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DEMAND SYSTEMS THAT ARE POLYNOMIAL IN INCOME

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by

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Summary

In this note we discuss the PUM, which is a demand system consisting of polynomials in income of degree N with coefficients being functions of prices. In particular, we derive the general forms of these coefficients for arbitrary value of N, when the system should display utility maximization under a budget restriction.

1. Introduction

If a polynomial demand system can be written as:

$$(1.1) \quad q_k(p,y) = c_{k0}(p) + c_{k1}(p)y + c_{k2}(p)y^2 + \ldots + c_{kN}(p)y^N$$

with $k=1,\ldots,K$,

where q_k = quantity consumed of commodity k,

p_k = price of k,

p = vector of all prices $p_1,\ldots,p_K$ and

y = "income" to be allocated to goods $1,\ldots,K$,

and the $c_{kn}(p)$ are functions of prices such that the conditions of additivity, homogeneity and utility maximization are satisfied, we call it a PUM system (Polynomial demand system based on Utility Maximization).

Additivity requires

$$(1.2) \qquad \sum_{k=1}^{K} p_k q_k = y.$$

For homogeneity the coefficients $c_{kn}(p)$ should be homogeneous of degree - n for all n = 0,1,...,N and all k. Utility maximization requires that the K*(N+1) matrix $C(p)=[c_{kn}(p)]$ is such that the Slutsky matrix $S=[s_{km}]$ with $s_{km}=\partial q_k/\partial p_m + q_m \cdot \partial q_k/\partial y$ is symmetric and negative semi-definite. Below we will show the implications of the conditions of additivity, homogeneity and symmetry for the functions $c_{kn}(p)$. The implications of the negative semi-definitiveness are not analyzed here.

Our line of reasoning is as follows:

First, we prove that for all n,r = 2,...,N the ratio $\alpha_{nr}=c_{kn}(p)/c_{kr}(p)$ is independent of k. This means that the last N-1 columns of $C(p)$ are proportional, given p. Hence $C(p)$ has a rank that is at most 3. This strong result is derived by Gorman [1981] in a more general context. For our special case (1.1) a much simpler proof is available.

Second, we simplify (1.1) in assuming, that all functions c_{k0} are zero. For this simplified system we shall derive the implications of the symmetry condition for functions c_{kn}. The K*N matrix of coefficients of the simplified system will be called $C^*(p)$.

Third, noticing that the matrix $C^*(p)$ contains at most two independent columns, the first, called $c^1(p)$, and, say, the last one, called $c^N(p)$, we provide a lemma (lemma 2) which gives us the specifications of these two columns.

Fourth, we specify the remaining columns of $C^*(p)$, i.e. we specify the factors of proportionality α_{nN} between each of these columns and $c^N(p)$.

Fifth, we return to the system (1.1).

In the next section we state the above-mentioned lemma, preceded by lemma 1 that helps to simplify our presentation. In section 3 we present our result; we end with a short discussion in section 4.

2. Two lemmas

In analyzing Slutsky symmetry lemma 1 appears to be useful.

Lemma 1. (Inverse Young theorem.) If $f_1, \ldots f_K$ are functions of the vector $p=(p_1, \ldots p_K)$ with continuous first derivatives such that for each function f_k with $k=1, \ldots, K$ and for all vectors p from a certain domain D and for all $k, m = 1, \ldots, K$.

$$(2.1) \quad \frac{\partial f_k(p)}{\partial p_m} = \frac{\partial f_m(p)}{\partial p_k} ,$$

then there exists a function F of $p_1, \ldots, p_K$ such that for all $p \in D$ and all $k = 1, \ldots, K$

$$(2.2) \quad f_k(p) = \frac{\partial F(p)}{\partial p_k}.$$

Proof: see Van Daal and Merkies [1984, pp 137-139].

The announced second lemma is:

Lemma 2. The demand system

$$(2.3) \quad q_k = \varphi_k(p)y + \psi_k(p)y^h$$

with $h = 2, 3, \ldots$, is compatible with utility maximization if and only if the φ_k and the ψ_k are such that (2.3) can be written as:

$$(2.4) \quad q_k = \frac{1}{a} \frac{\partial a}{\partial p_k} y + \frac{1}{a^{h-1}} \frac{\partial H}{\partial p_k} y^h ,$$

where $a(>0)$ and H are functions of prices that are homogeneous of degree 1 and 0 respectively and, furthermore, a and H are such that the corresponding cost of utility function

$$(2.5) \quad c(p,u) = a(\varepsilon H - u)^{\frac{1}{\varepsilon}} ,$$

with $\varepsilon = 1-h$, is concave.

For ease of exposition we omitted the arguments p of the functions a and H, just as we will do below for the functions φ_k and ψ_k. Note that, because of the budget restriction, the case that all functions φ_k are zero cannot occur; this means that the function a cannot be a constant.

Proof: The Slutsky element s_{km} of (2.3) is a polynomial in y of the form

(2.6) $\alpha_{km}y + \beta_{km}y^h + \gamma_{km} y^{2h-1}$,

it is symmetric in k and m if symmetry holds for all three coefficients. It turns out that γ_{km} is equal to $h\psi_k\psi_m$ and, therefore, is symmetric in k and m. Symmetry of the other two coefficients requires the fulfilment of the following two identities:

(2.7) $\dfrac{\partial \varphi_k}{\partial p_m} = \dfrac{\partial \varphi_m}{\partial p_k}$

(2.8) $\dfrac{\partial \psi_k}{\partial p_k} + (h-1)\varphi_m\psi_k = \dfrac{\partial \psi_m}{\partial p_k} + (h-1) \varphi_k \psi_m$

for all k and m.

Because of (2.7) and lemma 1, there is a function F of prices such that for all k = 1,...,K the functions $\varphi_1,...,\varphi_K$ can be written as:

(2.9) $\varphi_k = \dfrac{\partial F(p)}{\partial p_k}$,

or, taking F(p) = log a(p) with a(p) > 0 for all p,

(2.10) $\varphi_k = \dfrac{\partial \log a}{\partial p_k} = \dfrac{1}{a} \dfrac{\partial a}{\partial p_k}$.

Because of (2.3) and the budget restriction (1.2) we have:

(2.11) $\displaystyle\sum_k p_k\varphi_k = \sum_k \dfrac{p_k}{a} \dfrac{\partial a}{\partial p_k} = 1$,

hence a is homogeneous of degree 1 in prices. Inserting (2.10) into (2.8) and multiplying the result with a^{h-1} yields:

$$(2.12) \qquad a^{h-1} \frac{\partial \psi_k}{\partial p_m} + (h-1) a^{h-2} \frac{\partial a}{\partial p_m} \psi_k = a^{h-1} \frac{\partial \psi_m}{\partial p_k} + (h-1)a^{h-2} \frac{\partial a}{\partial p_k} \psi_m ,$$

or

$$(2.13) \qquad \frac{\partial}{\partial p_m} (a^{h-1} \psi_k) = \frac{\partial}{\partial p_k} (a^{h-1}\psi_m).$$

Hence, because of lemma 1, there is a function H such that for all $k=1,\ldots,K$ the functions ψ_k obey:

$$(2.14) \qquad a^{h-1}\psi_k = \frac{\partial H}{\partial p_k} .$$

For this function H the budget restriction implies:

$$(2.15) \qquad \sum_k p_k a^{h-1}\psi_k = \sum_k p_k \frac{\partial H}{\partial p_k} = 0 ,$$

hence H is homogeneous of degree zero. This leads to (2.4).
Shephard's theorem applied to (2.5) gives also (2.4).

Remarks

1. Note that (2.4) and (2.5) are not violated if we transform H first by some arbitrary differentiable function F of only one variable. Then we write $\partial H/\partial p_k .f(H)$ instead of $\partial H/\partial p_k$ where $f = F'$, or, more explicitly,

$$(2.16) \qquad \psi_k = \frac{1}{a^{h-1}} \frac{\partial H}{\partial p_k} f(H).$$

Although this does not entail more generality we shall need it in the next section.

2. In fact (2.4) holds for each $h \in R$ that is unequal to 1. The cost of utility function is, more general,

$$(2.17) \quad c(p,y) = a(\varepsilon H + \text{sgn } \varepsilon.u)^{\frac{1}{\varepsilon}},$$

where sgn $\varepsilon = + 1$ if $\varepsilon > 0$ and -1 if $\varepsilon < 0$. If $\varepsilon > 0$ then εH is the minimum level that u can attain and if $\varepsilon < 0$ then εH is 'bliss-level'.

3. Three theorems

<u>Theorem 1.</u> The rank of the matrix $C(p)$ of coefficients of the system (1.1) is at most 3.

<u>Proof.</u> Omitting the arguments of all functions from now on and indicating differentiation with respect to a price p_m by an additional index m preceded by a comma, we can express the Slutsky element s_{km} of (1.1) as follows

$$(3.1) \quad s_{km} = (c_{k0,m} + c_{k1,m} y + \ldots + c_{kN,m} y^N) +$$
$$+ \left(c_{m0} + c_{m1}y + \ldots + c_{mN}y^N\right) \left(c_{k1} + 2c_{k2} y + \ldots + Nc_{kN} y^{N-1}\right).$$

This is a polynomial of degree 2N-1. Slutsky symmetry requires that all coefficients are separately symmetric in k and m for all k,m = 1,...,K. For the proof of theorem 1 we only need the last N-1 coefficients. Therefore, we write s_{km} as follows

$$s_{km} = \sum_{r=0}^{N} \eta_{kmr} \, y^r +$$

$$+ y^{N+1} \left(Nc_{kN} \, c_{m2} + (N-1)c_{k(N-1)} \, c_{m3} + \ldots + 3c_{k3} \, c_{m(N-1)} + 2c_{k2}c_{mN} \right) +$$

$$\vdots$$

$$+ y^{N+n} \sum_{i=0}^{N-n-1} (N-i) \, c_{k(N-i)} \, c_{m(n+1+i)} +$$

$$\vdots$$

$$+ y^{2N-4} \left(Nc_{kN} \, c_{m(N-3)} + (N-1)c_{k(N-1)} \, c_{m(N-2)} + (N-2)c_{k(N-2)}c_{m(N-1)} + \right.$$

$$(3.2) \quad \left. + (N-3)c_{k(N-3)} \, c_{mN} \right) +$$

$$+ y^{2N-3} \left(Nc_{kN} \, c_{m(N-2)} + (N-1)c_{k(N-1)} \, c_{m(N-1)} + (N-2)c_{k(N-2)} \, c_{mN} \right) +$$

$$+ y^{2N-2} \left(Nc_{kN} \, c_{m(N-1)} + (N-1)c_{k(N-1)} \, c_{mN} \right) +$$

$$+ y^{2N-1} \, Nc_{kN} \, c_{mN} \ .$$

The coefficients of y^{2N-1} are always symmetric. Symmetry of the coefficients of y^{2N-2} requires

$$(3.3) \quad Nc_{kN} \, c_{m(N-1)} + (N-1)c_{k(N-1)} \, c_{mN} = Nc_{mN} \, c_{k(N-1)} + (N-1)c_{m(N-1)} \, c_{kN},$$

or

$$(3.4) \quad c_{kN} \, c_{m(N-1)} = c_{mN} \, c_{k(N-1)}.$$

This identity implies that we always must have, for all $k=1,\ldots,K$,

$$(3.5) \quad c_{k(N-1)} = \alpha_{(N-1)N} c_{kN},$$

where $\alpha_{(N-1)N}$ is a function of prices that is independent of k.
The coefficient of y^{2N-3} of (3.2) consists of three terms of which the middle one is symmetric; hence the sum of the other two terms has to be symmetric which leads to the requirement $c_{kN} \, c_{m(N-2)} = c_{mN}c_{k(N-2)}$, or

$$(3.6) \quad c_{k(N-2)} = \alpha_{(N-2)N} c_{kN},$$

for all $k=1,\ldots,K$ with $\alpha_{(N-2)N}$ another function of prices that is

independent of k. Inserting (3.5) and (3.6) into the coefficients of y^{2N-4} leads to the conclusion that only the first and the last term of these coefficients are non-symmetric. This leads to a ratio between the $c_{k(N-3)}$ and c_{kN} that is a function $\alpha_{(N-3)N}$ of prices independent of k. Continuing this process of substitution until the term with y^{N+1} leads to

$$(3.7) \qquad \frac{c_{kn}}{c_{kN}} = \alpha_{nN}$$

for all k=1,...,K and all n ≥ 2.
Alternatively written, we have for all n=2,...,N-1

$$(3.8) \qquad \begin{bmatrix} c_{1n} \\ \cdot \\ c_{Kn} \end{bmatrix} = \alpha_{nN} \begin{bmatrix} c_{1N} \\ \cdot \\ c_{KN} \end{bmatrix},$$

which shows that all columns of the matrix C(p) except the first two are dependent upon he last one. This means that each row k of C(p) is determined after c_{kN}, c_{k0}, c_{k1} and α_{nN} have been chosen, i.e. C(p) has a rank that is at most 3.

Theorem 2. The most general form of a utility consistent demand system that is a polynomial in income without a constant term is

$$(3.9) \qquad q_k(p,y) = \frac{1}{a} \frac{\partial a}{\partial p_k} y + \sum_{n=2}^{N} \frac{1}{a^{n-1}} \frac{\partial H}{\partial p_k} f_n(H)y^n ,$$

where a is homogeneous of degree 1 in prices, H of degree 0, and where the f_n (n=2,...,N) are arbitrary functions of H; in addition, all these functions have to be such that the second-order conditon of negative semi-definiteness of the Slutsky matrix is fulfilled.

Proof: Because of theorem 1 we can write $c_{kn} = \alpha_{nN}c_{kN}$ for k=2,...,N-1; hence (1,1) with $c_{k0} = 0$ becomes

$$(3.10) \qquad q_k(p,y) = c_{k1} y + \alpha_{2N} c_{kN} y^2 + \alpha_{3N} c_{kN} y^3 + \ldots + \alpha_{(N-1)N} c_{kN} y^{N-1} +$$
$$+ c_{kN} y^N .$$

If all functions α_{nN} are zero (3.10) coincides with (2.3) and according to lemma 2 also with the special case of (3.9) with $f_n(H)=0$ for $n=2,\ldots,N-1$, provided the c_{k1} and the c_{kN} are:

$$(3.11) \qquad c_{k1} = \frac{1}{a}\frac{\partial a}{\partial p_k}$$

and

$$(3.12) \qquad c_{kN} = \frac{1}{a^{N-1}}\frac{\partial H}{\partial p_k}f_N(H),$$

for all $k=1,\ldots,K$, where a (non-zero) and H are functions of prices, homogeneous of degree 1 and 0 respectively and f_N is an arbitrary function of only one argument with F_N as a primitive function; see remark 1 after lemma 2. So (3.9) cannot hold in general unless (3.11) and (3.12) can be satisfied. We will now show that, if this is the case, the values of α_{nN} must satisfy

$$(3.13) \qquad \alpha_{nN}c_{kN} = \frac{1}{a^{n-1}}\frac{\partial H}{\partial p_K}f_n(H)$$

for $n=2,\ldots,N-1$, or, in view of (3.12)

$$(3.14) \qquad \alpha_{nN} = a^{N-n}\frac{f_n(H)}{f_N(H)}.$$

In elaborating the Slutsky coefficient we will write α_n instead of α_{nN} from now on. As from (3.7) $c_{kn,m} \equiv \partial c_{kn}/\partial p_m = \partial(\alpha_n c_{kN})/\partial p_m = \alpha_n c_{kN,m} + \alpha_{n,m} c_{kN}$ we may write:

$$s_{km} = \sum_{n=1}^{N} c_{kn,m}y^n + \sum_{n=1}^{N} c_{mn}y^n \sum_{n=1}^{N} n\,c_{kn}\,y^{n-1}$$

$$= c_{k1,m}y + \left(\alpha_2 c_{kN,m} + \alpha_{2,m}\,c_{kN}\right)y^2 + \ldots +$$

$$+ \left(\alpha_{N-1}\,c_{kN,m} + \alpha_{N-1,m}\,c_{kN}\right)y^{N-1} + c_{kN,m}y^N + \left(c_{m1}y + \alpha_2 c_{mN}y^2 + \ldots + \right.$$

$$\left. + \alpha_{N-1}c_{mN}y^{N-1} + c_{mN}y^N\right).$$

$$\cdot \left(c_{k1} + 2\alpha_2 c_{kN} y + \ldots + (N-1)\alpha_{N-1} c_{kN} y^{N-2} + N c_{kN} y^{N-1} \right) =$$

$$= y \left(c_{k1,m} + c_{k1} c_{m1} \right) +$$

$$(3.15) \quad + y^2 \left(\alpha_2 c_{kN,m} + \alpha_{2,m} c_{kN} + 2\alpha_2 c_{kN} c_{m1} + \alpha_2 c_{mN} c_{k1} \right) +$$

$$+ y^3 \left(\alpha_3 c_{kN,m} + \alpha_{3,m} c_{kN} + 3\alpha_3 c_{kN} c_{m1} + 2(\alpha_2)^2 c_{kN} c_{mN} + \alpha_3 c_{mN} c_{k1} \right) +$$

$$\vdots$$

$$+ y^n \left(\alpha_n c_{kN,m} + \alpha_{n,m} c_{kN} + n \alpha_n c_{kN} c_{m1} + \sum_{i=1}^{n-2} (n-i) \alpha_{n-i} \alpha_{i+1}) c_{kN} c_{mN} + \right.$$

$$\left. + \alpha_n c_{mN} c_{k1} \right) +$$

$$\vdots$$

$$+ y^N \left(c_{kN,m} + N c_{kN} c_{m1} + \sum_{i=1}^{N-2} (N-i) \alpha_{N-i} \alpha_{i+1}) c_{kN} c_{mN} \right.$$

$$\left. + c_{mN} c_{k1} \right) + \sum_{r=N+1}^{2N-1} \sigma_{kmr} y^r .$$

Note that for all k,m and r=N+1,...,2N-1 we have $\sigma_{kmr} = \sigma_{mkr}$ because of theorem 1, hence we now only need to establish the symmetry of the first N terms, because the last N-1 are already symmetric. The first term of the last member, that with y, is symmetric in k and m because of (3.11). It can easily be seen that the terms with y^n for n=2,...,N-1 are symmetric if and only if for all these values of n

$$(3.16) \quad \alpha_n c_{kN,m} + \alpha_{n,m} c_{kN} + n\alpha_n c_{kN} c_{m1} + \alpha_n c_{mN} c_{k1} =$$

$$= \alpha_n c_{mN,k} + \alpha_{n,k} c_{mN} + n\alpha_n c_{mN} c_{k1} + \alpha_n c_{kN} c_{m1} .$$

Rearranging this and using (3.11) gives

$$(3.17) \quad \frac{\partial(\alpha_n c_{kN})}{\partial p_m} + \frac{n-1}{a} \frac{\partial a}{\partial p_m} (\alpha_n c_{kN}) = \frac{\partial(\alpha_n c_{mN})}{\partial p_k} + \frac{n-1}{a} \frac{\partial a}{\partial p_k} (\alpha_n c_{mN}) .$$

Multiplying this with a^{n-1} we see that (3.17) implies:

$$(3.18) \quad \frac{\partial(a^{n-1}\alpha_n c_{kN})}{\partial p_m} = \frac{\partial(a^{n-1}\alpha_n c_{mN})}{\partial p_k} .$$

This means, according to lemma 1, that there are some functions G_n of

prices such that for all $k=1,\ldots,K$ and $n=2,\ldots,N-1$:

(3.19) $\quad a^{n-1}\alpha_n c_{kN} = \dfrac{\partial G_n}{\partial p_k}$.

Combining this with (3.12) gives

(3.20) $\quad a^{n-1}\alpha_n \dfrac{1}{a^{N-1}} \dfrac{\partial F_N(H)}{\partial p_k} = \dfrac{\partial G_n}{\partial p_k}$.

Hence for all $n=2,\ldots,N-1$ we have

(3.21) $\quad \dfrac{\dfrac{\partial F_N(H)}{\partial p_1}}{\dfrac{\partial G_n}{\partial p_1}} = \dfrac{\dfrac{\partial F_N(H)}{\partial p_2}}{\dfrac{\partial G_n}{\partial p_2}} = \ldots = \dfrac{\dfrac{\partial F_N(H)}{\partial p_K}}{\dfrac{\partial G_n}{\partial p_K}}$.

Consequently, $F_N(H)$ and G_n are functionally dependent, i.e there are functions Φ_n and Ψ_n such that

(3.22) $\quad G_n = \Phi_n(F_n(H)) = \Psi_n(H)$;

see, e.g. Burkill an Burkill (1970). Let ψ_n be the derivative of Ψ_n with respect to H. Relations (3.20) then become

(3.23) $\quad \dfrac{\alpha_n}{a^{N-n}} f_N(H) \dfrac{\partial H}{\partial p_k} = \psi_n(H) \dfrac{\partial H}{\partial p_K}$,

for all $k=1,\ldots,K$. Hence

(3.24) $\quad \alpha_n = a^{N-n} f_n(H)/f_N(H)$

with $f_n(H) = \psi_n(H)$.

This proves the theorem.
The general case (1.1), with a constant term, can now easily be treated. Because every polynomial in y can also be written as a polynomial in y-z, where z is arbitrary, we can prove:

Theorem 3. The most general form of a utility consistent demand system whose equations are polynomials in income y is

$$(3.25) \qquad q_k = \frac{\partial \theta}{\partial p_k} + \frac{1}{a} \frac{\partial a}{\partial p_k} \, (y - \theta) + \sum_{n=2}^{N} \frac{1}{a_{n-1}} \, \frac{\partial H}{\partial p_k} \, f_n(H) \, (y - \theta)^n,$$

where a and θ are linear-homogeneous functions of prices, H is a zero homogenous function of prices and the f_n are functions of only one argument; all these functions have to be such that the matrix of Slutsky elements is negative semi-definite.

Proof

As (3.25) appears to satisfy additivity, homogeneity and the integrability conditions it is PUM. In order to prove the necessity of (3.25) consider the system

$$(3.26) \qquad q_k = \psi_k + \sum_{n=1}^{N} c_{kn} \, (y - \theta)^n,$$

where $k = 1, \ldots, K$, with ψ_k, c_{kn} and θ functions of p only. First, we shall show that for (3.26) being PUM it is necesary that θ is linear-homogeneous in p. Then we apply theorem 2 to show that the c_{kn} have to have the forms found in (3.25) and, subsequently, we show that each ψ_k has to be the derivative of θ with respect to p_k.

Applying Euler's theorem on homogeneous functions to (3.26) yields the following identity in p and y:

$$\sum_{m=1}^{K} p_m \, \psi_{k,m} + \sum_{m=1}^{K} \sum_{n=1}^{N} p_m \, c_{kn,m} \, (y - \theta)^n +$$

(3.27)

$$-\sum_{m=1}^{K} p_m \, \frac{\partial \theta}{\partial p_m} \sum_{n=1}^{N} n \, c_{kn} \, (y - \theta)^{n-1} + y \sum_{n=1}^{N} n \, c_{kn} \, (y - \theta)^{n-1} = 0,$$

where, again, differentiation of the ψ_k and the c_{kn} with respect to p_m is indicated by an index m preceded by a comma. The left-hand side of (3.27)

can be considered as a polynomial in y of degree N. The identity can, therefore, only be fulfilled if all coefficients of this polynomial are zero. The coefficient of y^N is equal to

$$(3.28) \quad \sum_m p_m \, c_{kN,m} + Nc_{kN} = 0.$$

This implies that c_{kN} is homogeneous of degree -N and, because c_{kN} may be identically zero, $c_{k(N-1)}$ must be homogeneous of degree -(N-1) and so on. About the coefficient of y^{N-1} we can state:

$$(3.29) \quad -N\theta \sum_m p_m \, c_{kN,m} - N \, c_{kN} \sum_m p_m \frac{\partial \theta}{\partial p_m} - N(N-1) \, \theta c_{kN} + \sum_m p_m c_{k(N-1),m} +$$
$$+ \quad (N-1) \, c_{k(N-1)} = 0.$$

Due to the homogeneity of c_{kN} and $c_{k(N-1)}$ this is equivalent to

$$(3.30) \quad -Nc_{kN} \sum_m p_m \frac{\partial \theta}{\partial p_m} + N \, \theta c_{kN} = 0,$$

or

$$(3.31) \quad \sum_m p_m \frac{\partial \theta}{\partial p_m} = \theta.$$

Hence θ is homogeneous of degree 1 in p.

As (3.26) is also PUM for $\theta=0$ and all $\psi_k=0$ (identically) we infer from theorem 2 that

$$(3.32) \quad c_{k1} = \frac{1}{a} \frac{\partial a}{\partial p_k}$$

and, for n=2,...,N,

$$(3.33) \quad c_{kn} = \frac{1}{a^{n-1}} \frac{\partial H}{\partial p_k} f_n(H).$$

To prove that $\psi_k = \partial \theta / \partial p_k$ we need Slutsky symmetry. The Slutsky element s_{km} for (3.26) obeys

$$(3.34) \quad s_{km} = \frac{\partial q_k}{\partial p_m} + q_m \frac{\partial q_k}{\partial y} = \psi_{k,m} + \sum_n c_{kn,m} (y-\theta)^n +$$

$$- \frac{\partial \theta}{\partial p_m} \sum_n n c_{kn} (y-\theta)^{n-1} +$$

$$+ (\psi_m + \sum_n c_{mn} (y-\theta)^n) \sum_n n c_{kn} (y-\theta)^{n-1} -$$

$$= \bar{s}_{km} + \psi_{k,m} + (\psi_m - \frac{\partial \theta}{\partial p_m}) \sum_n n c_{kn} (y-\theta)^{n-1}.$$

Note that $\bar{s}_{km}$ equals the second member of (3.15) with y replaced by (y-θ). Because of theorem 2 we must have $\bar{s}_{km} = \bar{s}_{mk}$. Hence the remainder of (3.34) must also be symmetric in k and m. This remainder is a polynomial in y of degree N-1, hence all its coefficients have to be symmetric. For the coefficient of y^{N-1} this means for all k and m = 1,...,K:

$$(3.35) \quad \left(\psi_m - \frac{\partial \theta}{\partial p_m}\right) \frac{N}{a^{N-1}} \frac{\partial H}{\partial p_k} f_N(H) = \left(\psi_k - \frac{\partial \theta}{\partial p_k}\right) \frac{N}{a^{N-1}} \frac{\partial H}{\partial p_m} f_N(H)$$

where we substituted for c_{kN} the form that it must have according to (3.33). From this we get

$$(3.36) \quad \left(\psi_m - \frac{\partial \theta}{\partial p_m}\right) \frac{\partial H}{\partial p_k} = \left(\psi_k - \frac{\partial \theta}{\partial p_k}\right) \frac{\partial H}{\partial p_m}.$$

This identitiy has to hold for any function H that is homogeneous of degree zero. This is only possible if for all m = 1,...,K

$$(3.37) \quad \psi_m = \frac{\partial \theta}{\partial p_m}.$$

This proves the theorem.

4. Concluding remarks

According to Weierstrasz' theorem every function can be approximated uniformly close by a polynomial. The advantage of the polynomial choice is that it satisfies the theorem of Nataf on aggregation. Above we have derived the constraints that should be imposed upon a polynomial demand

function if it must satisfy the requirements of utility maximization. The polynomials are not the only functions that satisfy Nataf. The functions of the Nataf class are of the form; see van Daal and Merkies [1984, p.33].

$$(4.1) \qquad q_{jk} = h_{jk} \left[\sum_m \varphi_{km} (x_{jkm}) \right]$$

where j refers to the individual, k to the commodity and m to the kind of input, whereas h_{jk} is an arbitrary monotonic function of one variable. A subclass of (4.1) is the Gorman class, where h_{jk} is the identity function, which may result after transformations $h^{-1}_{jk}(q_{jk})$ of the outputs. From this Gorman [1981] derived the integrable class as the class of functions that are generated by utility maximization and therefore restricted to be integrable. Gorman also presented all possible specifications of these integrable functions, see Gorman (1981, p.16). If homogeneity is also adopted, some of these functions drop out, see Merkies and Van Daal [1987]. The relation between the various possibilities is clarified in scheme 1 below.
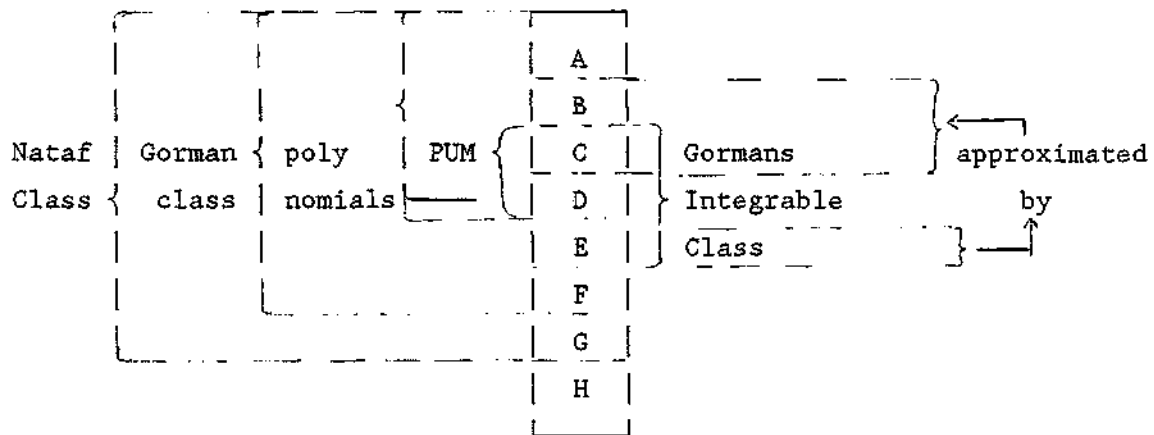
SCHEME 1

From the scheme the following relations appear:

Set of all functions   : A+B+C+D+E+F+G+H

Nataf's class   : A+B+C+D+E+F+G

Gorman's class   : A+B+C+D+E+F

Polynomial class   : A+B+C+D

PUM   : C+D

Gorman's integrable class   : C+D+E

Weierstrasz-class   : A+B+C+D

Weierstrasz-Gorman sub-class: B+C

Heineke and Shefrin class   : B

We have conjectured that the inverse of Weierstrasz' theorem ('any polynomial can act as an approximation of some non-polynomial') is true. If this is not the case we must split set A into two subsets one having and one missing this property and the latter is then not contained in the Weierstrasz' class. The Weierstrasz-Gorman sub-class is obtained after approximating each member of class E by a polynomial.

It should be stressed that we have imposed integrability requirements upon our functions after we have selected a polynomial. Hence the PUM class is a subset of the polynomial class, but as this in its turn is a subset of Gorman's class, the PUM is also a subset of Gorman's integrable class. We could also have started from the latter and derive from this with reference to Weierstrasz theorem the polynomial class B + C + D. Heineke and Shefrin (1986) show why we cannot guarantee to find an integrable member of the polynomial class that is sufficiently close to our demand function. In other words they show that the class, indicated by (B) is not necessarily empty. So if our PUM demand is only an approximated polynomial it may happen that it is not sufficiently close to our actual demand function. Therefore to complete the set of demand functions that are based upon utility maximization, we need to look for class B arising from non polynomial members of Gorman's integrable class, that -if approximated by a polynomial- end up in Heineke and Shefrins'class.

References

[1] Burkill, J.C. and H. Burkill, A second course in mathematical analysis, Cambridge University Press, Cambridge (U.K.), 1970.

[2] Daal, J. van and A.H.Q.M. Merkies, Aggregation in economic research, D. Reidel Publ. Cy., Dordrecht, 1984.

[3] Deaton, A.S. and J. Muellbauer, Economics and consumer behaviour, Cambridge University Press, Cambridge (U.K.), 1980.

[4] Gorman, W.M., On a class of preference fields, Metronomica 13 (1961), pp. 53-56.

[5] Gorman, W.M., Some Engel Curves, Chapter 1 in: Deaton, A.S. (1981). Essayes in the theory and measurement of consumer behaviour in honour of Sir Richard Stone, Cambridge University Press, Cambridge (U.K.), 1981.

[6] Heineke, J.M. and H.M. Shefrin, On an implication of a theorem due to Gorman, Economics Letters 21, 1986, pp. 321-323.

[7] Howe, H., R.A. Pollak and T.J. Wales, Theory and Time Series Estimation of the Quadratic Expenditure System, Econometrica, 47 (1979), pp. 1231-1248.

[8] Merkies, A.H.Q.M. and J. van Daal, "Nataf's Theorem, Taylor's Expansion and Homogeneity in Consumer Demand," in W. Eichhorn et al. (eds.), Theory and Measurement in Economics, Physica Verlag, Heidelberg-Vienna, 1987.

[9] Somermeyer, W.H., Delimination of the Class of Budget-Constrained Utility Maximizing Partially Linear Consumer expenditure Functions, an Alternative Approach, Zeitschrift fur Nationalokonomie, 34 (1974), pp. 309-326.

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