View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by DSpace at VU

TI 2006-110/3
Tinbergen Institute Discussion Paper

Last-In First-Out Oligopoly Dynamics

Jaap Abbring[1]
Jeffrey R. Campbell[2]

[1] *Vrije Universiteit Amsterdam, and Tinbergen Institute;*
[2] *Federal Reserve Bank of Chicago, and NBER.*

Tinbergen Institute
The Tinbergen Institute is the institute for economic research of the Erasmus Universiteit Rotterdam, Universiteit van Amsterdam, and Vrije Universiteit Amsterdam.

Tinbergen Institute Amsterdam
Roetersstraat 31
1018 WB Amsterdam
The Netherlands
Tel.: +31(0)20 551 3500
Fax: +31(0)20 551 3555

Tinbergen Institute Rotterdam
Burg. Oudlaan 50
3062 PA Rotterdam
The Netherlands
Tel.: +31(0)10 408 8900
Fax: +31(0)10 408 9031

Most TI discussion papers can be downloaded at http://www.tinbergen.nl.

Last-In First-Out Oligopoly Dynamics*

Jaap H. Abbring† Jeffrey R. Campbell‡

December, 2006

Abstract

This paper extends the static analysis of oligopoly structure into an infinite-horizon setting with sunk costs and demand uncertainty. The observation that exit rates decline with firm age motivates the assumption of last-in first-out dynamics: An entrant expects to produce no longer than any incumbent. This selects an essentially unique Markov-perfect equilibrium. With mild restrictions on the demand shocks, a sequence of thresholds describes firms' equilibrium entry and survival decisions. Bresnahan and Reiss's (1993) empirical analysis of oligopolists' entry and exit assumes that such thresholds govern the evolution of the number of competitors. Our analysis provides an infinite-horizon game-theoretic foundation for that structure.

*We are grateful to Eugene Amromin, Allan Collard-Wexler, Meredith Crowley, and Richard Rosen for their insightful comments and to Tom Holmes for his discussion at the 2006 Duke–Northwestern–Texas IO Theory Conference. The National Science Foundation supported this research through Grant 0137042 to the NBER. De Jonge Akademie of the Royal Netherlands Academy of Arts and Sciences supported this research through a travel grant. A replication file is available at http://www.nber.org/~jrc/lifo

†Vrije Universiteit Amsterdam and Tinbergen Institute. E-mail: jabbring@feweb.vu.nl

‡Federal Reserve Bank of Chicago and NBER. E-mail: jcampbell@frbchi.org

JEL Code: L13

Keywords: Sunk costs, Demand uncertainty, Markov-Perfect equilibrium, LIFO

1 Introduction

This paper develops and presents a simple and tractable model of oligopoly dynamics. The model's firms make entry and exit decisions in an infinite-horizon setting with stochastic demand. Calculating its equilibrium dynamics requires only a few *seconds* on a standard personal computer. With mild restrictions on the demand shocks, *threshold* rules govern firms' entry and exit decisions. That is, entry occurs whenever demand passes above one in a sequence of entry thresholds, and exit occurs if it subsequently passes below a corresponding exit threshold. Bresnahan and Reiss's (1993) empirical analysis of oligopolists' entry and exit assumes that such thresholds govern the evolution of the number of competitors. Our analysis provides an infinite-horizon game-theoretic foundation for that structure, which can be used to extend their earlier structural estimation of static oligopoly models to a fully dynamic setting. Because the model makes a unique equilibrium prediction, it can also be used for policy experiments. This paper's companion (Abbring and Campbell, 2006b) provides one example of such an experiment, which determines how raising a barrier to entry by increasing late entrants' sunk costs changes firms' short-run and long-run market participation decisions.

The model industry is a dynamic version of the static entry game used by Bresnahan and Reiss (1990). A random number of consumers demands the industry's services, and this state evolves stochastically. Entry possibly requires paying a sunk cost, and continued operation incurs fixed costs. The wish to avoid these per-period fixed costs in markets that are no longer profitable motivates firms to exit.

Bresnahan and Reiss (1991a) noted that the static version of this game can have multiple equilibria, which obviously complicates prediction. To select a unique equilibrium, both Bresnahan and Reiss (1990) and Berry (1992) assume that firms move sequentially. We take a similar approach by allowing older firms to commit to continuation before their younger counterparts. We also restrict attention to equilibria in which firms correctly believe that no firm will produce after an older rival exits. That is, the equilibria have a last-in first-out (LIFO) structure. Three considerations motivate this focus. First, it is consistent with the widespread observation that young firms exit more frequently than their older counterparts. Second, the equilibrium approximates the "natural" Markov-perfect equilibrium in an extension of the model in which firms' costs decrease with age and the most efficient firms survive. Third and perhaps most importantly, this restriction vastly simplifies the equilibrium analysis. We prove that there always exists such an equilibrium and that it is (essentially) unique.

The model's theoretical simplicity makes it well-suited for exploring how parameter changes impact equilibrium dynamics and long-run market structure. To show this, we

calculate the effects of increasing demand uncertainty on firms' equilibrium entry and exit thresholds. Non-strategic analysis of the firm life cycle suggests that additional uncertainty should raise the value of the option to exit and thereby substantially lower both entry and exit thresholds. The oligopolistic exit thresholds do indeed fall with uncertainty, but the entry thresholds do not. Their relative invariance to demand uncertainty reflects an offsetting effect that a monopolist does not face: Increasing demand uncertainty raises the probability of further entry and thereby reduces a new firm's value. We also calculate the population "estimates" of oligopoly profit margins using the ordered Probit procedure of Bresnahan and Reiss (1990) and data generated from the model's ergodic distribution. We find that the delay in exit arising from uncertainty (familiar from Dixit and Pindyck, 1994) biases these entry threshold estimates downwards, and this leads to a downward bias in the estimated rate that profits fall with additional competition. That is, the static "long-run" procedure that abstracts from relevant dynamic considerations can find "evidence" that profit margins decline with entry when in fact they are constant.

The remainder of this paper proceeds as follows. The next section presents the model's primitives and demonstrates the uniqueness of a Markov-perfect equilibrium with a LIFO structure. To clarify how the model's moving parts fit together, this section closes with an examination of a particular specification for the demand shocks that yields a pencil-and-paper solution. Section 3 gives sufficient conditions for firms to use threshold rules for their equilibrium entry and exit decisions, and Section 4 illustrates its application. Section 5 considers extensions of our analysis to include a learning curve and firm-specific technology shocks. Section 6 relates this paper with previous work on dynamic games with timing restrictions and with the extensive literature on oligopoly with Markov-perfect equilibrium. Section 7 contains some concluding remarks.

2 The Model

The model consists of a single oligopolistic market in discrete time $t \in \{0, 1, \ldots\}$. There is a large number of firms that are potentially active in the market. We index these firms by $\mathbb{N}$. At time 0, $N_0 = 0$ firms are active. Entry and subsequent exit of firms determines the number of active firms in each later period, N_t . The number of consumers in the market, C_t, evolves exogenously according to a nonnegative first-order Markov process bounded between $\hat{C}$ and $\check{C} < \infty$. We denote the conditional distribution of C_t with $Q(c|C_{t-1}) \equiv \Pr[C_t \leq c|C_{t-1}]$.

Figure 1 illustrates the sequence of events and actions within a period. It begins with

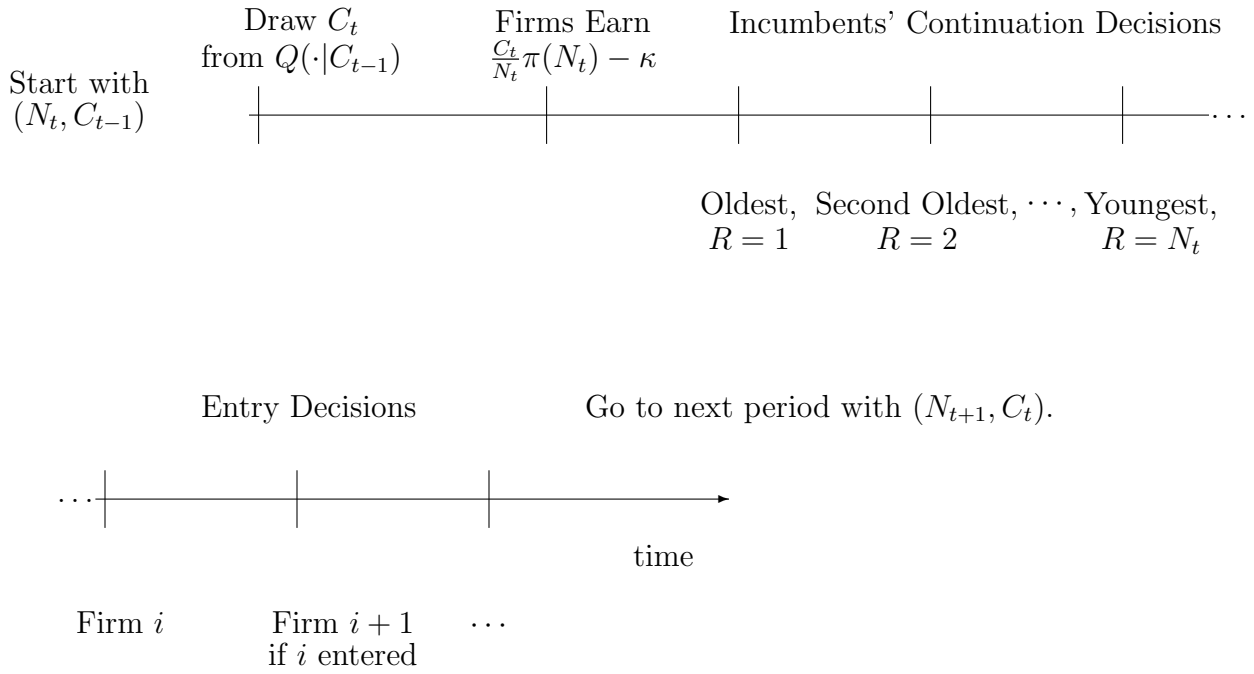

Figure 1: The Sequence of Actions within a Period

the inherited values of N_t and C_{t-1}. First, all participants observe the realization of C_t; and all active firms receive profits equal to $(C_t/N_t) \times \pi(N_t) - \kappa$. Here, each firm serves C_t/N_t consumers, and $\pi(N_t)$ is the producer surplus earned from each one. The term $\kappa > 0$ represents fixed costs of production.

After serving the market, active firms decide whether they will remain so. These decisions are sequential and begin with the oldest firm. After this, any remaining firms make the same decision in the order of their entry. If firm i is active, then R_t^i denotes its rank in this sequence. Exit is irreversible but otherwise costless. It allows the firm to avoid future periods' fixed production costs.

After active firms' continuation decisions, those firms that have not yet had an opportunity to enter make entry decisions in the order of their names. The cost of entry potentially depends on the number of firms already committed to serving the market in the next period. We denote the entry cost for a firm that would be the R'-th oldest among next period's active firms with $\varphi(R')$. We assume that $\varphi(R')$ is strictly positive and weakly increasing in R'. This allows for, but does not require, later entrants to face a "barrier to entry" in

the form of elevated sunk costs. The payoff to staying out of the industry is always zero, because a firm with an entry opportunity cannot delay its choice. The period ends when some potential entrant decides to stay out of the industry. Both active firms' and potential entrants' decisions maximize their expected stream of profits discounted with a factor $\beta < 1$.

2.1 Markov-Perfect Equilibrium

We choose as our equilibrium concept symmetric Markov-perfect equilibrium. When firm i decides whether to stay or exit, $N_t - R_t^i$ (the number of active firms following it in the sequence), C_t, and R_{t+1}^i (its rank in the next period's sequence of active firms) are available and payoff-relevant. Collect these into $H_{it} \equiv (N_t - R_t^i, C_t, R_{t+1}^i)$. Similarly, the payoff-relevant state to a potential entrant is $H_{it} \equiv (C_t, R_{t+1}^i)$. Note that H_{it} takes its values in $\mathcal{H}_S \equiv \mathbb{Z}_+ \times \left[\hat{C}, \check{C}\right] \times \mathbb{N}$ for firms active in period t and in $\mathcal{H}_E \equiv \left[\hat{C}, \check{C}\right] \times \mathbb{N}$ for potential entrants. Here and below, we use S and E to denote survivors and entrants.

A Markov strategy for firm i is a pair $(A_S^i(H_S), A_E^i(H_E))$ for each $H_S \in \mathcal{H}_S$ and $H_E \in \mathcal{H}_E$. These represent the probability of being active in the next period given that the firm is currently active $(A_S^i(\cdot))$ and given that the firm has an entry opportunity $(A_E^i(\cdot))$. A symmetric Markov-perfect equilibrium is a subgame-perfect equilibrium in which all firms follow the same Markov strategy.

When firms use Markov strategies, the payoff-relevant state variables determine an active firm's expected discounted profits, which we denote with $v(H_S)$. In a Markov-perfect equilibrium, this satisfies the Bellman equation

$$v(H_S) = \max_{a \in [0,1]} a\beta \mathbb{E}\left[\frac{C'}{N'}\pi(N') - \kappa + v(H_S') \,\middle|\, H_S\right], \tag{1}$$

Here and throughout, we adopt conventional notation and denote the variable corresponding to X in the next period with X'. In Equation (1), the expectation of N' is calculated using all firms' strategies conditional on the particular firm of interest choosing to be active.

It is well known that multiple Markov-perfect equilibria can exist in similar models.[1] To overcome this standard difficulty, we restrict attention to equilibria in which firms' entry and exit policies arise from a last-in first-out (LIFO) strategy.

Definition 1. *A LIFO strategy is a strategy* (A_S, A_E) *such that* $A_S(H_S) \in \{0, 1\}$, $A_E(H_E) \in \{0, 1\}$, *and* $A_S(N - R, C, R')$ *is weakly decreasing in* R.

[1] See Doraszelski and Satterthwaite (2005).

If all firms adopt a common LIFO strategy (A_S, A_E), then an active firm with rank $R \geq 2$ never stays if the predecessor in the sequence of active firms exits, because

$$A_S(N - R, C, R') = 0 \Rightarrow A_S(N - R - 1, C, R') = 0.$$

As a consequence, if firms adopt a common LIFO strategy, they exit in the reverse order of their entry. As we mentioned in the paper's introduction, this embodies in an extreme form the empirical regularity that young firms exit more frequently than their older counterparts. Conversely, if firms use a common strategy and always exit in the reverse order of their entry, then the common strategy is a LIFO strategy.

With this definition, we can demonstrate existence of a Markov-perfect equilibrium in a LIFO strategy.

Proposition 1. *There exists a symmetric Markov-perfect equilibrium in a LIFO strategy* (A_S, A_E) *such that* $A_S(N - R, C, R')$ *is invariant in* $N - R$ *and weakly decreasing in* R'.

The equilibrium survival probability in Proposition 1 decreases with the firm's rank in the next period and is invariant to the number of firms with unresolved continuation decisions.

This paper's appendix contains the proposition's constructive proof, which has two critical steps. First, we note that the upper bound on C implies that the number of firms that ever produce in a Markov-perfect equilibrium cannot exceed some bound, which we call $\check{N}$. Because a firm with rank $\check{N}$ expects none of its older competitors to cease production before it does, this firm's optimal exit rule corresponds to that from a simple dynamic programming problem. Second, we solve exit decision problems for firms with ranks $\check{N} - 1, \check{N} - 2, \ldots, 1$ that embody the assumption that other firms follow a LIFO strategy. A firm with rank R forms its expectations about the behavior of firms with higher ranks using the solutions of those firms' decision problems. With the solutions to these standard dynamic programming problems in hand, we construct a candidate LIFO strategy and then verify that it satisfies the proposition's conditions and forms a Markov-perfect equilibrium.

The existence proof strongly suggests that the Markov-perfect equilibrium in a LIFO strategy is unique, because the decision problems used in its construction have unique solutions to their Bellman equations. However, we can construct multiple LIFO equilibria by varying a firm's actions in states of indifference between activity and inactivity. We sidestep this difficulty by concentrating on equilibria in which a firm defaults to inactivity.

Definition 2. *A symmetric Markov-perfect equilibrium strategy* (A_S, A_E) defaults to inactivity *if* $A_S(H_S) = 0$ *whenever* $v(H_S) = 0$ *and* $A_E(C, R') = 0$ *whenever* $v(0, C, R') = \phi(R')$.

Proposition 2. *There exists a unique symmetric Markov-perfect equilibrium in a LIFO strategy that defaults to inactivity. This equilibrium's survival rule A_S is such that $A_S(N - R, C, R')$ is invariant in $N - R$ and weakly decreasing in R'.*

Other symmetric Markov-perfect equilibria that default to inactivity might exist, but in them the apparent advantage of early entrants to commit to continuation does not translate into longevity. Henceforth, we constrain our attention to the unique symmetric Markov-perfect equilibrium in a LIFO strategy that defaults to inactivity.

2.2 A Pencil-and-Paper Example

If we assume that $C_t = C_{t-1}$ with probability $1 - \lambda$ and that it equals a draw from a uniform distribution on $[\hat{C}, \check{C}]$ with the complementary probability, then we can calculate the model's equilibrium value functions and decision rules with pencil and paper. Before proceeding, we examine this special case to illustrate the model's moving parts. For further simplification, suppose that $\pi(N) = 0$ for $N \geq 3$, so at most two firms serve the industry. To ensure that the equilibrium dynamics are not trivial, we also assume that no firm will serve the industry if demand is low enough and that two firms will serve the industry if it is sufficiently high.[2]

To begin, consider an incumbent firm with rank 2. In an equilibrium in a LIFO strategy, its profit equals $(C/2)\pi(2) - \kappa$. It will earn this until the next time that C_t changes, at which point the new demand value will be statistically independent of its current value. It is straightforward to use these facts to show that this firm's value function is the following piecewise linear function of C.

$$v(0, C, 2) = \begin{cases} 0 & \text{if } C \leq \underline{C}_2, \\ \beta \frac{(1-\lambda)\left(\frac{C}{2}\pi(2) - \kappa\right) + \lambda \widetilde{v}(0,2)}{1 - \beta(1-\lambda)} & \text{if } C > \underline{C}_2, \end{cases}$$

where

$$\widetilde{v}(0, 2) = \frac{1}{2}\left(\frac{\hat{C} + \check{C}}{2}\right)\pi(2) - \kappa + \int_{\hat{C}}^{\check{C}} \frac{v(0, C', 2)}{(\check{C} - \hat{C})} dC'.$$

Here, $\widetilde{v}(0, 2)$ is the firm's average continuation value given a new draw of C_t and $\underline{C}_2$ is the largest value of C that satisfies $v(0, C, 2) = 0$. Optimality requires the firm to exit if $C < \underline{C}_2$.

[2] Sufficient conditions for these two properties are $(1 - \lambda)\left[\hat{C}\pi(1) - \kappa\right] + \lambda \frac{\frac{\hat{C}+\check{C}}{2}\pi(1) - \kappa}{1 - \beta} < 0$ and $\beta\left[\frac{(1-\lambda)\left(\frac{\check{C}}{2}\pi(2) - \kappa\right) + \frac{\lambda}{1-\beta}\left(\frac{1}{2}\frac{\hat{C}+\check{C}}{2}\pi(2) - \kappa\right)}{1 - \beta(1-\lambda)}\right] > \varphi(2)$.

This value function is monotonic in C, so there is a unique entry threshold $\overline{C}_2$ that equates the continuation value with the entry cost. Thus, a second duopolist enters whenever C_t exceeds $\overline{C}_2$ and exits if it subsequently falls at or below $\underline{C}_2$.

Next, consider the problem of an incumbent with rank 1. If this firm is currently a monopolist, it expects to remain so until $C_t > \overline{C}_2$; and if it is currently a duopolist, it expects to become a monopolist when C_t falls below $\underline{C}_2$. This firm's value function is also piecewise linear. If the firm begins the period as the sole incumbent, it is

$$v(0,C,1) = \begin{cases} 0 & \text{if } C \leq \underline{C}_1 \\ \beta\frac{(1-\lambda)(C\pi(1)-\kappa)+\lambda\widetilde{v}(0,1)}{1-\beta(1-\lambda)} & \text{if } \underline{C}_1 < C \leq \overline{C}_2, \\ \beta\frac{(1-\lambda)\left(\frac{C}{2}\pi(2)-\kappa\right)+\lambda\widetilde{v}(1,1)}{1-\beta(1-\lambda)} & \text{if } C > \overline{C}_2; \end{cases}$$

and if it begins as one of two incumbents it equals

$$v(1,C,1) = \begin{cases} 0 & \text{if } C \leq \underline{C}_1, \\ \beta\frac{(1-\lambda)(C\pi(1)-\kappa)+\lambda\widetilde{v}(0,1)}{1-\beta(1-\lambda)} & \text{if } \underline{C}_1 < C \leq \underline{C}_2, \\ \beta\frac{(1-\lambda)\left(\frac{C}{2}\pi(2)-\kappa\right)+\lambda\widetilde{v}(1,1)}{1-\beta(1-\lambda)} & \text{if } C > \underline{C}_2. \end{cases}$$

The exit threshold $\underline{C}_1$ is the greatest value of C such that $v(0,C,1) = 0$, and the average continuation values following a change in C_t for a monopolist and a duopolist are

$$\begin{aligned} \widetilde{v}(0,1) &= \left(\frac{\hat{C}+\check{C}}{2}\right)\pi(1) - \kappa + \int_{\hat{C}}^{\check{C}} \frac{v(0,C',1)}{(\check{C}-\hat{C})}dC', \\ \widetilde{v}(1,1) &= \frac{1}{2}\left(\frac{\hat{C}+\check{C}}{2}\right)\pi(2) - \kappa + \int_{\hat{C}}^{\check{C}} \frac{v(1,C',1)}{(\check{C}-\hat{C})}dC'. \end{aligned}$$

This value function does not always increase with C, because slightly raising C from $\overline{C}_2$ induces entry by the second firm and causes both current profits and the continuation value to discretely drop. Nevertheless we know that they drop to a value above $\varphi(1)$, because at this point the second firm chooses to enter. Hence, it is still possible to find a unique entry threshold $\overline{C}_1$ that equates the value of entering with rank 1 to the cost of doing so.

Figure 2 visually represents the equilibrium. In each panel, C runs along the horizontal axis. The vertical axis gives the value of a firm at the time that entry and exit decisions are made. The top panel plots the value of a firm with rank 1, while the bottom plots the value for a competitor with rank 2. For visual clarity, the two panels have different vertical scales. The value of a duopolist with rank 2 equals zero for $C < \underline{C}_2$, and thereafter increases linearly

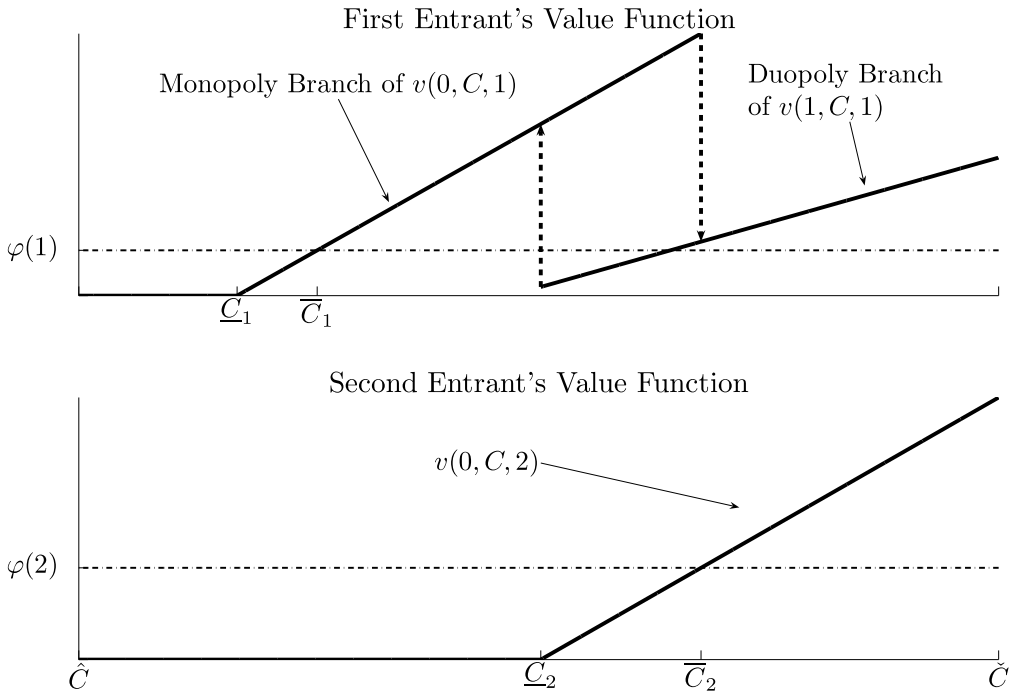

Figure 2: Equilibrium in the Pencil-and-Paper Example

with C. The entry threshold $\overline{C}_2$ equates this value with $\varphi(2)$. The value of an older firm with rank 1 has two branches. The upper monopoly branch gives the value of a monopolist expecting no further entry. If C increases above $\overline{C}_2$ and thus induces entry, the firm's value drops to the lower duopoly branch. This has the same slope as the value function in the lower panel. Its intercept is higher, because the incumbent expects to eventually become a monopolist the first time that C passes below $\underline{C}_2$.[3] When this occurs, the firm's value returns to the monopoly branch. The entry and exit thresholds for this firm occur where the monopoly branch intersects $\varphi(1)$ and 0.

The paper-and-pencil example provides a useful basic framework for analytically characterizing the effects of policy interventions in a dynamic duopoly. This paper's companion (Abbring and Campbell, 2006b) uses this framework to determine the effects of raising a barrier to entry in a monopoly by increasing a second entrant's sunk costs, and to explore the consequences of replacing the LIFO assumption with a first-in first-out (FIFO) assumption.

[3]The two panels' different vertical scales mask these results.

3 Threshold Entry and Exit Rules

In the paper-and-pencil example, all firms follow *threshold rules* for their entry and continuation decisions.

Definition 3. *A firm with rank R' follows a* threshold rule *if there exist real numbers $\underline{C}_{R'}$ and $\overline{C}_{R'} \geq \underline{C}_{R'}$ such that $A_S(N-R, C, R') = I\{C > \underline{C}_{R'}\}$ and $A_E(C, R') = I\{C > \overline{C}_{R'}\}$.*

With such a rule, a potential entrant into a market with $R'-1$ incumbents actually enters if and only if $C > \overline{C}_{R'}$, and this firm exits the first time that $C \leq \underline{C}_{R'}$.

There are three reasons to care about whether or not firms follow threshold rules. First, they pervade theoretical and empirical industrial organization. Second, they simplify the model's analysis, as the pencil-and-paper example illustrated. Third, as the next proposition shows, higher realizations of demand always result in more active firms if and only if all firms use threshold rules.

Proposition 3. *Consider a sequence of possible demand realizations, $C_1, C_2, \ldots, C_{t-1}$ and the corresponding number of operating firms from the equilibrium of Proposition 2, $N_1, N_2, \ldots, N_{t-1}$. Then increasing C_t weakly increases N_{t+j} for non-negative j, and* any *possible sequence of subsequent demand realizations $C_t, \ldots, C_{t+j}$ if and only if firms of all ranks follow threshold rules.*

This proposition's proof is obvious.

A monotonic influence of C_t on N_{t+j} appeals to us as "natural". It is straightforward to show that a firm with the highest possible rank always follows a threshold rule given stochastic monotonicity ($Q(\cdot|C)$ decreases with C). Hopenhayn (1992) imposes this condition on competitive firms' productivity shocks to demonstrate the existence of an optimal exit threshold. However, stochastic monotonicity does not guarantee that firms of all ranks use threshold rules in the LIFO equilibrium. Figure 3 illustrates this using a particular numerical example with $\check{N} = 2$. For this, we suppose that $\ln C_t \in [-1.5, 1.5]$ and that

$$Q(c|C) = \begin{cases} 0 & \text{if } \ln c < \max\{\ln C - 0.3, -1.5\} \\ 1/4 & \text{if } \max\{\ln C - 0.3, -1.5\} \leq \ln c < \ln C \\ 3/4 & \text{if } \ln C \leq \ln c < \min\{\ln C + 0.3, 1.5\} \\ 1 & \text{otherwise} \end{cases}$$

With this stochastic process, the probability of $\ln C_t$ remaining unchanged is $1/2$. With probability $1/4$ it falls to the maximum of $\ln C_t - 0.30$ and $\ln \hat{C}$, and with the same probability

it rises to the minimum of $\ln C_t + 0.30$ and $\ln \check{C}$. The model's other parameters in this example are $\varphi(1) = \varphi(2) = 10$, $\pi(N) = 2 \times I\{N \leq 2\}$, $\kappa = 1$, and $\beta = 1.05^{-1}$.[4]

The example's stochastic process displays stochastic monotonicity, so the value function for the second entrant increases with C_t. As with the paper and pencil example, the first firm's value function decreases to a point above $\varphi(1)$ at the second entrant's entry threshold. However, the value function also decreases at several points to the left of this threshold. The drops occur when increasing $\ln C_t$ moves one of the two extreme points in the support of $\ln C_{t+1}$ over another drop. The implication of this non-monotonicity is that this firm's value function crosses $\varphi(1)$ thrice. As a result, a firm with this entry opportunity would take it if $\ln C_t$ is in either of the disconnected sets labeled A and B but it would stay out of the market if $\ln C_t$ fell in the region between them. Intuitively, moving $\ln C_t$ from A into the region between A and B decreases the value of entry by increasing the probability of further entry without a compensating gain from increasing $\ln C_{t+1}$.

The above example illustrates that firms do not generically use threshold rules in equilibrium. In it, increasing the current value of C can discontinuously increase the likelihood of crossing $\overline{C}_2$ and thereby discontinuously decrease the incumbent's value. In contrast, increasing C in the pencil-and-paper example leaves the probability of future entry unchanged. Together, these examples suggest that firms will use threshold rules if the stochastic process limits the negative "potential entry" effect of increasing C on expected future profits. Here we present sufficient conditions for this to be so.

We rely on the following class of stochastic processes for C_t.

Definition 4. *The transition function $Q(\cdot|C)$ is a* mixture of uniform autoregressions with bounded growth *if (i) there exists a sequence of transition functions*

$$Q_k(c|C) = \begin{cases} 1 & \text{if } c > \mu_k(C) + \sigma_k/2 \\ (c - \mu_k(C) + \sigma_k/2)/\sigma_k & \text{if } \mu_k(C) - \sigma_k/2 \leq c \leq \mu_k(C) + \sigma_k/2 \\ 0 & \text{otherwise,} \end{cases}$$

with both $\mu_k(C) \leq C + \sigma_k/2$ and $\mu_k(C)$ weakly increasing in C; and (ii) there exists a sequence of positive real numbers p_k such that $\lim_{K\to\infty} \sum_{k=1}^{K} p_k = 1$ and

$$\lim_{K\to\infty} \sup_{c,C} \left| Q(c|C) - \sum_{k=1}^{K} p_k Q_k(c|C) \right| = 0.$$

[4]For the computation, we used the algorithm described in Section 4.1.

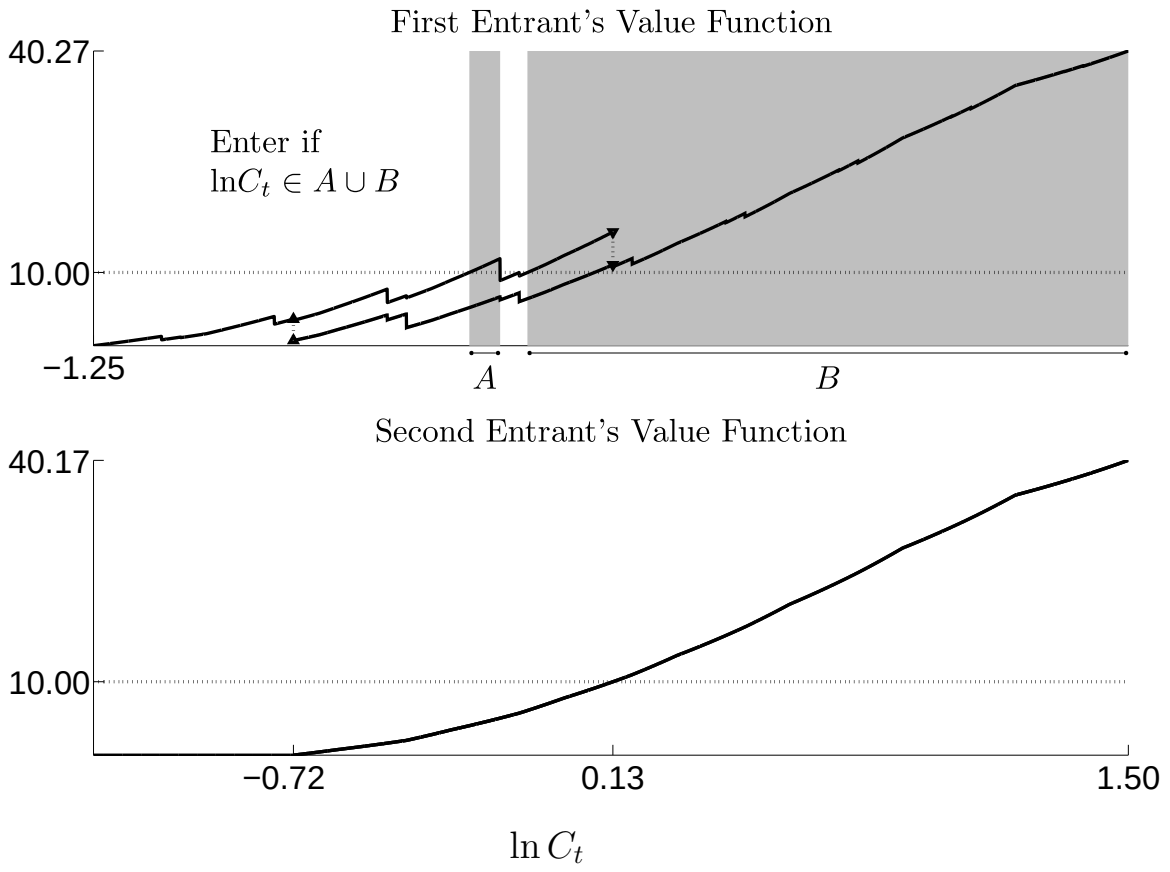

Figure 3: Example of LIFO Equilibrium with a Non-Monotonic Entry Rule

In this definition, each of the mixing distributions is a (possibly nonlinear) autoregression with conditional mean $\mu_k(C)$ and uniform innovations with variance $\sigma_k^2/12$. The coefficients p_k give the mixing probabilities. The condition that $\mu_k(C) \leq C + \sigma_k/2$ ensures that the current state is always in or above the support of each mixing distribution. This is the sense in which Definition 4 bounds the growth of C. With this definition in place, we can state this section's central result.

Proposition 4. *Let (A_S, A_E) be the unique symmetric Markov-perfect equilibrium in a LIFO strategy that defaults to inactivity. Assume that $Q(\cdot|C)$ is a mixture of uniform autoregressions with bounded growth. Then, firms with all ranks follow threshold policies.*

The key step in the proposition's proof is the demonstration that an increase in C that makes further entry more likely does not reduce the expected continuation value below the

firm's cost of entry. To appreciate the contribution of the restriction on $Q(\cdot|C)$ to this, notice that it requires the distribution of C' given C to have no modes to the right of C. Thus, increasing C cannot move a "substantial" probability mass over another firm's entry threshold as in our example of a non-monotonic exit rule.

A wide variety of demand processes are consistent with the requirements of Proposition 4. The stochastic process from the pencil-and-paper example satisfies the conditions with α and $1-\alpha$ serving as the mixing probabilities. In this case, one of the uniform distributions is degenerate at $\mu_1(C_t) = C_t$. To construct another example, consider a random walk reflected at $\hat{C}$ and $\check{C}$. That is, set

$$\mu(c) = \begin{cases} \hat{C} + \frac{\sigma}{2} & \\ c & \text{if } \hat{C} + \frac{\sigma}{2} \leq c \leq \check{C} - \frac{\sigma}{2}, \text{ and} \\ \check{C} - \frac{\sigma}{2} & \end{cases} \begin{array}{l} c < \hat{C} + \frac{\sigma}{2}, \\ \\ c > \check{C} - \frac{\sigma}{2}, \end{array}$$

for some $0 < \sigma < \check{C} - \hat{C}$. By mixing over such reflected random walks, we can approximate any symmetric and continuous distribution for the growth rate of demand in the region away from the boundaries of $[\hat{C}, \check{C}]$.

4 Entry and Exit with Uncertainty

This section applies our analysis to two related questions: How does adding uncertainty impact oligopolists' entry and exit thresholds? How do estimates of oligopolists' producer surplus per consumer based on static models of the "long-run" without both uncertainty and sunk costs differ from their actual values?

Dixit and Pindyck (1994) review a large literature that characterizes competitive firms' entry and exit decisions with sunk costs and uncertain profits. Such a firm's value equals its *fundamental value*, the expected discounted profits from perpetual operation, plus the value of an option to sell this stream of (potentially negative) profits at a strike price of zero. This literature's key insight is that uncertainty about future profits raises the value of this put option and thereby decreases the frequency of exit. Abbring and Campbell (2006a) estimated that this option value accounted for the majority of firm value in a particular competitive service industry. Our model allows us to investigate how the insights from this well-studied decision-theoretic problem apply to oligopolistic dynamics.

Our analysis of the second question follows a large literature based on static free-entry models of oligopoly structure, exemplified by Bresnahan and Reiss (1990, 1991b) and Berry

(1992). They determined empirically how changing market size influenced the number of competitors using observations from cross-sections of local retail (Bresnahan and Reiss) and airline (Berry) markets. The models they used to structure their analysis can be viewed as versions of ours in which either demand remains unchanged over time or firms incur no sunk costs. These papers point to current demand as the key determinant of the number of firms: A market will attain N firms if N entrants can recover their fixed costs but $N+1$ entrants cannot. These authors emphasize that the observed relationship between C and N depends on the rate at which $\pi(N)$ decreases (which Sutton, 1991, labeled the "toughness of competition") and the rate at which $\varphi(N)$ increases (which McAfee et al., 2004, define to be an economic barrier to entry). If both of these functions are constant, then the number of active firms is roughly proportional to demand, $\overline{C}_j = j \times \overline{C}_1$. However, if either $\pi(N)$ decreases or $\varphi(N)$ increases, then N/C declines with C. In this sense, increasing the toughness of competition or imposing a sunk barrier to entry increases concentration.

Our approach to answering these questions is computational. Accordingly, we begin this section with an explicit presentation of the algorithm for equilibrium computation. We then show how demand uncertainty impacts equilibrium entry and exit thresholds for a particular model parameterization. The section concludes with the calculation of the entry thresholds and the producer surplus per consumer calculated from feeding data generated by our model's ergodic distribution through a static Probit model of long-run equilibrium like that of Bresnahan and Reiss (1990, 1991b).

4.1 Equilibrium Computation

The proof of Proposition 1 outlines a simple algorithm for computing the Markov-perfect equilibrium of interest:

(i). Given values for the model's primitives, we choose an evenly spaced grid of values for C in the interval $[\hat{C}, \check{C}]$ and a Markov chain over this grid to approximate $Q(\cdot|C)$.

(ii). We set $\check{N}$ equal to the largest value of R such that

$$\frac{\check{C}}{R}\pi(R) - \kappa \geq 0.$$

(iii). We consider the entry and survival decision problem of a firm with rank $\check{N}$. This firm rationally expects no further entry and sets N' equal to $\check{N}$ in all states $(N-R, C)$. Under this supposition, we can solve the firm's dynamic programming problem by beginning

with a trial value for its value function $v(0, \cdot, \check{N})$ and iterating on the Bellman equation (1) for $N = R = \check{N}$. This gives the firm's expected discounted profits $v(0, C, \check{N})$ for all C on the chosen grid. In practice, this takes very, very little computer time. From $v(0, \cdot, \check{N})$, we can calculate the sets of demand states C in which the firm chooses to enter and survive. We refer to these as the entry and survival sets

$$\mathcal{E}_{\check{N}} \equiv \{C | v(0, C, \check{N}) > \varphi(\check{N})\} \qquad \text{and} \qquad \mathcal{S}_{\check{N}} \equiv \{C | v(0, C, \check{N}) > 0\}.$$

(iv). The rest of the computation proceeds recursively for $R = \check{N} - 1, \ldots, 1$. Suppose that we have computed entry sets $\mathcal{E}_{R+1}, \ldots, \mathcal{E}_{\check{N}}$ and survival sets $\mathcal{S}_{R+1}, \ldots, \mathcal{S}_{\check{N}}$. A firm with rank R rationally expects that these sets govern younger firms' entry and survival decisions, and that no firm will enter with rank larger than $\check{N}$. Hence, it expects that

$$N'_R(N - R, C) \equiv R + \sum_{\tilde{R}=R+1}^{\check{N}} \left[I\left\{\tilde{R} \leq N, C \in \mathcal{S}_{\tilde{R}}\right\} + I\left\{\tilde{R} > N, C \in \mathcal{E}_{\tilde{R}}\right\}\right]$$

governs the evolution of the number of firms. With this specification for N' in place, we can solve this firm's dynamic programming problem by iterating on the Bellman equation (1) for given R, starting with *e.g.* the value function for a firm with rank $R + 1$. This produces the expected discounted profits $v(N - R, \cdot, R)$, $N = R, \ldots, \check{N}$, and the entry and survival sets

$$\mathcal{E}_R \equiv \{C | v(0, C, R) > \varphi(R)\} \qquad \text{and} \qquad \mathcal{S}_R \equiv \{C | v(0, C, R) > 0\}$$

for a firm with rank R.

With the equilibrium entry and survival sets for all $\check{N}$ possible ranks in place, calculating observable aspects of industry dynamics (such as the ergodic distribution of N_t) is straightforward. Matlab programs and documentation are available in this project's replication file.

4.2 Equilibrium Entry and Exit Thresholds

With this algorithm, we have calculated the equilibrium entry and exit thresholds for a particular specification of the model that satisfies the sufficient conditions for firms to use threshold-based entry and exit policies. We set one model period to equal one year and chose β to replicate a 5% annual rate of interest. We set $\kappa = 1.75$ and $\varphi = 0.25(1 - \beta)/\beta$, so the fixed costs of a continuing establishment equal seven times sunk costs' rental equivalent value.

We also set $\pi(N) = 4$ for all N. With these parameter values and no demand uncertainty, the entry thresholds are twice the corresponding exit thresholds and the entry threshold for a second firm equals one. We set $\hat{C} = e^{-1.5}$, $\check{C} = e^{1.5}$, and $Q(\cdot|C)$ to equal a mixture over 51 reflected random walks in the *logarithm* of C with uniformly distributed innovations. The mixture approximates a normally distributed innovation. We denote the standard deviation of the normal distribution we seek to approximate with σ. Proposition 4 can be easily extended to the case where Definition 4 applies to a monotonic transformation of C_t, so the logarithmic specification for demand has no direct theoretical consequences. We choose it because population and income measures typically require a logarithmic transformation to display homoskedasticity across time.

The first two panels of Table 1 report the equilibrium entry and exit thresholds for this specification for four values of σ, 0, 0.10, 0.20, and 0.30. Given the support of C_t, up to eight firms could populate the industry when $\sigma = 0$. Because C_t is reflected off of $\check{C}$, demand displays mean reversion. Thus, such high states of demand are somewhat temporary when $\sigma > 0$ and the maximum number of firms observed in the ergodic distribution accordingly decreases with σ. The two panels' cells for those missing firms' entry and exit thresholds are blank.

Consider first the impact of increasing σ on the entry thresholds. At least one firm enters an empty market with no demand uncertainty if $C_t > 0.50$. This threshold hardly changes as σ increases. Likewise, the entry threshold for a second firm remains very close to 1.00 as σ rises. The thresholds for higher-ranked entrants all rise with σ with one exception (to be discussed further below). Apparently, increasing demand uncertainty makes entry into an oligopoly less likely for a given value of C_t. Demand uncertainty has exactly the opposite impact on the entry of a potential monopolist. For such a firm, increasing uncertainty increases the value of the put option associated with exit, thereby raising profitability and *lowering* the firm's entry threshold.

This difference between oligopolists' and monopolists' entry decisions arises from the threat of potential entry. A monopolist captures all of the increased profit from a favorable demand shock. For an oligopolist, further entry chops this right tail off of the profit distribution and thereby reduces the firm's option value. This explanation squares with the single exception to the rule that increasing σ increases the entry threshold. Increasing σ from 0.20 to 0.30 simultaneously eliminates the possibility that a sixth firm enters and reduces $\overline{C}_5$ from 2.72 to 2.56 .[5] The third panel of Table 1 further illustrates this effect. It reports the

[5]Entry by an eighth firm does not occur when there is demand uncertainty, so this discussion begs the

equilibrium entry thresholds for the case where $\pi(N) = 4 \times I\{N < 5\}$, so that no more than four firms will populate the market. The entry thresholds for the first, second, and third firms are nearly identical to their values in the first panel. However, the entry thresholds for the fourth firm (facing no further entry) decline with σ.

Next examine the exit thresholds in the table's second panel. Without demand uncertainty, these form a line out of the origin with a slope approximately equal to 0.44 . As expected, raising σ decreases all of the exit thresholds. This mimics the well-known effect of increased uncertainty on monopolists' exit decisions: Uncertainty raises the value of the firm's put option, and exit requires this option's exercise. For completeness, Table 1 reports the equilibrium exit thresholds when $\pi(N) = 4 \times I\{N < 5\}$. As expected, this change has almost no impact on the exit thresholds for firms with ranks less than four. For the fourth firm, eliminating the possibility of further entry makes survival more attractive and thereby lowers the exit threshold even further.

To summarize, adding uncertainty either leaves the equilibrium entry thresholds unchanged or raises them somewhat. This result embodies two effects: Increasing uncertainty alone would make entry more attractive, but the accompanying increase in the probability of future entry reduces expected future profits. On the other hand, adding uncertainty substantially reduces equilibrium exit thresholds.

4.3 Static Analysis of Market Size and Entry

We now characterize how a static "long-run" analysis of market size and entry interprets data generated by our dynamic model. For this, it is helpful to briefly review a stylized version of the entry model examined by Bresnahan and Reiss (1990, 1991b). As in our dynamic model, the producer surplus per firm equals $(C/N) \times \pi(N)$ and at most $\check{N}$ firms serve the industry. The fixed costs to a firm serving the market are $e^{\varepsilon}\kappa$, where ε is a normally distributed shock with mean 0 and variance ς^2. There are no sunk costs. Free entry requires that all active firms earn a positive profit and that an additional firm would earn a non-positive profit. That is

$$\frac{C}{N} \times \pi(N) > e^{\varepsilon}\kappa \qquad \text{and} \qquad \frac{C}{N+1} \times \pi(N+1) \leq e^{\varepsilon}\kappa.$$

question of why increasing σ from 0 to 0.10 raises $\overline{C}_7$ from 3.49 to 3.60 . This change reflects the mean reversion noted above. The same principle explains the rise of $\overline{C}_6$ from 3.13 to 3.19 when σ goes from 0.10 to 0.20.

Table 1: Equilibrium Entry and Exit Thresholds[(i,ii)]

$\pi(N) = 4$

Entry Thresholds

σ	1	2	3	4	5	6	7	8
0.00	0.50	1.00	1.49	1.99	2.48	2.97	3.49	3.97
0.10	0.48	0.98	1.51	2.03	2.56	3.13	3.60	
0.20	0.46	1.00	1.55	2.12	2.72	3.19		
0.30	0.46	1.02	1.62	2.20	2.56			

Exit Thresholds

σ	1	2	3	4	5	6	7	8
0.00	0.44	0.87	1.31	1.73	2.18	2.61	3.03	3.49
0.10	0.38	0.76	1.15	1.54	1.93	2.32	2.69	
0.20	0.33	0.69	1.05	1.40	1.77	2.12		
0.30	0.28	0.63	0.97	1.30	1.58			

$\pi(N) = 4 \times I\{N < 5\}$

Entry Thresholds

σ	1	2	3	4	5	6	7	8
0.00	0.50	1.00	1.49	1.99				
0.10	0.48	0.98	1.52	1.90				
0.20	0.46	0.99	1.57	1.77				
0.30	0.45	1.03	1.57	1.75				

Exit Thresholds

σ	1	2	3	4	5	6	7	8
0.00	0.44	0.87	1.31	1.73				
0.10	0.38	0.76	1.15	1.51				
0.20	0.33	0.69	1.05	1.31				
0.30	0.28	0.64	0.96	1.19				

(i) The parameter values used were $\kappa = 1.75$, $\beta = 1.05^{-1}$, $\varphi = 0.25 \times (1 - \beta)/\beta$, $\hat{C} = e^{-1.5}$, $\check{C} = e^{1.5}$, and $Q(\cdot|C)$ a mixture over reflected random walks in the logarithm of C with uniformly distributed innovations and approximate innovation variance σ^2. (ii) An empty cell indicates that the ergodic distribution of N_t puts zero probability on the given value of N. Please see the text for details.

For each $N = 1, \ldots, \check{N}$; define the deterministic entry threshold $C_N^\star$ to be the unique solution to $(C/N)\pi(N) - \kappa = 0$. Exactly N firms will serve the industry if $\ln C > \ln C_N^\star + \varepsilon$, and $\ln C \leq \ln C_{N+1}^\star + \varepsilon$. The probability that this occurs is $\Phi\left(\frac{\ln(C/C_N^\star)}{\varsigma}\right) - \Phi\left(\frac{\ln(C/C_{N+1}^\star)}{\varsigma}\right)$. In this expression, we set $C_0^\star = 0$ and $C_{\check{N}+1}^\star = \infty$.

Given observations of C and N from a cross section of markets, ordered Probit estimation immediately yields estimates for $C_1^\star, \ldots, C_{\check{N}}^\star$ and ς. With these we can estimate how the producer surplus per consumer falls with additional competitors. Specifically, the definition of $C_N^\star$ implies that $\pi(N)/\pi(1) = C_1^\star \times N/C_N^\star$. If the level of demand required to support N firms equals N times the level required for a monopolist, then we infer that the producer surplus per consumer does not fall with additional entry. On the other hand, if demand must exceed $N \times C_1^\star$ to induce N firms to enter, then the surplus per consumer must decline with N. In this way, the Probit analysis infers the toughness of competition from the relationship between market size and the number of competitors.

For a given joint distribution of C and N, we can define the population counterparts to the estimated thresholds by minimizing the population analogue of the ordered Probit's log-likelihood function.

$$\mathcal{L}(C_1^\star, \ldots, C_{\check{N}}^\star, \varsigma) \equiv \mathbb{E}\left[\sum_{R=0}^{\check{N}} I\{N = R\} \ln\left(\Phi\left(\frac{\ln(C/C_R^\star)}{\varsigma}\right) - \Phi\left(\frac{\ln(C/C_{R+1}^\star)}{\varsigma}\right)\right)\right]$$

Because the ordered Probit likelihood function is always concave, even if it does not represent the true data generating process, this function has a unique minimizer. Population "estimates" of $\pi(N)/\pi(1)$ correspond to these minimizing values for $C_1^\star, \ldots, C_{\check{N}}^\star$. Calculating these from data generated by our dynamic model and comparing them to their true values indicates whether abstraction from dynamic considerations substantially biases the static/long-run estimates of the toughness of competition.

The top panel of Table 2 reports ordered Probit estimates of $C_1^\star, \ldots, C_{\check{N}}^\star$ from the ergodic distribution of the dynamic model specification examined in Section 4.2, and its bottom panel gives the implied estimates of $\pi(N)/\pi(1)$. For all three of values of σ used, the static entry thresholds almost exactly equal the average of the dynamic model's corresponding entry and exit thresholds. That is, the static analysis "splits the difference" between them.

Recall that the true values of $\pi(N)/\pi(1)$ all equal one. That is, an additional competitor steals business from incumbents but does not lower the producer surplus earned per consumer. For the case with $\sigma = 0.10$, the implied values deviate little from the truth. However, raising σ further substantially lowers these "estimates". When $\sigma = 0.3$, the implied value of $\pi(2)/\pi(1)$ equals 0.85. Further increases in N change this little.

Table 2: Static Probit Analysis of Market Structure[(i)]

Implied Static Entry Thresholds

σ	1	2	3	4	5	6	7
0.10	0.43	0.87	1.32	1.78	2.24	2.71	3.13
0.20	0.39	0.84	1.29	1.74	2.21	2.62	
0.30	0.34	0.81	1.26	1.70	2.04		

Implied $\pi(N)/\pi(1)$

σ	1	2	3	4	5	6	7
0.10	1.00	0.98	0.97	0.96	0.95	0.95	0.96
0.20	1.00	0.94	0.92	0.90	0.89	0.90	
0.30	1.00	0.85	0.81	0.80	0.84		

(i) The table's top panel reports population values of Probit-based entry thresholds from the static model of Bresnahan and Reiss calculated using the ergodic distribution of the dynamic model specification of Section 4.2 and Table 1. The bottom panel reports the implied values of $\pi(N)$ normalized by $\pi(1)$. An empty cell indicates that the ergodic distribution of N_t puts zero probability on the given value of N. Please see the text for further details.

Apparently, the static Probit analysis can find evidence that $\pi(N)$ falls with data generated from a dynamic model in which $\pi(N)$ is constant. To gain some insight in the way sunk costs and uncertainty affect the static analysis of the toughness of competition, recall that the static Probit's thresholds are roughly the average of the dynamic entry and exit thresholds. Hence, without uncertainty both the static and the dynamic thresholds are evenly spaced if $\pi(N)$ is constant. For example, from the static analysis we may find that it takes 1000 consumers for one firm and 2000 consumers for two firms to be active in the market. With uncertainty, however, the option value sacrificed on exit will make firms of all ranks more reluctant to exit. Because this effect is not offset by a change in the entry thresholds (see the previous section and Table 1), the static thresholds will decrease as well. We may now find that that it takes only 500 consumers for one firm and 1500 consumers for two firms to be active. The fact that the number of consumers needs to triple, from 500 to 1500, to entice a second firm to join the first suggests that the producer surplus per consumer falls substantially when a second firm enters. However, uncertainty fully explains this effect; the

producer surplus per consumer is constant.[6]

In our analysis, the delay in exit arising from option-value considerations imparts a substantial downward bias to each estimated threshold. In this way, omitting dynamic considerations from a long-run analysis of industry structure can lead to a finding of falling producer surplus per consumer when in fact it is constant. This bias is large in the specification under consideration. Determining its importance for empirical work must proceed on a case-by-case basis, but we expect option-value considerations to pervade oligopolists' exit decisions. A comparison of the results of Bresnahan and Reiss (1991b) with "estimates" in Table 2 supports this view. Their abstract reports

> Our empirical results suggest that competitive conduct changes quickly as the number of incumbents increases. In markets with five or fewer incumbents, almost all variation in competitive conduct occurs with the entry of the second or third firm.

This is exactly the pattern displayed in Table 2.

5 Technology Dynamics

In the model, $\pi(N)$ depends neither on the identity of the firm nor on its history. The previous literature on industry dynamics suggests relaxing this in two ways. A firm's productivity could improve with experience, or it could be stochastic and require Bayesian learning on the part of owners. In this section, we show that the basic approach we follow can accommodate these two extensions.

5.1 Learning by Ageing

We begin with the learning curve. The most popular specification for technology which displays such intertemporal economies of scope is "learning-by-doing." That is, the production frontier shifts out with the level of previous cumulative output. Benkard (2000) estimated such a specification that also includes "forgetting-by-not-doing" using data from the production of a wide-bodied aircraft, and he investigated the technology's consequences

[6] Another logical possibility is that the fall in the implied static thresholds reflects mean reversion: Because C cannot fall below $\hat{C}$, a potential entrant does not expect extremely low values of C to persist. We examined whether this contributes to our results by changing $\hat{C}$ and $\check{C}$ from $e^{-1.5}$ and $e^{1.5}$ to e^{-2} and e^{2}. The results differ only minimally from those in Table 2.

for oligopolistic dynamics in Benkard (2004). Following this approach requires explicitly modeling firms' production decisions and incorporating them into the dynamic game. This would be an interesting extension, but it is overly ambitious for the present paper. Instead, we adopt here a specification of *learning-by-ageing*. That is, a firm's technology frontier expands deterministically with the passage of time. Bahk and Gort (1993) estimated such a specification for the learning curve using a panel of U.S. manufacturing plants, and Cabral (1993) examines how such learning impacts oligopoly dynamics in a model similar to ours but with constant demand and simultaneous entry and continuation decisions.

To incorporate learning-by-ageing into our model, we alter two of its assumptions. First, firms have heterogeneous fixed costs. Each period $\check{N}$ potential entrants have an opportunity to enter. The first has type $T = 1$, the second, $T = 2$, etc. A firm's fixed costs in its first period of operation are $\xi = \kappa + \nu(T)$, where $\nu(\cdot)$ is positive and strictly increasing. Thereafter, the firm's fixed costs evolve deterministically according to $\xi' - \kappa = \vartheta(\xi - \kappa)$, where $0 < \vartheta < 1$. So that older firms always have lower costs than their younger rivals, we assume that $\vartheta\nu(T) < \nu(1)$. Second, incumbent producers and potential entrants make their continuation decisions *simultaneously* instead of sequentially. The analysis of LIFO equilibrium assigns firms entering in the same period different ranks. In the model with the learning curve, the technology types distinguish simultaneous entrants.

In this environment, the payoff relevant state for entry and continuation decisions is C, and the vector of incumbents fixed costs. A Markov-perfect equilibrium is a pair of functions of this state giving the probabilities of survival and entry. We follow Cabral (1993) and focus on a "natural" equilibrium in which low-cost firms never exit while leaving behind a high-cost competitor and no high-cost potential entrant actually enters at the same time that a low-cost potential entrant remains inactive. Because a firm's cost decreases strictly with its age, any such equilibrium has a LIFO structure. Hence, it is straightforward to demonstrate analogous results to Propositions 1 and 2 using simple extensions of their proofs.

To see the relationship between the natural equilibrium in this setting and the LIFO equilibrium in our model, consider a sequence of specifications for firms' fixed costs in which $\nu(\cdot)$ converges to zero. Because the value functions are continuous in firms' fixed costs, the limits of the equilibrium value functions equal their counterparts from the equilibrium in LIFO strategies. In this specific sense, the assumption of sequential continuation decisions and a restriction to LIFO strategies "stand in" for a restriction to a "natural" equilibrium in a model with learning-by-ageing.

5.2 Bayesian Learning and Technology Shocks

Entry entails risk. Jovanovic (1982) modeled this risk as imperfect information about a time-invariant productivity parameter. A firm's owner optimally infers its value given noisy observations and makes continuation decisions based on this inference. Hopenhayn (1992) uses a similar specification with observable but continuously evolving productivity to generate a declining hazard rate for exit. Here, we demonstrate that versions of such technology shocks can be added to our model without destroying its simplicity.

To do so, we again focus on the case where a firm's fixed cost varies over time. Denote firm i's fixed cost at time t with κ_{it}. This can take on one of two values, $\hat{\kappa} \leq \check{\kappa}$. At the time of entry, this is drawn from a distribution with probability p on $\kappa_{it} = \hat{\kappa}$. Thereafter, it evolves according to a Markov chain. This fixed cost is observable to all market participants after production takes place. In this sense, this specification is closer to Hopenhayn's (1992) than Jovanovic's (1982), but the firm's owner does learn a substantial amount about productivity after its first period of production.

If the difference between $\check{\kappa}$ and $\hat{\kappa}$ is large enough, then there might not exist a symmetric Markov-perfect equilibrium in a LIFO strategy. To see this, suppose that p is close to one (so that the realization $\check{\kappa}$ is very unlikely), the probability of transiting from $\hat{\kappa}$ to $\check{\kappa}$ is small, and $\check{\kappa}$ is an absorbing state. If $\check{\kappa}$ greatly exceeds $\hat{\kappa}$ then an old firm facing younger competitors with lower costs might find continuation unprofitable, even given LIFO expectations. If such a firm exits, then its younger competitors' ranks decrease.

This difficulty disappears in two cases. In the first, the Markov transition matrix equals the identity matrix and survival as a perpetual monopolist is only profitable if $\kappa_{it} = \hat{\kappa}$. Allowing for such a possibility would add the realistic feature that adding a long-lived competitor potentially requires many entrepreneurs to try and fail first. In the second case, the difference between $\check{\kappa}$ and $\hat{\kappa}$ is small. Because the incumbents' value function is *strictly* decreasing in the firm's rank except in degenerate and uninteresting cases, we know that such small shocks will never induce a low ranked incumbent to exit before a high-ranked rival. Thus, the expectation that LIFO always holds is rational. Adding such a small technology shock would alter the timing of firms' exits but not their order.

6 Related Literature

This paper's analysis implicitly relies upon a great deal of previous work on the theory of dynamic games and the empirics of industry structure and dynamics. This section ac-

knowledges this dependence explicitly. There are two areas of previous research that are particularly important for us.

6.1 Timing and Expectational Assumptions

The sequential nature of firms' entry and exit decisions allows Markovian strategies to themselves depend on a firm's rank. This and the assumption that firms rationally expect LIFO dynamics substantially structures our analysis. In some previous work, the assumption that firms move sequentially gives early movers a form of commitment to their actions. Examples are Dixit's (1980) two-period Stackelberg investment game and Maskin and Tirole's (1988) infinite-horizon alternating-moves quantity game. In other work with finite-horizon games, ordering players' moves selects a unique Nash equilibrium for empirical analysis. As Berry (1992) notes, this approach is particularly useful when firm-specific observable variables are of substantial interest. Sequencing firms' actions need not select a single Markov-perfect equilibrium in an infinite-horizon setting like ours. In this case, researchers sometimes structure expectations with assumptions— such as LIFO— to select a "natural" equilibrium. Cabral's (1993) restriction that high-cost firms exit before their low-cost counterparts provides one example of such an expectational assumption.[7]

Amir and Lambson (2003) prove existence of a subgame-perfect equilibrium in an infinite-horizon model that is similar to ours, but in which firms move simultaneously in each stage game. They do so by constructing an equilibrium that is the limit of a sequence of LIFO equilibria in the finite-horizon versions of their model as the horizon grows to infinity. This suggests an alternative interpretation of our LIFO equilibrium as the limit of the sequence of equilibria from our model's finite-horizon analogues.

The most common defense of timing assumptions, that incumbents can take actions earlier simply by virtue of their incumbency, applies to our work as well. In the equilibrium we consider, these assumptions on timing and expectations make older firms more valuable than their otherwise identical younger counterparts. For this reason, we expect incumbent firms' to use the tools available to them to move potential entrants' expectations towards those we consider. A formal consideration of equilibrium selection is, however, well beyond the scope of this paper.

[7] Assumptions on agents' expectations can also select a "natural" equilibrium in finite-stage games of dynamic oligopoly with incomplete or private information. For example, Bagwell et al. (1997) assume that imperfectly informed consumers rationally expect the firms that charged the lowest price previously will do so again. This selects an equilibrium in which otherwise static price decisions have dynamic consequences.

6.2 Dynamic Empirical Analysis of Oligopolistic Markets

Ericson and Pakes (1995) proposed a framework for the empirical analysis of Markov-perfect dynamics that is particularly well-suited for modeling oligopolists' investment choices, and Benkard (2004) provides one example of its application. It allows for a wide variety of dynamic investment decisions, but there is no characterization of its equilibrium set beyond the existence proof due to Doraszelski and Satterthwaite (2005). Accordingly, the estimation of this framework's unknown parameters either occurs "off-line", as in Benkard (2004), or by considering each firm's decision problem in isolation and letting the data reveal which equilibrium was played in sample, as in Bajari et al. (2006) and Pesendorfer and Schmidt-Dengler (2003). Strictly speaking, the Ericson and Pakes framework encompasses our model, but we abstract from its most compelling feature, technological change arising from investment decisions. Nevertheless, we expect that the possibility of business-stealing future entry in those models will also severely reduce a new firm's option value and thereby reduce entry.

Bresnahan and Reiss (1993) take a different empirical approach to dynamic oligopoly to which our framework can also contribute. They consider panel observations of the numbers of consumers and producers from concentrated markets for dental services. Their goal is to estimate oligopolists' fixed and sunk costs (κ and $\varphi(R')$ in our model). They acknowledge the numerous theoretical difficulties associated with an infinite-horizon model of oligopolistic entry and exit (such as ours) and they then proceed to estimate a much more tractable two-period model in which entry and exit thresholds determine the number of operating firms given its previous value and the current number of consumers. Because structural estimates coming from such a finite-horizon model lack plausibility, Bresnahan and Reiss refrain from using the estimated thresholds to infer dentists' sunk costs. We believe that an extension of this paper's LIFO equilibrium model that includes econometric error could be appropriate for such a structural estimation.

7 Conclusion

Because there is essentially a unique Markov-perfect equilibrium in LIFO strategies, we can conduct comparative dynamics experiments such as that above. A companion paper to this (Abbring and Campbell, 2006b) applies this framework to another experiment of interest for industrial organization, raising late entrants' sunk costs. For the case of an industry with at most two firms, we prove that raising such a barrier to a second producer's entry increases the probability that some firm will serve the industry and decreases its long-run

entry and exit rates. We also show there that these conclusions are robust to assuming that the *oldest* firm exits first when C_t follows the simple stochastic process from the paper-and-pencil example. In numerical examples of LIFO equilibrium with more than two firms, imposing a barrier to entry stabilizes industry structure. Another natural application of this framework is the estimation of oligopoly entry and exit thresholds discussed above. This awaits future research.

References

Abbring, J. H. and J. R. Campbell (2006a). A firm's first year. Technical report, Federal Reserve Bank of Chicago. 12

Abbring, J. H. and J. R. Campbell (2006b). Oligopoly dynamics with barriers to entry. Technical report, Federal Reserve Bank of Chicago. 1, 8, 24

Amir, R. and V. E. Lambson (2003). Entry, exit, and imperfect information in the long run. *Journal of Economic Theory 110*, 191–203. 23

Bagwell, K., G. Ramey, and D. F. Spulber (1997). Dynamic retail price and investment competition. *RAND Journal of Economics 28*(2), 207–227. 23

Bahk, B. and M. Gort (1993). Decomposing learning by doing in new plants. *Journal of Political Economy 101*(4), 561–583. 21

Bajari, P., C. L. Benkard, and J. Levin (2006, October). Estimating dynamic models of imperfect competition. University of Minnesota Working Paper. 24

Benkard, C. L. (2000, September). Learning and forgetting: The dynamics of aircraft production. *American Economic Review 90*(4), 1034–1054. 20

Benkard, C. L. (2004). A dynamic analysis of the market for wide-bodied commercial aircraft. *Review of Economic Studies 71*(3), 581–611. 21, 24

Berry, S. (1992). Estimation of a model of entry in the airline industry. *Econometrica 60*(4), 889–917. 1, 12, 13, 23

Bresnahan, T. F. and P. C. Reiss (1990). Entry in monopoly markets. *Review of Economic Studies 57*(4), 531–553. 1, 2, 12, 13, 16

Bresnahan, T. F. and P. C. Reiss (1991a). Empirical models of discrete games. *Journal of Econometrics 48*(1-2), 57–81. 1

Bresnahan, T. F. and P. C. Reiss (1991b, October). Entry and competition in concentrated markets. *Journal of Political Economy 99*(5), 977–1009. 12, 13, 16, 19, 20

Bresnahan, T. F. and P. C. Reiss (1993). Measuring the importance of sunk costs. *Annales d'Economie et de Statistique 31*, 181–217. 0, 1, 24

Cabral, L. M. (1993). Experience advantages and entry dynamics. *Journal of Economic Theory 59*, 403–416. 21, 23

Dixit, A. K. (1980, March). The role of investment in entry-deterrence. *Economic Journal 90*(357), 95–106. 23

Dixit, A. K. and R. S. Pindyck (1994). *Investment under Uncertainty*. Princeton University Press. 2, 12

Doraszelski, U. and M. Satterthwaite (2005). Foundations of Markov-perfect industry dynamics: Existence, purification, and multiplicity. Technical report, Harvard University and Northwestern University. 4, 24

Ericson, R. and A. Pakes (1995). Markov-perfect industry dynamics: A framework for empirical work. *Review of Economic Studies 62*, 53–82. 24

Fudenberg, D. and J. Tirole (1991). *Game Theory*. Cambridge, MA: MIT Press. 31

Hopenhayn, H. A. (1992). Entry, exit, and firm dynamics in long run equilibrium. *Econometrica 60*(5), 1127–1150. 9, 22

Jovanovic, B. (1982). Selection and the evolution of industry. *Econometrica 50*(3), 649–670. 22

Maskin, E. and J. Tirole (1988). A theory of dynamic oligopoly, I: Overview and quantity competition with large fixed costs. *Econometrica 56*(3), 549–569. 23

McAfee, R. P., H. M. Mialon, and M. A. Williams (2004). What is a barrier to entry? *American Economic Review 94*(2), 461–465. 13

Pesendorfer, M. and P. Schmidt-Dengler (2003, May). Identification and estimation of dynamic games. Working Paper 9726, National Bureau of Economic Research, Cambridge, MA. 24

Sutton, J. (1991). *Sunk Costs and Market Structure*. Cambridge, MA: MIT Press. 13

Appendix

A Proofs of Results in Section 2

Proof of Proposition 1. The proof proceeds by first constructing a candidate equilibrium strategy and then verifying that it is a LIFO strategy that satisfies the conditions of Proposition 1 and forms an equilibrium.

To construct the candidate strategy, define $\check{N}$ as in Section 4.1. Because $\check{C}$ is finite, $\pi(R)$ is weakly decreasing in R, and $\kappa > 0$; $\check{N} < \infty$. This is an upper bound on the number of firms that would ever produce in a LIFO equilibrium.

Next, consider the exit decision problem of a firm that entered with rank $R \leq \check{N}$ and expects the number of firms to evolve according to the deterministic transition rule $N'_R : \mathbb{Z}_+ \times [\hat{C}, \check{C}] \to \{R, R+1, \ldots, \check{N}\}$. Here, $N'_R(X, C)$ is the number of firms that the firm with rank R expects to be active next period given a decision to continue, X younger firms are active this period, and the number of consumers equals C. The expected number of active firms next period is defined for the off-equilibrium-path event that $R + X > \check{N}$, but it never exceeds $\check{N}$. Define $\mathcal{W}$ to be the space of all functions

$$w : \{0, \ldots, \check{N} - 1\} \times \left[\hat{C}, \check{C}\right] \to \left[0, \frac{\beta\pi(1)\check{C}}{1-\beta}\right]$$

and define the Bellman operator $T_R : \mathcal{W} \to \mathcal{W}$ with

$$T_R(w)(X, C) = \max_{a \in [0,1]} a\beta\mathbb{E}\left[\frac{\pi(N'_R(X, C))C'}{N'_R(X, C)} - \kappa + w(N'_R(X, C) - R, C')\right]. \tag{2}$$

Note that T_R depends on the specification for N'_R. It satisfies Blackwell's sufficient conditions for a contraction mapping, and $\mathcal{W}$ is a complete metric space. Hence, T_R has a unique fixed point, the value function w_R that gives this firm's expected discounted profits at each state $(X, C) \in \{0, \ldots, \check{N} - 1\} \times \left[\hat{C}, \check{C}\right]$.

To construct the candidate equilibrium, begin with the decision problem for a firm with rank $\check{N}$ and the transition rule $N'_{\check{N}}(X, C) = \check{N}$ for all (X, C). This transition rule reflects the firm's expectations that it produces no longer than any earlier entrant, any younger active firms will exit, and no firms will enter. The fixed point $w_{\check{N}}$ of $T_{\check{N}}$ can be uniquely extended to a value function on the entire state space $\mathbb{Z}_+ \times \left[\hat{C}, \check{C}\right]$ by assigning $w_{\check{N}}(X, C) = w_{\check{N}}(0, C)$ for all (X, C). Denote the set $\{C | w_{\check{N}}(0, C) > 0\}$ with $\mathcal{S}_{\check{N}}$ and the set $\{C | w_{\check{N}}(0, C) > \varphi(\check{N})\}$

with $\mathcal{E}_{\check{N}}$. Under the maintained hypotheses of this maximization problem, this firm chooses to remain active if and only if $C \in \mathcal{S}_{\check{N}}$ and it chooses to enter the industry if and only if $C \in \mathcal{E}_{\check{N}}$.[8]

Next, iterate the following argument for $R = \check{N} - 1, \ldots, 1$. Suppose that we have determined value functions $w_{R+1}, \ldots, w_{\check{N}}$, entry sets $\mathcal{E}_{R+1}, \ldots, \mathcal{E}_{\check{N}}$, and survival sets $\mathcal{S}_{R+1}, \ldots, \mathcal{S}_{\check{N}}$. Suppose that we have established that

(i). $\mathcal{E}_{R+1} \supseteq \cdots \supseteq \mathcal{E}_{\check{N}}$,

(ii). $\mathcal{S}_{R+1} \supseteq \cdots \supseteq \mathcal{S}_{\check{N}}$,

(iii). for all $\tilde{R} \geq R+1$, $w_{\tilde{R}}(X, C) = w_{\tilde{R}}(\check{N} - \tilde{R}, C)$ if $X > \check{N} - \tilde{R}$, and

(iv). for all $\tilde{R} \geq R+1$, $w_{\tilde{R}}(X, C) > 0$ if and only if $C \in \mathcal{S}_{\tilde{R}}$.

Consider the decision problem for a firm with rank R and transition rule

$$N'_R(X, C) = R + \sum_{j=1}^{\infty} \left[I\left\{ j \leq X, C \in \mathcal{S}_{R+j} \right\} + I\left\{ j > X, C \in \mathcal{E}_{R+j} \right\} \right], \tag{3}$$

where $\mathcal{E}_{\tilde{R}} = \mathcal{S}_{\tilde{R}} = \emptyset$ for $\tilde{R} > \check{N}$. This transition rule reflects the firm's expectations that it produces no longer than any earlier entrant and that $\mathcal{E}_{R+j}$ and $\mathcal{S}_{R+j}$, $j \in \mathbb{N}$, govern younger firms' entry and survival. The specification for N'_R implies that $N'_R(X, C) = N'_R(\check{N} - R, C)$ if $X > \check{N} - R$. Therefore, we can uniquely extend the fixed point w_R of T_R to a value function on the entire state space $\mathbb{Z}_+ \times \left[\hat{C}, \check{C}\right]$ by assigning $w_R(X, C) = w_R(\check{N} - R, C)$ for all $X > \check{N} - R$.

We first prove some properties of this value function. Consider the complete subspace $\mathcal{W}_R \subseteq \mathcal{W}$ of functions w such that $w(X+1, C) \geq w_{R+1}(X, C)$, $X = 0, \ldots, \check{N} - 2$, and $w(X, C)$ is weakly decreasing in X, for all C. To show that the Bellman operator T_R maps $\mathcal{W}_R$ into itself, note that

(i). $N'_R(X, C)$ is weakly increasing in X, so that $T_R(w)(X, C)$ is weakly decreasing in X if $w \in \mathcal{W}_R$;

(ii). we have that

(a) $0 \leq N'_{R+1}(X, C) - N'_R(X+1, C) = I\{C \notin \mathcal{S}_{R+1}\} \leq 1$, so that

$$\frac{\pi(N'_R(X+1, C))C'}{N'_R(X+1, C)} \geq \frac{\pi(N'_{R+1}(X, C))C'}{N'_{R+1}(X, C)}; \text{ and}$$

[8] This specification of $\mathcal{S}_{\check{N}}$ and $\mathcal{E}_{\check{N}}$ ensures that the firm defaults to inactivity in the case of indifference.

(b) for $w \in \mathcal{W}_R$,

$$\begin{aligned} w(N'_R(X+1,C)-R,C') &\geq w(N'_{R+1}(X,C)-(R+1)+1,C') \\ &\geq w_{R+1}(N'_{R+1}(X,C)-(R+1),C'), \end{aligned}$$

so that we can write

$$\begin{aligned} T_R(w)(X+1,C) &= \max_{a\in[0,1]} a\beta\mathbb{E}\left[\frac{\pi(N'_R(X+1,C))C'}{N'_R(X+1,C)}-\kappa+w(N'_R(X+1,C)-R,C')\right] \\ &\geq \max_{a\in[0,1]} a\beta\mathbb{E}\left[\frac{\pi(N'_{R+1}(X,C))C'}{N'_{R+1}(X,C)}-\kappa+w_{R+1}(N'_{R+1}(X,C)-(R+1),C')\right] \\ &= w_{R+1}(X,C). \end{aligned}$$

Since T_R maps $\mathcal{W}_R$ into itself, $w_R \in \mathcal{W}_R$. That is,

(i). $w_R(X+1,C) \geq w_{R+1}(X,C)$ for all $X = 0,\ldots,\check{N}-2$ and all C, and

(ii). $w_R(X,C)$ is weakly decreasing in X for all C.

These properties extend to the entire state space $\mathbb{Z}_+ \times \left[\hat{C},\check{C}\right]$, because, for $X \geq \check{N}-R$,

(i). $w_R(X+1,C) = w_R(\check{N}-R,C) \geq w_{R+1}(\check{N}-R-1,C) = w_{R+1}(X,C)$ and

(ii). $w_R(X,C) = w_R(\check{N}-R,C)$.

The firm chooses to enter the industry if and only if $C \in \mathcal{E}_R \equiv \{C|w_R(0,C) > \varphi(R)\}$. If the firm is active and $X = 0$, it stays in the industry if and only if $C \in \mathcal{S}_R \equiv \{C|w_R(0,C) > 0\} \supseteq \mathcal{E}_R$. To show that it is also optimal for an active firm with $X \geq 1$ to stay in the industry if and only if $C \in \mathcal{S}_R \equiv \{C|w_R(0,C) > 0\}$, note that

(i). if $C \in \mathcal{S}_R$ then survival is optimal because either

  (a) $C \notin \mathcal{S}_{R+1}$, so that $w_R(X,C) = w_R(0,C) > 0$, or

  (b) $C \in \mathcal{S}_{R+1}$, so that $w_R(X,C) \geq w_{R+1}(X-1,C) > 0$;

(ii). if $C \notin \mathcal{S}_R$ then exit is optimal because $v(X,C,R) \leq v(0,C,R) \leq 0$.

Finally, $w_R(0,C) \geq w_R(1,C) \geq w_{R+1}(0,C)$ for all C, so that $\mathcal{E}_R \supseteq \mathcal{E}_{R+1}$ and $\mathcal{S}_R \supseteq \mathcal{S}_{R+1}$.

With the value functions in hand and their properties established, consider the strategy

$$A_S(X, C, R) = \begin{cases} 1 & \text{if } C \in \mathcal{S}_R \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$A_E(C, R) = \begin{cases} 1 & \text{if } C \in \mathcal{E}_R \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

By construction, this strategy is a LIFO strategy that satisfies the conditions of Proposition 1. It forms a symmetric Markov-perfect equilibrium if no firm can gain by a one-shot deviation from the strategy (e.g. Fudenberg and Tirole, 1991, Theorem 4.2). By construction, the strategy prescribes the optimal action in each state if all other firms follow the same strategy. Hence, no firm can profit from a one-shot deviation and the strategy forms an equilibrium. □

Proof of Proposition 2. The LIFO strategy constructed in the proof of Proposition 1 defaults to inactivity. Thus, a symmetric Markov-perfect equilibrium in a LIFO strategy that defaults to inactivity exists.

Uniqueness can be proven recursively, following the recursive construction of a candidate equilibrium strategy in the proof of Proposition 1. First note that, in any equilibrium in a LIFO strategy that defaults to inactivity,

(i). the expected discounted profits $v(X, C, R)$ equal 0 and the entry and survival sets equal $\mathcal{E}_R = \mathcal{S}_R = \emptyset$ in all states (X, C, R) such that $R > \check{N}$; and

(ii). therefore, $N'_{\check{N}}(X, C)$ gives the expected number of firms in the next period in all states $(X, C, \check{N})$, so that the expected discounted profits $v(X, C, \check{N})$ equal $w_{\check{N}}(X, C)$, the entry set equals $\mathcal{E}_{\check{N}}$, and the survival set equals $\mathcal{S}_{\check{N}}$ in all states $(X, C, \check{N})$.

Next, iterate the following argument for $R = \check{N}-1, \ldots, 1$. Suppose that, in any equilibrium in a LIFO strategy that defaults to inactivity, the expected discounted profits $v(X, C, \tilde{R})$ equal $w_{\tilde{R}}(X, C)$, the entry set equals $\mathcal{E}_{\tilde{R}}$, and the survival set equals $\mathcal{S}_{\tilde{R}}$ in all states $(X, C, \tilde{R})$ such that $\tilde{R} > R$. Then, $N'_R(X, C)$ defined by equation (3) gives the expected number of firms in the next period in state (X, C, R). Hence, in all such equilibria, the expected discounted profits $v(X, C, R)$ equal $w_R(X, C)$, the entry set equals $\mathcal{E}_R$, and the survival set equals $\mathcal{S}_R$ in all states (X, C, R).

Finally, note that the corresponding survival rule A_S is such that $A_S(N - R, C, R')$ is invariant in $N - R$ and weakly decreasing in R'. □

B Proofs of Results in Section 3

We develop three auxiliary results before the proof's presentation.

Definition 5. *A function $f : [\hat{C}, \check{C}] \to \mathbb{R}$ is* $\tilde{C}$-separable, *$\tilde{C} \in [\hat{C}, \check{C}]$, if (i) $f(C) \geq f(\tilde{C})$ for all $C > \tilde{C}$ and (ii) $f(C) \leq f(\tilde{C})$ for all $C < \tilde{C}$.*

Lemma 1. *Let $f : [\hat{C}, \check{C}] \to \mathbb{R}$ be integrable with respect to a uniform measure over its domain, $\tilde{C}$-separable, and non-decreasing on $[\hat{C}, \tilde{C}]$, for some $\tilde{C} \in [\hat{C}, \check{C}]$. Given a conditional probability distribution $Q(\cdot|C)$ for C' with non-decreasing expectation $\mu(C)$ that satisfies either*

(i). $Q(\cdot|C)$ is degenerate at $\mu(C) \leq C$ for all $C \in [\hat{C}, \check{C}]$, or

(ii). $Q(\cdot|C)$ is uniform on $[\mu(C) - \frac{\sigma}{2}, \mu(C) + \frac{\sigma}{2}] \subseteq [\hat{C}, \check{C}]$ with $\sigma > 0$ and $\mu(C) - \frac{\sigma}{2} \leq C$ for all $C \in [\hat{C}, \check{C}]$

then $g(C) \equiv \int_{\hat{C}}^{\check{C}} f(C')dQ(C'|C)$ is non-decreasing in C on $[\hat{C}, \tilde{C}]$.

Proof. In Case (i) , the result follows immediately from $g(C) = f(\mu(C))$. Now consider Case (ii). First, note that $g(C) = \sigma^{-1} \int_{\mu(C)-\sigma/2}^{\mu(C)+\sigma/2} f(u)du$. Because f is non-decreasing on $[\hat{C}, \tilde{C}]$, it immediately follows that g is non-decreasing on $\{C \in [\hat{C}, \tilde{C}] | \mu(C) + \sigma/2 \leq \tilde{C}\}$. Next, for $C^\star \leq C \leq \tilde{C}$ such that $\mu(C^\star) + \sigma/2 \geq \tilde{C}$, we have that

$$\begin{aligned}
\sigma\left(g(C) - g(C^\star)\right) &= \int_{\mu(C^\star)+\sigma/2}^{\mu(C)+\sigma/2} f(u)du - \int_{\mu(C^\star)-\sigma/2}^{\mu(C)-\sigma/2} f(u)du \\
&\geq \int_{\mu(C^\star)+\sigma/2}^{\mu(C)+\sigma/2} f(\tilde{C})du - \int_{\mu(C^\star)-\sigma/2}^{\mu(C)-\sigma/2} f(\tilde{C})du \\
&= 0.
\end{aligned}$$

Taken together, this implies that g is non-decreasing on $[\hat{C}, \tilde{C}]$. □

Lemma 2. *Let $f : [\hat{C}, \check{C}] \to \mathbb{R}$ and $\tilde{C}$ satisfy the conditions of Lemma 1. If $Q^K(\cdot|C) = \sum_{k=1}^K p_k Q_k(\cdot|C)$ for some positive $p_1, \ldots, p_K$ and $Q_1(\cdot|C), \ldots, Q_K(\cdot|C)$ that each individually satisfy the conditions of Lemma 1, then $g^K(C) \equiv \int_{\hat{C}}^{\check{C}} f(C')dQ^K(C'|C)$ is non-decreasing in C on $[\hat{C}, \tilde{C}]$.*

Proof. Lemma 1 implies that $g_k(C) \equiv \int_{\hat{C}}^{\check{C}} f(C')dQ_k(C'|C)$ is non-decreasing on $[\hat{C}, \tilde{C}]$, $k = 1, \ldots, K$. In turn, because $g^K(C) = \sum_{k=1}^K p_k g_k(C)$, this implies that $g^K(C)$ is non-decreasing on $[\hat{C}, \tilde{C}]$. □

Lemma 3. *Let $f : [\hat{C}, \check{C}] \to \mathbb{R}$ be bounded, $\tilde{C}$-separable, and non-decreasing on $[\hat{C}, \tilde{C}]$, for some $\tilde{C} \in (\hat{C}, \check{C}]$. Let $Q^1, Q^2, \ldots$ be a sequence of mixture Markov transition functions satisfying the conditions of Lemma 2 such that $\sup |Q^K - Q| \to 0$ for some Markov transition distribution function Q as $K \to \infty$. Then, $g(C) \equiv \int_{\hat{C}}^{\check{C}} f(C')dQ(C'|C)$ is non-decreasing in C on $[\hat{C}, \tilde{C}]$.*

Proof. Lemma 2 implies that the function g^K corresponding to each Q^K, $K = 1, 2, \ldots$, is non-decreasing on $[\hat{C}, \tilde{C}]$. Because f is bounded, $g^K \to g$ as $K \to \infty$ and g is non-decreasing on $[\hat{C}, \tilde{C}]$. □

We are now prepared to present the proof of Proposition 4.

Proof of Proposition 4. The proof begins with a characterization of $\mathcal{S}_{\check{N}} = \{C | v(0, C, \check{N}) > 0\}$ and $\mathcal{E}_{\check{N}} = \{C | v(0, C, \check{N}) > \varphi(\check{N})\}$. Recall from the proof of Proposition 2 that $v(0, C, \check{N}) = w_{\check{N}}(0, C)$, with $w_{\check{N}}$ the unique fixed point of the Bellman operator $T_{\check{N}}$ defined by Equation (2). This operator maps the space of functions in $\mathcal{W}$ that are non-decreasing in C into itself, so the value function $v(0, C, \check{N})$ is non-decreasing in C. It immediately follows that there exist thresholds $\underline{C}_{\check{N}}$ and $\overline{C}_{\check{N}}$ such that $\mathcal{S}_{\check{N}} = \{C \,|\, C > \underline{C}_{\check{N}}\}$ and $\mathcal{E}_{\check{N}} = \{C \,|\, C > \overline{C}_{\check{N}}\}$. Note that either of these thresholds might equal $\hat{C}^-$, for some $\hat{C}^- < \hat{C}$, or $\check{C}$.

Next, iterate the following argument for $R = \check{N} - 1, \ldots, 1$. Suppose that, for all $\tilde{R} = R + 1, \ldots, \check{N}$, there exist thresholds $\overline{C}_{\tilde{R}}$ and $\underline{C}_{\tilde{R}}$ such that $\mathcal{S}_{\tilde{R}} = \{C \,|\, C > \underline{C}_{\tilde{R}}\}$ and $\mathcal{E}_{\tilde{R}} = \{C \,|\, C > \overline{C}_{\tilde{R}}\}$ and $v(0, C, \tilde{R})$ is non-decreasing in C for all $C < \overline{C}_{\tilde{R}}$. Consider the characterization of $\mathcal{S}_R = \{C | v(0, C, R) > 0\}$ and $\mathcal{E}_R = \{C | v(0, C, R) > \varphi(R)\}$. There are two cases to consider.

(i). In the first, $\overline{C}_{\tilde{R}} = \check{C}$ for all $\tilde{R} = R + 1, \ldots, \check{N}$, so that a firm entering with rank R expects no further entry to occur during its lifetime. This case is identical to the case where $R = \check{N}$, so there exist thresholds $\underline{C}_R$ and $\overline{C}_R$ such that $\mathcal{S}_R = \{C \,|\, C > \underline{C}_R\}$ and $\mathcal{E}_R = \{C \,|\, C > \overline{C}_R\}$.

(ii). In the second case, $\overline{C}_{R+1} < \check{C}$. Here, there are two sub-cases to consider.

(a) In the first, $v(0, C, R) > \varphi(R)$ for all C, so we can set $\underline{C}_R = \overline{C}_R = \hat{C}^-$.

(b) In the second sub-case, $v(0, C, R) \leq \varphi(R)$ for some C. The argument for this sub-case requires the construction of an auxiliary sequence of value functions by iterating on the Bellman operator T_R. To this end, recall that $v(0, C, R + 1) = w_{R+1}(0, C)$ (where w_{R+1} is the unique fixed point of T_{R+1}), and initialize $w_R^1 \equiv$

w_{R+1}. Then, for $j = 2, 3, \ldots$, set $w_R^j \equiv T_R(w_R^{j-1})$. From Equation (3), it follows that

$$N'_R(X,C) - R - \left[N'_{R+1}(X,C) - (R+1)\right] \\ = I\{C \in \mathcal{S}_{R+1}\} + I\{C \in \mathcal{E}_{R+X+1}\} - I\{C \in \mathcal{S}_{R+X+1}\}.$$

Because $\mathcal{S}_{R+1} \supseteq \mathcal{S}_{R+X+1} \supseteq \mathcal{E}_{R+X+1}$, this implies that

$$0 \leq N'_R(X,C) - R - \left[N'_{R+1}(X,C) - (R+1)\right] \leq 1.$$

From this, $w_{R+1} \in \mathcal{W}_{R+1}$, and $N_R(X,C) \leq N_{R+1}(X,C)$; it follows that $w_R^2 = T_R(w_R^1) \geq w_R^1 = w_{R+1}$. Because T_R is monotonic, this implies that $w_R^j \geq w_R^{j-1}$ for all $j \geq 2$.

Define $v^j(0,C,R) \equiv w_R^j(0,C)$ for all j and C. We first show with induction that $v^j(0,C,R)$ and $\overline{C}_R^j \equiv \inf\{C | v^j(0,C,R) > \varphi(R)\} \leq \overline{C}_R^{j-1}$ together satisfy the conditions for $f(C)$ and $\tilde{C}$ in Lemma 3. By assumption, this is the case for $v^1(0,C,R)$ and $\overline{C}_R^1$, because $\overline{C}_R^1 \leq \overline{C}_{R+1}$. Next, suppose that $v^{j-1}(0,C,R)$ and $\overline{C}_R^{j-1}$ satisfy Lemma 3's requirements for $f(C)$ and $\tilde{C}$. Then this Lemma implies that $\mathbb{E}[v^{j-1}(0,C',R)|C]$ is non-decreasing in C on $[\hat{C}, \overline{C}_R^{j-1}]$. Therefore, inspection of Equation (2) determines that $v^j(0,C,R)$ is non-decreasing in C on the same interval. Because $v^j(0,C,R) \geq v^{j-1}(0,C,R)$, we have that $\overline{C}_R^j \leq \overline{C}_R^{j-1}$. Thus, $v^j(0,C,R)$ and $\overline{C}_R^j$ satisfy Lemma 3's requirements of $f(C)$ and $\tilde{C}$.

Define $\overline{C}_R = \lim_{j\to\infty} \overline{C}_R^j$. We wish to show that

(A) $v(0,C,R) \leq \varphi(R)$ and non-decreasing in C for all $C \in [\hat{C}, \overline{C}_R]$ and

(B) $v(0,C,R) > \varphi(R)$ for all $C \in (\overline{C}_R, \check{C}]$.

To show (A), first note that it holds trivially if $\overline{C}_R = \hat{C}$ and focus on the case that $\overline{C}_R > \hat{C}$. Note that $v^j(0,C,R)$ is non-decreasing in C and weakly less than $\varphi(R)$ on $[\hat{C}, \overline{C}_R^j)$ for all j. Because $\overline{C}_R \leq \overline{C}_R^j$, it must be that for all $C^\star \leq C \leq \overline{C}_R$ that $\lim_{j\to\infty} v^j(0,C,R) \leq \varphi(R)$ and $\lim_{j\to\infty} v^j(0,C^\star,R) \leq \lim_{j\to\infty} v^j(0,C,R)$.

We demonstrate (B) inductively. Because $\varphi(R) \leq \varphi(R+1)$ and $v^1(0,C,R) = w_R(0,C)$ is non-decreasing in C on $[\overline{C}_R^1, \overline{C}_{R+1}]$, we know that $v^1(0,C,R) > \varphi(R)$ for $C \in (\overline{C}_R^1, \check{C}]$. Suppose that $v^{j-1}(0,C,R) > \varphi(R)$ for all $C \in (\overline{C}_R^{j-1}, \check{C}]$. Then, $v^j(0,C,R) \geq v^{j-1}(0,C,R) > \varphi(R)$ for all $C \in (\overline{C}_R^{j-1}, \check{C}]$ as well. Furthermore, because $v^j(0,C,R)$ is non-decreasing in C on $[\hat{C}, \overline{C}_R^{j-1}]$, the definition of $\overline{C}_R^j$ implies that $v^j(0,C,R) > \varphi(R)$ for all $C \in (\overline{C}_R^j, \overline{C}_R^{j-1}]$. Because the sequence $\{v^j(0,C,R)\}$ is non-decreasing, $v(0,C,R) > \varphi(R)$ for all $C \in (\overline{C}_R, \check{C}]$.

With this established, it is clear that $\mathcal{S}_R = \{C|C > \overline{C}_R\}$. Define

$$\underline{C}_R \equiv \sup\{C|v(0, C, R) \leq 0\}$$

if $\{C|v(0, C, R) \leq 0\} \neq \emptyset$, and $\underline{C}_R \equiv \hat{C}^-$ otherwise. By construction, $\underline{C}_R \leq \overline{C}_R$. Because $v(0, C, R)$ is non-decreasing for $C \leq \overline{C}_R$, we can write $\mathcal{E}_R = \{C|C > \underline{C}_R\}$.

□