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Consistency, Monotonicity and Implementation of Egalitarian Shapley Values

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Abstract

One of the main issues in economics is the trade-off between marginalism and egalitarianism. In the context of cooperative games this trade-off can be framed as one of choosing to allocate according to the Shapley value or the equal division solution. In this paper we provide tools that make it possible to study this trade-off in a consistent way by providing three types of results on *egalitarian Shapley values* being convex combinations of the Shapley value and the equal division solution. First, we show that all these solutions satisfy the same reduced game *consistency*. Second, we characterize this class of solutions using *monotonicity* properties. Finally, we provide a non-cooperative *implementation* for these solutions which only differ in the probability of breakdown at a certain stage of the game.

Keywords: Shapley value, Equal division solution, Egalitarian Shapley value, Reduced Game Consistency, Monotonicity, Implementation

JEL code: C71; C72; D60

AMS subject classification: 91A12; 91A10

1 Introduction

One of the main issues in economics is the trade-off between marginalism and egalitarianism. In this paper we consider this issue in the context of cooperative games with transferable utility. For these games the trade-off between marginalism and egalitarianism can be seen as the trade-off between allocating according to the *Shapley value* or the *equal division solution*. Both solutions consider situations where eventually the 'grand coaliton' consisting of all players forms. The Shapley value (Shapley (1953)) assigns to every player its expected marginal contribution to this coalition assuming that all possible orders of entrance of the players occur with equal probability. Here, the marginal contribution of a player to a coalition is the increase in transferable utility when this player joins the coalition. The marginalist characteristic of the Shapley value is most clearly formulated by Young (1985) who characterized it as the unique solution that is efficient, symmetric and strongly monotonic. This *strong monotonicity* states that the payoffs of a player do not decrease if its marginal contributions to coalitions do not decrease. Since symmetry (i.e. equal treatment of equals) is usually considered a desirable property, and under complete information also efficiency is widely accepted, this result says that the Shapley value is the only solution in which the utility payoff of a player is fully determined by its marginal contributions to the transferable utility of coalitions[1].

On the other hand, the *equal division solution* which allocates the worth of the grand coalition equally among all players, can be seen as the most egalitarian solution for such games. The trade-off between marginalism and egalitarianism can be made by considering convex combinations of the Shapley value and the equal division solution. This class of solutions is introduced by Joosten (1996) and are called *egalitarian Shapley values*. In order to consider these solutions as making a trade-off between marginalism and egalitarianism we need to provide results that not only show the difference between these solutions, but also their similarities. We do this by providing three types of results that are very common in game theory: consistency, monotonicity and implementation.

First, we show that all these solutions satisfy the *reduced game consistency* that is used by Sobolev (1973) to characterize the Shapley value. In these reduced games, after a particular player leaves the game with its payoff, the remaining coalitions assume with a probability that is proportional to their cardinality that the leaving player cooperates with them or not. Reduced game consistency requires that players respect the recommendations made by the solution in the sense that the solution assigns to the players in the reduced game the same payoffs as it assigns to those players in the original game. However, the

[1] As already noticed by Young (1985) it is sufficient to weaken strong monotonicity to *marginalism* stating that the payoff of a player in two games is equal if all its marginal contributions are equal in both games.

egalitarian Shapley values differ with respect to the *standardness for two-player games* that they satisfy. The usual standardness for two-player games which is satisfied by the Shapley value states that in a two player game every player earns its own worth plus half of what remains of the worth of the two-player ('grand') coalition (see, e.g. Hart and Mas-Colell (1988, 1989)). Egalitarian standardness states that in two player games the worth of the grand coalition is split equally among the two players. For general egalitarian Shapley values the 'sharing of the surplus' depends on the weights put on the Shapley value and equal division solution. Since all these solutions share the same reduced game consistency, the difference thus boils down to the allocation that is applied in two-player games.

The second characterization is based on monotonicity properties. Above we already refered to the characterization of the Shapley value by efficiency, symmetry and strong monotonicity (or marginalism) in Young (1985). Since all egalitarian Shapley values satisfy efficiency and symmetry, this implies that the Shapley value is the unique egalitarian Shapley value that is strongly monotonic. However, it turns out that all egalitarian Shapley values satisfy the weaker property which states that the payoff of a player does not decrease if its marginal contributions do not decrease and, moreover, the worth of the 'grand coalition' does not decrease. This is a considerable weakening of the strong monotonicity property. Since the worth of the 'grand coalition' is what is to be allocated by any efficient solution, strong monotonicity requires that the payoff of a player does not decrease if its marginal contributions do not decrease *irrespective* of what is to be allocated, which is a very strong requirement. However, if the worth to be allocated is not decreasing then this requirement on the payoffs seems reasonable for a marginalistic solution. We show that the class of egalitarian Shapley values is characterized by this *weak monotonicity* together with the well-known properties of efficiency, linearity and local monotonicity.

Whereas the above two characterizations of the egalitarian Shapley values give a cooperative foundation (on a variable, respectively, fixed player set), our third result provides a non-cooperative foundation by *implementing* the egalitarian Shapley values as the unique subgame perfect equilibrium outcome in an extensive form bidding mechanism. This bidding mechanism generalizes the one for the Shapley value given in Pérez-Castrillo and Wettstein (2001), differing only in an additional possibility of breakdown of the negotiations. The bidding mechanism of Pérez-Castrillo and Wettstein (2001) starts with all players and proceeds in various rounds that each have four stages. In the first stage all players that are still 'in the game' make bids to all other bidders showing their willingness to become the proposer. A player with the highest 'net bid' becomes the proposer. In stage 2 the proposer makes (additional) payoff offers to the other players. In stage 3 these other players either accept or reject the proposal. Finally, in stage 4 payoffs or the continuation of the game is determined. If all others accept the proposal then they all get their offers and the proposer earns what remains. If at least one of them rejects then the proposer leaves the game with its stand-alone payoff and the others continue the bidding game without the proposer, starting again at stage 1 (of the next round). The bidding game thus ends when either all remaining players accept the proposal or the last player leaves the game. Our bidding mechanism only differs from that of Pérez-Castrillo and Wettstein (2001) in the sense that at the end of the first round, after the offer is rejected there is an additional possibility that the negotiations breakdown and all players earn zero payoff. The probability of breakdown is determined by the weights put on the Shapley value and equal division solution.

The paper is organized as follows. Section 2 discusses some preliminaries on cooperative games with transferable utility and solutions. In Section 3 we provide the characterization using standardness and reduced game consistency. In Section 4 we provide a characterization of the class of egalitarian Shapley values using monotonicity properties. In Section 5 we provide an implementation of the egalitarian Shapley values. Finally, Section 6 contains some concluding remarks.

2 Preliminaries

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game, being a pair (N, v), where $N \subset \mathbb{N}$ is a finite set of players and $v\colon 2^N \to \mathbb{R}$ is a characteristic function on N such that $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S)$ is called the *worth* of coalition S. This is the transferable utility that the members of coalition S can obtain by agreeing to cooperate. We denote the class of all TU-games by $\mathcal{G}$. A TU-game (N, v) is *monotone* if $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$. The *unanimity game* of coalition $T \subseteq N$, $T \neq \emptyset$, on N is the monotone game (N, u_T) given by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise. In the sequel we denote $n = |N|$ for the number of players in N. For generic coalitions $S \subseteq N$ we denote $s = |S|$.

A *payoff vector* of game (N, v) is an n-dimensional real vector $x \in \mathbb{R}^n$ which represents a distribution of the payoffs that can be earned by cooperation over the individual players. A (point-valued) *solution* for TU-games is a function ψ which assigns a payoff vector $\psi(N, v) \in \mathbb{R}^n$ to every TU-game $(N, v) \in \mathcal{G}$ such that $\psi_i(\{i\}, v) = v(\{i\})$ for all $i \in \mathbb{N}$. Two well-known solutions are the Shapley value and the equal division solution. The *Shapley value* (Shapley (1953)) is the solution that assigns to every TU-game (N, v)

the payoff vector

$$Sh_i(N,v) = \sum_{\substack{S\subseteq N\\ i\in S}} \frac{(n-s)!(s-1)!}{n!}(v(S)-v(S\setminus\{i\})) \text{ for all } i\in N.$$

The *equal division solution* is the solution that distributes the worth $v(N)$ of the 'grand coalition' equally among all players and thus assigns to every TU-game (N,v) the payoff vector

$$ED_i(N,v) = \frac{v(N)}{n} \text{ for all } i\in N.$$

Joosten (1996) introduced a new class of solutions that are obtained as convex combinations of the Shapley value and the equal division solution. For every $\alpha\in[0,1]$, he defines the *α-egalitarian Shapley value* φ^α as the solution given by

$$\varphi^\alpha(N,v) = \alpha Sh(N,v) + (1-\alpha)ED(N,v).$$

By $\Phi = \{\varphi^\alpha \mid \alpha\in[0,1]\}$ we denote the class of all α-egalitarian Shapley values and refer to a generic solution in this class as an *egalitarian Shapley value*. Some well-known properties of solutions for TU-games are the following. Solution ψ

- is *efficient*[2] if $\sum_{i\in N}\psi_i(N,v) = v(N)$ for all $(N,v)\in\mathcal{G}$.
- is *linear* if $\psi(N,av+bw) = a\psi(N,v)+b\psi(N,w)$ for all $(N,v),(N,w)\in\mathcal{G}$, where $av+bw$ is given by $(av+bw)(S) = av(S)+bw(S)$ for all $S\subseteq N$.
- is *symmetric* if $\psi_i(N,v) = \psi_j(N,v)$ for all $(N,v)\in\mathcal{G}$ and $i,j\in N$ such that $v(S\cup\{i\}) = v(S\cup\{j\})$ for all $S\subseteq N\setminus\{i,j\}$.
- satisfies *local monotonicity* if $\psi_i(N,v)\geq\psi_j(N,v)$ for all $(N,v)\in\mathcal{G}$ and $i,j\in N$ such that $v(S\cup\{i\})\geq v(S\cup\{j\})$ for all $S\subseteq N\setminus\{i,j\}$.
- satisfies *strong monotonicity* if $\psi_i(N,v)\geq\psi_i(N,w)$ for every pair of games $(N,v),(N,w)$ and $i\in N$ such that $v(S\cup\{i\})-v(S)\geq w(S\cup\{i\})-w(S)$ for all $S\subseteq N\setminus\{i\}$.
- satisfies *α-standardness for two-player games*, $\alpha\in[0,1]$, if for every $(N,v)\in\mathcal{G}$ with $N=\{i,j\},\ i\neq j$, it holds that $\psi_i(N,v) = \frac{\alpha}{2}(v(\{i\})-v(\{j\}))+\frac{1}{2}v(N)$.

Specific choices of $\alpha\in[0,1]$ give different versions of standardness for two-player games as encountered in the literature. Taking $\alpha=1$ yields *standardness for two-player games* as considered in, e.g. Hart and Mas-Colell (1988,1989): $\psi_i(N,v) = \frac{1}{2}v(\{i\})-\frac{1}{2}v(\{j\})+$

[2] Efficient solutions are often called *values*.

$\frac{1}{2}v(N) = v(\{i\}) + \frac{1}{2}(v(N) - v(\{i\}) - v(\{j\}))$ with $N = \{i, j\}$. Taking $\alpha = 0$, yields *egalitarian standardness for two-player games*: $\psi_i(N, v) = \frac{1}{2}v(N)$ for $i \in N$.

It is known that the Shapley value satisfies standardness and the equal division solution satisfies egalitarian standardness for two-player games. Joosten (1996) showed that for $\alpha \in [0, 1]$, the α-egalitarian Shapley value φ^α satisfies the corresponding α-standardness for two-player games.

3 Consistency

In the literature various reduced game properties are discussed. Suppose that one player leaves a game with some payoff. *Reduced games* describe what games are played between the remaining players, i.e. what is earned by coalitions of the remaining players after one player has left the game. The worths of coalitions in the reduced games usually depend on (i) the worths that these coalitions could earn on their own in the original game, (ii) what these coalitions could earn with the leaving player and (iii) the payoff with which the leaving player left the game. One of the oldest reduced games is the following that is introduced by Sobolev (1973). Consider a game (N, v), a payoff vector $x \in \mathbb{R}^n$, and a player $j \in N$. Suppose player j leaves the game with its payoff x_j. In the reduced game between the remaining players the coalition $N \setminus \{j\}$, being the 'grand coalition' in the reduced game, earns the worth of N minus the payoff x_j. Clearly, this is what is left to be allocated among the players in $N \setminus \{j\}$ after player j has left the game with payoff x_j. In Sobolev's reduced game any coalition $S \subseteq N \setminus \{j\}$ either has (or thinks it has) the cooperation of the leaving player j but must pay for j's payoff and thus earns $v(S \cup \{j\}) - x_j$, or is on its own and earns its own worth $v(S)$. The probability with which a coalition S has the cooperation of j depends on the cardinality of S. More precisely, with probability $\frac{s}{n-1}$ coalition $S \subset N \setminus \{j\}$ has j's cooperation and thus earns $v(S \cup \{j\}) - x_j$, and with probability $1 - \frac{s}{n-1} = \frac{n-1-s}{n-1}$ coalition $S \subset N \setminus \{j\}$ is on its own and earns $v(S)$.

Definition 3.1 *Given game $(N, v) \in \mathcal{G}$, player $j \in N$, and efficient payoff vector $x \in \mathbb{R}^N$, the* **reduced game with respect to** j **and** x *is the game $(N \setminus \{j\}, v^x)$ given by*

$$v^x(S) = \frac{s}{n-1}(v(S \cup \{j\}) - x_j) + \frac{n-1-s}{n-1}v(S) \text{ for all } S \subseteq N \setminus \{j\}.$$

Note that indeed $v^x(N \setminus \{j\}) = v(N) - x_j$. Consistency with respect to a particular reduced game means that given a game (N, v), if x is a solution payoff vector for (N, v), then for every player $j \in N$, the payoff vector $x_{N \setminus \{j\}}$ with payoffs for the players in $N \setminus \{j\}$, must be a solution payoff vector of the reduced game $(N \setminus \{j\}, v^x)$. It is a kind of internal consistency requirement to guarantee that players respect the recommendations made by the solution. In the following we refer to consistency with respect to the reduced game defined in Definition 3.1 just as consistency[3].

Definition 3.2 *Let ψ be a solution on $\mathcal{G}$. Solution ψ satisfies* **consistency** *on $\mathcal{G}$ if and only if for every $(N,v) \in \mathcal{G}$ with $n \geq 2$, $j \in N$, and $x = \psi(N,v)$ it holds that $\psi_i(N \setminus \{j\}, v^x) = \psi_i(N,v)$ for all $i \in N \setminus \{j\}$.*

Sobolev (1973) showed that the Shapley value is the unique solution that satisfies this consistency and is standard for two-player games. Surprisingly, also the equal division solution satisfies this consistency. We can even state the following more general result for all egalitarian Shapley values.

Proposition 3.3 *Every egalitarian Shapley value φ^α, $\alpha \in [0,1]$, satisfies consistency.*

Proof

Take $j \in N$, $S \subseteq N \setminus \{j\}$ and any efficient payoff vector z. Recall that $n = |N|$ and $s = |S|$. First, for a reduced game $(N \setminus \{j\}, v^z)$, for any $i \in N \setminus \{j\}$ and $S \subseteq N \setminus \{i,j\}$, it follows that

$$\begin{aligned}
v^z(S \cup \{i\}) - v^z(S) &= \frac{s+1}{n-1}(v(S \cup \{i,j\}) - z_j) + \frac{n-s-2}{n-1}v(S \cup \{i\}) \\
&\quad - \frac{s}{n-1}(v(S \cup \{j\}) - z_j) - \frac{n-s-1}{n-1}v(S) \\
&= \frac{s+1}{n-1}(v(S \cup \{i,j\}) - v(S \cup \{j\})) + \frac{1}{n-1}(v(S \cup \{j\}) - z_j) \\
&\quad + \frac{n-s-1}{n-1}(v(S \cup \{i\}) - v(S)) - \frac{1}{n-1}v(S \cup \{i\}) \\
&= \frac{s+1}{n-1}(v(S \cup \{i,j\}) - v(S \cup \{j\})) + \frac{n-s-1}{n-1}(v(S \cup \{i\}) - v(S)) \\
&\quad + \frac{1}{n-1}(v(S \cup \{j\}) - v(S \cup \{i\})) - \frac{1}{n-1}z_j.
\end{aligned}$$

Then, for all $i, j \in N$, $i \neq j$, the Shapley value for player i in the reduced game with respect to j and z can be written as

$$Sh_i(N \setminus \{j\}, v^z) = \sum_{S \subseteq N \setminus \{i,j\}} \frac{s!(n-s-2)!}{(n-1)!}(v^z(S \cup \{i\}) - v^z(S))$$

[3] We only consider the class $\mathcal{G}$ of all TU-games. If one considers subclasses $\mathcal{C} \subset \mathcal{G}$, then in the definition of consistency one should aditionally require that the reduced games $(N \setminus \{j\}, v^x)$ also belong to $\mathcal{C}$.

$$
\begin{aligned}
&= \sum_{S\subseteq N\setminus\{i,j\}} \frac{s!(n-s-2)!}{(n-1)!}\cdot\frac{s+1}{n-1}(v(S\cup\{i,j\})-v(S\cup\{j\}))\\
&\quad+\sum_{S\subseteq N\setminus\{i,j\}} \frac{s!(n-s-2)!}{(n-1)!}\cdot\frac{n-s-1}{n-1}(v(S\cup\{i\})-v(S))\\
&\quad+\sum_{S\subseteq N\setminus\{i,j\}} \frac{s!(n-s-2)!}{(n-1)!}\cdot\frac{1}{n-1}(v(S\cup\{j\})-v(S\cup\{i\}))\\
&\quad-\sum_{S\subseteq N\setminus\{i,j\}} \frac{s!(n-s-2)!}{(n-1)!}\cdot\frac{1}{n-1}z_j\\
&= \frac{1}{n-1}\left(\sum_{S\subseteq N\setminus\{i,j\}} \frac{(s+1)!(n-s-2)!}{(n-1)!}(v(S\cup\{i,j\})-v(S\cup\{j\}))\right.\\
&\quad+\sum_{S\subseteq N\setminus\{i,j\}} \frac{s!(n-s-1)!}{(n-1)!}(v(S\cup\{i\})-v(S))\\
&\quad\left.+\sum_{S\subseteq N\setminus\{i,j\}} \frac{s!(n-s-2)!}{(n-1)!}(v(S\cup\{j\})-v(S\cup\{i\}))-z_j\right)\\
&= \frac{1}{n-1}\left(\sum_{S\subseteq N, S\ni j} \frac{s!(n-s-1)!}{(n-1)!}(v(S\cup\{i\})-v(S))\right.\\
&\quad\left.+\sum_{S\subseteq N, S\not\ni j} \frac{s!(n-s-1)!}{(n-1)!}(v(S\cup\{i\})-v(S))\right)\\
&\quad\left.+\sum_{S\subseteq N\setminus\{i,j\}} \frac{s!(n-s-2)!}{(n-1)!}(v(S\cup\{j\})-v(S\cup\{i\}))-z_j\right) \qquad (3.1)\\
&= \frac{1}{n-1}\left(Sh_i(N,v)+\sum_{S\subseteq N\setminus\{i,j\}} \frac{s!(n-s-2)!}{(n-1)!}(v(S\cup\{j\})-v(S\cup\{i\}))-z_j\right).
\end{aligned}
$$

Take $y = Sh(N,v)$. By Sobolev (1973) we have $Sh_i(N,v) = Sh_i(N\setminus\{j\}, v^y)$ for all $i,j \in N,\ i \neq j$. Thus, with (3.1) it follows that

$$
Sh_i(N,v)+\frac{y_j}{n-1} = \frac{1}{n-1}\left(Sh_i(N,v)+\sum_{S\subseteq N\setminus\{i,j\}} \frac{s!(n-s-2)!}{(n-1)!}(v(S\cup\{j\})-v(S\cup\{i\}))\right). \qquad (3.2)
$$

Let $x = \varphi^\alpha(N,v)$. By applying (3.1) and (3.2) to x, we have, for any $i,j\in N,\ i\neq j$,

$$
\varphi_i^\alpha(N\setminus\{j\}, v^x) \;=\; \alpha Sh_i(N\setminus\{j\}, v^x) + (1-\alpha)\frac{v^x(N\setminus\{j\})}{n-1}
$$

$$
\begin{aligned}
&= \frac{\alpha}{n-1}\left(Sh_i(N,v) + \sum_{S\subseteq N\setminus\{i,j\}} \frac{s!(n-s-2)!}{(n-1)!}(v(S\cup\{j\}) - v(S\cup\{i\})) - x_j\right) \\
&\quad +(1-\alpha)\frac{v(N)-x_j}{n-1} \\
&= \alpha\left(Sh_i(N,v) + \frac{y_j}{n-1} - \frac{x_j}{n-1}\right) + (1-\alpha)\frac{v(N)-x_j}{n-1} \\
&= \alpha Sh_i(N,v) + \alpha\frac{y_j}{n-1} - \frac{x_j}{n-1} + (1-\alpha)\frac{v(N)}{n-1} \\
&= \alpha Sh_i(N,v) + \alpha\frac{y_j}{n-1} - \frac{\alpha y_j + (1-\alpha)\frac{v(N)}{n}}{n-1} + (1-\alpha)\frac{v(N)}{n-1} \\
&= \alpha Sh_i(N,v) + (1-\alpha)\frac{v(N)}{n} = \varphi_i^\alpha(N,v),
\end{aligned}
$$

where the first equality follows by definition of φ^α, the second equality follows from (3.1) and the third equality follows from (3.2). □

Sobolev (1973) characterized the Shapley value as the unique solution that satisfies consistency and standardness for two-player games. Since in Proposition 3.3 we showed that all egalitarian Shapley values satisfy the same consistency, the Shapley value is the unique egalitarian Shapley value satisfying standardness for two-player games. However, the α-egalitarian Shapley value satisfies the corresponding α-standardness for two-player games. For any $\alpha \in [0,1]$ these two axioms characterize the corresponding egalitarian Shapley value.

Theorem 3.4 *Take any $\alpha \in [0,1]$. A solution ψ satisfies consistency and α-standardness for two-player games if and only if $\psi = \varphi^\alpha$.*

Proof

Since it is straightforward that φ^α satisfies α-standardness for two-player games, by Proposition 3.3 we are left to show uniqueness. Suppose that solution ψ satisfies the two properties of the theorem. We will show that $\psi(N,v) = \varphi^\alpha(N,v)$ for any game $(N,v) \in \mathcal{G}$. For $n=1$ by definition of a solution we have $\psi_i(\{i\},v) = v(\{i\})$ for all $i \in \mathbb{N}$. For $n=2$, α-standardness of ψ and φ^α implies that they are equal. Proceeding by induction, suppose that $\psi(N',v') = \varphi^\alpha(N',v')$ whenever $2 \le |N'| < n$.

Let $x = \psi(N, v)$ and $y = \varphi^\alpha(N, v)$. Take any $i, j \in N,\ i \neq j$, and consider the two reduced games $(N \setminus \{j\}, v^x)$, $(N \setminus \{j\}, v^y)$. Then,

$$\begin{aligned}
x_i - y_i &= \psi_i(N, v) - \varphi_i^\alpha(N, v) \\
&= \psi_i(N \setminus \{j\}, v^x) - \varphi_i^\alpha(N \setminus \{j\}, v^y) \\
&= \varphi_i^\alpha(N \setminus \{j\}, v^x) - \varphi_i^\alpha(N \setminus \{j\}, v^y) \\
&= \alpha \left(Sh_i(N \setminus \{j\}, v^x) - Sh_i(N \setminus \{j\}, v^y) \right) \\
&\quad + (1 - \alpha) \left(ED_i(N \setminus \{j\}, v^x) - ED_i(N \setminus \{j\}, v^y) \right) \\
&= \alpha \left(-\frac{(x_j - y_j)}{n - 1} \right) + (1 - \alpha) \frac{y_j - x_j}{n - 1} = \left(\frac{y_j - x_j}{n - 1} \right),
\end{aligned}$$

where the second equality follows since both solutions ψ and φ^α satisfy consistency, the third equality follows by the induction hypothesis, the fourth equality follows by definition of φ^α and the fifth equality follows from (3.1). Thus, for all $i, j \in N,\ i \neq j$, we have

$$x_i - y_i = \frac{y_j - x_j}{n - 1}.$$

Since $n \geq 3$, it must hold that $x_i - y_i = 0$ for all $i \in N$, and thus $\psi(N, v) = \varphi^\alpha(N, v)$. □

As a corollary we obtain an axiomatization of the class of egalitarian Shapley values with axioms that do not depend on α. We say that a solution ψ satisfies *weak standardness for two-player games* if there exists an $\alpha \in [0, 1]$ such that for every $(N, v) \in \mathcal{G}$ with $n = 2$, ψ satisfies α-standardness for two-player games.

Corollary 3.5 *A solution ψ satisfies consistency and weak standardness for two-player games if and only if ψ is an egalitarian Shapley value.*

Note that, although the egalitarian Shapley values satisfy Sobolev's reduced game consistency for any game with at least two players, from the proof above it follows that for the axiomatization it is sufficient to require this consistency only for games with at least three players since the specific standardness property sets the payoffs in two player games.

Note that the reduced game of Sobolev (1973) is not the only reduced game with respect to which the Shapley value is consistent. For example, the Shapley value satisfies consistency with respect to the reduced game of Hart and Mas-Colell (1988, 1989). However, the equal division solution is not Hart and Mas-Colell consistent. Joosten (1996) characterized the α-egalitarian Shapley value using α-standardness for two-player games and an adapted version of Hart and Mas-Colell's reduced game consistency where the reduced game depends on the parameter α. The main disadvantage of that characterization therefore is that the parameter α appears both in the standardness and in the reduced game property. Also the interpretation of the parameter α in the reduced game is problematic. However, we have shown here that all α-egalitarian Shapley values have the *same* reduced game consistency property in common when considering Sobolev (1973)'s reduced game.

Uniqueness of a solution satisfying α-standardness for two-player games and consistency also follows from Yanovskaya and Driessen (2002) who show uniqueness of a solution satisfying α-standardness for two-player games and a general reduced game consistency where, similar to the above mentioned generalization of Hart and Mas-Colell's reduced game, parameters also enter the reduced game[4]. Thus, every solution in their class is characterized by a different reduced game consistency. However, our main purpose here has been to show that all egalitarian Shapley values satisfy the same reduced game consistency.

4 Monotonicity

Several mononoticity properties of solutions have been discussed in the literature. For example, Young (1985) showed that the Shapley value is characterized by efficiency, symmetry and strong monotonicity[5]. Since all egalitarian Shapley values are efficient and symmetric, the Shapley value thus is the only egalitarian Shapley value that satisfies strong monotonicity. However, it turns out that all egalitarian Shapley values satisfy the weaker monotonicity property which requires that the payoff of a player does not decrease if the worth of the 'grand coalition' as well as all his marginal contributions do not decrease.

Axiom 4.1 (Weak monotonicity) *A solution ψ satisfies weak monotonicity if $\psi_i(N,v) \geq \psi_i(N,w)$ whenever $v(N) \geq w(N)$ and $v(S) - v(S \setminus \{i\}) \geq w(S) - w(S \setminus \{i\})$ for all $S \subseteq N$ with $i \in S$.*

Note that this is a considerable weakening of the strong monotonicity property. Since the worth of the 'grand coalition' is what is to be allocated by any efficient solution requiring

[4]We gave the explicit proof of uniqueness in Theorem 3.4 since this is much shorter with the explicit reduced game that we consider here.

[5]In van den Brink (2006) it is shown that the equal division solution is characterized by efficiency, symmetry and *coalitional monotonicity*, where this last property states that the payoff of a player does not decrease if the worths of all coalitions it is a member of do not decrease.

that the payoff of a player does not decrease if its marginal contributions do not decrease *irrespective* of what is to be allocated is a very strong requirement. However, if the worth to be allocated is not decreasing then this requirement on the payoffs seem reasonable for a marginalistic solution.

It turns out that the class of egalitarian Shapley values is characterized by efficiency, linearity, local monotonicity and this weak monotonicity[6].

Theorem 4.2 *A solution ψ satisfies efficiency, linearity, local monotonicity and weak monotonicity if and only if it is an egalitarian Shapley value.*

Since the proof for games with at least three players is different from that for games with at most two players, we prove this result in two lemmas. Let $\mathcal{G}_3$ and $\mathcal{G}_2$ be the class of games with at least three, respectively with two, players.

Lemma 4.3 *A solution ψ on $\mathcal{G}_3$ satisfies efficiency, linearity, local monotonicity and weak monotonicity if and only if it is an egalitarian Shapley value.*

Proof

It is straightforward to verify that all egalitarian Shapley values satisfy these four properties on $\mathcal{G}_3$. Now, suppose that solution ψ satisfies these properties on $\mathcal{G}_3$. If (N, v) is a null game given by $v(S) = 0$ for all $S \subseteq N$, then efficiency and local monotonicity imply that $\psi_i(N, v) = 0$ for all $i \in N$. (Note that local monotonicity implies symmetry.)

Next, we consider unamimity games $(N, u_T) \in \mathcal{G}_3,\ T \subseteq N,\ T \neq \emptyset$. We prove uniqueness for unanimity games (N, u_T) by induction on $|T|$. First, suppose that $|T| = 1$. Local monotonicity implies that there is a $c^* \in \mathbb{R}$ such that $\psi_j(N, u_T) = c^*$ for all $j \in N \setminus T$. For $i \in T$, local monotonicity further implies that $\psi_i(N, u_T) \geq c^*$. So, for $i \in T$ we can write $\psi_i(N, u_T) = c^* + \alpha$ for some $\alpha \geq 0$. For $i \in T$, efficiency then implies that $\psi_i(N, u_T) = 1 - (n-1)c^*$, and thus $\alpha = 1 - (n-1)c^* - c^* = 1 - nc^*$. We obtain $c^* = \frac{1-\alpha}{n}$, and thus

$$\psi_i(N, u_T) = \begin{cases} c^* + \alpha = \frac{1-\alpha}{n} + \alpha & \text{if } i \in T \\ c^* = \frac{1-\alpha}{n} & \text{if } i \in N \setminus T, \end{cases} \tag{4.3}$$

with $\alpha \geq 0$. Weak monotonicity and the fact that all players in a null game earn zero payoff, implies that $\psi_j(N, u_T) \geq 0$ for all $j \in N \setminus T$, and thus $\alpha \leq 1$. Since $ED_i(N, u_T) = \frac{1}{n}$

[6] Note that this is different from weak monotonicity in Malawski (2005) which is some kind of non-negativity property stating that in any monotone TU-game all players earn a non-negative payoff. He characterizes a class of solutions that contains the egalitarian Shapley values by efficiency, linearity, local monotonicity and this non-negativity. However, not all solutions in his class satisfy Sobolev's reduced game consistency.

for all $i \in N$, $Sh_i(N,u_T) = 1$ for $i \in T$, and $Sh_i(N,u_T) = 0$ for $i \in N \setminus T$, with (4.3) we have $\psi(N,u_T) = (1-\alpha)ED(N,u_T) + \alpha Sh(N,u_T) = \varphi^\alpha(N,u_T)$, $\alpha \in [0,1]$.
Next we show that the weight put on the Shapley value is the same α for all singleton unanimity games. Consider $\overline{T} \subset N$, $\overline{T} \neq T$ and $|\overline{T}| = |T| = 1$. Let $T = \{t\}$, $\overline{T} = \{\overline{t}\}$ and $j \in N \setminus \{t, \overline{t}\}$. (Note that such a j exists since $n \geq 3$.) Similar as above for (N,u_T), it follows that $\psi(N,u_{\overline{T}}) = \varphi^{\overline{\alpha}}(N,u_T)$ for some $\overline{\alpha} \in [0,1]$. Since weak monotonicity implies that $\psi_j(N,u_{\overline{T}}) = \psi_j(N,u_T)$, we then have that $\varphi^\alpha_j(N,u_T) = \varphi^{\overline{\alpha}}_j(N,u_{\overline{T}})$. Since $Sh_j(N,u_T) = Sh_j(N,u_{\overline{T}}) = 0$ and $ED_j(N,u_T) = ED_j(N,u_{\overline{T}}) = \frac{1}{n}$, it must hold that $\alpha = \overline{\alpha}$.

Now, consider (N,u_T), $2 \leq |T| < n$. Proceeding by induction assume that we determined $\psi(N,u_{T'}) = \varphi^\alpha(N,u_{T'})$ for some $\alpha \in [0,1]$ whenever $|T'| < |T|$. If $j \in N \setminus T$ then weak monotonicity implies that $\psi_j(N,u_T) = \psi_j(N,u_{T\setminus\{i\}})$ for any $i \in T$. With the induction hypothesis it then follows that $\psi_j(N,u_T) = \varphi^\alpha_j(N,u_{T\setminus\{i\}}) = \frac{1-\alpha}{n} = \varphi^\alpha_j(N,u_T)$. With efficiency it then follows that $\sum_{i\in T}\psi_i(N,u_T) = 1 - \sum_{j\in N\setminus T}\psi_j(N,u_T) = 1-(n-|T|)\frac{1-\alpha}{n}$. Since local monotonicity implies symmetry it then follows that $\psi_i(N,u_T) = \frac{1-(n-|T|)\frac{1-\alpha}{n}}{|T|} = \frac{1-\alpha}{n} + \frac{\alpha}{|T|} = (1-\alpha)ED_i(N,u_T) + \alpha Sh_i(N,u_T) = \varphi^\alpha_i(N,u_T)$ for $i \in T$. Thus, $\psi(N,u_T) = \varphi^\alpha(N,u_T)$.

For the unanimity game (N,u_N), local monotonicity and efficiency imply that $\psi_i(N,v) = \frac{1}{n}$ for all $i \in N$.

For arbitrary $(N,v) \in \mathcal{G}_3$, uniqueness follows from linearity of ψ and the fact that $v = \sum_{\substack{T\subseteq N\\ T\neq\emptyset}} \Delta_v(T)u_T$ for every $(N,v) \in \mathcal{G}$, with $\Delta_v(T) = \sum_{H\subseteq T}(-1)^{|T|-|H|}v(H)$ the Harsanyi dividend of coalition $T \subseteq N$, $T \neq \emptyset$ (see Harsanyi (1959)). □

In proving that the payoff of player $t \in N$ in the unanimity game $(N,u_{\{t\}})$ is equal to the payoff of player $\overline{t} \in N$ in the unanimity game $(N,u_{\{\overline{t}\}})$, in the proof of Lemma 4.3 we needed to compare these payoffs with those of another player $i \in N \setminus \{t,\overline{t}\}$. Therefore we need at least three players to apply this proof. Although we cannot apply this proof for two player games, also on that class the four axioms of Lemma 4.3 characterize the egalitarian Shapley values.

Lemma 4.4 *A solution ψ on $\mathcal{G}_2$ satisfies efficiency, linearity, local monotonicity and weak monotonicity if and only if it is an egalitarian Shapley value.*

PROOF
It is straightforward to verify that all egalitarian Shapley values satisfy efficiency, linearity, local monotonicity and weak monotonicity on $\mathcal{G}_2$. To prove uniqueness, let $(N,v) \in \mathcal{G}_2$. If v is a null game then efficiency and local monotonicity again imply that $\psi_i(\{i,j\},v) =$

$\psi_j(\{i,j\},v) = 0$. For two-player unanimity games $(\{i,j\},u_T)$, $i \neq j$, we distinguish the following two cases. (i) If $T = \{i,j\}$ then efficiency and local monotonicity imply that $\psi_i(\{i,j\},u_{\{i,j\}}) = \psi_j(\{i,j\},u_{\{i,j\}}) = \frac{1}{2}$. (ii) Consider the case that $|T| = 1$. Let $\alpha = 2\psi_i(\{i,j\},u_{\{j\}}) - 1$, i.e. α is defined such that $\psi_i(\{i,j\},u_{\{j\}}) = \frac{1-\alpha}{2} = \varphi_i^\alpha(\{i,j\},u_{\{j\}})$. Then by efficiency we have $\psi_j(\{i,j\},u_{\{j\}}) = 1 - \psi_i(\{i,j\},u_{\{j\}}) = 1 - \left(\frac{1-\alpha}{2}\right) = \alpha + \frac{1-\alpha}{2} = \varphi_j^\alpha(\{i,j\},u_{\{j\}})$. Similar as in the proof of Lemma 4.3, local monotonicity implies that $\alpha \geq 0$ and weak monotonicity and the nulll game case imply that $\alpha \leq 1$. In van den Brink and Funaki (2004) it is shown that on $\mathcal{G}_2$ any solution that satisfies efficiency, linearity, local monotonicity and weak monotonicity also satisfies anonymity[7]. But then $\psi_i(\{i,j\},u_{\{i\}}) = \psi_j(\{i,j\},u_{\{j\}})$ and $\psi_i(\{i,j\},u_{\{j\}}) = \psi_j(\{i,j\},u_{\{i\}})$. Thus, $\psi(\{i,j\},u_T) = \varphi^\alpha(\{i,j\},u_T)$ for $|T| = 1$. The proof of uniqueness for any two player unanimity game then follows from linearity. □

Since for one-player games $(\{i\},v)$ the payoff $\psi_i(\{i\},v) = v(\{i\})$ follows by definition of a solution, Lemmas 4.3 and 4.4 yield the main result of this section as stated in Theorem 4.2. Note that, similar to Corollary 3.5, Theorem 4.2 characterizes the class of egalitarian Shapley values using axioms that do not depend on the parameter α. Logical independence of the axioms of Theorem 4.2 is shown by the following alternative solutions:

1. The solution ψ given by $\psi_i(N,v) = 0$ for all $i \in N$ and $(N,v) \in \mathcal{G}$ satisfies the axioms of Theorem 4.2 except efficiency.

2. The normalized Banzhaf value $\overline{\beta}$ given by $\overline{\beta}_i(N,v) = \frac{\beta_i(N,v)}{\sum_{j\in N}\beta_j(N,v)}v(N)$ with $\beta_i(N,v) = \sum_{\substack{S\subseteq N\\ i\in S}} \frac{1}{2^{n-1}}(v(S) - v(S\setminus\{i\}))$, $i \in N$, satisfies the axioms of Theorem 4.2 except linearity[8].

3. For $N \subset \mathbb{N}$, let $\overline{n}(N)$ be the lowest labeled player in N, i.e. $i \geq \overline{n}(N)$ for all $i \in N$. Then the solution ψ given by $\psi_{\overline{n}(N)}(N,v) = v(N)$, and $\psi_i(N,v) = 0$ for all $i \in N \setminus \{\overline{n}(N)\}$ satisfies the axioms of Theorem 4.2 except local monotonicity.

4. The CIS-value given by $CIS_i(N,v) = v(\{i\}) + \frac{v(N)-\sum_{j\in N}v(\{j\})}{n}$ for all $i \in N$ (see Driessen and Funaki (1991)) satisfies the axioms of Theorem 4.2 except weak monotonicity.

We end this section by remarking that in the proof of Theorem 4.2 weak monotonicity is used for two purposes. First, it is used to show that a player who is a null player in two

[7] A solution ψ satisfies anonymity on $\mathcal{C} \subset \mathcal{G}$ if for every permutation $\pi : N \to N$ it holds that $\psi_i(N,v) = \psi_{\pi(i)}(N,\pi v)$ for every $(N,v) \in \mathcal{C}$ such that $(N,\pi v) \in \mathcal{C}$, where the permuted game $(N,\pi v)$ is defined by $\pi v(S) = v(\cup_{i\in S}\{\pi(i)\})$ for all $S \subseteq N$.

[8] An axiomatization of this solution is given in van den Brink and van der Laan (1998).

distinct unanimity games gets the same payoff in both games. This is also established by the *null player constant property* which states that the payoff of null players is always the same for games that have the same worth of the 'grand coalition', i.e. a solution ψ on $\mathcal{G}^N$ satisfies the null player constant property if $\psi_i(N, v) = \psi_i(N, w)$ whenever $i \in N$ is a null player in (N, v) and (N, w) with $v(N) = w(N)$. Second, weak monotonicity is used to show that null players in a unanimity game earn a nonnegative payoff. This also follows from *nonegativity* which states that players in monotone games earn a nonnegative payoff. Therefore, in Theorem 4.2 weak monotonicity can be replaced by the null player constant property and nonnegativity[9].

5 Implementation

In the previous two sections we characterized the egalitarian Shapley values from a cooperative viewpoint. Next we study these solutions from a non-cooperative perspective. In the literature various implementations of the Shapley value can be found, see e.g. Gul (1989), Hart and Mas-Colell (1996) and Pérez-Castrillo and Wettstein (2001). In this section we adapt the last mechanism and obtain a two-level bidding mechanism implementing any egalitarian Shapley value as the unique subgame perfect equilibrium outcome[10].

This bidding mechanism is defined recursively as follows. When there is only one player, say i, in the game, this player simply gets its stand-alone payoff, $v(\{i\})$. Given the rules of the mechanism for games with at most $k-1 < n$ players, the bidding game for a set of k players proceeds in rounds. Dependent upon the strategies of the corresponding players, the bidding game may have up to n rounds, each consisting of four stages. Let N_t be the player set of the game with which the bidding game of each round $t \in \{1, ..., n\}$ will start.

Round 1 : $N_1 = N$. Goto Stage 1.

Stage 1: Each player $i \in N$ makes bids $b^i_j \in \mathbb{R}$ for every $j \neq i$. For each $i \in N$, let $B^i = \sum_{j \neq i} \left(b^i_j - b^j_i\right)$ be the *net bid* of player i measuring its 'relative' willingness to be the proposer. Let i^*_1 be the player with the highest net bid in this round. (In case of a non-unique maximizer we choose any of these maximal bidders to be the 'winner' with equal probability.) Once the winner i^*_1 has been determined, player i^*_1 pays every other player $j \in N \backslash \{i^*_1\}$ its offered bid $b^{i^*_1}_j$. The 'winner' i^*_1 becomes the proposer in the next stage. Goto Stage 2.

[9] Note that the null player constant property is weaker than weak monotonicity but nonegativity is not.

[10] A general approach to bidding mechanisms implementing solutions for cooperative games is discussed in Ju and Wettstein (2006).

Stage 2: Player i_1^* proposes an offer $y_j^{i_1^*} \in \mathbb{R}$ to every player $j \neq i_1^*$. (This offer is additional to the bids paid at stage 1.) Goto Stage 3.

Stage 3: The players other than i_1^*, sequentially, either accept or reject the offer. If at least one player rejects it, then the offer is rejected. Otherwise, the offer is accepted. Goto Stage 4.

Stage 4: If the offer is accepted, then each player $j \in N \backslash \{i_1^*\}$ receives $y_j^{i_1^*}$ and player i_1^* obtains $v(N) - \sum_{j \neq i_1^*} y_j^{i_1^*}$. Hence, in this case the final payoff to player $j \neq i_1^*$ is $y_j^{i_1^*} + b_j^{i_1^*}$, while player i_1^* receives $v(N) - \sum_{j \neq i_1^*}(y_j^{i_1^*} + b_j^{i_1^*})$. Stop.

If the offer is rejected then with probability $\alpha \in [0,1]$ player i_1^* leaves the game and obtains her stand-alone payoff $v(\{i_1^*\})$, while the players in $N \backslash \{i_1^*\}$ proceed to round 2 to bargain over $v(N \backslash \{i_1^*\})$. With probability $(1-\alpha) \in [0,1]$ the game breaks down and all players, including the proposer i_1^*, get zero payoffs at this stage (and thus only the bids of stage 1 are transferred, i.e. the payoff to player $j \neq i_1^*$ is $b_j^{i_1^*}$ and the payoff to the proposer i_1^* is $-\sum_{j \neq i_1^*} b_j^{i_1^*}$.). Stop.

In the following rounds, the stages 1, 2 and 3 are the same as in round 1 but with the reduced player set where the proposer in the previous round has left. However, at stage 4 of the following rounds there is no possibility of breakdown. To be complete we describe the following rounds.

Round t**,** $t \in \{2, ..., n-1\}$: $N_t = N_{t-1} \backslash \{i_{t-1}^*\}$. Goto Stage 1.

Stage 1: Each player $i \in N_t$ makes bids $b_j^i \in \mathbb{R}$ for every $j \neq i$. For each $i \in N_t$, let $B^i = \sum_{j \in N_t \backslash \{i\}} \left(b_j^i - b_i^j\right)$, be the *net bid* of player i. Let i_t^* be the player with the highest net bid of round t. (In case of a non-unique maximizer we choose any of these maximal bidders to be the 'winner' with equal probability.) Once the winner i_t^* has been determined, player i_t^* pays every other player $j \in N_t \backslash \{i_t^*\}$, its offered bid $b_j^{i_t^*}$. The 'winner' i_t^* becomes the proposer in the next stage. Goto Stage 2.

Stage 2: Player i_t^* proposes an offer $y_j^{i_t^*} \in \mathbb{R}$ to every player $j \in N_t \setminus \{i_t^*\}$. (This offer is additional to the bids paid at stage 1.) Goto Stage 3.

Stage 3: The players other than i_t^*, sequentially, either accept or reject the offer. If at least one player rejects it, then the offer is rejected. Otherwise, the offer is accepted. Goto Stage 4.

Stage 4: If the offer is accepted, then each player $j \in N_t \backslash \{i_t^*\}$ receives $y_j^{i_t^*}$ and player i_t^* obtains $v(N_t) - \sum_{j \in N_t \backslash \{i_t^*\}} y_j^{i_t^*}$ at this stage. Hence, in this case the final payoff to player $j \in N_t \setminus \{i_t^*\}$ is $y_j^{i_t^*} + b_j^{i_t^*} + \sum_{k=1}^{t-1} b_j^{i_k^*}$, while player i_t^* receives $v(N_t) - \sum_{j \in N_t \setminus \{i_t^*\}} (y_j^{i_t^*} + b_j^{i_t^*}) + \sum_{k=1}^{t-1} b_{i_t^*}^{i_k^*}$. Stop.

If the offer is rejected then player i_t^* leaves the game and obtains its stand-alone payoff $v(\{i_t^*\})$, while the players in $N_t \setminus \{i_t^*\}$ proceed to round $t+1$ to bargain over $v(N_t \setminus \{i_t^*\})$.

Round n : $N_n = N_{n-1} \setminus \{i_{n-1}^*\}$. Apparently, N_n is a singleton coalition so that it is a one-player game in this round. The game immediately stops such that player $i \in N_n$ gets its stand-alone payoff $v(N_n)$. So, its final payoff is $v(N_n) + \sum_{k=1}^{n-1} b_i^{i_k^*}$.

Note that this bidding mechanism is the same as that of Pérez-Castrillo and Wettstein (2001), except for the possibility of breakdown after the offer is rejected in the first round, and thus for the specific value $\alpha = 1$ it is exactly the same as their mechanism. If the game continues after this first rejection then this possibility of breakdown does not occur anymore, and the whole game can only be stopped by acceptance of all relevant players in a certain round or rejection in a two-player bidding subgame.

Given the characteristic function v, we can calculate the final payoffs of the players who are assumed to be risk neutral in the mechanism. In case of rejection in the first round, the expected final gain of proposer i_1^* is $\alpha v(\{i^*\}) - \sum_{j \neq i^*} b_j^{i^*}$, whereas every other player $j \neq i_1^*$ finally obtains $b_j^{i^*}$ plus the expected payoff due to the contingent (with probability α) outcome of the mechanism continuing with player set $N \setminus \{i_1^*\}$. In case of acceptance of the proposal in the first round, the final gain of i_1^* is $v(N) - \sum_{j \neq i_1^*} (b_j^{i_1^*} + y_j^{i_1^*})$, whereas the final gain of every player $j \neq i_1^*$ is $b_j^{i_1^*} + y_j^{i_1^*}$.

Next we generalize the result of Pérez-Castrillo and Wettstein (2001) who showed that for $\alpha = 1$ this bidding mechanism implements the Shapley value for zero-monotonic games. A TU-game (N, v) is *zero-monotonic* if $v(S) \geq v(S \setminus \{i\}) + v(\{i\})$ for all $S \subseteq N$ and all $i \in S$. It turns out that for any zero-monotonic game such that the 'grand coalition' earns a nonnegative worth, the given bidding mechanism implements the α-egalitarian Shapley values as subgame perfect equilibrium (SPE) outcomes. For $T \subset N$ the restricted game $(T, v_T) \in \mathcal{G}$ is given by $v_T(S) = v(S)$ for all $S \subseteq T$.

Theorem 5.1 *Let $\alpha \in [0,1]$ be the probability that the bidding continues after rejection in the first round, and let $v \in \mathcal{G}$ be a zero monotonic game with $v(N) \geq 0$. Then the outcome in any subgame perfect equilibrium of the bidding mechanism coincides with the payoff vector $\varphi^\alpha(N, v)$.*

Proof

Let (N, v) be a zero-monotonic with $v(N) \geq 0$, and let $\alpha \in [0,1]$. We first show that the α-egalitarian Shapley value payoff $\varphi^\alpha(N, v)$ is indeed an SPE outcome. We do this in three steps.

1. We first explicitly construct an SPE that yields the α-egalitarian Shapley value $\varphi^\alpha(N,v)$ as an SPE outcome. Consider the following strategy adopted by a player $i \in N$.

Round 1:
At stage 1, player $i \in N$ announces $b^i_j = \alpha(Sh_j(N,v) - Sh_j(N\backslash\{i\}, v_{N\backslash\{i\}})) + (1-\alpha)\frac{v(N)}{n}$ for every $j \neq i$.
At stage 2, if i is chosen as the proposer, i offers $y^i_j = \alpha Sh_j(N\backslash\{i\}, v_{N\backslash\{i\}})$ to every $j \neq i$.
At stage 3, if $j \neq i$ is the proposer, i accepts any offer that is greater than or equal to $\alpha Sh_i(N\backslash\{j\}, v_{N\backslash\{j\}})$ and rejects any offer strictly smaller than $\alpha Sh_i(N\backslash\{j\}, v_{N\backslash\{j\}})$.[11]

At stage 1 of any round t, $t \in \{2,\ldots,n-1\}$, player $i \in N_t$ (i.e. i was not proposer in any of the previous rounds) announces $b^i_j = Sh_j(N_t, v_{N_t}) - Sh_j(N_t\backslash\{i\}, v_{N_t\backslash\{i\}})$ for every $j \in N_t \setminus \{i\}$. (Clearly, players in $N \setminus N_t$ do not have to choose an action in round t and henceforth.)
At stage 2, if i is chosen as the proposer, i offers $y^i_j = Sh_j(N_t\backslash\{i\}, v_{N_t\backslash\{i\}})$ to every $j \in N_t \setminus \{i\}$.
At stage 3, if $j \in N_t \setminus \{i\}$ is the proposer, then i accepts any offer that is greater than or equal to $Sh_i(N_t\backslash\{j\}, v_{N_t\backslash\{j\}})$ and rejects any offer strictly smaller than $Sh_i(N_t\backslash\{j\}, v_{N_t\backslash\{j\}})$.

Round n: If $i \in N_n$ is in this round, i will be the only player and gets its stand-alone payoff $v(\{i\})$.

2. This profile of strategies of all players in N yields acceptance in round 1. Since $b^{i^*_1}_j + y^{i^*_1}_j = \alpha(Sh_j(N,v) - Sh_j(N\backslash\{i^*_1\}, v_{N\backslash\{i^*_1\}})) + (1-\alpha)\frac{v(N)}{n} + \alpha Sh_j(N\backslash\{i^*_1\}, v_{N\backslash\{i^*_1\}}) = \alpha Sh_j(N,v) + (1-\alpha)\frac{v(N)}{n} = \varphi^\alpha_j(N,v)$ for all $j \in N\backslash\{i^*_1\}$, any player who is not the proposer obtains its α-egalitarian Shapley value payoff. Moreover, given that following the strategies the grand coalition is formed, the proposer also obtains her α-egalitarian Shapley value payoff $\varphi^\alpha_{i^*_1}(N,v) = v(N) - \sum_{j\in N\backslash\{i^*_1} \varphi^\alpha_j(N,v)$.

3. To check that the above strategies constitute an SPE, first recall Theorem 1 in Pérez-Castrillo and Wettstein (2001), which implies that the subgame perfect equilibrium outcome of each subgame starting from round 2 (irrespective of the proposer, its bids and proposals in round 1) is the Shapley value payoff vector of the corresponding game. Then we further apply backward induction to verify that the given strategy profile yields subgame perfectness at the stages of round 1. Note that at stage 4 no actions by players are

[11] Note that at stages 2 and 3 the non-proposers, respectively, the proposer do not have an action. The same holds for the following rounds. At stage 4 no player (besides 'nature' in round 1) has an action but only payoffs are transferred.

chosen, but only payoffs are paid and (possibly) nature chooses whether the game continues or breaks down. Given the SPE outcome from round 2, if non-proposer $j \neq i_1^*$ chooses according to the given strategy profile in stage 3 then (after all preceding non-proposers accept the offer of the proposer) j accepts any offer from proposer i_1^* if and only if the offer is greater or equal to $\alpha Sh_j(N\backslash\{i_1^*\}, v_{N\backslash\{i_1^*\}})$ which is its expected payoff after rejection, showing that the given strategy profile yields an SPE in the subgame that starts when non-proposer $j \neq i_1^*$ has to choose to accept or reject the offer at stage 3. By $v(N) \geq 0$ and zero-monotonicity of v, $v(N) - \alpha v(N \setminus \{i_1^*\}) \geq \alpha(v(N) - v(N \setminus \{i_1^*\})) \geq \alpha v(\{i_1^*\})$, and thus the proposer prefers acceptance paying the minimal offers for the non-proposers to accept at stage 3, so $y_j^{i_1^*}$ will be as just determined for stage 3.

To verify that the bids at stage 1 complete an SPE, note that all net bids are equal to zero because for all $i, j \in N$ the balanced contributions property of the Shapley value (see Myerson (1980))[12] implies that

$$\begin{aligned} b_i^j &= \alpha(Sh_i(N,v) - Sh_i(N\backslash\{j\}, v_{N\backslash\{j\}})) + (1-\alpha)\frac{v(N)}{n} \\ &= \alpha(Sh_j(N,v) - Sh_j(N\backslash\{i\}, v_{N\backslash\{i\}})) + (1-\alpha)\frac{v(N)}{n} = b_j^i. \end{aligned}$$

Therefore, according to this strategy profile at stage 1 all players are chosen as the proposer with equal probability. Consider player $j \in N$. Decreasing (at least one of) its bids, j will be chosen as proposer with probability zero. Which other player will be the proposer depends on the way j decreases its bids. But given that all other players do not change strategies, i.e. making bids and offers according to the above strategies, player j would still obtain its α-egalitarian Shapley value payoff. Hence, j cannot increase its payoff by decreasing its bids. In order to further elaborate this, one can readily check that a decreasing bid of j will not be part of an equilibrium because it will induce some (the players whom j makes decreased bids to) of the others to decrease the bids without changing the level of their net bids. If j increases (at least one of) its bids, then j will be the proposer with probability one. But since the continuation of the game from round 2 (with this proposer) does not change, the earnings of j in the subgame that is 'entered' does not change. But j's bid increased and therefore j's total payoff decreases. So, no player can increase its payoff by changing its bid, showing that the given strategy profile is an SPE.

Thus, we have shown that the α-egalitarian Shapley value payoff vector is indeed an SPE outcome.

To prove that any SPE yields the α-egalitarian Shapley value as outcome, note that it follows along the same lines as in Pérez-Castrillo and Wettstein (2001) that for any player,

[12] This balanced contributions property states that $Sh_i(N,v) - Sh_i(N\backslash\{j\}, v_{N\backslash\{j\}})) = Sh_j(N,v) - Sh_j(N\backslash\{i\}, v_{N\backslash\{i\}}))$ for all $(N,v) \in \mathcal{G}$ and $i, j \in N$.

his or her final payoffs are the same regardless of the identity of the proposer. Next we only have to show that in any SPE the final payoff received by each player coincides with his or her α-egalitarian Shapley value payoff. Note that if i is the proposer in round 1, her final payoff will be $v(N) - \alpha v(N\backslash\{i\}) - \sum_{j\neq i} b^i_j$ whereas if $j \neq i$ is the proposer, i will obtain final payoff $\alpha Sh_i(N\backslash\{j\}, v_{N\backslash\{j\}}) + b^j_i$. Hence, since all net bids are zero and thus all players are proposer with equal probability (which follows similar as in Pérez-Castrillo and Wettstein (2001)), using a recursive formula of the Shapley value given in Maschler and Owen (1989)[13], the expected payoff of player i equals

$$\begin{aligned}
&\frac{1}{n}\left(v(N) - \alpha v(N\backslash\{i\}) - \sum_{j\neq i} b^i_j + \sum_{j\neq i}\left(\alpha Sh_i(N\backslash\{j\}, v_{N\backslash\{j\}}) + b^j_i\right)\right) &(5.4)\\
= &\frac{1}{n}\left(v(N) - \alpha v(N\backslash\{i\}) + \sum_{j\neq i}\alpha Sh_i(N\backslash\{j\}, v_{N\backslash\{j\}})\right)\\
= &\alpha Sh_i(N,v) + (1-\alpha)\frac{v(N)}{n}\\
= &\varphi^\alpha_i(N,v).
\end{aligned}$$

□

It can be seen from the proof of Theorem 5.1 that the condition of a nonnegative worth for the 'grand coalition' and zero-monotonicity of the game (N, v) can be weakened to *α-zero-monotonic*, $\alpha \in [0, 1]$, meaning that $v(N) \geq \alpha\left(v(N\backslash\{i\}) + v(\{i\})\right)$ for all $i \in N$, and $v(S) \geq v(S\backslash\{i\}) + v(\{i\})$ for all $S \subset N$, $S \neq N$, and all $i \in S$.

Further, note that (5.4) at the end of the proof above yields a recursive formula for all egalitarian Shapley values that is constructed as follows:

$$\begin{aligned}
\varphi^\alpha_i(N,v) = \quad &\frac{1}{n}\sum_{j\neq i}\varphi^\alpha_i(N\backslash\{j\}, v_{N\backslash\{j\}})\\
+ \quad &\frac{1}{n}\left(v(N) - \alpha v(N\backslash\{i\}) - (1-\alpha)\frac{\sum_{j\neq i} v(N\backslash\{j\})}{n-1}\right), \quad \text{for all } i \in N.
\end{aligned}$$

Taking $\alpha = 1$ yields the recursive formula of the Shapley value given in Maschler and Owen (1989).

6 Concluding remarks

In this paper we formulated the trade-off between marginalism and egalitarianism in cooperative games by considering convex combinations of the Shapley value and the equal

[13] This recursive formula is $Sh_i(N,v) = \frac{1}{n}\left(v(N) - v(N\backslash\{i\}) + \sum_{j\neq i} Sh_i(N\backslash\{j\}, v_{N\backslash\{j\}})\right)$ for all $i \in N$.

division solution. We provided three main characterizations of these egalitarian Shapley values. We first showed that all these solutions satisfy the same reduced game consistency that is introduced by Sobolev (1973). Second we showed that all egalitarian Shapley values satisfy weak monotonicity (which is a weaker version of strong monotonicity), and that together with efficiency, linearity and local monotonicity this weak monotonicity characterizes the class of egalitarian Shapley values. Finally, we showed that all egalitarian Shapley values have a similar strategic implementation as the unique subgame perfect equilibrium outcome in an extensive form bidding mechanism which is an adaptation of the one for the Shapley value given in Pérez-Castrillo and Wettstein (2001), and only differs in the probability of breakdown at a certain stage of the negotiations. Since these solutions have these important types of properties in common we consider the egalitarian Shapley values as an important concept to make the trade-off between marginalism and egalitarianism.

Malawski (2005) obtains the egalitarian Shapley values by a procedure where for every order of entrance to the 'grand coalition' player $i \in N$ gets a share α in its marginal contribution and the predecessors of i equally share the remainder of i's marginal contribution. Taking the average over all orders of entrance yields the corresponding α-egalitarian Shapley value as expected payoffs. In a similar spirit, Ju, Borm and Ruys (2007) allocate for every order of entrance the surplus $v(S) - v(S \setminus \{i\}) - v(\{i\})$ instead of the marginal contribution among the entrant i and its predecessor, and assign the worth $v(\{i\})$ fully to player i. This yields the convex combinations of the Shapley value and the CIS-value (see the fourth example at the end of Section 4), the so-called generalized consensus values, as expected payoffs. All generalized consensus values satisfy standardness for two-player games but, except fot the Shapley value, do not satisfy Sobolev (1973)'s reduced game consistency. Finding a reduced game consistency for the generalized consensus values is a plan for future research.

In van den Brink and Funaki (2004) a class of equal surplus sharing solutions is studied that includes the equal division solution, the CIS-value, the ENSC-value (i.e. the dual of the CIS-value) and all their convex combinations. Further generalizations of the egalitarian Shapley values and generalized consensus values consider convex combinations of any of these equal surplus sharing solutions with the Shapley value (or even the more general semi-values, see Dubey, Neyman and Weber (1981)).

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