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SOJOURN TIME DISTRIBUTION IN DATA NETWORKS WITH INDEPENDENT EXPONENTIAL SERVICES TIMES

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Researchmemorandum 1982-24   October '82

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CONTENTS page

1. INTRODUCTION

A data network is a collection of switching nodes connected together by a set of communication channels. It provides a message switching service to the users at the various nodes. Messages in the network are routed from one node to another in a store-and-forward manner until they reach their destinations. A key performance measure of the data network is the sojourn time from the arrival of the message at its source to the successfull delivery of this message at its destination. Kleinrock (1964) developed an open queueing network model for data networks and derived an expression for the mean sojourn time. This expression has been used extensively for performance analysis and network design.

Kleinrock's result is the mean sojourn time over all the messages delivered by the network, but he does not give results about the distribution of the sojourn time.

In this paper we treat messages with the same source-destination pair as belonging to a particular message class and derive the distribution of the sojourn time of each class. We consider a network with fixed routing, and assume that there is one unique path for each message class between any pair of channels in the network.

Knowledge of the distribution of the sojourn time allows us to determine statistics as the mean, variance and 90-percentile of the sojourn time.
Our derivation is based on Kleinrock's model with emphasis given to classes of messages. A description of this model is given in section 2. The model is a special case of queueing network model studied by Jackson (1957). We consider this model in section 3. In section 4, our basic result on the distribution of the sojourn time for a class of messages is given. This basic result is then generalized to the whole network.

Finally section 5 is devoted to a numerical example and application of the results to data networks.

2. MODEL DESCRIPTION

We first assume, that the sojourn time experienced by a message in a data network is approximated by the queueing time and the data transfer time in the channels. The processing time at the switching nodes and the propagation delay in the channels are assumed to be negligible.

Let M be the number of channels and C(i) be the capacity of channel i, i=1,2,...,M. In our open queueing network model, each of the M channels is represented by a single server queue. The queueing discipline at each channel is first-come-first-served. We assume that all channels are error free and all the nodes have unlimited buffer space.

Messages are classified according to source-destination pairs. In particular, a message is said to belong to class (s,d) if its source node is s and its destination node is d. Let R be the total number of message classes. In a network with N switching nodes, R=N.(N-1). For convenience, we assume that message classes are numbered from 1 to R, and we use r instead of (s,d) to denote a message class. The arrival process of class r messages from outside the network is assumed to be Poisson with mean rate g(r). Message lengths for all classes are assumed to have the same exponential distribution and we use 1/m to denote the mean message length.

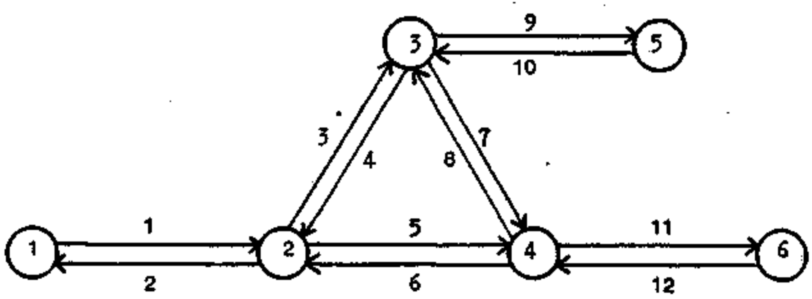

Fig.1. Hypothetical data network.

It follows from this last assumption that the data transfer time of all messages at channel i is exponential with mean 1/(m.C(i)). For the mathematical analysis to be tractable, Kleinrock's independence assumption is used. This assumption states that each time a message enters a switching node, a new length is chosen from the exponential message length distribution.

The route of a message through the network may be described by an ordered set of nodes or an ordered set of channels between these nodes. We assume that the routing in a data network is in general along the shortest possible route. If there are alternatives, one route is chosen. This means that the route of each message class is unique. We use a(r) to denote the ordered set of channels over which class r messages are routed.

In fig. 1 we show a hypothetical data network with 6 nodes and 12 channels. The matching queueing network is shown in fig. 2.

We note that a full duplex channel between two nodes in the data network consists of two independent channels in the queueing network.

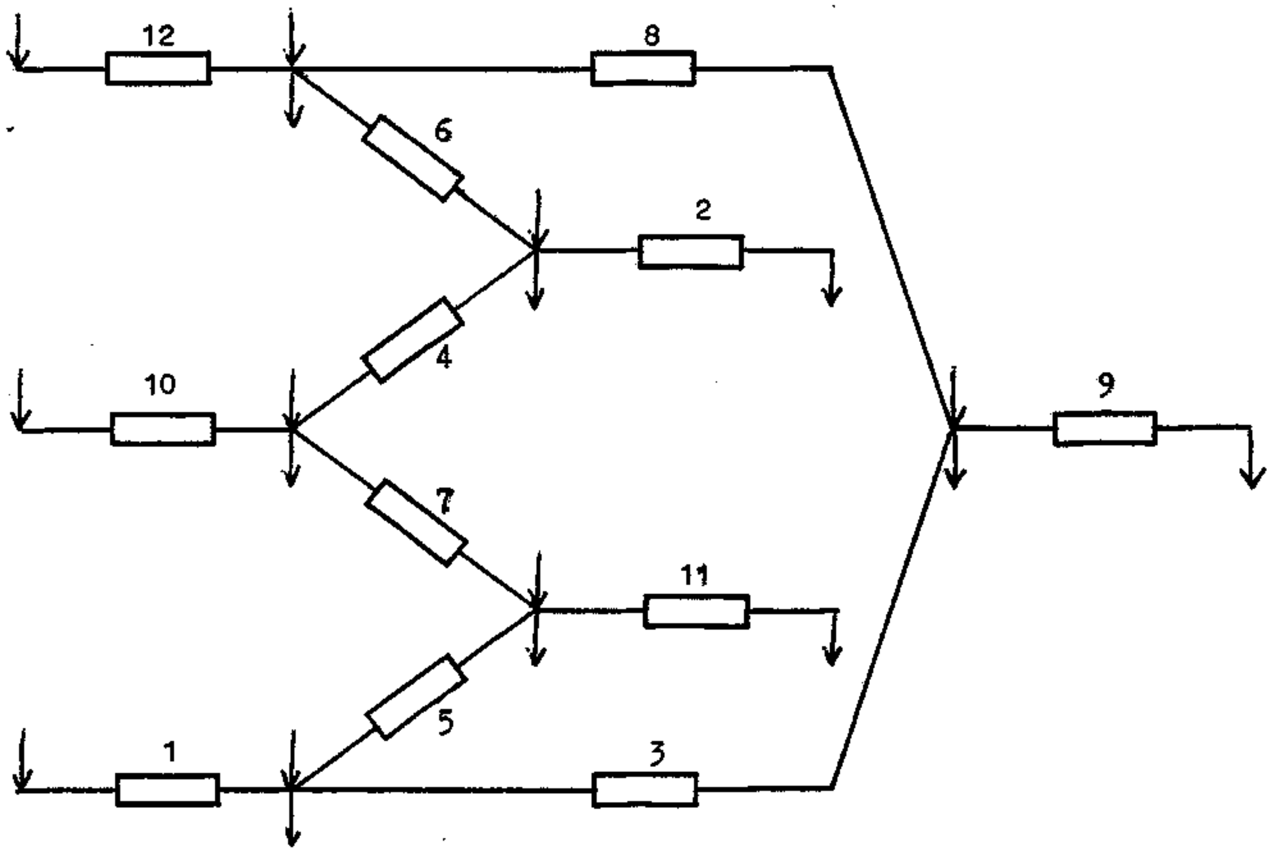

Fig.2. Matching queueing network.

3. OPEN JACKSON NETWORKS

The model of a data network and the matching queueing network, which we described in the previous section has much in common with an open Jackson network with single server queueing stations.

We will briefly discuss this system and use the vocabulary of the previous section.
The model introduced by Jackson (1957) is a generalisiation of a M/M/1 queue to an arbitrarily interconnected open network of single server channels with exponential service and Poisson external input. Every channel has a first-come-first-served queue discipline and a waiting room of unlimited capacity. The external input stream to channel i is Poisson with rate b(i) and these external input streams are assumed to be independent. The service times at channel i are independent and have a common exponential distribution with parameter m.C(i) and are also independent of the message arrivals at channel i. A message leaving channel i is immediately and independently routed to channel j with probability p(i,j) and the message departs the network from channel i with probability

$$1 - \sum_{j=1}^{N} p(i,j). \qquad (1)$$

Let b=(b(1),...,b(N)) be the row vector of the external input intensities, let P be the N x N-matrix of the p(i,j)'s and let a=(a(1),a(2),...,a(N)) be the row vector solution of the traffic equation

$$a=b+a.P. \qquad (2)$$

Since we are assuming the network is open, that is, any message in any channel eventually leaves the system, it follows that each entry of the matrix P^n converges to Ø if $n \to \infty$. Thus the matrix I-P is invertible and there is an unique solution for the traffic equation for a given b. In row form (2) is equivalent to:

$$a(i)=b(i)+\sum_{j=1}^{N} p(i,j).a(j) \qquad i=1,...,N. \qquad (3)$$

That is, the equilibrium rate of flow through channel i a(i) is the sum of the external input rate b(i) and the total rate of internal transfers to channel i. It is plausible that equilibrium conditions will be obtained if the traffic intensity q(i) is less than one at channel i.

$$q(i)=a(i)/(m.C(i)) < 1 \qquad i=1,...,N. \qquad (4)$$

Under these assumptions, Jackson showed that if P(C) is the equilibrium probability of the network of being in state C=(c(1),..., c(N)) then

$$P(C)=P_1(c(1)).P_2(c(2))...P_N(c(N)) \quad (5)$$

where $P_i(c(i))$ is the equilibrium probability of having c(i) messages in a M/M/1 queue with input rate a(i) and service rate m.C(i).

The product form of the right hand side of (5) reveals the mutually independence of the states of the various channels in the network. However we must be careful. The form of the equilibrium solution has misled many researchers in believing that the channels behave generally like individual M/M/1 queues. Also the conjecture that the traffic transfer between channels consists of mutual independent Poisson processes is not true. We will have a closer look at these networks with regard to the transfer processes.

By a result of Burke (1956) and Reich (1957) we know that the output of an M/M/1 queue in equilibrium is Poisson with the same intensity as the input process. This result, when applied to a tandem queue of exponential servers shows each input and output process to be Poisson.

From Kelly (1976) we know that the message streams leaving a Jackson network form mutual independent Poisson processes.

Beutler and Melamed (1978) generalised the above results. They proved that traffic on all exit paths of channels in an open Jackson network in equilibrium is Poisson and moreover, the message streams leaving any exit set are mutually independent. Here an exit path of a channel is a path from that channel such that a message moving along that path cannot return to that channel and a path is an ordered set (u(1),..., u(i), u(i+1),...,u(m)) of channels with

$$P(u(i),u(i+1)) > 0 \qquad \text{for } 1 \leq i,i+1 \leq m.$$

An exit set is a set of channels such that messages departing from this set can never visit this set again.

As a special case, we get Kelly`s result. In contrast, traffic on non-exit paths is non-Poisson and indeed non-renewal.

Beutler and Melamed defined a canonical decomposition of the network as a partition of the set of channels into components, each component consisting of communicating channels.

Channels i and j communicate if channel i is accessible from channel j and if channel j is accessible from channel i. Channel i is accessible from j if $P^{(n)}(j,i) > 0$ for some n.
Beutler and Melamed showed that the set of all exit paths coincides with the paths emanating from the components. This means that all the traffic streams between the components are mutually independent Poisson streams. If in a queueing network all the channels are non-communicating then the set of components is the same as the set of channels.
The network can be decomposed into single channels and all the streams in the network are mutual independent Poisson processes. It looks like we may consider the channels in these networks on their own. However Simon and Foley (1979) proved that in general there are dependencies in the sojourn times of the different channels. Therefore we will now consider the sojourn times in network.

Reich proved that, in equilibrium, the sojourn times of a message in each of M/M/1 queues in tandem are independent and he extended this result to an arbitrary number of such queues in tandem (Reich(1963)). This result was extended by Lemoine (1979) to the case of acyclic Jackson networks in which any two channels are connected by at most one path. Since such networks have no parallel paths, and since the service discipline is first-come-first-served, every path in the network has the so called non- overtaking property. This property says that a message travelling along the path cannot be overtaken by the effects of subsequent arrivals.

However a path need not be a message route in the sense that it may not be possible for any single message to follow the successive channels in a path. The non-overtaking property means that all paths from i(u) to i(v) must go through i(u+1). Hence a message which traverses i(1),.., i(u),..,i(v),..,i(m) cannot be overtaken either directly by any message which inters i(1) after him or indirectly by subsequent arriving messages. Thus it is information or influence as well as physical presence which is not allowed to pass a message.

Walrand and Varaiya (1980) showed very recently that in any open Jackson network, the sojourn times of a message at the various channels of a non- overtaking path are all mutually independent. Since the distribution of the sojourn times at each channel is known, it is possible to calculate the sojourn times for non-overtaking paths.

Walrand and Varaiya showed that the non-overtaking property cannot be generally relaxed and they also showed that for any network the sojourn times along any path which permits overtaking cannot be independent at least under light traffic.

If a path in a network has the non-overtaking property then certainly the channels on this path do not communicate in the sense of the definition of Beutler and Melamed.

If in a network all the paths have the non-overtaking property then there are no communicating channels in network and we can decompose the network into independent M/M/1 channels.

We note that if we are only interested in the mean sojourn time of messages in a channel, on a route or in the whole network we only need to use Jackson's results.

4. SOJOURN TIME DISTRIBUTION

In section 2 we assumed that the routing in a data network is along the shortest possible route and if there exist alternatives one of these is chosen.

If a route has the non-overtaking property then it is possible to derive the distribution of the sojourn time along this route.

Because according to the result of Walrand and Varaiya the sojourn times of a message at the various channels of this route are mutually independent. This means that the sojourn time of message of class r in the network is given by the sum of /a(r)/ independent random variables. Each variable in this sum is the sojourn time in a channel on path a(r). The sojourn time of channel i is given by the sojourn time of a M/M/1 queue.

Let b(i,r) i=1,2,...,M and r=1,2,...,R be the mean arrival rate of class r messages to channel i. Then

$$b(i,r)=\begin{cases} g(r) & \text{if channel } i \in a(r) \\ 0 & \text{otherwise.} \end{cases} \qquad (6)$$

Let q(i,r) be the utilization of channel i by class r messages, then

$$q(i,r)=b(i,r)/(m.C(i)). \qquad (7)$$

The total utilization of channel i, q(i), can then be written as

$$q(i)=\sum_{r=1}^{R} q(i,r) \qquad (8)$$

We require that q(i) < 1 for i=1,2,...,M, the equilibrium condition.
According to queueing theory, the density function of the sojourn time of channel i $d_i(x)$ is

$$d_i(x)=m.C(i).(1-q(i)).\exp(-m.C(i).(1-q(i).x) \qquad x \geq 0. \qquad (9)$$

The mean E(T(i)) and variance Var(T(i)) of the sojourn time in channel i are

$$E(T(i))=1/(m.C(i).(1-q(i))) \qquad (10)$$

and

$$Var(T(i))=1/(m.C(i).(1-q(i))) \quad . \tag{11}$$

Let $t_r(x)$ be the density function of the sojourn time of class r messages and $T_r(s)$ be its Laplace transform, i.e.

$$T_r^{x}(s) = \int_0^{\infty} \exp(-s.x).t_r(x).dx. \tag{12}$$

The Laplace transform $T_r^{x}(s)$ of the sum of /a(r)/ independent random variables with density functions $d_i(x)$ and $i \in a(r)$, is given by the product of their Laplace transforms, i.e.

$$T_r^{x}(s) = \prod_{i \in a(r)} \frac{m.C(i).(1-q(i))}{s + m.C(i).(1-q(i))}. \tag{13}$$

$T_r(s)$ can easily be inverted, by using partial fractions, to give $t_r(x)$.

The mean $E(T_r)$ and variance $Var(T_r)$ of the sojourn time T_r of messages of class r are given by

$$E(T_r) = \sum_{i \in a(r)} 1/(m.C(i).(1-q(i))) \tag{14}$$

and

$$Var(T_r) = \sum_{i \in a(r)} 1/(m.C(i).(1-q(i))) \quad . \tag{15}$$

If all the routes in a network have the non-overtaking property then it is possible to derive the distribution of the sojourn time of all the messages in the network.

If we let $g = \sum_{r=1}^{R} g(r)$ then it is easily seen that the Laplace transform of the sojourn time of all messages in the network T (s) is given by

$$T^{x}(s) = \sum_{r=1}^{R} \frac{g(r)}{g}.T_r^{x}(s). \tag{16}$$

For the mean E(T) of the sojourn time of messages in the network we obtain

$$E(T)=\sum_{r=1}^{R} \frac{g(r)}{g} \cdot \sum_{i \in a(r)} 1/(m.C(i).(1-q(i))). \qquad (17)$$

If $b(i)=\sum_{r=1}^{R} b(i,r)$ and $q(i)=b(i)/(m.C(i))$

$$E(T)=\frac{1}{g} \cdot \sum_{i=1}^{M} b(i)/(m.C(i).(1-q(i))). \qquad (18)$$

For the variance Var(T) of the sojourn time of messages in the network we can write

$$\mathrm{Var}(T)=E(\mathrm{Var}(T_r|r)) + \mathrm{Var}(E(T_r|r)), \qquad (19)$$

so

$$\mathrm{Var}(T)=\sum_{r=1}^{R} \frac{g(r)}{g} \cdot \mathrm{Var}(T_r) + \sum_{r=1}^{R} \frac{g(r)}{g} \cdot (E(T_r) - E(E(T_r)))^2 \qquad (20)$$

and

$$E(E(T_r))=\sum_{r=1}^{R} \frac{g(r)}{g} \cdot E(T_r)=E(T). \qquad (21)$$

Hence

$$\mathrm{Var}(T)=\sum_{r=1}^{R} \frac{g(r)}{g} \cdot \mathrm{Var}(T_r)+\sum_{r=1}^{R} \frac{g(r)}{g} \cdot \left[E^2(T_r)-2.E(T_r).E(T)+E^2(T)\right] \qquad (22)$$

so

$$\mathrm{Var}(T)=\sum_{r=1}^{R} \frac{g(r)}{g} \cdot \mathrm{Var}(T_r) + \sum_{r=1}^{R} \frac{g(r)}{g} \cdot E^2(T_r) - E^2(T). \qquad (23)$$

Eq.(18) and (23) may also be obtained by using

$$E(T^n) = (-1)^n . T^{*(n)}(0).$$

5. EXAMPLES

Our numerical examples are based on the hypothetical network shown in fig. 1. The external arrival rate of messages belonging to each source-destination pair is given by the traffic matrix in fig. 3.

source \ destination	1	2	3	4	5	6
1	0	2	2	2	1	1
2	2	0	2	2	2	2
3	2	2	0	2	2	2
4	2	2	2	0	2	2
5	1	2	2	2	0	1
6	1	2	2	2	1	0

Fig. 3. Traffic matrix.

All channels are assumed to have the same capacity. The capacity and the mean message length are chosen that $1/(m.C(i)) = 0.05$ for channel i, i=1,...,12.

We first consider the case that the routing is based on the shortest path between each pair of nodes. Suppose we are interested in the sojourn time from node 1 to 5. Messages routed along the shortest path between this source and destination pair we denote by class 1.

The channels on this path are 1, 3 and 9.

In fig. 4 we give the traffic streams that pass these channels.

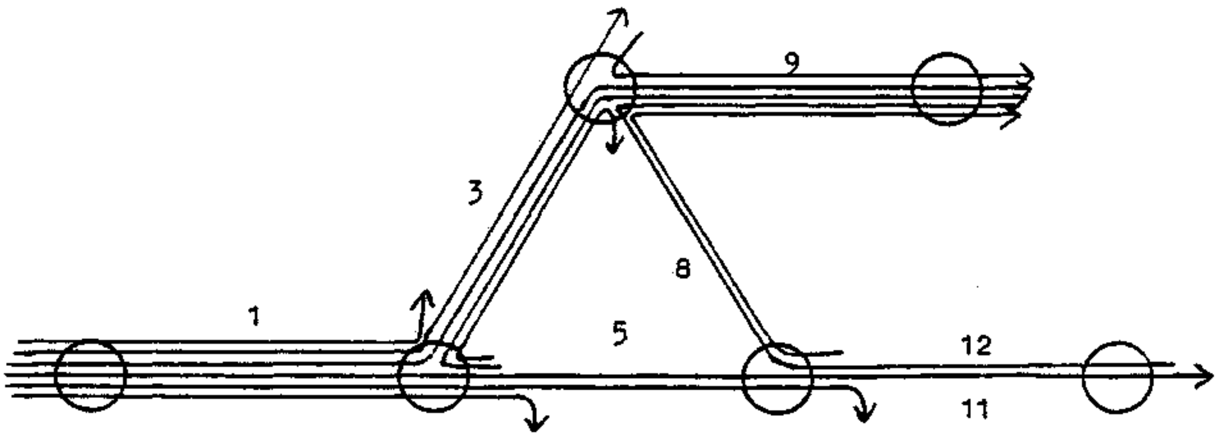

Fig. 4. Traffic streams.

From the traffic matrix we get the total arrival rate for

channel 1: 10
channel 3: 6
channel 9: 8.

We apply eq.(13) and get:

$$T_1^x(s) = \frac{10}{s+10} \cdot \frac{14}{s+14} \cdot \frac{12}{s+12}.$$

We may express this as the following sum

$$T_1^x(s) = \frac{210}{s+10} + \frac{210}{s+14} - \frac{420}{s+12}.$$

This Laplace transform can be inverted to give

$$t_1(x) = 210.\exp(-10x) + 210.\exp(-14x) - 420.\exp(-12x) \qquad x \geq 0$$

and of course $t_1(x) = 0$ for $x < 0$.

A plot of $t_1(x)$ is shown in fig. 5.

The n-th moment of X is calculable from

$$E(X^n)=(-1)^n \cdot T_1^{x(n)}(\emptyset).$$

For the mean E(X) and the variance Var(X) of the sojourn time along the path (1,3,9) we may compute

E(X)=0.255

and

Var(X)=0.022.

This result may also be obtained by using eq.(14) and eq.(15).

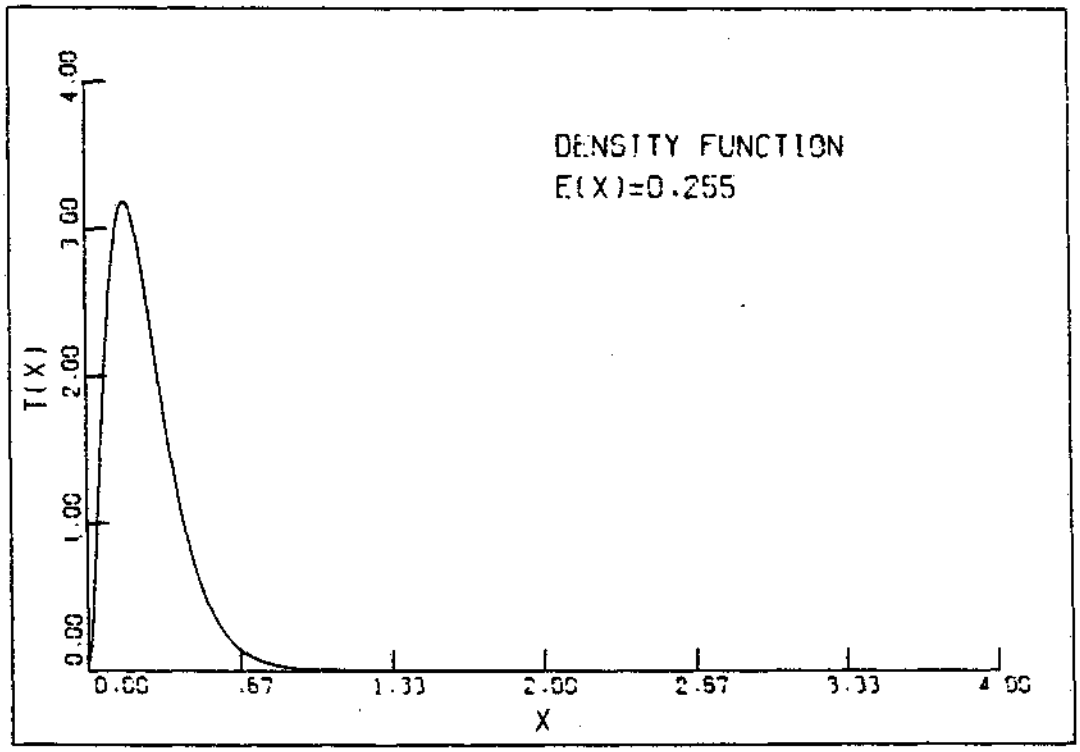

Fig. 5. Sojourn time distribution of class 1 messages.

Now consider the case that we do not have routing based on the shortest path from node 1 to node 5. Instead of this the messages from node 1 to node 5 go through the channels 1,5,8,9 but all other messages are still routed through their shortest path.
In fig. 6 we give the traffic streams that pass these channels.

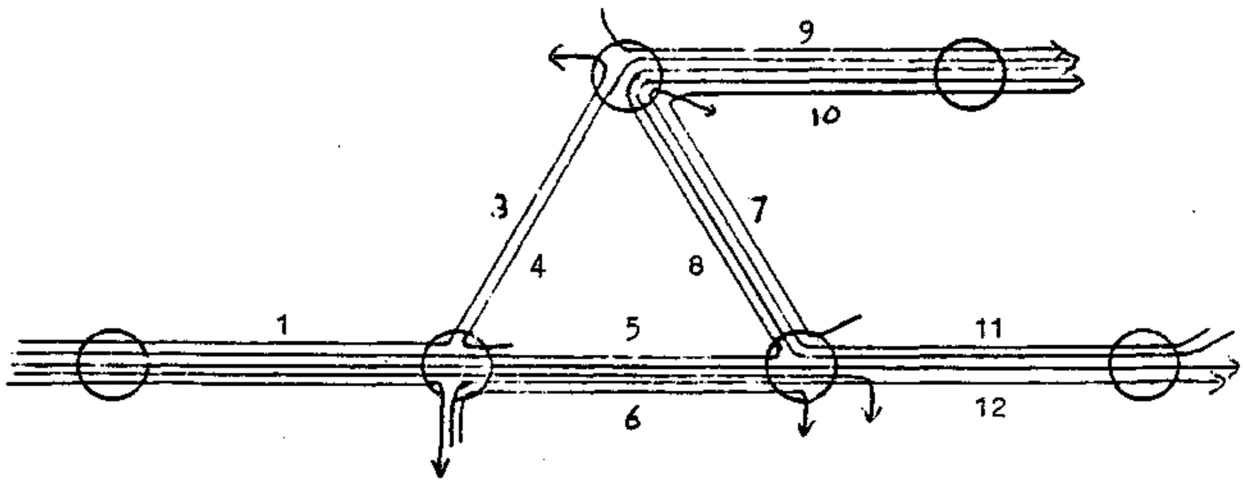

Fig. 6. Traffic streams.

We see that path (1,5,8,9) can be overtaken by path (1,3,9). So it is impossible to derive the distribution of the sojourn time along this path.

In this network with traffic between all nodes we will have dependencies if the routing of the message is so that

$$p(3,7)+p(8,4)+p(4,5)+p(6,3)+p(5,8)+p(7,6) > 0.$$

We will have dependencies between the sojourn time in channels of a data network if the matching queueing network allows overtaking.
It will only be possible to derive the distribution of the sojourn time of messages in a data network if in the matching queueing network any two channels are only connected by one path.

In fig. 7 we give some examples of such data networks. We assume shortest path routing of messages and if there are alternative routes: take the route that turns clockwise.

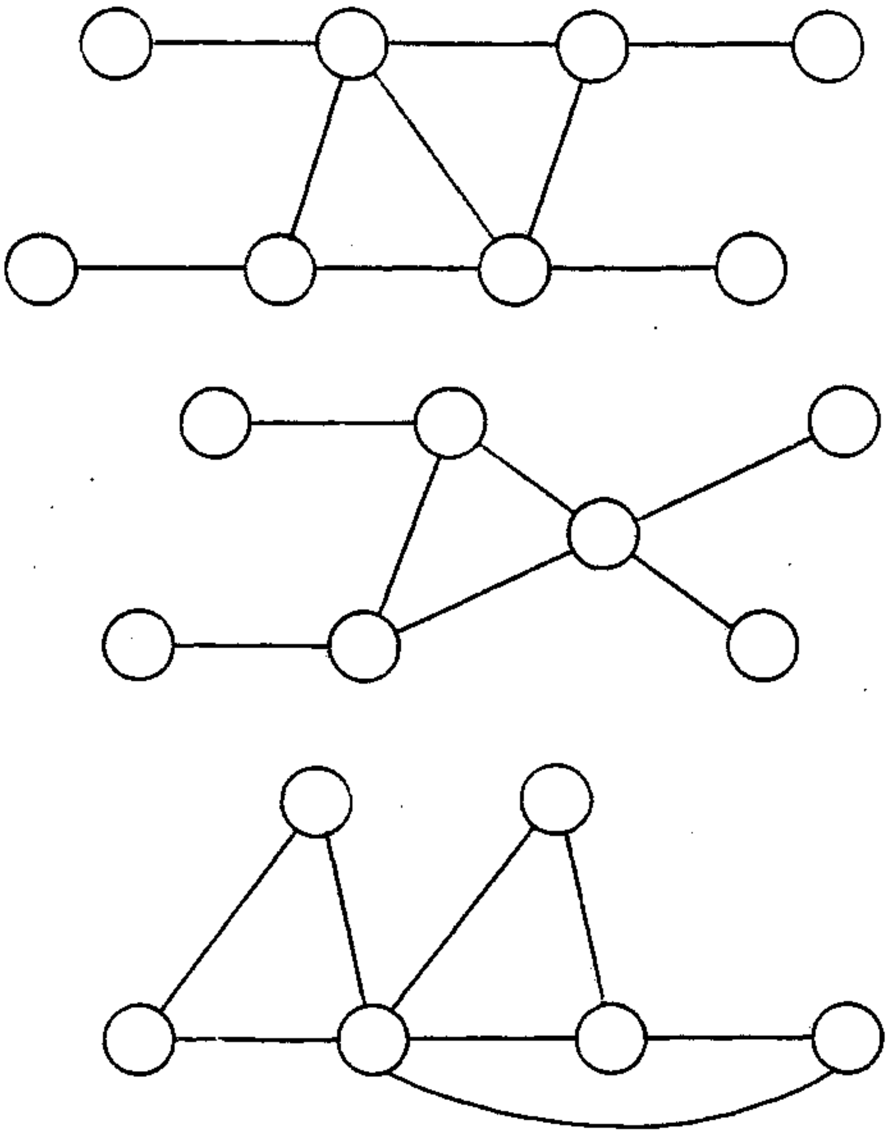

Fig. 7. Data networks.

REFERENCES

Beutler and Melamed (1978), F.J. Beutler and B. Melamed, Decomposition and Customer Streams of Feedback Networks of queues in Equilibrium, Operations Research, Vol.26, no. 6, 1978.

Burke (1956), P.J. Burke, Output of a Queueing System, Operations Research, Vol.4, pp.699-704, 1956.

Jackson (1957), J.R. Jackson, Networks of Waiting Lines, Operations Research, Vol.5, pp.518-521, 1957.

Kelly (1976), F.P. Kelly, Networks of Queues, Adv. Applied Probability, Vol.8, pp.416-432, 1976.

Kleinrock (1964), L. Kleinrock, Communication Nets, Stochastic Message Flow and Delay, McGraw-Hill, New York, 1964.

Lemoine (1979), A.J. Lemoine, On Total Sojourn Time in Networks of Queues, Management Science, Vol.25, No.10, 1979.

Reich (1957), E. Reich, Waiting Times when Queues are in Tandem, Annals of Math. Statistics, Vol.28, pp.768-773, 1957.

Reich (1963), E. Reich, Note on Tandem Queues, Annals of Math. Statistics, Vol.34, pp.338-341, 1963.

Simon and Foley (1979), B. Simon and R.D. Foley, Some Results on Sojourn times in acyclic Jackson Networks, Management Science, Vol.25, No.10, 1979.

Walrand and Varaiya (1980), J. Walrand and P. Varaiya, Sojourn Times and the Overtaking Condition in Jacksonian Networks, Adv.Applied Probability, Vol.12, pp.1000-1018, 1980.