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Extreme Equilibria in a General Negotiation Model

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Abstract

We study a bargaining model with a disagreement game between offers and counteroffers. In order to characterize the set of its subgame perfect equilibrium payoffs, we provide a recursive technique that relies on the Pareto frontier of equilibrium payoffs. When players have different time preferences, reaching an immediate agreement may not be Pareto efficient. The recursive technique developed in this paper generalizes that of Shaked and Sutton (1984) by incorporating the possibility of making unacceptable proposals into the backward induction analysis. Results from this paper extend all the previous findings and resolve some open issues in the current literature.

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1 Introduction

Endogenous threats are an essential constituent of bargaining problems, as emphasized in Nash (1953) at the dawn of modern bargaining theory. This paper develops a general negotiation model that incorporates endogenous threats into the alternating-offer bargaining model of Rubinstein (1982). The early studies on this type of model, such as Fernandez and Glazer (1991), Haller (1991), Haller and Holden (1990) and Bolt (1995), formally introduce the union's decision to strike in contract negotiations. Busch and Wen (1995, 2001), Houba (1997), and Slantchev (2003) allow for more general forms of endogenous threats, modeled as a normal-form game, called the disagreement game, to be played between offers and counteroffers.[1] Despite of the fact that this class of games has complete information, it generally admits multiple equilibria, including inefficient ones with delayed agreements or even no agreement at all. The set of the equilibrium payoffs is fully characterized by the so-called extreme equilibria that yield the lowest and highest equilibrium payoffs to each player. The backward induction technique of Shaked and Sutton (1984) is commonly used to derive these extreme equilibrium payoffs in this class of models.

The model studied in this paper allows for a generic disagreement game in normal form and a general set of possible agreements that might not even be convex. Despite of our well-understanding of this model under common time preferences, we cannot directly adopt the technique of Shaked and Sutton (1984) when players have different time preferences. Characterizing extreme equilibrium payoffs requires the Pareto frontier of equilibrium payoffs. Under common time preferences and a convex set of possible agreements, all possible payoffs are bounded by the bargaining frontier so that the Pareto frontier of equilibrium payoffs must be a subset of the given bargaining frontier. In other words, any Pareto efficient equilibrium must be achieved by immediate agreement. Consequently, making unacceptable proposals would not be effective in obtaining extreme equilibria.

When players have different time preferences, however, it is possible to have equilibrium

[1] See also surveys by Muthoo (1999) and Houba and Bolt (2002).

payoffs above the bargaining frontier even when all disagreement payoffs are bounded by the bargaining frontier. Players may receive payoffs above the bargaining frontier through intertemporal trade when they have different time preferences. It has been realized in other dynamic problems that Pareto improvement is possible through intertemporal trade among agents with different time preferences, see e.g., Ramsey (1928), Bewley (1972) and, more recently, Lehrer and Pauzner (1999). In repeated games, Lehrer and Pauzner (1999) demonstrate that many equilibrium payoffs are outside the conventionally defined set of feasible payoffs. The same phenomenon happens in the negotiation model when players have different time preferences. As a result, the Pareto frontier of equilibrium payoffs is no longer a subset of the bargaining frontier. Therefore, we must incorporate the possibility of making unacceptable proposals in the analysis of the extreme equilibria.[2]

Unlike in a repeated game where the Pareto frontier of equilibrium payoffs is determined by the given stage game, such a frontier in a negotiation game depends on the extreme equilibrium payoffs, which in turn depend on the Pareto frontier of equilibrium payoffs. Due to this interdependency of extreme equilibrium payoffs and the Pareto frontier of equilibrium payoffs, it is not trivial to extend the technique of Shaked and Sutton (1984) in this general setup. We show that the lowest equilibrium payoff to the proposing player is the most crucial extreme equilibrium payoff since it determines not only the other extreme equilibrium payoffs but also the Pareto frontier of equilibrium payoffs. The lowest equilibrium payoff to the proposing player is characterized by the least fixed point of a minimax problem when players are sufficiently patient. Except for some special cases, an analytical solution to the proposing player's least equilibrium payoff is not available in general due to the complicated nature of the problem.

Our analysis confirms Fudenberg and Tirole (1991) who show that including unacceptable proposals into the analysis would not change the insights obtained by Rubinstein (1982). Excluding the possibility of making unacceptable proposals may have serious consequences in

[2]Unacceptable proposals are also necessary in studying the stochastic bargaining model of Merlo and Wilson (1995), which is different from the negotiation model studied in this paper.

a negotiation model with a non-degenerate disagreement game. This issue first surfaced when Bolt (1995) demonstrated that the strategy profile supporting the firm's worst equilibrium in Fernandez and Glazer (1991) fails to be an equilibrium when the firm is less patient than the union. Our analysis in this paper identifies the root of this problem in general and resolves some issues left open in the literature.

This paper is organized as follows. In Section 2, we present a general negotiation model with a generic disagreement game and a generic set of possible agreements. The analysis is partitioned into three subsections. In Section 3.1, we derive a set of necessary and sufficient conditions for extreme equilibrium payoffs. In order to solve the extreme equilibrium payoffs, we need to know the Pareto frontier of continuation payoffs, which is studied in Section 3.2. In Section 3.3, we derive the worst equilibrium to the proposing player. All our results can be related to the literature, as we discuss in Section 3.4. In Section 4 we apply our results to the special case with a common interest disagreement game to illustrate our findings and resolve some open questions.

2 The Model

Two players, called 1 and 2, negotiate for an agreement in the presence of a disagreement game. In any period before an agreement is reached, one player makes a proposal and the other player decides whether to accept the proposal. If the proposal is accepted, then the negotiations end with the accepted proposal as the agreement. Otherwise, the players play the disagreement game once before the negotiations proceed to the following period.

More specifically, there are infinitely many periods and two players alternate in making proposals. Let $B \subseteq \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq 1\}$ be the non-empty, closed, and strictly comprehensive set of possible agreements in terms of average present values to the two players. The Pareto frontier of B is referred to as the bargaining frontier. For convenience, let $\beta^i : \mathbb{R} \to \mathbb{R}$ be the continuous and strictly decreasing function that describes the bargaining frontier; $(x_1, x_2) \in B$ is on the bargaining frontier if and only if $x_i = \beta^i(x_j)$ for $i, j = 1, 2$, and $i \neq j$. The disagreement game is given in normal form: $G = \langle A_1, A_2, d_1(\cdot), d_2(\cdot) \rangle$, where A_i is the set of player i's disagreement actions that is assumed to be non-empty and compact, and $d_i(\cdot) : A \to \mathbb{R}$ is player i's disagreement payoff function that is assumed to be continuous, where $A = A_1 \times A_2$ is the set of disagreement outcomes. We assume that G has at least one Nash equilibrium. Without loss of generality, every player's minimax value in G is normalized to 0. Lastly, assume that every disagreement outcome is weakly dominated by some agreement; $d(a) \in B$ for all $a \in A$.

A generic outcome path, denoted by $\pi = \left(a^1, a^2, \cdots, a^T, x\right)$ for $T \geq 0$, consists of all disagreement outcomes ($a^t \in A$ in period t for $t \leq T$) before the agreement $x \in B$ is reached in period $T + 1$. Such an outcome path specifies an immediate agreement with $T = 0$, and perpetual disagreement with $T = \infty$. Player i's intertemporal time preference on the set of all possible outcome paths is represented by his average discounted payoffs from the disagreement game before the agreement and the agreement itself afterward:

$$v(\pi) = (1 - \delta_i) \sum_{t=1}^{T} \delta_i^{t-1} d_i\left(a^t\right) + \delta_i^T x_i, \tag{1}$$

where $\delta_i \in (0, 1)$ is player i's discount factor per period.

The negotiation model described so-far is a well-defined noncooperative game of complete information. A history is a complete description of how the game has been played up to a period. A player's strategy specifies one appropriate action for every finite history. For technical convenience, we allow for publicly correlated strategies where players can coordinate their continuation play based on public coordination devices. Every strategy profile induces a unique distribution on outcome paths and players evaluate their strategies based on their expected average discounted payoffs. The equilibrium concept applied throughout this paper is subgame perfect equilibrium (SPE).

Many important and influential studies in the literature are special cases of this general negotiation model. For example, one may interpret the models of Rubinstein (1982), Herrero (1989), and van Damme (1991) having a degenerate disagreement game. Fernandez and Glazer (1991) study the case of a specific common interest disagreement game and a linear bargaining frontier, while Haller (1991) and Haller and Holden (1990) further impose a common discount factor. With a general disagreement game and a common discount factor, Busch and Wen (1995, 2001) analyze the case with a linear bargaining frontier, while Houba (1997) generalizes the analysis to a concave and piecewise-linear bargaining frontier. More recently, Slantchev (2003) studies this model with a general common interest disagreement game and a linear bargaining frontier. One important message from these studies is that this negotiation model generally admits multiple SPEs, including inefficient delay despite of the fact that the players are completely informed. What is less clear is a full characterization of the set of SPE payoffs. More importantly, we must reexamine the backward induction method used in finding the extreme SPE payoffs in this model, in particular, when the two players have different time preferences.

3 The Set of SPE Payoffs

Given a Nash equilibrium in the disagreement game, the negotiation model has a stationary SPE that specifies the Nash equilibrium in every disagreement game. Standard arguments apply to the establish existence of such a stationary SPE, which we omit. The existence of a Nash equilibrium in the disagreement game ensures that set of SPE payoffs in the negotiation model is non-empty. The key to characterize the set of SPE payoffs is then to derive each player's lowest and highest SPE payoffs, referred to as extreme SPE payoffs. We first provide a set of necessary and sufficient conditions for these extreme SPE payoffs in Section 3.1. In applying these conditions to derive the extreme SPE payoffs in one period, we need to know the Pareto frontier of SPE payoffs in the following period. We then focus on these effective continuation payoffs in Section 3.2. It turns out that the Pareto frontier of SPE payoffs depends on both the discount factors and the extreme SPE payoffs. As shown in Section 3.3, this inter-dependency between the extreme SPE payoffs and the Pareto frontier of SPE payoffs requires a new set of techniques to analyze the negotiation model with different time preferences. Our analysis is sufficient to characterize the extreme SPE payoffs, and hence the set of SPE payoffs in any negotiation game. Finally, we will tie our results to the existing literature on the negotiation model in Section 3.4.

3.1 Extreme SPE Payoffs

Let E^i, for $i = 1$ and 2, be the non-empty set of SPE payoffs in any period in which player i makes a proposal to player j for $j \neq i$. For simplicity, we suppress all the other parameters that E^i may depend on, such as the discount factors. Given the model setup, E^i is a bounded subset of $\mathbb{R}^2_+$. Applying the technique of self-generating payoffs for a repeated game by Abreu et al. (1986, 1990), and for a bargaining game by Shaked and Sutton (1984) and Binmore (1987),[3] we can prove that E^i is also compact and convex.[4] Given the compactness of E^i, for $l = i$ and j, player l's lowest and highest SPE payoffs when player i proposes are

$$m^i_l = \min_{v \in E^i} v_l \quad \text{and} \quad M^i_l = \max_{v \in E^i} v_l. \tag{2}$$

In any period in which player i proposes, after player j rejects player i's proposal, the players have to play a disagreement outcome $a \in A$ in the current period and a continuation SPE with payoff vector $v = (v_i, v_j) \in E^j$ in the following period when player j proposes. Playing $a \in A$ in the current period and $v \in E^j$ in the following period is a SPE if and only if, for $l = i$ and j,

$$(1 - \delta_l)d_l(a) + \delta_l v_l \geq (1 - \delta_l)g_l(a) + \delta_l m^j_l, \tag{3}$$

where $g_l(a) = \max_{a'_l \in A_l} d_l(a'_l, a_{-l})$. Inequality (3) states that player l's payoff from complying is at least what he could obtain by deviating from $a \in A$ in the current period followed by his lowest SPE in the following period. Obviously, any Nash equilibrium of G satisfies (3) for all discount factors and all continuation payoffs. By incorporating the possibility of unacceptable proposals explicitly in the backward induction technique of Shaked and Sutton (1984), we obtain the following result:

[3] See e.g., Mailath and Samuelson (2006) and van Damme (1991) for more comprehensive treatments of self-generating sets of SPE payoffs.

[4] Upon request, a detailed proof is available from the authors.

Proposition 1 *For all $(\delta_i, \delta_j) \in (0,1)^2$, we have*

$$m_i^i = \min_{a\in A, v\in E^j} \max \begin{cases} (1-\delta_i)d_i(a) + \delta_i v_i, \\ \beta^i\left((1-\delta_j)\, d_j(a) + \delta_j v_j\right), \end{cases} \quad s.t.\ (3), \tag{4}$$

$$M_i^i = \max_{a\in A, v\in E^j} \max \begin{cases} (1-\delta_i)d_i(a) + \delta_i v_i, \\ \beta^i\left((1-\delta_j)\, d_j(a) + \delta_j v_j\right), \end{cases} \quad s.t.\ (3), \tag{5}$$

$$m_j^i = \min_{a\in A, v\in E^j} (1-\delta_j)d_j(a) + \delta_j v_j, \quad s.t.\ (3), \tag{6}$$

$$M_j^i = \max_{a\in A, v\in E^j} (1-\delta_j)d_j(a) + \delta_j v_j, \quad s.t.\ (3). \tag{7}$$

Proof. In any period in which player i proposes, player j may either accept or reject player i's proposal. A strategy profile also requires how the game will be played after player j rejects every possible proposal by player i, which generally depends on the rejected proposal. More specifically, a SPE must specify a proposal $\hat{x} = (\hat{x}_i, \hat{x}_j) \in B$ by player i, and for all $x \in B$, player j's response to x and a continuation SPE that consists of $a(x) \in A$ and $v(x) \in E^j$ after player j rejects x. Denote player i's payoff from such a general SPE as

$$v_i^* = \begin{cases} (1-\delta_i)d_i(a(\hat{x})) + \delta_i v_i(\hat{x}), & \text{if player } j \text{ rejects } \hat{x}, \\ \hat{x}_i, & \text{if player } j \text{ accepts } \hat{x}. \end{cases} \tag{8}$$

First, consider any sequence of proposals $\{x^n\}_{n=1}^{\infty} \subset B$ such that $x_i^n = \beta^i\left(x_j^n\right) > v_i^*$ for all $n \geq 1$ and $\lim_{n\to\infty} x_i^n = v_i^*$. For v_i^* to be player i's SPE payoff, player j must reject x^n and player i must receive no more than v_i^* after player j rejects x^n for all $n \geq 1$ (otherwise, player i would have an incentive to propose x^n instead). In other words, for all $n \geq 1$, we have

$$(1-\delta_i)d_i(a(x^n)) + \delta_i v_i(x^n) \leq v_i^* \quad \text{and} \quad x_j^n \leq (1-\delta_j)d_j(a(x^n)) + \delta_j v_j(x^n). \tag{9}$$

Since A is compact by assumption, sequence $\{a(x^n)\}_{n=1}^{\infty} \subset A$ has a convergent subsequence, say (without loss of generality) $\lim_{n\to\infty} a(x^n) = \bar{a} \in A$. The compactness of E^j then implies that $\{v(x^n)\}_{n=1}^{\infty} \subset E^j$ also has a convergent subsequence, say (without loss of generality) $\lim_{n\to\infty} v(x^n) = \bar{v}$. Since $(a(x^n), v(x^n))$ satisfies (3) for all $n \geq 1$, so does $(\bar{a}, \bar{v})$ due to the continuity of $d(\cdot)$. As $n \to \infty$, the two inequalities in (9) become

$$(1-\delta_i)d_i(\bar{a}) + \delta_i \bar{v}_i \leq v_i^* \qquad \text{and} \qquad x_j^* \leq (1-\delta_j)d_j(\bar{a}) + \delta_j \bar{v}_j.$$

Note that $\beta^i\left(x_j^*\right) = v_i^*$ due to the continuity of $\beta^i(\cdot)$ and $x_i^n = \beta^i\left(x_j^n\right) > v_i^*$ for all $n \geq 1$. By the monotonicity of $\beta^i(\cdot)$, we obtain

$$v_i^* \geq \max\left\{(1-\delta_i)d_i(\bar{a}) + \delta_i\bar{v}_i, \beta^i\left((1-\delta_j)\,d_j\,(\bar{a}) + \delta_j\bar{v}_j\right)\right\},$$

which is bounded from below by the right-hand side of (4). Let $(\hat{a},\hat{v})$ be a solution to the optimization problem (4). Consider the following strategy profile $\hat{\sigma}$:

— player i proposes $\hat{x}$ where $\hat{x}_i = \max\left\{\begin{array}{l}(1-\delta_i)d_i(\hat{a}) + \delta_i\hat{v}_i, \\ \beta^i\left(1-(1-\delta_j)d_j(\hat{a}) - \delta_j\hat{v}_j\right),\end{array}\right.$

— player j accepts x if and only if $x_j \geq (1-\delta_j)d_j(\hat{a}) + \delta_j\hat{v}_j$, and

— the continuation SPE is $(a(x), v(x)) = (\hat{a},\hat{v})$ after player j rejects x for all $x \in B$.

Strategy profile $\hat{\sigma}$ constitutes a SPE, from which player i receives exactly (4). Hence, (4) characterizes player i's lowest SPE payoff m_i^i.

Second, accepting $\hat{x}$ is a best response for player j if and only if $\hat{x}_j \geq (1-\delta_j)d_j(a(\hat{x})) + \delta_j v_j(\hat{x})$. Together with (8), we have

$$v_i^* \leq \left\{\begin{array}{ll}(1-\delta_i)d_i(a(\hat{x})) + \delta_i v_i(\hat{x}), & \text{if player } j \text{ rejects } \hat{x}, \\ \beta^i\left((1-\delta_j)d_j(a(\hat{x})) + \delta_j v_j(\hat{x})\right), & \text{if player } j \text{ accepts } \hat{x},\end{array}\right.$$

which is bounded from above by the right-hand side of (5). The strategy profile $\hat{\sigma}$ above with $(\hat{a},\hat{v})$ being a solution to the optimization problem in (5) supports player i's highest SPE payoff M_i^i.

Third, player j certainly rejects any $x \in B$ such that x_j is less than the right-hand side of (6), because player i cannot receive less than his lowest continuation payoff after rejecting any proposal. Therefore, player j's SPE payoffs are bounded from below by the right-hand side of (6). Furthermore, player j receives exactly the right-hand side of (6) in the SPE $\hat{\sigma}$ above with $(\hat{a},\hat{v})$ being a solution to the optimization problem in (6).

Lastly, since player j certainly accepts any $x \in B$ such that x_j is greater than the right-hand side of (7), player i will never propose $x \in B$ such that x_j is more than player j's highest continuation payoff. In other words, $\hat{x}_j$ must be less than or equal to the right-hand side of

(7). Whether player j accepts $\hat{x}$ or not, player j cannot obtain more than the right-hand side of (7). Again, player j receives exactly the right-hand side of (7) in the SPE $\hat{\sigma}$ above with $(\hat{a}, \hat{v})$ being a solution to the optimization problem in (7). ■

Note that (4) and (5) are significantly different from what we know from the current literature. The first component in these two bounds represents what the proposing player receives from making an unacceptable proposal. The possibility of making unacceptable proposals has often been overlooked, which could affect final conclusions as we have seen in the current literature. Proposition 1 revises what has been commonly used by including the possibility of making unacceptable proposals into the analysis.

Although the objective functions in (4)—(7) are well-defined and continuous, A is compact, we know nothing about E^j at this stage other than its non-emptiness, compactness, and convexity. In order to fully understand the issues involved, we have to discuss effective continuation SPE payoffs in solving these extreme SPE payoffs. We next show that the most effective continuation SPE payoffs in solving (4)—(7) are those that are on the Pareto frontier of E^j. Accordingly, denote the Pareto frontier of E^j as

$$\varphi^j(v_i) = \max_{v' \in E^j} v'_j \quad \text{s.t. } v'_i \geq v_i \qquad \text{and} \qquad \varphi^i(v_j) = \max_{v' \in E^j} v'_i \quad \text{s.t. } v'_j \geq v_j. \tag{10}$$

Since E^j is a non-empty, compact, and convex subset of $\mathbb{R}^2_+$, both $\varphi^i(\cdot)$ and $\varphi^j(\cdot)$ are continuous and non-increasing. Given Proposition 1, the following conditions on the responding player j's extreme SPE payoffs are immediate:

Proposition 2 *For all* $(\delta_i, \delta_j) \in (0,1)^2$, *we have*

$$m^i_j \geq \delta_j m^j_j, \tag{11}$$

$$M^i_j \leq \max_{a \in A} \left[(1-\delta_j) d_j(a) + \delta_j \varphi^j \left(\frac{1-\delta_i}{\delta_i} [g_i(a) - d_i(a)] + \delta_i m^i_i \right) \right]. \tag{12}$$

Proof. Substituting (3) into (6), we have

$$m^i_j \geq \min_{a \in A} \left[(1-\delta_j) g_i(a) + \delta_j m^j_j \right] = (1-\delta_j) \min_{a \in A} \max_{a'_i} d_i(a'_i, a_j) + \delta_j m^j_j,$$

which is (11) since player j's minimax value in G is normalized to be zero. For $l = j$, (3) and (10) imply that

$$v_j \leq \varphi^j(v_i) \leq \varphi^j\left(\frac{1-\delta_i}{\delta_i}\left[g_i(a) - d_i(a)\right] + m_i^j\right). \tag{13}$$

Substituting (13) into (7), we obtain (12). ■

For sufficiently large $(\delta_i, \delta_j) \in (0,1)^2$, Proposition 2 implicitly describes how the players behave in the responding player's worst and best SPEs. In player j's worst SPE, if player j rejects any proposal, he will receive his minimax value of 0 when playing G in the current period followed by his lowest SPE payoff m_j^j in the following period. In player j's best SPE, on the other hand, if player j rejects any proposal, he will receive his highest continuation payoff, taking into account that player i must be compensated in the following period after complying in the disagreement game. In fact, when the players are sufficiently patient, (11) and (12) hold with equalities for the responding player's lowest and highest SPE payoffs. These results generalize those of Busch and Wen (1995).

We now turn to the proposing player's extreme SPE payoffs.

Proposition 3 *For all $(\delta_i, \delta_j) \in (0,1)^2$, we have*

$$m_i^i \geq \min_{a\in A}\max\left\{\begin{array}{l}(1-\delta_i)g_i(a) + \delta_i^2 m_i^i,\\ \beta^i\left((1-\delta_j)d_j(a) + \delta_j\varphi^j\left(\frac{1-\delta_i}{\delta_i}\left[g_i(a)-d_i(a)\right] + \delta_i m_i^i\right)\right),\end{array}\right. \tag{14}$$

$$M_i^i \leq \max\left\{\begin{array}{l}\max_{a\in A}\left[(1-\delta_i)d_i(a) + \delta_i\varphi^i\left(\frac{1-\delta_j}{\delta_j}\left[g_i(a)-d_j(a)\right] + m_j^j\right)\right],\\ \beta^i(\delta_j m_j^j).\end{array}\right. \tag{15}$$

Proof. Substituting (3) and (13) into (4) yields

$$m_i^i \geq \min_{a\in A}\max\left\{\begin{array}{l}(1-\delta_i)g_i(a) + \delta_i m_i^j,\\ \beta^i\left((1-\delta_j)d_j(a) + \delta_j\varphi^j\left(\frac{1-\delta_i}{\delta_i}\left[g_i(a)-d_i(a)\right] + m_i^j\right)\right),\end{array}\right.$$

which implies (14) due to $m_i^j \geq \delta_i m_i^i$ by Proposition 2. For $l = i$, (3) and (10) imply that

$$v_i \leq \varphi^i\left(v_j\right) \leq \varphi^i\left(\frac{1-\delta_j}{\delta_j}\left[g_j(a) - d_j(a)\right] + m_j^j\right).$$

Substituting the last inequality and (3) into (7) yields

$$M_i^i \le \max \left\{ \begin{array}{l} \max_{a \in A} \left[(1-\delta_i) d_i(a) + \delta_i \varphi^i \left(\frac{1-\delta_j}{\delta_j} \left[g_j(a) - d_j(a) \right] + m_j^j \right) \right], \\ \max_{a \in A} \beta^i \left((1-\delta_j) g_j(a) + \delta_j m_j^j \right), \end{array} \right.$$

which is equivalent to (15) due to the monotonicity of $\beta^i(\cdot)$ and the normalization of player j's minimax value in G. ■

Propositions 2 and 3 imply that proposing player's lowest SPE is essential to determine the other extreme SPE payoffs. In order to solve m_i^i from (14), we need to know the Pareto frontier of E^j, which contains the effective continuation payoffs.

3.2 Effective Continuation Payoffs

First, we discuss how E^j is determined by the players' lowest SPE payoffs. Whenever player j proposes, $\pi = \left(a^1, a^2, \cdots, a^T, x\right)$ can be supported as a SPE outcome path if and only if for all $t \le T+1$,

$$(1-\delta_l) \sum_{s=t}^{T} \delta_l^{s-t} d_l(a^s) + \delta_l^{T+1-t} x_l \ge \left\{ \begin{array}{ll} m_l^i, & \text{if } t \text{ is even,} \\ m_l^j, & \text{if } t \text{ is odd,} \end{array} \right. \qquad \text{for } l = i \text{ and } j, \tag{16}$$

and for all $t \le T$,

$$(1-\delta_l) \sum_{s=t}^{T} \delta_l^{s-t} d_l(a^s) + \delta_l^{T+1-t} x_l \ge (1-\delta_l) g_l(a^t) + \left\{ \begin{array}{ll} \delta_l m_l^j, & \text{if } t \text{ is even,} \\ \delta_l m_l^i, & \text{if } t \text{ is odd.} \end{array} \right. \tag{17}$$

For $t = T+1$, (16) implies that no matter who proposes the final agreement $x \in B$, it needs to be a SPE agreement in period $T+1$. Condition (17) states that if player l deviates from a^t in period $t \le T$, then this player will be punished by his lowest SPE payoff, either m_l^i or m_l^j, in the following period. With publicly correlated strategies, E^j is the convex hull of $\{v(\pi) : (16) \text{ and } (17)\}$, where $v(\pi)$ is defined in (1). Note that for $T = 0$, (16) and (17) imply that for any $x_i \in \left[m_i^j, \beta^i(m_j^j)\right]$ immediate agreement on $\left(x_i, \beta^j(x_i)\right)$ belongs to E^j.[5] Hence, $\varphi^j(v_i) \ge \beta^j(v_i)$ for all $(\delta_i, \delta_j) \in (0,1)^2$ and $v_i \in \left[m_i^j, \beta^i(m_j^j)\right]$.

Due to Proposition 2, when the discount factors are sufficiently large, we can rewrite (16) and (17) in terms of m_i^i and m_j^j only. Consequently, the set E^j depends on m_i^i and

[5] This generalizes the range of SPE payoffs with immediate agreement in Haller and Holden (1990).

m_j^j only. For $i = 1$ and 2, substituting E^j ($j \neq i$) in terms of m_i^i and m_j^j into (4) provides two equations, one for m_1^1 and one for m_2^2. The solution m_i^i from such an implicit equation system can be supported as player i's (lowest) SPE payoff for $i = 1$ and 2 when the discount factors are sufficiently large.

In Proposition 3, we show that only the Pareto frontier of E^j is effective in solving m_i^i from (14). In the rest of this subsection, we provide specific structures on the continuation paths that achieve the Pareto frontier of E^j, which requires the insights in Lehrer and Pauzner (1999) derived for repeated game with different time preferences. The key issue is that there may be many SPE payoffs in the negotiation model that can be above the bargaining frontier. Lehrer and Pauzner (1999) investigate in great detail the Pareto frontier of SPE payoffs in a repeated game under different time preferences. There are many obstacles in directly applying their results to a negotiation game. SPE payoffs in a repeated game are bounded from below by players' stage-game minimax payoffs that are invariant with respect to the discount factors and time periods. In a negotiation game, however, players' lowest SPE payoffs depend on the discount factors and also on who proposes. A typical outcome path in a negotiation game ends with an agreement that ceases any future payoff variation. In a repeated game, it may not be possible to have a SPE in which a player receives exactly his minimax payoff, so it is often sufficient to provide a SPE where a player's payoff is sufficiently close to his minimax value. In a negotiation game, however, we need the SPE where a player receives *exactly* his lowest SPE payoff. In order to derive the Pareto frontier of E^j, we have to modify Lehrer and Pauzner's technique for these differences between a repeated game and a negotiation game.

According to Lehrer and Pauzner (1999), in order to characterize the Pareto frontier of E^j in the direction of $\lambda = (\lambda_i, \lambda_j) \in \Delta$, where $\Delta \subset \mathbb{R}_+^2$ denotes the unit simplex, we need to solve the following optimization problem:

$$\max_{\pi} \quad \lambda \cdot v(\pi), \qquad \text{subject to (16) and (17).} \tag{18}$$

In other words, (18) provides the payoff vectors on the Pareto frontier E^j in the direction of

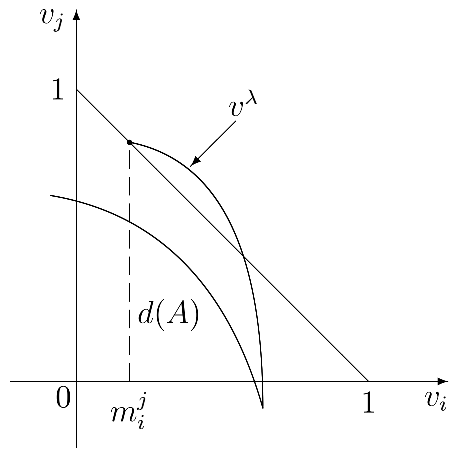

Figure 1: The curve of payoff vectors v^λ for all $\lambda \in \Delta$.

$\lambda \in \Delta$. When G is a finite game, E^j is a polygon in $\mathbb{R}_+^2$ and (18) provides us all the vertices in the direction of $\lambda \in \Delta$. Note that under (16) and (17), we can write (18) as

$$\max_{T \geq 0} \sum_{t=1}^{T} \max_{y^t} \left[\lambda_i (1-\delta_i) \delta_i^{t-1} y_i^t + \lambda_j (1-\delta_j) \delta_j^{t-1} y_j^t \right],$$

where $y^t = d(a^t) \in d(A)$ for all $t \leq T$ and $y^T = x \in B \cap E^j$ at $t = T+1$. In the rest of this subsection, we will solely focus on $\delta_i < \delta_j$, while similar arguments apply for $\delta_i > \delta_j$.

When $\delta_i < \delta_j$, for all $\lambda = (\lambda_i, \lambda_j) \in \Delta$, the weight ratio

$$\frac{\lambda_j (1-\delta_j) \delta_j^{t-1}}{\lambda_i (1-\delta_i) \delta_i^{t-1}}$$

is monotonically increasing with respect to $t \geq 0$. Therefore, in any potential solution to (18) with $T > 0$, we must have

$$d_i(a^t) \geq d_i(a^{t+1}) > x_i \quad \text{and} \quad d_j(a^t) \leq d_j\left(a^{t+1}\right) < x_j = \beta^j\left(x_i\right), \tag{19}$$

whenever it is possible under (16) and (17). Given $\delta_i < \delta_j$, the weight ratio will be greater than one for sufficiently large t. This implies that the two players must reach an agreement within finite periods in any potential solution to (18). These arguments narrow down the potential solutions to (18). Given (19), sequential rationality (16) simplifies to $x_i \geq m_i^i$ if T is even, and $x_i \geq m_i^j$ if T is odd. For all $t \leq T$, $d(a^t)$ should be as close as possible to

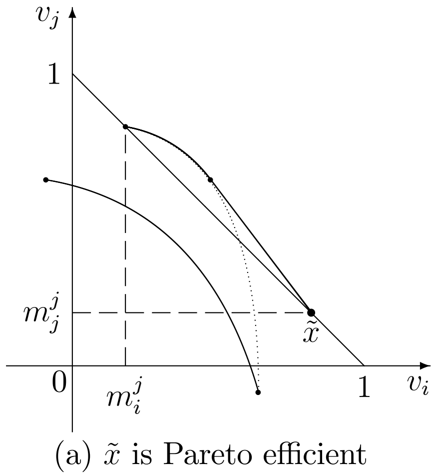

Figure 2: The Pareto frontier of E^j where $\tilde{x} = (\beta^i(m_j^j), m_j^j)$.

the Pareto frontier of $d(A)$, provided that (16) and (17) hold. As in a repeated game, player j's per-period payoff during the early phase of such an outcome path could be lower than his minimax value. Since player j's payoff increases over time, his average payoff from the entire path will not be less than his lowest SPE payoff. Let v^λ, for all $\lambda \in \Delta$, denote the payoff vector resulting from a solution to (18) under (19). Figure 1 illustrates the curve of payoff vectors v^λ for all $\lambda \in \Delta$ under a linear bargaining frontier. Similar as in Lehrer and Pauzner (1999), this curve is continuous in all $d\left(a^t\right)$ and the final agreement that is either $(m_i^i, \beta^j(m_i^i))$ or $(\delta_i m_i^i, \beta^j(\delta_i m_i^i))$.

Figure 1 does not fully specify the Pareto frontier of E^j, because the outcome path $\pi = (\tilde{x})$ with an immediate agreement on $\tilde{x}_j = m_j^j$ and $\tilde{x}_i = \beta^i(m_j^j)$ (with $T = 0$) may also solve (18), in particular when λ_j/λ_i is sufficiently close to 0. With publicly correlated strategies, the Pareto frontier of E^j is completely characterized by $\tilde{x}$ and v^λ for all $\lambda \in \Delta$. Figure 2 illustrates the two possible cases. Given m_j^j, E^j (as a correspondence of m_i^i) is convex-valued and continuous with respect to m_i^i. Moreover, the function φ^j, which is implicitly a function of m_i^i and m_j^j, is also continuous in both m_i^i and m_j^j.

To summarize, whenever time preferences are sufficiently different, the Pareto frontier of

E^j is generally above the bargaining frontier.[6] This will affect how m_i^i is determined. As we have shown above, the Pareto frontier of E^j is rather complicated, which prevents us from obtaining a closed-form solution for m_i^i. Nevertheless, our analysis provides a general technique on how to solve the players' lowest SPE payoffs, and, hence, how to characterize the set of SPE payoffs in the negotiation model when $\delta_i \neq \delta_j$. In the next section, we will demonstrate how this technique works for a common interest disagreement game and a linear bargaining frontier.

3.3 Proposing Player's Lowest SPE Payoff

In the previous subsections, we have established that m_i^i and m_j^j are the key in finding the other extreme SPE payoffs. Note that condition (14) depends on m_i^i directly and on m_i^i and m_j^j indirectly. In this subsection, we will show how to solve m_i^i as the least fixed-point to (14) and provide an important range for its value. Figure 2 suggests that we need to analyze two distinct cases.

Case 1: $\delta_i < \delta_j$

Instead of solving m_i^i and m_j^j simultaneously from the implicit equation system implied by (14), we can first find m_i^i independently of m_j^j when δ_i and δ_j are sufficiently large. Condition (14) can be rewritten as $m_i^i \geq \Lambda(m_i^i)$, where

$$\Lambda(m_i^i) = \min_{a \in A} \max \left\{ \begin{array}{l} (1-\delta_i) g_i(a) + \delta_i^2 m_i^i, \\ \beta^i \left((1-\delta_j) d_j(a) + \delta_j \varphi^j \left(\delta_i m_i^i + \frac{1-\delta_i}{\delta_i} \left[g_i(a) - d_i(a) \right] \right) \right) \end{array} \right. . \tag{20}$$

Since $\varphi^j \left(\delta_i m_i^i + \frac{1-\delta_i}{\delta_i} \left[g_i(a) - d_i(a) \right] \right)$ is continuous with respect to $a \in A$, m_i^i and (indirectly) m_j^j, (20) is a well-defined and continuous function of m_i^i. To solve (20) when δ_i is sufficiently close to 1, we only need to know $\varphi^j (v_i)$ for v_i sufficiently close to $\delta_i m_i^i$, i.e.,

$$v_i \in \left[\delta_i m_i^i, \delta_i m_i^i + \frac{1-\delta_i}{\delta_i} \max_{a \in A} \left[g_i(a) - d_i(a) \right] \right].$$

[6] Similar as in Lehrer and Pauzner (1999), for every $\delta_i < 1$ and δ_j approximately equal to 1, we have that v^λ is almost rectangular in Figure 1.

As we have shown in Section 3.2 for the case of $\delta_i < \delta_j$, $\varphi^j(\cdot)$ in this part of its domain is independent of m_j^j for sufficiently large (δ_i, δ_j).[7]

In order to present our next main proposition, we need additional notation. For every $a \in A$, define

$$F(x_i; a) = \beta^i \left((1-\delta_j)d_j(a) + \delta_j \beta^j \left(\delta_i x_i + \frac{1-\delta_i}{\delta_i} [g_i(a) - d_i(a)] \right) \right), \tag{21}$$

which is a monotonic increasing and continuous function of x_i. Given $a \in A$, $F(x_i; a)$ has, at least, a fixed-point over the following interval:

$$[\underline{x}_i(a), \overline{x}_i(a)] \equiv \left[\min \left\{ \beta^i(d_j(a)), \frac{g_i(a) - d_i(a)}{\delta_i} \right\}, \max \left\{ \beta^i(d_j(a)), \frac{g_i(a) - d_i(a)}{\delta_i} \right\} \right].$$

Because both $\beta^i(\cdot)$ and $\beta^j(\cdot)$ are monotonically decreasing functions, we have

$$\begin{aligned} F(\underline{x}_i(a); a) &\geq \beta^i \left((1-\delta_j)d_j(a) + \delta_j \beta^j (\underline{x}_i(a)) \right) \geq \beta^i (d_j(a)) \geq \underline{x}_i(a), \\ F(\overline{x}_i(a); a) &\leq \beta^i \left((1-\delta_j)d_j(a) + \delta_j \beta^j (\overline{x}_i(a)) \right) \leq \beta^i (d_j(a)) \leq \overline{x}_i(a). \end{aligned}$$

The last two inequalities and the monotonicity of function $F(\cdot; a)$ imply that $F(\cdot; a)$ maps from $[\underline{x}_i(a), \overline{x}_i(a)]$ into itself. By Brouwer's fixed point theorem, $F(\cdot; a)$ has a least fixed-point in $[\underline{x}_i(a), \overline{x}_i(a)]$. In many cases, such as when the bargaining frontier is linear, $F(\cdot; a)$ has a unique fixed-point for all $a \in A$. However, this may not the case in general. Let $X_i(a) = \{x_i \in [\underline{x}_i(a), \overline{x}_i(a)] : F(x_i; a) = x_i\}$ denote the set of all fixed-points of (21). Since $F(\cdot; a)$ is continuous, $X_i(a)$ is a closed subset of a compact interval, hence it is compact. Now define

$$\hat{m}_i^i = \min_{a \in A} \max \left\{ \tfrac{g_i(a)}{1+\delta_i}, \min_{x_i \in X_i(a)} x_i \right\}. \tag{22}$$

Our next proposition identifies the least fixed-point of $\Lambda(\cdot)$ in the refined interval of $[0, \hat{m}_i^i]$ as a lower bound on m_i^i.

Proposition 4 *For sufficiently large $(\delta_i, \delta_j) \in (0,1)^2$ and $\delta_i < \delta_j$, m_i^i is bounded from below by the least fixed-point of $\Lambda(\cdot)$ in $[0, \hat{m}_i^i]$.*

[7] In fact, this part of $\varphi^j(\cdot)$ is the curve of vectors v^λ in Figure 1, which is independent of m_j^j.

Proof. Since both $\beta^i(\cdot)$ and $\varphi^j(\cdot)$ are continuous and monotonically decreasing, $\Lambda(\cdot)$ is a well-defined, continuous and monotonically increasing function of m_i^i. First, observe from (20) that $\Lambda(0) \geq \min_{a\in A}[(1-\delta_i)g_i(a)] = 0$. Since $\phi^j(v_i) \geq \beta^j(v_i)$ for all $v_i \in [0, \beta^i(0)]$, (20) implies that

$$\Lambda(m_i^i) \leq \min_{a\in A} \max \begin{cases} (1-\delta_i)g_i(a) + \delta_i^2 m_i^i, \\ F(m_i^i, a). \end{cases} \tag{23}$$

Let $\hat{a} \in A$ be a solution to (22) and $\hat{x}_i(\hat{a}) = \min_{x_i \in X_i(\hat{a})} x_i$. There are two possibilities:

$$\text{either} \quad \hat{m}_i^i = \tfrac{g_i(\hat{a})}{1+\delta_i} \geq F(\hat{x}_i(\hat{a}), \hat{a}) = \hat{x}_i(\hat{a}) \quad \text{or} \quad \hat{m}_i^i = F(\hat{m}_i^i, \hat{a}) \geq \tfrac{g_i(\hat{a})}{1+\delta_i}.$$

Evaluate (23) at $\hat{m}_i^i$, we have

$$\Lambda(\hat{m}_i^i) \leq \max \begin{cases} (1-\delta_i)g_i(\hat{a}) + \delta_i^2 \hat{m}_i^i \\ F(\hat{m}_i^i, \hat{a}) \end{cases} \leq \hat{m}_i^i.$$

To summarize, we have shown that $\Lambda(0) \geq 0$ and $\Lambda(\hat{m}_i^i) \leq \hat{m}_i^i$. Due to its monotonicity, $\Lambda(\cdot)$ must map from $[0, \hat{m}_i^i]$ into itself. Brouwer's fixed point theorem then asserts that $\Lambda(\cdot)$ has at least one fixed-point in $[0, \hat{m}_i^i]$. Since any value m_i^i that is less than the least fixed point of $\Lambda(\cdot)$ violates $m_i^i \geq \Lambda(m_i^i)$, m_i^i must be bounded from below by the least fixed-point of $\Lambda(\cdot)$ in $[0, \hat{m}_i^i]$. ■

Our next proposition asserts that when the discount factors are sufficiently close to 1, the least fixed-point of $\Lambda(\cdot)$ can be supported as a SPE payoff of player i. Therefore, proposing player i's lowest SPE payoff m_i^i indeed coincides with the least fixed-point of $\Lambda(\cdot)$. Since the proof is rather long, we defer it to the Appendix.

Proposition 5 *There exists a $\hat{\delta} \in (0,1)$ such that for all $\delta_j > \delta_i \geq \hat{\delta}$, there is a SPE in which player i receives the least fixed-point of $\Lambda(\cdot)$.*

Case 2: $\delta_i > \delta_j$

First, we solve m_j^j independently as we described in Case 1 by switching i and j. Once the value of m_j^j is given, $\varphi^j\left(\delta_i m_i^i + \frac{1-\delta_i}{\delta_i}[g_i(a) - d_i(a)]\right)$ is a continuous function of $a \in A$

and m_i^i only. This allows us to establish similar results for m_i^i as in Proposition 4 and 5 for this case, which we omit. The proofs of Propositions 4 and 5 do not rely on the fact that $\delta_i < \delta_j$, except by requiring that m_i^i be the only unknown variable in the part of $\varphi^j(\cdot)$ we need. Similar to the case of $\delta_i < \delta_j$, m_i^i is the least fixed-point of (20) for large enough $(\delta_i, \delta_j) \in (0,1)^2$. This ends our discussion on Case 2.

3.4 Discussion and Related Literature

We have provided a complete procedure for deriving the proposing player's lowest SPE payoff for sufficiently large discount factors. We are then able to characterize all other extreme SPE payoffs and, hence the set of SPE payoffs in the negotiation model when the discount factors are sufficiently large. The Pareto frontier of SPE payoffs is rather complicated under different time preferences, which makes it impossible to obtain a closed-form solution to the proposing player's lowest SPE payoff. Although our discussions in the rest of this section refer to *Case 1* in Section 3.3, they apply to both cases.

Proposition 4 provides an upper bound on the proposing player's lowest SPE payoff, namely $\hat{m}_i^i$ given by (22). The value of $\hat{m}_i^i$ is directly related to studies on this issue in the existing literature. It combines and balances two sets of reasonings that have been identified. The first term in (22) closely resembles player i's no-concession payoff in Bolt (1995), resulted from receiving alternately $g_i(a)$ (from making unacceptable proposals) and 0 (from being minimaxed after rejecting any proposal). The second term in (22) extends to what has been identified by Busch and Wen (1995) and Houba (1997) for two subclasses of the current model where both players always make acceptable proposals in player i's worst SPE.

There are situations where $\hat{m}_i^i$ is in fact player i's lowest SPE payoff. When the frontier of SPE payoffs is a subset of the bargaining frontier, such as when the two players have a common discount factor and B is convex, it can be verified that $\hat{m}_i^i$ is the least fixed point of $\Lambda(\cdot)$. This result generalizes Haller (1991), Haller and Holden (1990), Busch and Wen

(1995) and Houba (1997) to an arbitrary convex set B. Note however that common time preferences alone are not sufficient to warrant this conclusion, we also need B to be convex.

We can take this insight one step further. In player i's worst SPE, if we never need any SPE whose payoff vector is above the bargaining frontier, $\hat{m}_i^i$ will be player i's lowest SPE payoff. Suppose that $\hat{a} \in A$ is a solution to (20) at the least fixed point of $\Lambda(\cdot)$ and $g_i(\hat{a}) = d_i(\hat{a})$. Then the only effective continuation payoff in solving (20) is $\varphi^j(\delta_i m_i^i)$, which is always equal to $\beta^j(\delta_i m_i^i)$ when $\delta_i \leq \delta_j$. When $\delta_i > \delta_j$, it is also likely the case that $\varphi^j(\delta_i m_i^i) = \beta^j(\delta_i m_i^i)$. Consequently, $\hat{m}_i^i$ is the least fixed point of $\Lambda(\cdot)$, and hence player i's lowest SPE payoff. Although requiring $g_i(\hat{a}) = d_i(\hat{a})$ might seem of limited interest, it is trivial if player i has only one disagreement action, such as the firm in the wage bargaining model analyzed by Fernandez and Glazer (1991), Haller (1991), Haller and Holden (1990) and Bolt (1995).

In the case of a linear bargaining frontier $\beta^j(x_i) = 1 - x_i$ as in most of the current literature, $F(\cdot; a)$ is a contraction mapping, hence has a unique fixed-point for all $a \in A$. Accordingly, (22) simplifies to

$$\hat{m}_i^i = \min_{a \in A} \max \left\{ \frac{g_i(a)}{1+\delta_i}, \frac{1-\delta_j}{1-\delta_i \delta_j} \left[1 - d_j(a) + \frac{\delta_j}{\delta_i} \frac{1-\delta_i}{1-\delta_j} \left[g_i(a) - d_i(a) \right] \right] \right\}, \tag{24}$$

where the second term is the expression identified by Muthoo (1999) when only acceptable proposals are considered. Furthermore, when $\delta_i = \delta_j = \delta$, together with the assumption that $d(a) \in B$ for all $a \in A$, $m_i^i = \hat{m}_i^i$ can be further simplified to what Busch and Wen (1995) obtain, $\frac{1}{1+\delta} \min_{a \in A} [1 - d_i(a) - d_j(a) + g_i(a)]$. Despite of the fact that $\hat{m}_i^i$ is player i's lowest SPE payoff in all these instances we have discussed, player i's lowest SPE payoff is strictly less than $\hat{m}_i^i$ in general, such as in the class of negotiation games we study in Section 4.

Our study strengthens the findings by Rubinstein (1982), Fudenberg and Tirole (1991), Herrero (1989), van Damme (1991), and many others in the standard alternating-offer bargaining model, which can be considered as a special case of our model with a degenerate disagreement game such that $g_i(a) = d_i(a) = 0$ for all $a \in A$. In this case, even if there are multiple SPEs, the frontier of SPE payoffs must coincide with the bargaining frontier, and hence $m_i^i = \hat{m}_i^i$. Since $0 \leq \delta_i^2 x_i \leq x_i < \beta^i\left(\delta_j \beta^j\left(\delta_i x_i\right)\right)$ for all $x < m_i^i$, which is the least fixed-point of $\Lambda(\cdot)$, (20) reduces to

$$\Lambda(m_i^i) = \max\left\{\delta_i^2 m_i^i, \beta^i\left(\delta_j \beta^j\left(\delta_i m_i^i\right)\right)\right\} = \beta^i\left(\delta_j \beta^j\left(\delta_i m_i^i\right)\right).$$

This result theoretically underpins the commonly held wisdom for the standard alternating-offer bargaining model; it is without loss of generality to assume that only acceptable proposals support the extreme SPE payoffs, which is also formally examined by Fudenberg and Tirole (1991) in this simple case.

4 Common Interest Disagreement Games

In this section, we focus on an important class of negotiation games that contains the models studied in Fernandez and Glazer (1991), Haller and Holden (1990), Bolt (1995), and Slantchev (2003). The set of possible agreements is assumed to be $B = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq 1\}$. The disagreement game is a common interest game where there exists a unique Pareto dominant disagreement outcome.[8] Formally, there is an $a^* \in A$ such that $d(a^*) \geq d(a)$ for all $a \in A$. Without loss of too much generality, we assume that $d(a^*)$ is on the bargaining frontier, i.e., $d_1(a^*) + d_2(a^*) = 1$. Note that $a^* \in A$ is a Nash equilibrium in G, and also a serious candidate to (24). In fact, the value of the objective function in (24) at $a^* \in A$,

$$x_i(a^*) \equiv \max\left\{\frac{1}{1+\delta_i} d_i\left(a^*\right), \frac{1-\delta_j}{1-\delta_i\delta_j}\left[1 - d_j\left(a^*\right)\right]\right\},$$

can be supported as a player i's SPE payoff when the discount factors are sufficiently large. The proof of the following proposition is similar to but much simpler than that of Proposition 5. It is straightforward to verify the following SPE. The strategy profile specifies a^* after player i's proposal is rejected and player i's minimax outcome in G after player i rejects any

[8] Common interest games are studied in other dynamic settings, see, e.g., Farrel and Saloner (1985) and Takahashi (2005).

proposal. Player i demands $x_i(a^*)$ whenever player i proposes, and player j accepts player i's demand $x_i(a^*)$ if and only if $\delta_i \geq \delta_j$. Accordingly, we have

Proposition 6 *For sufficiently large $(\delta_i, \delta_j) \in (0,1)^2$, there is a SPE where player i receives $x_i(a^*)$ and player j receives no less than $1 - x_i(a^*)$ in a period when player i proposes.*

With the linear bargaining frontier and the common interest disagreement game, most results in Section 3 can be further refined. First, (7) implies that for all $(\delta_i, \delta_j) \in (0,1)^2$,

$$M_j^i \leq (1-\delta_j) \max_{a \in A} d_j(a) + \delta_j \max_{v \in E^j} v_j = (1-\delta_j) d_j(a^*) + \delta_j M_j^j. \tag{25}$$

For sufficiently large $(\delta_i, \delta_j) \in (0,1)^2$, (25) implies that in player j's best SPE, if player j rejects any proposal, player j will receive his highest disagreement payoff $d_j(a^*)$ in the current period followed by his highest SPE payoff M_j^j in the following period. Our next proposition characterizes the proposing player's highest SPE payoff.

Proposition 7 *For sufficiently large $(\delta_i, \delta_j) \in (0,1)^2$, we have*

$$M_i^i \leq 1 - m_j^i. \tag{26}$$

Proof. In this class of negotiation games, (5) becomes

$$\begin{aligned}
M_i^i &= \max \left\{ \begin{array}{ll} \max_{a,v} \left[(1-\delta_i) d_i(a) + \delta_i v_i \right], & \text{s.t. (3)}, \\ \max_{a,v} \left[1 - (1-\delta_j) d_j(a) - \delta_j v_j \right], & \text{s.t. (3)}, \end{array} \right. \\
&\leq \max \left\{ \begin{array}{ll} (1-\delta_i) \max_{a \in A} d_i(a) + \delta_i \max_{v \in E^j} v_i, & \\ 1 - \min_{a,v} \left[(1-\delta_j) d_j(a) - \delta_j v_j \right], & \text{s.t. (3)}, \end{array} \right. \\
&\leq \max \left\{ (1-\delta_i) d_i(a^*) + \delta_i M_i^j, 1 - m_j^i \right\}.
\end{aligned}$$

For sufficiently large $(\delta_i, \delta_j) \in (0,1)^2$, however, it cannot be the case that

$$1 - m_j^i \leq (1-\delta_i) d_i(a^*) + \delta_i M_i^j.$$

Suppose not, then $M_i^i \leq (1-\delta_i) d_i(a^*) + \delta_i M_i^j$ and (25) would imply that $M_i^i \leq d_i(a^*)$, which contradicts the fact that $M_i^i \geq 1 - x_j(a^*) > d_i(a^*)$ by Proposition 6. Consequently, (26) must prevail. ■

Proposition 7 implies that Case (b) in Figure 2 is impossible when the disagreement game is a common interest game. (11), (25), and (26) provide three of the four conditions for the extreme SPE payoffs. With common interest disagreement games, these three conditions are relatively simple because they are not affected by the complications of the Pareto frontier of SPE payoffs. In this class of negotiation games, we are able to further refine Proposition 4:

Proposition 8 *For sufficiently large $(\delta_i, \delta_j) \in (0,1)^2$, we have*

$$
\begin{aligned}
m_i^i &= \frac{1-\delta_j}{1-\delta_i\delta_j}[1-d_j(a^*)], && \text{if } \delta_i \geq \delta_j, \\
m_i^i &\in \left[\frac{1-\delta_j}{1-\delta_i\delta_j}[1-d_j(a^*)], \frac{1}{1+\delta_i}d_i(a^*)\right], && \text{if } \delta_i < \delta_j.
\end{aligned}
$$

Proof. Since $d_j(a) \leq d_j(a^*)$ for all $a \in A$ and $v_j \leq M_j^j$ for all $v \in E^j$, (26) and (11) imply that $M_j^j \leq 1 - \delta_i m_i^i$. From (4), we have

$$
\begin{aligned}
m_i^i &\geq \min_{a \in A, v \in E^j} 1 - (1-\delta_j)d_j(a) - \delta_j v_j, \qquad \text{s.t. (3)}, \\
&\geq 1 - (1-\delta_j)d_j(a^*) - \delta_j M_j^j \\
&\geq 1 - (1-\delta_j)d_j(a^*) - \delta_j(1-\delta_i m_i^i) \\
&= (1-\delta_j)[1-d_j(a^*)] + \delta_j\delta_i m_i^i, \\
&\Rightarrow \; m_i^i \geq \frac{1-\delta_j}{1-\delta_i\delta_j}[1-d_j(a^*)].
\end{aligned}
$$

Together with Proposition 6, i.e., $m_i^i \leq x_i(a^*)$, we conclude this proof. ■

When $\delta_i \geq \delta_j$, we pin down the value of m_i^i at $x_i(a^*)$. Therefore, the SPE of Propositions 6 is indeed player i's worst SPE when $\delta_i \geq \delta_j$. This conclusion confirms the findings in the literature if and only if the proposing player is at least as patient as the responding player.

When $\delta_i < \delta_j$, Proposition 8 confines the value of m_i^i between the two bounds that have been studied. Solving m_i^i from (14) requires the Pareto frontier of E^j, that must be generated from the following type of paths:

$$
\pi^T = (\underbrace{a^*, \ldots, a^*}_{T}, x^*), \qquad \text{for all } T \geq 0, \tag{27}
$$

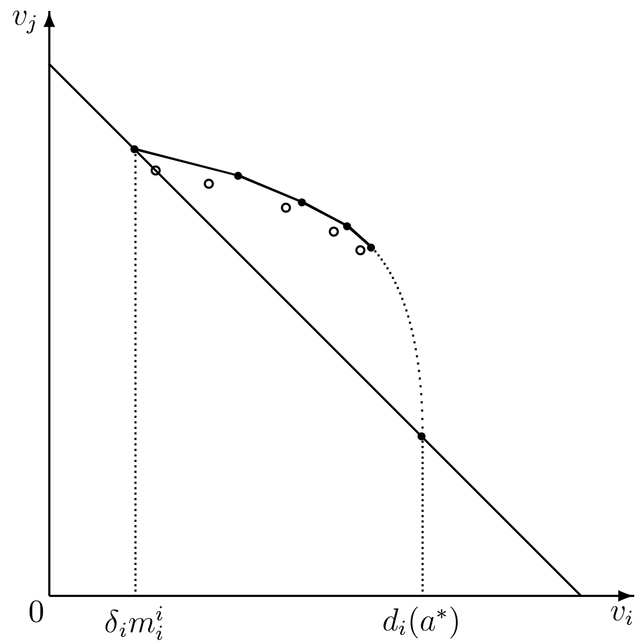

Figure 3: The payoff vectors $v(\pi^T)$ for $T \leq 8$.

where $x_i^* = m_i^j$ for even T and $x_i^* = m_i^i$ for odd T. With Proposition 8, it is straightforward to verify that for any even $T \geq 0$,

$$v_i(\pi^T) \quad < \quad v_i(\pi^{T+1}) \leq v_i(\pi^{T+2}) < d_i(a^*), \tag{28}$$

$$v_j(\pi^T) \quad > \quad v_j(\pi^{T+2}) < v_j(\pi^{T+1}) < d_j(a^*). \tag{29}$$

(28) and (29) can be best illustrated by Figure 3, where $v(\pi^T)$ is represented by solid dots for even $T \leq 8$ and open dots for odd $T < 8$. It implies that for any even $T \geq 0$, $v(\pi^{T+1})$ is dominated by some convex combination of $v(\pi^T)$ and $v(\pi^{T+2})$. Intuitively speaking, if the continuation path were associated with an odd T, then player i would make a proposal along such a continuation, from which player i could exploit his advantage of being the proposer. Consequently, such a continuation can never be effective in solving (4). For all even T, any convex combination of $v(\pi^T)$ and $v(\pi^{T+2})$ can be achieved by a publicly correlated strategy between π^T and π^{T+2}.

For sufficiently large $\delta_i < \delta_j$, when v_i is sufficiently close to $\delta_i m_i^i$,

$$\varphi^j(v_i) = \min_{T \in 2\mathbb{N}} \left\{ v_j(\pi^T) + \frac{v_j(\pi^{T+2}) - v_j(\pi^T)}{v_i(\pi^{T+2}) - v_i(\pi^T)} \left[v_i - v_i(\pi^T) \right] \right\}, \tag{30}$$

which is a concave, decreasing and piecewise-linear function of v_i. Substituting (30) into (20), one can solve the least fixed point of $m_i^i = \Lambda(m_i^i)$.

We now present an example to demonstrate how to solve m_i^i when $\delta_i < \delta_j$. Consider a negotiation game with the following 2×2 disagreement game for $\varepsilon \geq 0$:

Player 1 \ Player 2	L	R
U	0.5, 0.5	$-\varepsilon$, 0.5
D	0.5, 0	0, -1

where $a^* = (U, L)$. For simplicity, we consider pure actions only. Note that in this example, (22) yields

$$\hat{m}_1^1 = \min\left\{\frac{1}{1+\delta_1}\frac{1}{2}, \frac{1-\delta_2}{1-\delta_1\delta_2}\left[\frac{1}{2} + \frac{\delta_2}{\delta_1}\frac{1-\delta_1}{1-\delta_2}\varepsilon\right]\right\}.$$

Contrary to what the current literature suggests, $\hat{m}_1^1$ is not player 1's lowest SPE payoff in this example when δ_2 is significantly higher than δ_1. Finding m_1^1 in this example is still a daunting task. First, we need to solve the least fixed-point of $\Lambda(\cdot)$ for each linear segment of $\varphi^2(\cdot)$ as in the first linear segment demonstrated below. Then, we need to identify the minimum of these fixed-points for all linear segments of $\varphi^2(\cdot)$.

We now demonstrate a SPE where the continuation payoffs are on the first linear segment of the Pareto frontier $\varphi^2(\cdot)$, i.e., continuations involve at most two periods of delay in reaching an agreement. Consider the following strategy profile:

- In an odd period, player 1 demands

$$x_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2}\left[\frac{1}{2} + \frac{\delta_2}{\delta_1}\frac{1+\delta_2}{1+\delta_1}\varepsilon\right] \tag{31}$$

  and player 2 will reject if and only if player 1 demands more than x_1^*.

  If player 1 demands more than x_1^* and player 2 rejects, then (U, R) will be played.

- In an even period, if player 1 deviates from U in the last (odd) period, player 2 will offer $\delta_1 x_1^*$ and player 1 will accept.

- Otherwise, with probability $1-p$, player 2 will offer $\delta_1 x_1^*$ in the current even period, and with probability p, (U, L) will be played for two periods, followed by player 2's offer $\delta_1 x_1^*$. Player 1 accepts in both cases. In this equilibrium,

$$p = \frac{1}{\delta_1(1-\delta_1)} \cdot \frac{\varepsilon}{0.5 - \delta_1 x_1^*}. \tag{32}$$

- In an even period, if player 1 rejects $\delta_1 x_1^*$ (that should be accepted), then (D, R) will be played once followed by player 1's demand x_1^*.

- If player 2 deviates from the strategies described above, then continuation will switch immediately to the stationary SPE from which player 1 receives 0.5.

To verify that the above strategy profile constitutes a SPE, first note that player 1 has no incentive to deviate from (U, R) if his payoff from deviation is the same as what player 1 receives if he does not:

$$\delta_1^2 x_1^* = (1-\delta_1) \cdot (-\varepsilon) + \delta_1 \left[(1-p)\delta_1 x_1^* + p\left(0.5(1-\delta_1^2) + \delta_1^3 x_1^*\right)\right]. \tag{33}$$

One can show that (33) holds for p as given by (32). Next, player 1 should demand x_1^* rather than making an unacceptable proposal,

$$x_1^* \geq (1-\delta_1) \cdot (-\varepsilon) + \delta_1 \left[(1-p)\delta_1 x_1^* + p\left(0.5(1-\delta_1^2) + \delta_1^3 x_1^*\right)\right] = \delta_1^2 x_1^*,$$

which follows from (33). Lastly, player 1 cannot demand more than x_1^* since $1 - x_1^*$ is exactly equal to player 2's continuation payoff after rejecting any demand higher than x_1^*:

$$1 - x_1^* = 0.5(1-\delta_2) + \delta_2[(1-p)(1-\delta_1 x_1^*) + p\left[0.5(1-\delta_2^2) + \delta_2^2(1-\delta_1 x_1^*)\right]]. \tag{34}$$

In fact, (33) and (34) yield x_1^* and p as given by (31) and (32), respectively.

For $\delta_1 = 0.8$ and $\varepsilon = 0.15$, Figure 4 shows $x_1^* < \hat{m}_1^1$ for all $\delta_2 \in (0.877, 1)$. When the difference between the players' time preferences is not significant enough such as $\delta_2 \in (0.8, 0.877)$, it would be too costly to compensate player 1 during the delay in the continuation. When

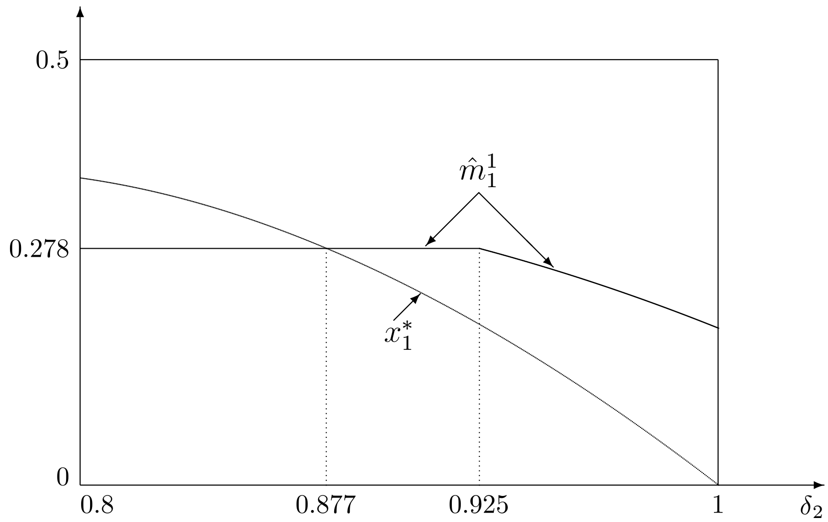

Figure 4: Plot of x_1^* with respect to $\delta_2 \in (\delta_1, 1)$ for $\delta_1 = 0.8$ and $\varepsilon = 0.15$.

this happens, the SPE of Proposition 6 is likely to be player 1's worst SPE. However, such incidence diminishes as the value of ε decreases.

Appendix: Proof of Proposition 5

Without loss of generality, assume that when $\delta_i \geq \delta'$ and $\delta_j \geq \delta'$ for some $\delta' \in (0,1)$, there exists a SPE from which player j's payoff is less than $\beta^j(\hat{m}_i^i) - \varepsilon$ for some $\varepsilon > 0$, i.e., $m_j^j \leq \beta^j(\hat{m}_i^i) - \varepsilon$. Let m_i^i be the least fixed-point of $\Lambda(\cdot)$, which generally depends on $(\delta_i, \delta_j) \in (0,1)^2$. For sufficiently large δ'' and $\delta_j > \delta_i \geq \delta''$, we have $\varphi^j(v_i)$ depends only on m_i^i for all $v_i \in \left[\delta_i m_i^i, \delta_i m_i^i + \frac{1-\delta_i}{\delta_i} \max_{a \in A}[g_i(a) - d_i(a)]\right]$ and

$$\varphi^j\left(\delta_i m_i^i + \frac{1-\delta_i}{\delta_i} \max_{a \in A}[g_i(a) - d_i(a)]\right) \geq m_j^j + \frac{\varepsilon}{2}. \tag{35}$$

From the definition of $\varphi^j(\cdot)$, $(v_i, \varphi^j(v_i))$ is a SPE payoff vector for all

$$v_i \in \left[\delta_i m_i^i, \delta_i m_i^i + \frac{1-\delta_i}{\delta_i} \max_{a \in A}[g_i(a) - d_i(a)]\right]$$

as long as $\delta_j > \delta_i \geq \max\{\delta', \delta''\}$ and m_i^i can be supported as player i's SPE payoff. Choose $\hat{\delta} \geq \max\{\delta', \delta''\}$ sufficiently large so that $\frac{1-\hat{\delta}}{\hat{\delta}} \max_{a \in A}[g_j(a) - d_j(a)] \leq \frac{\varepsilon}{2}$. In other words, if

$\delta_j \geq \hat{\delta}$ then player j will not deviate in the disagreement game as long as player j's average loss in the continuation payoffs is no less than $\frac{\varepsilon}{2}$.

For all $\delta_j > \delta_i \geq \hat{\delta}$, let m_i^i be the least fixed-point of $\Lambda(\cdot)$ and $\tilde{a} \in A$ be the corresponding solution to (20). We have the following two cases to examine:

Case 1: $m_i^i = (1-\delta_i)g_i(\tilde{a}) + \delta_i^2 m_i^i$.

Consider the following strategy profile: Player i makes an unacceptable offer (such as demands m_i^i or more). Player j rejects if and only if player i offers less than

$$(1-\delta_j)d_j(\tilde{a}) + \delta_j \varphi^j \left(\delta_i m_i^i + \frac{1-\delta_i}{\delta_i} \left[g_i(\tilde{a}) - d_i(\tilde{a}) \right] \right) \geq \beta^j \left(m_i^i \right),$$

followed by $\tilde{a}$ once. If player i deviates from $\tilde{a}$, player j will offer $\delta_i m_i^i$ and player i will accept in the following period. Otherwise, the continuation SPE in the following period will be on the Pareto frontier of E^j from which player i will receive $\delta_i m_i^i + \frac{1-\delta_i}{\delta_i} \left[g_i(\tilde{a}) - d_i(\tilde{a}) \right]$. If player j deviates from what is described above, player j will be punished by the SPE provided at the beginning of this proof from which his payoff will not be higher than $\beta^j \left(\hat{m}_i^i \right) - \varepsilon$.

We now verify sequential rationality. It is clear from the construction that no one deviates in the proposing and responding stages. For example, player i has to offer at least

$$(1-\delta_j)d_j(\tilde{a}) + \delta_j \varphi^j \left(\delta_i m_i^i + \frac{1-\delta_i}{\delta_i} \left[g_i(\tilde{a}) - d_i(\tilde{a}) \right] \right)$$

in order to induce player j to accept, from which player i receives at most m_i^i. Player i will not deviate from $\tilde{a}$ because

$$(1-\delta_i)g_i(\tilde{a}) + \delta_i \left(\delta_i m_i^i \right) = (1-\delta_i)d_i(\tilde{a}) + \delta_i \left(\delta_i m_i^i + \frac{1-\delta_i}{\delta_i} \left[g_i(\tilde{a}) - d_i(\tilde{a}) \right] \right).$$

Case 2: $m_i^i = \beta^i \left((1-\delta_j)d_j(\tilde{a}) + \delta_j \varphi^j \left(\delta_i m_i^i + \frac{1-\delta_i}{\delta_i} \left[g_i(\tilde{a}) - d_i(\tilde{a}) \right] \right) \right)$.

Consider the following strategy profile: Player i demands m_i^i. Player j rejects if and only if player i demands more than m_i^i. If player i demands more and player j rejects (which should not occur), the two players will play $\tilde{a}$, and the continuations are the same as those in Case 1 for the corresponding histories.

Similar to Case 1, no one will deviate after player i demands more than m_i^i and player j rejects. If player i demands more than m_i^i at the beginning, player j will reject, and player i will receive

$$(1-\delta_i)d_i(\tilde{a}) + \delta_i \left[\delta_i m_i^i + \frac{1-\delta_i}{\delta_i}\left[g_i(\tilde{a}) - d_i(\tilde{a})\right]\right] \leq m_i^i.$$

Therefore, player i will demand m_i^i, which will be accepted by player j. In summary, no one has an incentive to deviate when player i is supposed to demand m_i^i.

We have shown that in either case, there is an equilibrium where player i receives m_i^i, the least fixed-point of $\Lambda(\cdot)$, when making a proposal. ■

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