ET 69

05348-.

1987

SERIE RESEARCH MEMORANDA

DYNAMIC SPATIAL INTERACTION MODELS:
NEW DIRECTIONS

P. Nijkamp
A. Reggiani

Researchmemorandum 1987-69 december 1987

VRIJE UNIVERSITEIT
FACULTEIT DER ECONOMISCHE WETENSCHAPPEN
EN ECONOMETRIE
AMSTERDAM

DYNAMIC SPATIAL INTERACTION MODELS:
NEW DIRECTIONS

Peter Nijkamp
Dept. of Economics
Free University
P.O. Box 7161
1007 HC Amsterdam
The Netherlands

Aura Reggiani
Dept. of Mathematics
University of Bergamo
Via Salvecchio, 19
24100 Bergamo
Italy

Abstract

Spatial interaction models have received a great deal of attention in the past decade. In recent years, also various approaches have been developed to take into account dynamic aspects of spatial interaction models, for instance, by means of optimal control theory, bifurcation theory or catastrophe theory.

The present paper deals with new directions in dynamic spatial interaction research. It will focus on a general dynamic interaction model analyzed in the framework of optimal control theory. The objective function used is a bi-criterion utility model, to be maximized subject to a set of differential equations which bear some resemblance to those used by Wilson in a shopping centre context.

Next, we investigate the link between our model and a catastrophe type of model. It will be demonstrated that catastrophe behaviour may emerge as a particular case of this optimal control model.

Finally, it will also be shown how external influences (e.g., stochastic impacts of the Brownian motion type) will affect the optimal trajectory.

1. Prologue

In the past decade, regional economics and geography have exhibited an enormous interest in the analysis of dynamic spatial systems. Space-time patterns have not only been studied at the micro level of individual actors, but also at the meso or macro level of groups (or regions) in a whole system. And consequently dynamic analysis has become a focal point in migration analysis, residential choice analysis, transportation analysis, labour market analysis and locational analysis.

The models used range from simple comparative static models to continuous space-time models (see Beckmann and Puu, 1985). Their contents ranges from exploratory to confirmatory analyses, while their statistical properties vary between empirical estimation and simulation.

In recent years, the attention has increasingly been focused on dynamic spatial models incorporating multiple equilibria and even discontinuities (see also Fischer and Nijkamp, 1987). Following Prigogine (1981), the following classification of models may be made: (a) macro-phenomenological (in which macro variables represent the average dynamic pattern of underlying micro variables), (b) micro-stochastic (in which the behaviour of a system replicates in a stochastic sense the behaviour of micro variables), (c) based on dynamic laws (in which the system's trajectory is governed by fundamental laws of motion). It is interesting to observe that all 3 types are present in the wide spectrum of dynamic spatial interaction models that have been published in the recent literature. However, the use of optimal control theory (especially in combination with multi-objective analysis and stochasticity) is still underrepresented in the field of dynamic spatial interaction analysis. The present paper aims at filling to some extent this gap.

2. Dynamic Spatial Interaction Models: a Brief Survey

In this section a brief overview of recent developments related to Spatial Interaction (S.I.) systems will be presented. First of all, it is appropriate to put forward the definition of a dynamical system as << any set of equations giving the time evolution of the state of a system from a knowledge of its previous history >> (Ott, 1981).

Such systems are in many disciplines often modelled in comparative static terms (e.g., in economics and ecology). This implies the assumption of the existence of equilibrium points without any reference to when or how the equilibrium is reached. Discontinuities are attributed to random external influences rather than to the structure of spatial systems (see, e.g., Varaiya and Wiseman, 1984).

In this framework we will classify dynamic S.I. models according to the principal methodologies (or approaches) used to analyze them, i.e., bifurcation and catastrophe theory, the theory of stochastic processes, optimal control theory and dynamic programming.

2.1 Bifurcation and Catastrophe Theory

Significantly important advances in dynamic modelling have been achieved by means of Bifurcation Theory (B.T.), which has been developed to describe dynamic systems characterized by multiple equilibria in which shifts from one equilibrium to another may involve discontinuities. Obviously these discontinuities are properties of the system rather than the result of external shocks (see again Varaiya and Wiseman, 1984). Catastrophe Theory (C.T.) (see among others, Gilmore, 1981; Poston and Stewart, 1978; Saunders, 1980; Thom, 1975; Zeeman, 1977) may be considered as a special form of B.T. (see also Casti, 1983), so that B.T. seems to offer a general framework for the analysis and classification of dynamic systems with structural discontinuities.

In the context of dynamic S.I. models, we may identify three important branches of research using B.T.. The first one arises from the modelling framework of Allen et al. (1979) related to the evolution of cities on the basis of Prigogine's well-known analysis of self-organization from bifurcation through fluctuations (Nicolis and Prigogine, 1977). It should be pointed out that Allen et al.'s model contains stochastic elements (related to the random perturbation of exogenous parameters) that affect the simulated behaviour of the system (see also Dendrinos, 1980a).

The second class is the one developed along the lines of the well-known Harris and Wilson (1978) model of retail location. It should be noted that no stochastic aspect appears in this non-linear dynamic model.

The third type stems from the approaches of Dendrinos (1978) and Dendrinos and Mullally (1980), related to urban development, on the basis of Zeeman and Thom's analysis. The main distinguishing feature in the work of these authors is the a priori specification of discontinuity, so that their analysis simply rediscovers the hypothesized catastrophe (see also Varaiya and Wiseman, 1984).

In contrast to the latter authors, (other authors such as Amson (1974) and Papageorgiou (1980)), dealing with urban growth, seek the equilibrium manifold directly and interpret then the results by means of C.T. (see Wilson, 1981).

We may also regard models based on Volterra-Lotka equations as members of the latter class; a prototype is essentially found in Dendrinos' model (1980b) on the evolution of cities.

All above-mentioned models have been so frequently described that there is no need to present them again here. We just draw attention to some interesting surveys on this field such as Andersson and Kuenne (1986); Barentsen and Nijkamp (1986a); Day (1985); Dendrinos (1980c); Griffith and Lea (1983); Nijkamp et al. (1985); Rabino (1985); Varaiya and Wiseman (1984); Wegener et al. (1986); Wilson (1981).

It is worth noting that almost all above-mentioned models involve static bifurcations (i.e., bifurcations that refer only to the solution of the "static" equation), so that we find hardly any example of "dynamic" bifurcations in the literature on urban systems.

Finally, it is necessary to mention the Chaos Theory (CH.T.) approach, closely related to B.T. and C.T., which is receiving a great deal of attention at present; CH.T. originates from turbulent-type motions in physical systems (for a review see Eckmann and Ruelle, 1985; Ott, 1981), while recently it has also been used by Dendrinos (1986) to model spatial movements of labour.

There is no doubt a need for a further elaboration of CH.T., especially in order to identify other spatial systems which can exhibit chaotic motions associated with strange attractors.

2.2 The Theory of Stochastic Processes

This category refers primarily to the class of Markov chain analyses that give rise to two different classes.

The first one is related to the class of the so-called evolutionary models based either on compartmental analysis (see, e.g., De Palma and Lefèvre, 1984; Leonardi and Campisi, 1981) or on master equation analysis (cf. Barentsen and Nijkamp, 1986b; Haken, 1983; Weidlich and Haag, 1983). These "evolutionary" models describe the evolution of the transition probabilities depending on the state of the system, so that their dynamics is far richer than the simple dynamics of stationary Markov chains (Kanaroglou et al., 1986a). Obviously also here the stationary distribution (where the spatial interaction system obtains a stable mode of operation) corresponds to the static version of the differential equations.

The second class refers to the class of discrete choice theory established by McFadden (1974). Dynamic discrete choice models have been developed so far only by a few authors (see, e.g., Ben-Akiva and De Palma, 1986; De Palma and Lefèvre, 1983, 1985; Leonardi, 1985; Sonis, 1984) owing to the difficulties involved. The most interesting feature of these models is the incorporation into the static logit form of social interaction, in addition to time.

There are also various attempts at combining panel data approaches with discrete choice theory, but the resulting methodology is essentially more descriptive rather than explanatory (see for some arguments, Fischer and Nijkamp, 1987).

Other appealing approaches link evolutionary models also to discrete choice analysis (Haag, 1986; Kanaroglou et al., 1986a, 1986b; De Palma and Lefèvre, 1983; Leonardi, 1985); in general, however, these contributions are mainly theoretical in nature or based on simulation experiments, because the empirical aspects still pose numerical problems as well as data problems.

2.3 Optimal Control and Dynamic Programming

Optimal Control Theory (O.C.T.), and in general Dynamic Programming, is a very useful and popular tool in dynamic economic systems analysis (see, among others, Kamien and Schwartz, 1981; Nijkamp,1980; Miller, 1979; Tan and Bennett, 1984), but only in recent years O.C.T. has been applied to S.I. analysis; an example can be found in Wilson (1981) where an O.C.T. approach has been proposed for controlling the evolution of shopping centres.

In the particular context of a multiperiod cumulative entropy, in recent years a fundamental result emerged by means of O.C.T., which demonstrated the dynamic connection between S.I. models and discrete choice theory (Leonardi, 1983; Nijkamp and Reggiani, 1987a).

Subsequently it has also been shown how urban decline phenomena may originate from a solution of a particular optimal control S.I. model (Nijkamp and Reggiani, 1987b). In the context of uncertainty, it has also been shown by means of stochastic O.C.T. what the effect is of a stochastic white noise process in spatial interaction and input-output analysis (Nijkamp and Reggiani, 1986). However, thus far a coherent blend of the previous approaches has not yet been achieved. At this point it might be worth extending the preceding methodology either by introducing multiple objective functions for different driving mechanisms of a dynamic S.I. system (leading to multi-criteria optimal control models) or by linking O.C.T. not only to stochastic processes but also to catastrophe and bifurcation theory. An attempt at developing such a more "integrated" approach between B.T., O.C.T. and stochasticity will be indicated in the next sections with the aim to offer new perspectives in the analysis and description of dynamic S.I. systems.

3. A Bi-Criterion Optimal Control Formulation of a Dynamic Spatial Interaction Model

In this section we will design an optimal control model for dynamic spatial interactions. The objective function is assumed to be a multiperiod cumulative entropy function (cf. Sonis, 1986). This entropy function may be regarded as a general utility (or social welfare) function for a spatial system through time. Here we use a new formulation as a generalization of Wilson's (1981) specification:

$$\max U = \int_o^T \{-\frac{1}{\beta} \sum_{i,j} T_{ij} (\log T_{ij}-1) + \sum_{i,j} T_{ij} (\frac{\alpha}{\beta} \log W_j - c_{ij})\} dt \tag{3.1}$$

where in this case the following definitions hold:

T_{ij} = volume of flows of commuters from residence i to labour market j

c_{ij} = travel costs for commuters from i to j
W_j = number of workplaces in labour market j.
Thus our model is essentially dealing with journey-from-home-to-work trips, although it can - without loss of generality - also easily be applied to other spatial interaction problems.
This formulation is essentially a bi-criterion optimal control model, where the coefficient $1/\beta$ is essentially the relative weight attached to the first criterion (i.e., the first term in brackets) (see also Nijkamp and Reggiani, 1986). This multi-temporal objective function maximizes essentially the aggregate consumers' surplus of people living in i and working in j within a certain planning horizon T. The first term in brackets represents the entropy (interaction) of our spatial system, while the second term denotes the aggregate net benefits of people living in i and travelling to j. In this bi-criterion formulation α is in fact a scale parameter.

The bi-criterion optimal control model will be maximized subject to a set of relevant constraints. Here we will assume a single (production-) constrained spatial interaction system, i.e.,

$$\sum_j T_{ij} = O_i \tag{3.2}$$

Next, we assume a simple evolution of the number of workplaces as a linear function of the volume of all inflows into j and of the initial volume of workplaces in j :

$$\dot{W}_j = \varepsilon\left(\sum_i T_{ij} - \kappa W_j\right) , \tag{3.3}$$

where the parameter ϵ represents the response rate of the system. This assumption implies an upward pressure on the number of workplaces if the attractiveness of j increases (i.e., if the capacity $\sum_i T_{ij}$ is growing) and a downward pressure if the labour force is growing (i.e., if W_j is increasing). The plausibility of this assumption rests on the idea that a rise in workplaces in j leads to (both internal and external) competition, this competition being stronger if certain capacity limits (i.e., $\sum_i T_{ij}$) are reached (thus affecting the growth in workplaces in a negative way via the accessibility parameter κ). In a formal way, this assumption is equal to Wilson's (1981) shopping model.

Now we assume that the number of workplaces, W_j, may be regarded as state variables, while the transportation flow T_{ij} will be conceived of as control variables (e.g. via price and tax regulations).

Finally, we have to impose some boundary conditions:

$$W_j(o) = W_j^o \qquad (3.4)$$

Given (3.1)-(3.4), we may now solve this bi-criterion optimal control model for dynamic spatial interactions.

4. <u>Solution of the Bi-Criterion Optimal Control Model</u>

The Lagrangean function associated with (3.1)-(3.4) is:

$$\begin{aligned} L = & -\frac{1}{\beta} \sum_{i,j} T_{ij} (\log T_{ij} - 1) + \sum_{i,j} T_{ij} \left(\frac{\alpha}{\beta} \log W_j - c_{ij}\right) \\ & + \sum_i \lambda_i \left(O_i - \sum_j T_{ij}\right) + \sum_j \Psi_j \hat{W}_j , \end{aligned} \qquad (4.1)$$

where Ψ_j is the co-state variable in our optimal control model.

The necessary conditions for a constrained maximum are:

$$\frac{\partial L}{\partial T_{ij}} = -\frac{1}{\beta} \log T_{ij} + \frac{\alpha}{\beta} \log W_j - c_{ij} - \lambda_i + \varepsilon \Psi_j = 0 \qquad (4.2)$$

or

$$T_{ij} = \exp (\alpha \log W_j - \beta c_{ij} - \beta \lambda_i + \beta \varepsilon \Psi_j) \qquad (4.3)$$

so that the following solution can be found:

$$T_{ij} = A_i \, O_i \, W_j^{\alpha} \exp (\beta \varepsilon \Psi_j - \beta c_{ij}) , \qquad (4.4)$$

with:

$$A_i = \exp (-\beta \lambda_i)/O_i \qquad (4.5)$$

or - in view of constraint (3.2) - :

$$A_i = 1/\sum_j W_j^{\alpha} \exp (\beta \varepsilon \Psi_j - \beta c_{ij}) \qquad (4.6)$$

It is easily seen that if we impose the boundary condition $\Psi_j(T)=0$, we find for $t = T$ the original Wilson model (analyzed in a shopping centre context).

Now we have to interpret our optimal control outcome. Expression (4.4) incorporates essentially a social cost-benefit measure; the term $\beta \varepsilon \dot{\Psi}_j$ embodies the imputed (shadow) price of employment growth in place j , while the term $\beta\ C_{ij}$ refers to distance friction costs. In this formulation, β is related to the sensitivity of commuters for bridging the distance between home and workplace. Finally, the parameter α occurs as a exponent to W_j , so that it reflects scale economies of labour market j (in the form of the elasticity of the inflows with respect to the size of the market).

Next we will try to analyse the equilibrium properties of solution (4.4). In previous publications (see Nijkamp and Reggiani, 1987b) it has already been shown that the stability analysis of dynamic spatial interaction models is far from easy, especially because for the non-linear dynamic expressions usually no analytical solution can be found.

It is noteworthy that if the impose the equilibrium condition $\dot{W}_j = 0$ at time period $t = T$, we find as a particular result the equilibrium conditions analyzed by Wilson (1981) in the context of a retail model. It is well-known that by varying the parameters α, β and κ various types of catastrophic behaviour may emerge. In case of $\dot{W}_j = 0$ at $t = T$, we find:

$$\sum_i T_{ij} = \kappa W_j \tag{4.7}$$

and

$$T_{ij} = A_i\ O_i\ W_j^{\alpha} \exp\ (-\ \beta c_{ij}) \tag{4.8}$$

or

$$\sum_i T_{ij} = \sum_i \left\{ \frac{O_i\ W_j^{\alpha} \exp\ (-\ \beta c_{ij})}{\sum_j W_j^{\alpha} \exp\ (-\ \beta c_{ij})} \right\} \tag{4.9}$$

This type of model has been studied quite extensively in the literature (see, among others, Beaumont et al., 1981; Chudzyńska and Slodkowski, 1984; Harris and Wilson, 1978; Harris et al., 1982; Rijk and Vorst, 1983; Wilson and Kirkby, 1980). The stability conditions for this particular case in the plane of the control variable T_{ij} and the state variable W_j , based on variations in α, β and κ , will be shown in Annex A.

However, in our general optimal control case it is more complicated to derive conditions for the whole time trajectory $0 \leq t \leq T$, as will be shown now.

First, we have to add the second necessary condition for our optimal control model, viz the co-state conditions:

$$\frac{\partial L}{\partial W_j} = - \dot{\Psi}_j \tag{4.10}$$

or

$$\frac{\alpha}{\beta} (\sum_i T_{ij}) \frac{1}{W_j} - \varepsilon \kappa \Psi_j + \dot{\Psi}_j = 0 \tag{4.11}$$

This set of J equations forms together with (3.3) a system of 2 J differential equations containing - after substitution of (4.4) - the unknowns Ψ_j and W_j :

$$\left.\begin{aligned} &\dot{\Psi}_j + \frac{\alpha}{\beta} \sum_i \left\{\frac{O_i W_j^{\alpha} \exp(\varepsilon\beta\Psi_j - \beta c_{ij})}{\sum_j W_j^{\alpha} \exp(\varepsilon\beta\Psi_j - \beta c_{ij})}\right\} \frac{1}{W_j} - \varepsilon \kappa \Psi_j = 0 \\ &\hat{W}_j - \varepsilon \sum_i \left\{\frac{O_i W_j^{\alpha} \exp(\varepsilon\beta\Psi_j - \beta c_{ij})}{\sum_j W_j^{\alpha} \exp(\varepsilon\beta\Psi_j - \Psi c_{ij})}\right\} + \varepsilon \kappa W_j = 0 \end{aligned}\right\} \tag{4.12}$$

System (4.12) is a set of complicated non-linear dynamic equations which cannot be solved analytically, so that numerical solution procedures are necessary here. However, it is possible to introduce the optimality conditions for an equilibrium:

$$\left.\begin{aligned} \dot{\Psi}_j &= 0 \\ \hat{W}_j &= 0 \end{aligned}\right\} \tag{4.13}$$

Then we find on the basis of (4.12):

$$\Psi_j = \frac{\alpha}{\beta\varepsilon} \tag{4.14}$$

as coordinates of the optimal point. The related value of W_j cannot be found analytically, but has to be derived from (4.12) in a numerical way. Thus, the solution of this problem has a surprisingly simple value. Whether or not this is a stable solution, has to checked by means of second-order conditions.

5. A Stochastic Dynamic Optimal Control Version

In this section we will introduce some external influences (e.g., a stochastic impact of the Brownian motion type), so that our optimal control problem (3.1)-(3.3) will incorporate some random components. In particular we assume that these stochastic components, representing the statistical uncertainty at the supply side of our eq. (3.3), will be of the so-called "white noise" type, in the following way:

$$d W_j = \{\varepsilon(\sum_i T_{ij} - \kappa W_j)\} dt + W_j \sigma_j dz_j \tag{5.1}$$

Eq. (5.1), which represents the stochastic version of the deterministic eq. (3.3), bears some resemblance to that used by Vorst (1985) in a shopping centre context. A further difference here is that we incorporate the stochastic differential equation (5.1) in an optimal control model. While stochastic differential equations, in the context of S.I. models, have been used so far only by Sikdar and Karmeshu (1982) and Vorst (1984), we have to notice that their use in an optimal control S.I. framework is rather novel (see Nijkamp and Reggiani, 1986). The last term at the right-hand side of eq. (5.1) represents the total stochastic perturbating force which is assumed to be proportional to the number of workplaces W_j.

We also notice that the elements σ_j are the diffusion components of the stochastic process, while the elements dz_j are the incremental changes (white noise) in a stochastic process z_j that satisfies a Wiener process (called also Brownian motion). All other relevant properties about stochastic differential equations are discussed in Arnold (1974), Kamien and Schwartz (1981), and Malliaris and Brock (1982).

Therefore we may specify the following stochastic optimal control model:

$$U^* = \max E \int_0^T \{- \frac{1}{\beta} \sum_{ij} T_{ij} (\log T_{ij} - 1) + \sum_{ij} T_{ij} (\frac{\alpha}{\beta} \log W_j - c_{ij})\} dt \tag{5.2}$$

s.t.

$$\sum_j T_{ij} = O_i \tag{5.3}$$

$$d\,W_j = g_j\,dt + W_j\,\sigma_j\,dz_j \tag{5.4}$$

where $g_j = \varepsilon(\sum_i T_{ij} - \kappa\,W_j)$ represents the deterministic part of eq. (5.4), and where E is the mathematical expectation of the weighted objective functions which has to be maximized.

The solution of (5.2)-(5.4), which follows directly from the well-known Bellman's Principle of Optimality, will be illustrated in the next section.

6. Solution of the Stochastic Dynamic Optimal Control Version

According to Malliaris and Brock (1982) the Hamilton-Jacobi-Bellman equation associated with (5.2)-(5.4) is the following:

$$-\frac{\partial U^*}{\partial t} = \max_{T_{ij}} \{ [-\frac{1}{\beta}\sum_{ij} T_{ij}\,(\log T_{ij}-1) + \sum_{ij} T_{ij}\,(\frac{\alpha}{\beta}\log W_j - c_{ij})] + \sum_j \frac{\partial U^*}{\partial W_j}\,g_j + \frac{1}{2}\sum_j \frac{\partial^2 U^*}{\partial W_j^{\,2}}\,\sigma_j^2\,W_j^{\,2}\} \tag{6.1}$$

Next, by defining now the co-state variables Ψ_j^* as:

$$\Psi_j^* = \frac{\partial U^*}{\partial W_j} \tag{6.2}$$

it is easily seen that eq. (6.1) can also be written as :

$$-\frac{\partial U^*}{\partial t} = \max_{T_{ij}} \{ [-\frac{1}{\beta}\sum_{ij} T_{ij}\,(\log T_{ij}-1) + \sum_{ij} T_{ij}\,(\frac{\alpha}{\beta}\log W_j - c_{ij})] + \sum_j \Psi_j^*\,g_j + \frac{1}{2}\sum_j \frac{\partial \Psi_j^*}{\partial W_j}\,\sigma_j^{\,2}\,W_j^{\,2}\} = \max_{T_{ij}}\ \{H^*\} \tag{6.3}$$

If we introduce now the constraints (5.3) on the control variables we get the following Lagrangean L* :

$$L^* = H^* + \sum_i \lambda_i\,(O_i - \sum_j T_{ij}) \ , \tag{6.4}$$

so that we may apply the Pontryagin Stochastic Maximum Principle (see Malliaris and Brock, 1982).

It is easily seen that the optimal solution T_{ij}^* which maximizes the Lagrangean (6.4) is the following:

$$T_{ij}^* = \exp(\alpha \log W_j - \beta c_{ij} - \beta\lambda_i + \beta\varepsilon\Psi_j^*) \qquad (6.5)$$

or

$$T_{ij}^* = A_i^* O_i W_j^\alpha \exp(\beta\varepsilon\Psi_j^* - \beta c_{ij}) \qquad (6.6)$$

where

$$\begin{aligned} A_i^* &= \exp(-\beta\lambda_i)/O_i = \\ &= 1/\sum_j W_j^\alpha \exp(\beta\varepsilon\Psi_j^* - \beta c_{ij}) \end{aligned} \qquad (6.7)$$

Eq. (6.6), which represents the optimal control solution, is formally equivalent to eq. (4.4), but here the Ψ_j^*'s satisfy the following stochastic differential equations:

$$d\Psi_j^* = -\frac{\partial L^*}{\partial W_j} dt + \sum_j \frac{\partial \Psi_j^*}{\partial W_j} W_j \sigma_j dz_j \qquad (6.8)$$

and also the following transversality conditions:

$$\left.\begin{aligned} &\Psi_j^* \{W_j(T), T\} = \frac{\partial U^*}{\partial W_j} \{W_j(T), T\} \geqq 0 \\ &\Psi_j^*(T) W_j(T) = 0 \end{aligned}\right\} \qquad (6.9)$$

so that eq. (6.6) is a stochastic expression owing to the terms Ψ_j^*. Obviously the elements Ψ_j^* do not have analytical (explicit) solutions because of the difficulties involved in the calculation of (6.8) and (6.9).

7. Epilogue

In this paper it has been shown, after a brief survey of studies dealing with dynamic S.I. models, that a formal connection between catastrophe theory and optimal control theory may exist. In particular it has been illustrated how catastrophe theory can analytically emerge from particular equilibrium solutions of an optimal control model. Furthermore random components of the Brownian motion type have been introduced in order to investigate their influence in the previous dynamic system.

The interesting result is a stochastic movement stemming from deterministic equations which are formally similar to the usual S.I. models.

As a next step, it might be worth studying whether this kind of system with initial conditions on time could lead to chaotic behaviour and consequently to a strange attractor. This latter analysis, which is still an underdeveloped field in the class of dynamic S.I. models, deserves no doubt in the future full-scale attention in efforts analyzing dynamic systems, in order to achieve a better understanding of space-time patterns.

Acknowledgement

A first draft of this paper was written while the second author was in Berkeley (USA), at the Department of Geography, under a Fullbright Grant. She would like to thank prof. A. Pred, prof. G. Leitman, prof. P. Varaiya and dr. M. Ferrari for the meaningful and stimulating discussions and all the facilities offered.

ANNEX A. Conditions of Stability and Bifurcations Based on Parameters α, β, κ in the Control Variable/State Variable Plane.

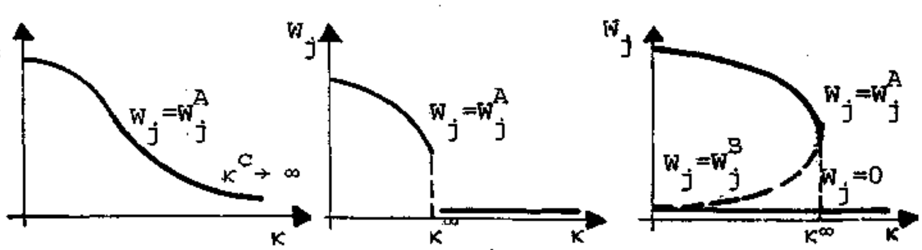
It is easy to see that the type of model (4.7) and (4.9) is formally equal to the one analyzed by Harris and Wilson (1978) for a retail trade model. Let us repeat here eqs. (4.7) and (4.9) for the sake of convenience:

$$\sum_i T_{ij} = \kappa W_j \tag{A.1}$$

$$\sum_i T_{ij} = \sum_i \left\{ \frac{O_i W_j^{\alpha} \exp(-\beta c_{ij})}{\sum_j W_j^{\alpha} \exp(-\beta c_{ij})} \right\} \tag{A.2}$$

We recall that in our transportation system the elements $\sum_i T_{ij}$ (T_{ij} are the control variables) represent the total inflows in j, the W_j's (the state variables) are the number of workplaces in j, and the O_i's are the outflows from i.

Therefore, by following Harris and Wilson, we can deduce the functional form of the inflows $\sum_i T_{ij}$ towards the workplaces W_j for different values of α (see Fig. 1).



N.B. $W_j^A \rightarrow$ stable points
$W_j^B \rightarrow$ unstable points

Fig. 1. The inflow curves and equilibrium points.

If in a way analogous to the urban retail model of Harris and Wilson, we consider the effect of varying κ (recall that κ is here an accessibility parameter which converts inflow units into workplace units), we obtain a set of equilibrium points linked to the fold catastrophe (see Fig. 2).

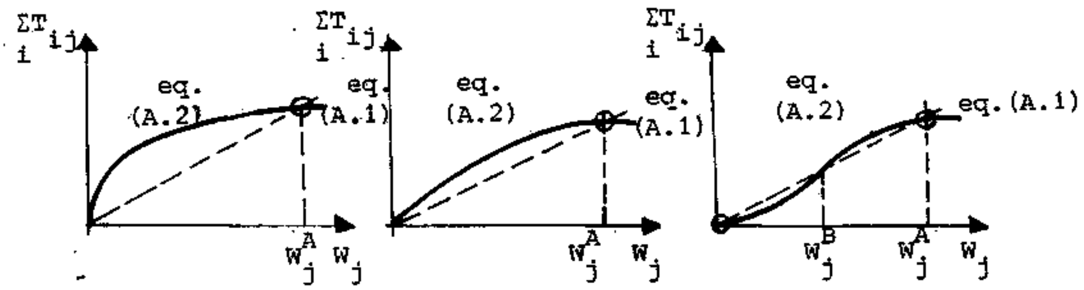


Fig. 2. The equilibrium points as a function of κ.

It is clear that there are critical values of $\kappa(\kappa^c)$, beyond which W_j will jump from W_j^A to zero (for $\alpha \geq 1$). This means that for the case $\alpha \geq 1$ there is a critical value in the accessibility beyond which the number of workplaces jumps to zero. This fact can be explained, e.g., through the phenomenon of congestion. Therefore only for $\alpha < 1$, we do not have a catastrophic behaviour, since in this case there is a unique positive equilibrium (see also Kaashoek and Vorst, 1984). It should be noted that also changes of β could lead to possible jumps in the workplaces (see Wilson, 1981).

The conclusion is that at time $t = T$ and under the condition that eq. (3.2) is in equilibrium, certain smooth parameter changes could lead to discrete changes in the state variables W_j.

References

Allen, P.M., J.L. Deneubourg, M. Sanglier, F. Boon, and A. De Palma, The Dynamics of Urban Evolution, Final Report to the U.S. Department of Transportation, Washington D.C., 1978.

Amson, J.C., Equilibrium and Catastrophe Modes on Urban Growth, Space-Time Concepts in Urban and Regional Models (E.L. Cripps, ed.), Pion, London, 1974, pp. 108-128.

Andersson, Å.E. and R.E. Kuenne, Regional Economic Dynamics, Handbook of Regional and Urban Economics, (P. Nijkamp, ed.), North-Holland, Amsterdam, 1986, pp. 201-253.

Arnold, L., Stochastic Differential Equations: Theory and Applications, J. Wiley & Sons, New York, 1974.

Barentsen, W. and P. Nijkamp, Modelling Non-Linear Processes in Time and Space, Research Memorandum 1986-50, Dept. of Economics, Free University, Amsterdam, 1986a.

Barentsen, W. and P. Nijkamp, Spatial Synergetics and Spatial Multipliers in Dynamic Models, Paper presented at the Thirty-Third North American Meetings of the Regional Science Association, Columbus, Ohio, 1986b.

Beaumont, J.R., M. Clarke and A.G. Wilson, The Dynamics of Urban Spatial Structure: Some Exploratory Results Using Difference Equations and Bifurcation Theory, Environment and Planning A, vol. 13, 1981, pp. 1473-1483.

Beckmann, M. and T. Puu, Spatial Economics: Density, Potential and Flow, North-Holland Publ. Co., Amsterdam, 1985.

Ben-Akiva, M. and A. De Palma, Analysis of a Dynamic Residential Location Choice Model with Transaction Costs, Journal of Regional Science, vol. 26, no. 2, 1986, pp. 321-341.

Casti, J., Topological Methods for Social and Behavioural Systems, RR-83-3, International Institute for Applied Systems Analysis, Laxenburg, Austria, 1983.

Chudzynska, I. and Z. Stodkowski, Equilibria of a Gravity Demand Model, Environment and Planning A, 1984, vol. 16, pp. 185-200.

Day, R.H., Dynamical Systems Theory and Complicated Economic Behaviour, Environment and Planning B, vol. 12, 1985, pp. 55-64.

Dendrinos, D.S., Urban Dynamics and Urban Cycles, Environment and Planning A, vol. 10, 1978, pp. 43-49.

Dendrinos, D.S., On Urban Evolution and Bifurcation Theory, Catastrophe Theory in Urban and Transport Analysis (D.S. Dendrinos, ed.), U.S. Department of Transportation, Washington D.C., 1980a, pp. 3-20.

Dendrinos, D.S., A Basic Model of Urban Dynamics Expressed as a Set of Volterra-Lotka Equations, Catastrophe Theory in Urban and Transport Analysis (D.S. Dendrinos, ed.), U.S. Department of Transportation, Washington D.C., 1980b, pp. 79-103.

Dendrinos, D.S., Catastrophe Theory in Urban and Transport Analysis, U.S. Department of Transportation, Washington D.C., 1980c.

Dendrinos, D.S., On the Incongruous Spatial Employment Dynamics, Technological Change, Employment and Spatial Dynamics (P.Nijkamp, ed.), Springer-Verlag, Berlin, 1986, pp. 321-339.

Dendrinos, D.S. and H. Mullally, Fast and Slow Equations: the Development Patterns of Urban Settings, Catastrophe Theory in Urban and Transport Analysis (D.S. Dendrinos, ed.), U.S. Department of Transportation, Washington D.C., 1980, pp. 59-77.

De Palma, A. and C. Lefèvre, Individual Decision-Making in Dynamic Collective Systems, Journal of Mathematical Sociology, 9, 1983, pp. 103-124.

De Palma, A. and C. Lefèvre, The Theory of Deterministic and Stochastic Compartmental Models and its Applications: The State of the Art, Sistemi Urbani, 3, 1984, pp. 281-323.

De Palma, A. and C. Lefèvre, Residential Change and Economic Choice Behaviour, Regional Science and Urban Economics, 15, 1985, pp. 421-434.

Eckmann, J.P. and D. Ruelle, Ergodic Theory of Chaos and Strange Attractors, Reviews of Modern Physics, vol. 57, no. 3, 1985, pp. 617-656.

Fischer, M.M. and P. Nijkamp, From Static Towards Dynamic Discrete Choice Modelling: A State of the Art Review, Regional Science and Urban Economics, vol. 17, no. 1, 1987, pp. 3-28.

Gilmore, R., Catastrophe Theory for Scientists and Engineers, Wiley, New York, 1981.

Griffith, D. and A.C. Lea (eds.), Evolving Geographical Structures, Martinus Nijhoff, The Hague, 1983.

Haag, G., A Stochastic Theory for Residential and Labour Mobility, Technological Change, Employment and Spatial Dynamics, (P. Nijkamp, ed.), Springer-Verlag, Berlin, 1986, pp. 340-357.

Haken, H., Synergetics, Springer-Verlag, Berlin, 1983.

Harris, B. and A.G. Wilson, Equilibrium Values and Dynamics of Attractiveness Terms in Production-Constrained-Spatial-Interaction Models, Environment and Planning A, vol. 10, 1978, pp. 371-388.

Harris, B., J.H. Choukroun and A.G. Wilson, Economies of Scale and the Existence of Supply-Side Equilibria in a Production-Constrained Spatial-Interaction Model, Environment and Planning A, vol. 14, 1982, pp. 823-837.

Kaashoek, J.F. and A.C.F. Vorst, The Cusp Catastrophe in the Urban Retail Model, Environment and Planning A, vol. 16, 1984, pp. 851-862.

Kamien, M.I. and N.L. Schwartz, Dynamic Optimization, North-Holland Publ. Co., Amsterdam, 1981.

Kanaroglou, P., K.L. Liaw and Y.Y. Papageorgiou, An Analysis of Migratory Systems: 1. Theory, Environment and Planning A, vol. 18, 1986a, pp. 913-928.

Kanaroglou, P., K.L. Liaw and Y.Y. Papageorgiou, An Analysis of Migratory Systems: 2. Operational Framework, Environment and Planning A, vol. 18, 1986b, pp. 1039-1060.

Leonardi, G., An Optimal Control Representation of a Stochastic Multistage Multiactor Choice Process, Evolving Geographical Structures (D.A. Griffith and A.C. Lea, eds.), Martinus Nijhoff, The Hague, 1983, pp. 61-72.

Leonardi, G., A Stochastic Multi-Stage Mobility Choice Model, Optimization and Discrete Choice in Urban Systems (G. Hutchinson, P. Nijkamp and M. Batty, eds.), Springer-Verlag, Berlin, 1985, pp. 132-147.

Leonardi, G. and D. Campisi, Dynamic Multistage Random Utility Choice Processes: Models in Discrete and Continuous Time, Paper presented at the Second Meeting of the Italian Section of the Regional Science Association, Naples, 1981.

Malliaris, A.G. and W.A. Brock, Stochastic Methods in Economics and Finance, North-Holland Publ. Co., Amsterdam, 1982.

McFadden, D., Conditional Logit Analysis of Qualitative Choice Behaviour, Frontiers in Econometrics (P. Zarembka, ed.), Academic Press, New York, 1974, pp. 105-142.

Miller, R.E., Dynamic Optimization and Economic Applications, McGraw-Hill, New York, 1979.

Nicolis, G. and I. Prigogine, Self-Organization in Non-Equilibrium Systems, J. Wiley, New York, 1977.

Nijkamp, P., Environmental Policy Analysis, J. Wiley, New York, 1980.

Nijkamp, P. and A. Reggiani, Spatial Interaction and Input-Output Models: A Dynamic Stochastic Multi-Objective Framework, Paper presented at the Thirty-Third North American Meetings of the Regional Science Association, Columbus, Ohio, 1986.

Nijkamp, P. and A. Reggiani, Spatial Interaction and Discrete Choice: Statics and Dynamics, Contemporary Developments in Quantitative Geography (J. Hauer, H. Timmermans and N. Wrigley, eds.), D. Reidel Publishing Company, Dordrecht, 1987a (forthcoming).

Nijkamp, P. and A. Reggiani, Analysis of Dynamic Spatial Interaction Models by means of Optimal Control, Geographical Analysis, 1987b (forthcoming).

Nijkamp, P., A. Rima and L. van Wissen, Spatial Mobility in Models for Structural Urban Dynamics, Transport and Mobility in an Era of Transition (G.R.M. Jansen, P. Nijkamp and C.J. Ruijgrok, eds.), North-Holland Publ. Co., Amsterdam, 1985, pp. 121-137.

Ott, E., Strange Attractors and Chaotic Motions of Dynamical Systems, Reviews of Modern Physics, vol. 53, no. 4, 1981, pp. 655-671.

Papageorgiou, Y.Y., On Sudden Urban Growth, Environment and Planning A, vol. 12, 1980, pp. 1035-1050.

Poston, T. and I. Stewart, Catastrophe Theory and its Applications, Pitman Publ., New York, 1978.

Prigogine, I., Time, Irreversibility and Randomness, The Evolutionary Vision (E. Jantsch, ed.), Westview Press, Boulder, 1981, pp. 36-52.

Rabino, G., Modelli Dinamici del Sistema Integrato Territorio e Trasporti, Territorio e Trasporti: Modelli Matematici per l'Analisi e le Pianicazione (A. Reggiani, ed.), Franco Angeli, Milano, 1985, pp. 67-82.

Rijk, F.J.A. and A.C.F. Vorst, Equilibrium Points in an Urban Retail Model and their Connection with Dynamical Systems, Regional Science and Urban Economics, vol. 13, 1983, pp. 383-399.

Saunders, P., An Introduction to Catastrophe Theory, Cambridge University Press, Cambridge, 1980.

Sikdar, P.K. and Karmeshu, On Population Growth of Cities in a Region: a Stochastic Nonlinear Model, Environment and Planning A, vol. 14, 1982, pp. 585-590.

Sonis, M., Dynamic Choice of Alternatives, Innovation Diffusion and Ecological Dynamics of Volterra-Lotka, London Papers in Regional Science 14. Discrete Choice Models in Regional Science (D. Pitfield, ed.), Pion, London, 1984, pp. 29-43.

Sonis, M., A Unified Theory of Innovation Diffusion, Dynamic Choice of Alternatives, Ecological Dynamics and Urban/Regional Growth and Decline, Paper presented at the Conference on Innovation Diffusion, Venice, 1986.

Tan, K.C. and R.J. Bennett, Optimal Control of Spatial Systems, Allen & Unwin, London, 1984.

Thom, R., Structural Stability and Morphogenesis, Addison-Wesley, Reading, MA, 1975.

Varaiya, P. and M. Wiseman, Bifurcation Models of Urban Developments, Regional and Industrial Development, 1984.

Vorst, A.C.F., A Stochastic Version of the Urban Retail Model, Environment and Planning A, vol. 17, 1985, pp. 1569-1580.

Wegener, M., F. Gnad and H. Vannahme, The Time Scale of Urban Change, Advances in Urban Systems Modelling (B. Hutchinson and M. Batty, eds.), North-Holland, Amsterdam, 1986.

Weidlich, W. and G. Haag, Concepts and Models of a Quantitative Sociology, Springer-Verlag, Berlin, 1983.

Wilson, A.G., Catastrophe Theory and Bifurcation, Croom Helm, London, 1981.

Wilson, A.G. and M.J. Kirkby, Mathematics for Geographers and Planners, Oxford University Press, Oxford, 1980.

Zeeman, E.C., Catastrophe Theory: Selected Papers 1972-1977, Addison-Wesley, Reading, Mass., 1977.