queueing system satisfying the conditions stated in theorem 3.3. the constants as defined in lemma 2.1. and theorem 2.2. can be found by applying lemma 3.2. to a system where the interarrival time distribution $F(.)=A(.)$ and the Laplace-Stieltjes transform of the service time distribution $\hat{H}(s)=\hat{B}^{SC}(s)$ with $\hat{B}^{SC}(s)$ as in (3.8).

In the remainder of this section we give detailed solutions for a number of important special cases of the $M^X/G/1$ queue. First we specify some foregoing results for the $M^X/G/1$ system with constant and geometrically distributed batch sizes. For the case of a constant batch size the generating function of the batch size distribution is given by $G(s)=s^\gamma$ while for the case of a geometrically distributed batch size $G(s)=s/(\gamma-(\gamma-1)s)$. In the latter case it is easily verified from (3.1) that for the particular constant θ: $\hat{B}(-\theta)=\gamma(\lambda+\theta)/(\gamma\lambda+(\gamma-1)\theta)$.

The $M^X/D/1$ queue.

(a) Constant batch size. Then θ (cf. theorem 3.1.) is the unique positive solution to the following equation in s:

$$\lambda e^{\gamma sD} = \lambda+s,$$

where the mean service time $\mu=D$. The constants γ_θ and δ_θ are given by

$$\gamma_\theta = \frac{1-\rho}{\gamma D(\lambda+\theta)-1}, \quad \text{and} \quad \delta_\theta = \frac{\theta\gamma_\theta}{\gamma\lambda(e^{\theta D}-1)}.$$

(b) Geometric batch size. Then θ is the unique positive solution to the following equation in s:

$$e^{sD} = \frac{\gamma(\lambda+s)}{\gamma\lambda+(\gamma-1)s},$$

and the constants γ_θ and δ_θ are given by

$$\gamma_\theta = \frac{(1-\rho)\lambda}{D(\lambda+\theta)(\gamma\lambda+(\gamma-1)\theta)-\lambda}, \quad \text{and} \quad \delta_\theta = \frac{(1-\rho)(\gamma\lambda+(\gamma-1)\theta)}{D\gamma(\lambda+\theta)(\gamma\lambda+(\gamma-1)\theta)-\lambda\gamma}.$$

The $M^X/E_{k,n}/1$ queue. Let $b(t)=(\, q\varphi^k t^{k-1}/(k-1)!+(1-q)\varphi^n t^{n-1}/(n-1)!\,)e^{-\varphi t}$, with mean $\mu=(qk+n(1-q))/\varphi$. Here it is no restriction to assume that $k\le n$.

(a) Constant batch size. Then θ from (3.1) is the smallest positive solution to the following equation in s:

$$(\lambda+s)(\varphi-s)^{\gamma n} = \lambda\varphi^{\gamma k}(q(\varphi-s)^{n-k}+(1-q)\varphi^{n-k})^\gamma.$$

The constant γ_θ and δ_θ are given by

$$\gamma_\theta = \frac{(1-\rho)}{\gamma(\lambda+\theta)(\varphi-\theta)\left(\dfrac{kq(\varphi-\theta)^{n-k}+n(1-q)\varphi^{n-k}}{q(\varphi-\theta)^{n-k}+(1-q)\varphi^{n-k}}\right)-1}$$

and

$$\delta_\Theta = \frac{\gamma_\Theta}{\gamma\lambda}\left(\frac{\Theta(\varphi-\Theta)^n}{\varphi^k[q(\varphi-\Theta)^{n-k}+(1-q)\varphi^{n-k}]-(\varphi-\Theta)^n}\right).$$

(b). Geometric batch size. Then Θ from (3.1) is the smallest, positive solution to the following equation in s:

$$(\gamma+(\gamma-1)s)\varphi^k(q(\varphi-s)^{n-k}+(1-q)\varphi^{n-k}) = \gamma(\lambda+s)(\varphi-s)^n,$$

and γ_Θ and δ_Θ are given by

$$\gamma_\Theta = \frac{(1-\rho)\lambda}{(\gamma\lambda+(\gamma-1)\Theta)(\lambda+\Theta)(\varphi-\Theta)\left(\frac{kq(\varphi-\Theta)^{n-k}+n(1-q)\varphi^{n-k}}{q(\varphi-\Theta)^{n-k}+(1-q)\varphi^{n-k}}\right)-\lambda}$$

and

$$\delta_\Theta = \frac{\gamma_\Theta(\gamma\lambda+(\gamma-1)\Theta)}{\gamma\lambda}.$$

<u>The $M^X/C_2/1$ queue.</u> Let $b(t)=q\mu_1 e^{-\mu_1 t}+(1-q)\mu^{-\mu_2 t}$, with mean $\mu=q/\mu_1+(1-q)/\mu_2$. Here it is no real restriction to assume that $\mu_1\neq\mu_2$ and $q\neq 0,1$ (otherwise the exponential distribution applies). Put for abbreviation $\alpha=q\mu_1+(1-q)\mu_2$.
(a) Constant batch size. Then Θ from (3.1) is the smallest positive root to the following equation in s:

$$\lambda(\mu_1\mu_2-\alpha s)^\gamma = (\lambda+s)(\mu_1-s)^\gamma(\mu_2-s)^\gamma .$$

The constants γ_Θ and δ_Θ are given by

$$\gamma_\Theta = \frac{(1-\rho)}{\gamma(\lambda+\Theta)\left[\frac{\alpha\Theta^2-2\mu_1\mu_2\Theta+\mu_1^2\mu_2^2\mu}{(\mu_1\mu_2-\alpha\Theta)(\mu_1-\Theta)(\mu_2-\Theta)}\right]-1}$$

and

$$\delta_\Theta = \frac{\gamma_\Theta}{\gamma\lambda}\left(\frac{(\mu_1-\Theta)(\mu_2-\Theta)}{\mu_1\mu_2\mu-\Theta}\right)$$

b) Geometric batch size. Then Θ is explicitly given by

$$\Theta = \frac{-1}{2\gamma}(\gamma\lambda-\alpha-\gamma\mu_1\mu_2\mu)-\frac{1}{2\gamma}[(\gamma\lambda-\alpha-\gamma\mu_1\mu_2\mu)^2-4\gamma\mu_1\mu_2(1-\rho)]^{1/2}$$

The constant γ_Θ and δ_Θ are found by applying corollary 3.3. and are given by

$$\gamma_\Theta = \frac{\lambda(\mu_1\mu_2-\alpha\Theta)}{\mu_1\mu_2(\lambda+\Theta)} \cdot \frac{\chi}{\chi-\Theta}, \qquad \text{and} \qquad \delta_\Theta = \frac{\gamma_\Theta(\gamma\lambda+(\gamma-1)\Theta)}{\gamma\lambda}.$$

where $\chi=-(\Theta+\frac{1}{\gamma}(\gamma\lambda-\alpha-\gamma\mu_1\mu_2\mu))$.

Appendix A. The arithmetic batch size distribution.

In this appendix we focus on a $GI^X/G/1$ queueing system with batch arrivals where the batch size distribution is arithmetic. For this model we give a definition of the waiting time distribution of an arbitrary customer arriving when the system has reached statistical equilibrium. The analysis below uses the results derived for queueing systems with a non-arithmetic batch size distribution.

Consider an arithmetic batch size distribution $\{g_k, k=1,2,\ldots\}$ with period $P>1$, that is P is the largest integer with $\sum_{n=0}^{\infty} g_{nP}=1$. In case of an arrival of a batch of size KP, let us split this batch into K groups of consecutive customers. These groups are called composed customers. We assume that customers belonging to the same group are served consecutively. Further, the groups are numbered in order of commencement of the service of their customers. Denote the waiting time of the customer who is the i-th one served from the n-th composed customer by $W_{n,i}$. It is obvious that

(A.1) $W_{n,i} = W_{n,0} + S_{n,i}$

where $S_{n,i}$ is the total service time of the first (i-1) individual customers from the composed customer. Because the service order is assumed to be independent of the service time requirement it follows that

(A.2) $\Pr\{S_{n,i} \le x\} = B^{(i-1)*}(x),\ i=1,\ldots,P.$

In order to analyze $W_{n,0}$ we apply the same technique as in section 2. By considering a composed customer as one entity we create a modified $GI^X/G/1$ queueing system with batch arrivals. Then, by the definition of the composed customers, the batch size distribution for this modified system is given by

$\bar{g}_k = g_{kP}$, $k=0,1,\ldots,$

and the service time distribution of a composed customer is given by $B^{P*}(x)$. Note that the batch size distribution in this modified system is non-arithmetic. Let W_n^{cc} be the waiting time of the n-th composed customer. It can be seen that $W_n^{cc}=W_{n,0}$. By the same arguments as used to derive (2.4) and (2.5) we find that the limiting distribution of the W_n^{cc}, denoted by $W^{cc}(.)$, satisfies

(A.3) $W^{CC}(x) = W^{SC} * W_B^{CC}(x)$,

where

$$W_B^{CC}(x) = \sum_{j=1}^{\infty} B^{P(j-1)*}(x) \frac{1}{(\gamma/P)} \sum_{k=j}^{\infty} \bar{g}_k.$$

Here $W^{sc}(.)$ is defined as the limiting distribution of the waiting time of a super customer in the modified batch arrival model. A little thought shows that a super customer consisting of composed customers in the modified batch arrival model is nothing else than a super customer consisting of individual customers in the original batch arrival model. Hence $W^{sc}(.)$ is exactly the function $W^{sc}(.)$ defined in (2.2).

Now, by (A.1), (A.2) and (A.3) we find by applying the Helly-Bray theorem, that for $i=1,\ldots,P$,

$$\lim_{n\to\infty} \Pr(W_{n,i} \le x) = W^{CC} * B^{(i-1)*}(x),$$

and we denote this limit by $W_i(x)$. Next consider an arbitrary customer of a batch arriving when the system has reached statistical equilibrium. This customer is with probability 1/P served at the i-th position from a composed customer. So the following definition of the waiting time distribution of an arbitrary customer is justified:

(A.5) $W(x) = \frac{1}{P} \sum_{i=1}^{P} W_i(x)$.

Combining (A.3), (A.4) and (A.5) and $\Sigma_{k=K}^{\infty}\bar{g}_k = \Sigma_{j=L}^{\infty} g_j$, for $L=(K-1)P+1,\ldots,KP$, we obtain after some algebra involving interchange of summation

(A.6) $W(x) = {}_0\int^{\infty} W^{SC}(x-y)dW_B(y)$,

with

$$W_B(y) = \sum_{k=1}^{\infty} B^{(k-1)*}(y) . \frac{1}{\gamma} \sum_{j=k}^{\infty} g_j .$$

Now (A.6) is exactly equal to (2.4). Therefore we can apply the same analysis for systems with respectively an arithmetic and a non-arithmetic batch size distribution. This proves the following theorem.

Theorem A.1. Theorem 2.2. also holds for systems with batch arrivals when the batch size distribution is arithmetic, provided W(.) is defined by (A.5).

Appendix B. The assumptions 2.1. and 2.2.

In this appendix we show that the assumptions 2.1. and 2.2. are equivalent to the assumption Iglehart [1972] made to prove lemma 2.1. We first state the assumption used by Iglehart [1972].

Assumption B.1. Define C(.) by

(B.1) $C(x) = {}_0\int^{\infty} (1-A(y-x))dB^{SC}(y)$.

Let a $\theta>0$ exists which satisfies

(B.2) $\hat{C}(\theta) = {}_{-\infty}\int^{\infty} e^{\theta x}dC(x) = 1$,

and

(B.3) $d_\theta = {}_{-\infty}\int^{\infty} xe^{\theta x}dC(x) < \infty$.

In the following lemma we prove that the assumptions 2.1. and 2.2. are sufficient and necessary for the model with super customers to satisfy assumption B.1.

Lemma B.2. Assumption B.1. is equivalent to the assumptions 2.1. and 2.2.

Proof. In the first part we show that assumption B.1. implies the assumptions 2.1. and 2.2. while in the second part the converse implication is shown. For now, let assumption B.1. be given. From (B.1) and (B.2) we may conclude that

$$\text{(B.4)}\quad \hat{C}(s) = {}_0\int^{\infty} e^{-sx} dA(x)\ {}_0\int^{\infty} e^{sx} dB^{SC}(x),$$

is defined on $[0,\theta]$. Let α_0 be the real part of the axis of convergence of ${}_0\int^{\infty} e^{-sx} dB^{SC}(x)$. Then $\theta \leq -\alpha_0$ and so, by Widder [1946, chapter II, theorem 5.b], $\hat{B}^{SC}(s)$ is analytic for s with $Re(s) > \theta$. Therefore $\hat{C}(s)$ is analytic for s with $Re(s) \in (0,\theta)$. Let $M = \sup\{e^{\theta x}(1-B^{SC}(x) | x \in R\}$. By (B.1) and (B.4) ${}_0\int^{\infty} e^{\theta x} dB^{SC}(x) < \infty$, so

$$e^{\theta x}\ {}_x\int^{\infty} dB^{SC}(y) \leq {}_x\int^{\infty} e^{\theta y} dB^{SC}(y) = o(1),\ x \to \infty,$$

and hence $M < \infty$. Next, from (2.1),

$$e^{\theta x}[1-B^{SC}(x)] = \sum_{k=1}^{\infty} g_k [1-B^{k*}(x)] e^{\theta x} \geq \sum_{k=1}^{\infty} g_k (1-B(x)) e^{\theta x},$$

and so $(1-B(x)) = o(e^{-\alpha x})$ as $x \to \infty$ for any $\alpha \in (0,\theta)$. Denote the Laplace-Stieltjes transform of $B^{SC}(.)$ by $\hat{B}^{SC}(.)$. Then, from (2.1), (B.2) and (B.4) we get

$$\text{(B.5)}\quad \hat{B}^{SC}(-s) = G(\hat{B}(-s)) < \infty \quad \text{for } s \geq -\theta.$$

Because $\hat{B}(-s) > 1$ for $s < 0$, the generating function $G(.)$ must have a radius of convergence $R > 1$ and hence $g_k = o(\beta^k)$ as $k \to \infty$ for any β with $1/R < \beta < 1$. This proves the condition of assumption 2.1. To prove that assumption 2.2. holds we define

$$\varphi_\theta = \lim_{s \uparrow \theta} \int_0^{\infty} \frac{e^{\theta x} - e^{sx}}{\theta - s}\ dB^{SC}(x).$$

Then, by (B.5),

$$\text{(B.6)}\quad \varphi_\theta = \lim_{s \uparrow \theta} [G(\hat{B}(-\theta)) - G(\hat{B}(-s))]/\theta - s = \lim_{s \uparrow \theta} [\frac{\hat{B}(-\theta) - \hat{B}(-s)}{\theta - s}] \sum_{n=1}^{\infty} n g_n [\hat{B}(-\theta)]^{n-1},$$

where the last equality holds by monotone convergence. Next, we may write, by (B.2), (B.3), (B.4) and (B.6),

$$\text{(B.7)}\quad d_\theta = \lim_{s \uparrow \theta} \frac{\hat{C}(\theta) - \hat{C}(s)}{\theta - s} = -\hat{B}^{SC}(-\theta)\ {}_0\int^{\infty} x e^{-\theta x} dA(x) + \varphi_\theta\ {}_0\int^{\infty} e^{-\theta x} dA(x),$$

where the first equality can be proven in the following way. Write

$$\frac{\hat{C}(\Theta)-\hat{C}(s)}{\Theta-s} = {}_{-\infty}\int^{0} \frac{e^{\Theta x}-e^{sx}}{\Theta-s} \; dC(x) + {}_{0}\int^{\infty} \frac{e^{\Theta x}-e^{sx}}{\Theta-s} \; dC(x).$$

For $x>0$ we have $0\leq(e^{\Theta x}-e^{sx})/(\Theta-s)\uparrow xe^{\Theta x}$ for $s\uparrow\Theta$ and so, by monotone convergence,

$$\text{(B.8)} \quad \lim_{s\uparrow\Theta} {}_{0}\int^{\infty} \frac{e^{\Theta x}-e^{sx}}{\Theta-s} \, dC(x) = {}_{0}\int^{\infty} xe^{\Theta x} dC(x).$$

For $x<0$ we have $0\geq(e^{\Theta x}-e^{sx})/(\Theta-s)\downarrow xe^{\Theta x}$ for $s\uparrow\Theta$ and so (B.8) also holds if we replace the integration interval by $(-\infty,0]$. Furthermore, by the definition of $C(.)$, we find ${}_{-\infty}\int^{0} x dC(x)>-\infty$ and so ${}_{-\infty}\int^{0} xe^{sx} dC(x)>-\infty$ for $s\geq0$. Together with (B.8) this proves the first equality in (B.7). Now, because ${}_{0}\int^{\infty} e^{-sx} dA(x)$ is analytic for s with $Re(s)>0$ and $\hat{B}^{sc}(-\Theta)$ is finite we obtain assumption (2.2) by combining (B.3), (B.5) and (B.7). This concludes the first part of the proof.

In the following part of the proof we show that the assumptions 2.1. and 2.2. imply assumption B.1. We first remark that (2.7) implies that $R>1$. Then it follows from (2.6) that a $\xi>0$ exists with ${}_{0}\int^{\infty} e^{\xi x} dB(x)<R$ and so $\Psi>0$. Hence $\hat{B}(s)$ is analytic for s with $Re(s)>-\Psi$. Let us define $C(.)$ as in (B.1). Then $\hat{C}(.)$ as defined in (B.2) is, by (B.4), analytic for s with $Re(s)\in(0,\Psi)$ and from (B.2) it follows that $\hat{C}''(s)>0$ for $s\in(0,\Psi)$. Hence $\hat{C}(.)$ is convex on $(0,\Psi)$. Then, since $\hat{C}'(0)=\mu-1/\lambda<0$ and $\hat{C}(0)=1$, we have by (2.8) a unique solution $\Theta>0$ to the equation

$$G(\hat{B}(-\Theta))\,{}_{0}\int^{\infty} e^{-\Theta t} dA(t) = 1.$$

Now, first consider the case that $\lim_{s\uparrow\Psi} G(\hat{B}(-s))\,{}_{0}\int^{\infty} e^{-st} dA(t)=1$. Then by the convexity of $\hat{C}(.)$ on $(0,\Psi)$ and $\hat{C}'(0)<0$ we must have $\lim_{s\uparrow\Psi}\hat{C}'(s)>0$ and hence $\Psi<\infty$. By (2.9) and (B.7) $\Theta=\Psi$ satisfies the condition (B.3). Next consider the case that $\lim_{s\uparrow\Psi} G(\hat{B}(-s))\,{}_{0}\int^{\infty} e^{-st} dA(t)>1$. By assumption 2.2. $\Psi\leq-\alpha_0$, where α_0 is the real part of the axis of convergence of ${}_{0}\int^{\infty} e^{-sx} dB(x)$. Because $\Theta\in(0,\Psi)$ we easily find

$$\lim_{s\uparrow\Theta} \frac{\hat{B}(-\Theta)-\hat{B}(-s)}{\Theta-s} < \infty.$$

Furthermore, by the definition of Ψ and because $\Theta<\Psi$, we have $\hat{B}(-\Theta)<R$. Hence $G(.)$ is analytic in the point $\hat{B}(-\Theta)$ and so

$$\sum_{n=1}^{\infty} ng_n[\hat{B}(-\theta)]^{n-1} < \infty.$$

Together with (B.6) and (B.7) this yields (B.3). This concludes the second part of the lemma. Q.E.D.

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