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# **SERIE RESEARCH MEMORANDA**

A NEW METHODOLOGY FOR THE ANALYSIS  
OF DYNAMIC SPATIAL INTERACTION MODELS

P. Nijkamp  
A. Reggiani

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A New Methodology for the Analysis  
of Dynamic Spatial Interaction Models

Peter Nijkamp

Aura Reggiani

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## 1. Introduction

In recent years a wide variety of dynamic models has been developed for analyzing the evolutionary pattern of spatial systems. Both in the field of discrete choice theory (see Fischer and Nijkamp, 1987) and of spatial interaction analysis (see Nijkamp and Reggiani, 1987) and in the field of urban and regional modeling (see Rima and van Wissen, 1987) many advances have been made in the last decade.

A new development has emerged in the analysis of dynamic evolutions of spatial systems, based on the competition between various places in space (see among others Dendrinos and Sonis, 1984, Griffith and Haining, 1986, and Johansson and Nijkamp, 1987). In this context, various models based on Volterra-Lotka dynamics or predator-prey dynamics have been developed (see among others Goh and Jennings, 1977, Jeffries, 1979, Pimm, 1982, and Wilson, 1981).

The present paper is devoted to a further analysis of the relevance of predator-prey models in the context of dynamic spatial interaction analysis. The assumption will be made that inflows and outflows related to a certain work place reflect essentially pull and push forces that may be interpreted as predator-prey phenomena. Then an optimal control model will be developed in order to study the stability conditions of the spatial systems concerned. The analysis will be carried out on the basis of a simple conventional utility function.

## 2. A Predator-Prey Model for Spatial Interaction

The predator-prey model has originally been developed by Lotka and Volterra earlier this century. In the two species case (with only one prey and one predator, represented by  $x_1$  and  $x_2$  respectively), a typical Volterra-Lotka model becomes:

$$\begin{aligned} \dot{x}_1 &= x_1 (b_1 - a_{11}x_1 - a_{12}x_2) \\ \dot{x}_2 &= x_2 (-b_2 + a_{21}x_1) \end{aligned} \tag{2.1}$$

The coefficients  $b_1$  and  $b_2$  are related to the endogenous dynamics of each corresponding variable, while the parameters  $a_{11}$ ,  $a_{12}$  and  $a_{21}$  reflect the interaction between species. Model (2.1) has two equilibrium solutions, a trivial one (viz.  $x_1 = x_2 = 0$ ) and a more complicated one:

$$\begin{aligned} x_1^* &= b_2/a_{21} \\ x_2^* &= b_1/a_{12} - a_{11}/a_{12} x_1^* \end{aligned} \quad (2.2)$$

The Volterra-Lotka equations (2.1) cannot be solved in an analytical way due to their non-linearity, but solutions can be found for the linearized approximations (see Brouwer and Nijkamp, 1985).

However, if we restrict ourselves to the exploration of the optimal trajectories in a phase space diagram, we can plot the solution lines of model (2.1) for one given configuration of the coefficients (see Fig. 2.1).

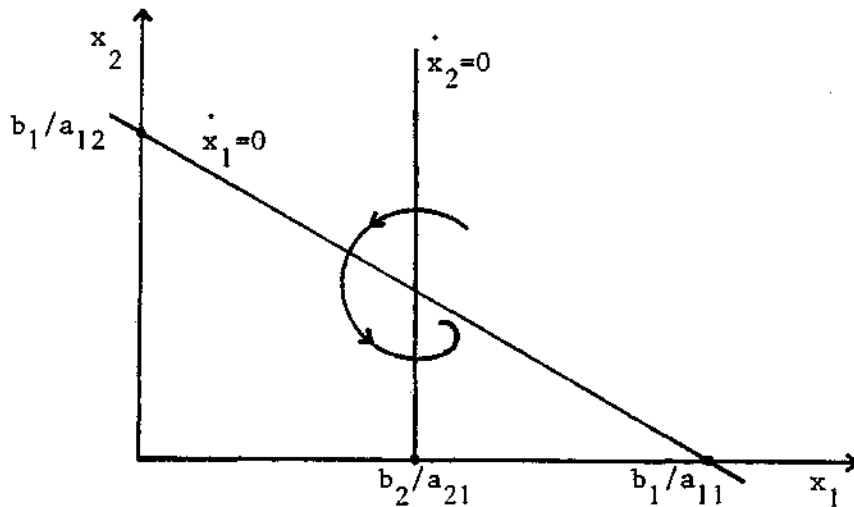


Fig. 2.1 Stable equilibrium for prey-predator equations.

It is well known that in this case (i.e.,  $b_1/a_{11} > b_2/a_{21}$ ) the equilibrium is stable (see e.g. Hirsch and Smale, 1974, Wilson, 1981, and Wilson and Kirkby, 1980). But by varying the sign of the coefficient  $a_{11}$  (in particular by supposing  $a_{11}$  to be negative), we may obtain an unstable solution (see Fig. 2.2).

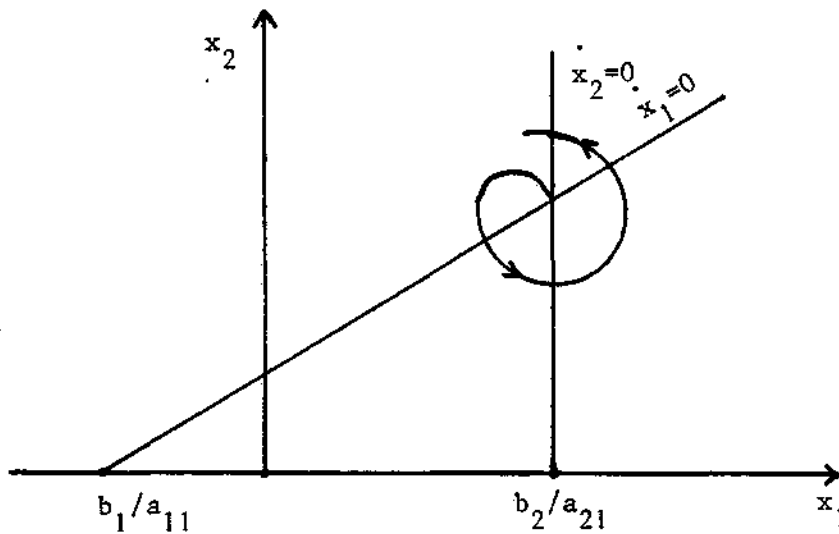


Fig. 2.2 Unstable equilibrium for prey-predator equations

Therefore it is clear that the coefficient  $a_{11}=0$  represents a critical value at which a bifurcation may emerge - from stability ( $a_{11}>0$ ) to instability ( $a_{11}<0$ ). In particular it can be shown that when  $a_{11}=0$  the solutions to the system of differential equations (2.1) are closed orbits (see also Haken, 1983; Hirsch and Smale, 1974) as depicted in Fig. 2.3.

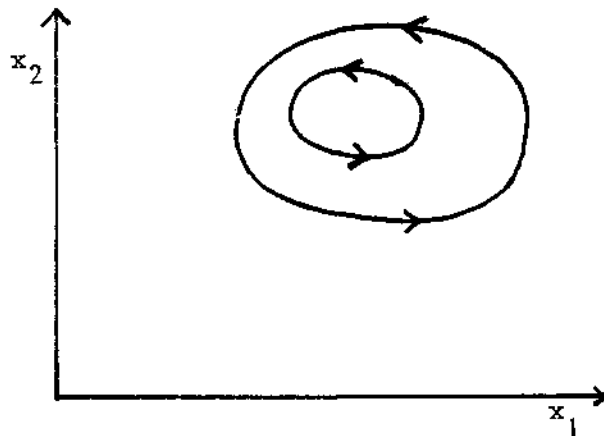


Fig. 2.3 Closed orbits as solutions

This particular case will be the object of our study, where in the context of a dynamic spatial interaction analysis we will assume here a predator-prey model of the following form:

$$\dot{W}_j = (\alpha_j - \beta_j D_j) W_j \quad , j=1, \dots, J \quad (2.3)$$

and

$$\dot{D}_j = (\epsilon_j W_j - \phi_j) D_j \quad , j=1, \dots, J \quad (2.4)$$

Equation (2.3) states that the number of vacancies on the labour market in place  $j$ ,  $W_j$ , exhibits a growth pattern upon which the

number of inflows of people into place  $j$ ,  $D_j$ , exerts a (multiplicative) negative impact. Thus this equation is typically related to a prey phenomenon. Similarly, equation (2.4) assumes that the growth of inflows is positively influenced by the number of vacant workplaces, so that this equation is related to a predator phenomenon.

It should also be noted that  $\alpha_j$  in (2.3) indicates the growth rate of the vacant work places, in the absence of inflows. For example  $\alpha_j$  might depend on new technological regimes. Analogously,  $\phi_j$  in (2.4) represents the decline rate of inflows in the absence of work places. It is interesting to observe that models of the form (2.3) and (2.4) bear some resemblance to the economic cycle model formalized by Goodwin (1967). In fact this author analyzes a system of non-linear differential equations (of the (2.3) and (2.4) type) which describes the motion of the employment rate (prey) and of the workers' income share (predator). Despite some criticism on the realism of this model approach, it still remains the base for many theoretical contributions and empirical applications (see, among others, Maresi and Ricci, 1976, Balducci et al., 1984).

However, predator-prey models of the (2.1) type (i.e., with limited growth) occur much more frequently in the economic literature, mostly in fishery and other renewable resources (see, e.g. Chauduri, 1987, and Ragozin and Brown, 1985). A reason for this popularity (see Hannesson, 1983) is that eqs. (2.1) are capable of producing a stable equilibrium while the system described by eqs. (2.3) and (2.4) exhibits oscillating behaviour, as it has been illustrated previously.

Since we are also interested in an optimal control policy, we prefer to use in our analysis specifications of the (2.3) and (2.4) type which are certainly simpler from a mathematical viewpoint, even though they may present the so-called 'neutral' stability.

### 3. A Simple Optimal Control Predator-Prey Model

In recent publications (see among others Nijkamp and Reggiani, 1988) the use of optimal control theory for dynamic spatial interaction analysis has been advocated. Given the non-linear nature of the predator-prey model, it is clear that fairly complicated mathematical expressions may emerge in this framework. Therefore, we will start here with a simple optimal control model based on a frequently used (concave) logarithmic utility function:



$$\begin{aligned}
 \max u &= \int_0^T \sum_j \gamma_j (\ln \beta_j D_j + \ln \epsilon_j W_j) dt \\
 \text{s.t.} & \\
 \dot{W}_j &= (\alpha_j - \beta_j D_j) W_j \\
 \dot{D}_j &= (\epsilon_j W_j - \phi_j) D_j
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} \max u \\ \text{s.t.} \\ \dot{W}_j \\ \dot{D}_j \end{aligned}} \right\} (3.1)$$

We will assume here that  $W_j$  ( $j=1, \dots, J$ ) and  $D_j$  ( $j=1, \dots, J$ ) are state variables, while  $\beta_j$  and  $\epsilon_j$  are control variables.

The parameters  $\beta_j$  and  $\epsilon_j$  may be interpreted as accessibility measures. In our paper we will analyze the optimal control problem of selecting  $\beta_j$  and  $\epsilon_j$  in order to maximize utility function  $u$  (i.e., a maximum interaction between places of inflows and workplaces).

The Hamiltonian  $H$  related to (3.1) is equal to:

$$\begin{aligned}
 H(\beta_j, \epsilon_j, W_j, D_j, \lambda_j, \psi_j) &= \sum_j \gamma_j (\ln \beta_j D_j + \ln \epsilon_j W_j) + \\
 &+ \sum_j \lambda_j (\alpha_j - \beta_j D_j) W_j + \sum_j \psi_j (\epsilon_j W_j - \phi_j) D_j,
 \end{aligned}
 \quad (3.2)$$

Where  $\lambda_j$  and  $\psi_j$  are the costate variables related to the  $\dot{W}_j$  constraints and  $\dot{D}_j$  constraints respectively.

Therefore the first-order (necessary) conditions for the control variables become:

$$\begin{aligned}
 \frac{\partial H}{\partial \beta_j} &= 0 \\
 \frac{\partial H}{\partial \epsilon_j} &= 0
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} \frac{\partial H}{\partial \beta_j} \\ \frac{\partial H}{\partial \epsilon_j} \end{aligned}} \right\} (3.3)$$

or:

$$\begin{aligned}
 \gamma_j / \beta_j - \lambda_j D_j W_j &= 0 \\
 \gamma_j / \epsilon_j + \psi_j W_j D_j &= 0
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} \gamma_j / \beta_j - \lambda_j D_j W_j \\ \gamma_j / \epsilon_j + \psi_j W_j D_j \end{aligned}} \right\} (3.4)$$

or:

$$\beta_j = \frac{\gamma_j}{\lambda_j D_j W_j} \quad \beta_j, \epsilon_j, \gamma_j, W_j, D_j > 0 \quad (3.5)$$

$$\epsilon_j = \frac{-\gamma_j}{\psi_j D_j W_j} \quad (3.6)$$

From (3.5) we can also derive that the optimal values  $\beta_j$  are a decreasing function of the shadow prices  $\lambda_j$  ( $\frac{\partial \beta_j}{\partial \lambda_j} < 0$ ).

Next we find from (3.6) that, since  $\epsilon_j$ ,  $D_j$ ,  $W_j$ ,  $\gamma_j$  are positive, the shadow prices  $\psi_j$  are negative and also that  $\epsilon_j$  is an increasing function of the corresponding shadow price

$$\psi_j \left( \frac{\partial \epsilon_j}{\partial \psi_j} > 0; \psi_j < 0 \right).$$

Next a link between the shadow prices  $\lambda_j$  and  $\psi_j$  can be found. In fact by dividing eq. (3.5) by eq. (3.6) it follows that:

$$\frac{\psi_j}{\lambda_j} = - \frac{\beta_j}{\epsilon_j} \quad (3.7)$$

In other words: the ratio of the shadow prices of the  $\dot{W}_j$  and  $\dot{D}_j$  constraints is inversely related to the corresponding optimal control variables (i.e. the interaction terms  $\beta_j$  and  $\epsilon_j$ ) in the predator-prey model (3.1).

Then substitution of (3.5) and (3.6) into the predator-prey relationships gives:

$$\begin{aligned} \dot{W}_j &= \left( \alpha_j - \frac{\gamma_j}{\lambda_j W_j} \right) W_j = \alpha_j W_j - \frac{\gamma_j}{\lambda_j} \\ \dot{D}_j &= \left( \frac{-\gamma_j}{\psi_j D_j} - \phi_j \right) D_j = -\phi_j D_j - \frac{\gamma_j}{\psi_j} \end{aligned} \quad (3.8)$$

In case of a stationary solution the following conditions also hold since  $\dot{W}_j = \dot{D}_j = 0$ ):

$$W_j = \frac{\gamma_j}{\lambda_j \alpha_j} \quad \text{for } \dot{W}_j = 0 \quad (3.9)$$

$$D_j = \frac{-\gamma_j}{\psi_j \phi_j} \quad \text{for } \dot{D}_j = 0 \quad (3.10)$$

Therefore eqs. (3.9) and (3.10) represent the optimal paths of  $W_j$  and  $D_j$ . We observe that in particular (3.9) and (3.10) depict two hyperbolas in the planes  $(W_j; \lambda_j)$  and  $(D_j; \psi_j)$  as follows:

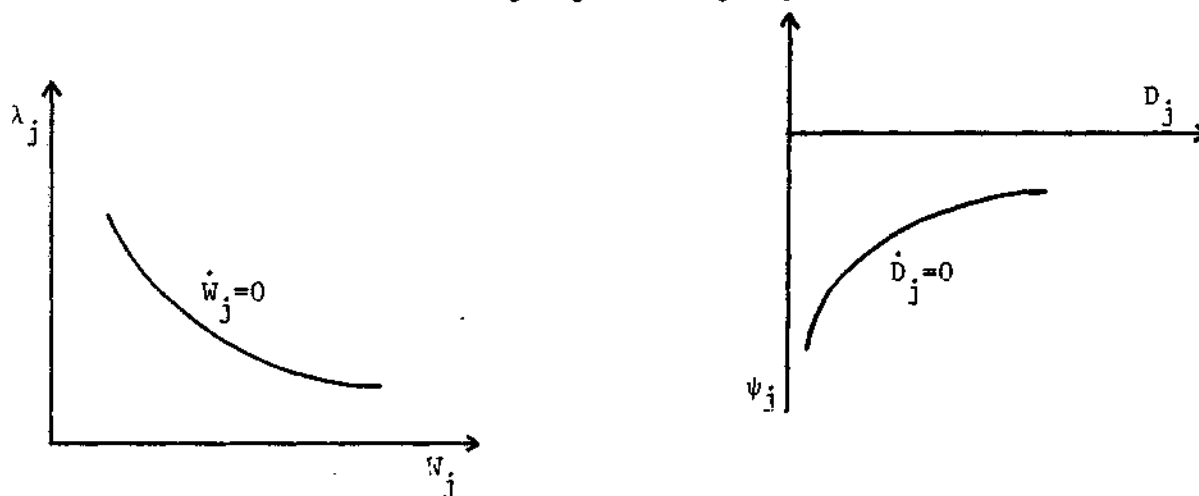


Fig. 3.1 The optimal paths consistent with the differential equations (3.8).

Next we will deal with the costate equations. Here the following conditions hold (see, e.g., Kamien and Schwartz, 1981, and Miller, 1979):

$$\left. \begin{aligned} \dot{\lambda}_j &= \frac{-\partial H}{\partial W_j} \\ \dot{\psi}_j &= \frac{-\partial H}{\partial D_j} \end{aligned} \right\} \quad (3.11)$$

or

$$\begin{aligned} \dot{\lambda}_j &= -\frac{\gamma_j}{W_j} - \lambda_j \alpha_j + \lambda_j \beta_j D_j - \psi_j \epsilon_j D_j \\ \dot{\psi}_j &= -\frac{\gamma_j}{D_j} + \psi_j \phi_j + \lambda_j \beta_j W_j - \psi_j \epsilon_j W_j \end{aligned} \quad (3.12)$$

Then if we substitute the optimal values (3.5) and (3.6) in (3.12), we obtain:

$$\dot{\lambda}_j = \frac{\gamma_j}{W_j} - \alpha_j \lambda_j \quad (3.13)$$

and

$$\dot{\psi}_j = \frac{\gamma_j}{D_j} + \phi_j \psi_j \quad (3.14)$$

It is surprising that (3.13) is only a function of  $\lambda_j$ ,  $W_j$ , while (3.14) is only a function of  $\psi_j$ ,  $D_j$ . Therefore the final system:

$$\left. \begin{aligned} \dot{W}_j &= \alpha_j W_j - \frac{\gamma_j}{\lambda_j} \\ \dot{\lambda}_j &= \frac{\gamma_j}{W_j} - \alpha_j \lambda_j \\ \dot{D}_j &= -\phi_j D_j - \frac{\gamma_j}{\psi_j} \\ \dot{\psi}_j &= \frac{\gamma_j}{D_j} + \phi_j \psi_j \end{aligned} \right\} \quad (3.15)$$

can easily be divided into two independent subsystems, as follows:

$$\left. \begin{aligned} \dot{W}_j &= \alpha_j W_j - \frac{\gamma_j}{\lambda_j} \\ \dot{\lambda}_j &= \frac{\gamma_j}{W_j} - \alpha_j \lambda_j \end{aligned} \right\} \quad (3.16)$$

and

$$\left. \begin{aligned} \dot{D}_j &= -\phi_j D_j - \frac{\gamma_j}{\psi_j} \\ \dot{\psi}_j &= \frac{\gamma_j}{D_j} + \phi_j \psi_j \end{aligned} \right\} \quad (3.17)$$

We will now first analyze system (3.16). The steady state is defined by  $\dot{W}_j = \dot{\lambda}_j = 0$ . The optimal path  $\dot{W}_j = 0$  has been defined

in (3.9). The optimal path  $\dot{\lambda}_j = 0$  is defined by:

$$\lambda_j = \frac{\gamma_j}{\alpha_j W_j} \quad (3.18)$$

It is clear that (3.18) represents the same hyperbola as the one defined in (3.9) and depicted in Fig. 3.1. This means that all points of (3.9) (being equal to (3.19)) are steady states! It is straightforward to see that the same happens regarding the condition  $\dot{\psi}_j = 0$ , or

$$\psi_j = -\frac{\gamma_j}{\phi_j D_j} \quad (3.19)$$

Hence: all points of (3.10) (being equal to (3.20)) are steady states in the plane  $(D_j, \psi_j)$ .

To catalogue the nature of these steady states, we will take the linear terms of Taylor series expansion of the right hand side (around all steady states) to obtain the approximate linear differential equation system:

$$\left. \begin{aligned} \dot{W}_j &= \alpha_j + \frac{\gamma_j}{\lambda_j^2} \\ \dot{\lambda}_j &= \frac{-\gamma_j}{W_j^2} - \alpha_j \end{aligned} \right\} \quad (3.20)$$

The characteristic equation of (3.20) is the following:

$$\begin{vmatrix} \alpha_j - r & + \frac{\gamma_j}{\lambda_j^2} \\ \frac{-\gamma_j}{W_j^2} & - \alpha_j - r \end{vmatrix} = 0 \quad (3.21)$$

or

$$r^2 - p r + q = 0 \quad (3.22)$$

with

$$p = r_1 + r_2 = 0 \quad (3.23)$$

$$q = r_1 \cdot r_2 = \alpha_j^2 + \frac{\gamma_j^2}{W_j^2 \lambda_j^2} > 0 \quad (3.24)$$

Now, the emerging result is indeed interesting: as  $q > 0$  and  $p = 0$  (i.e., the trace of the coefficient matrix equal to zero), every point of eq. (3.9) (or of eq. (3.19)) is a center. (For a further exposition on the classification of singularities, see, e.g., Hirsch and Smale, 1979, Kaplan, 1958, and Ku, 1958).

Therefore, the motion of  $W_j, \lambda_j$  is periodic around the center, so that it has the same qualitative properties as the original model  $W_j, D_j$  defined in (2.3) and (2.4).

We can then depict the time variation of  $W_j, \lambda_j$ , corresponding to a closed orbit as follows (see Fig. 3.2).

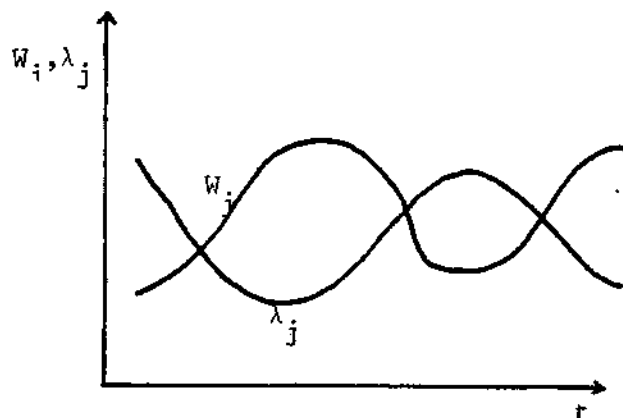


Fig. 3.2. Time variations of  $W_j, \lambda_j$ .

Fig. 3.2 shows that an increase of workplaces leads to a decrease of the corresponding shadow prices  $\lambda_j$ . But this depreciation will bring about a decrease of workplaces, so that  $\lambda_j$  will increase again, and so forth.

Next it should be noted that since the center describes a curve (and in particular a hyperbola) in the phase plane, system (3.17) generating closed orbits represents some sort of persistent cyclicity which however is not constant (see Balducci et al., 1984).

Obviously the same pattern will result in the plane  $D_j, \psi_j$ . It can easily be seen from (3.18) that for the relative linearized system the trace of the coefficient matrix is equal to zero. Next, since also here the product of the eigenvalues is positive, it is straightforward to see that we have again a situation where the center describes the hyperbola (3.20) and where we consequently face cyclical motions.

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