# Bounds on Transverse Momentum Dependent Distribution and Fragmentation Functions

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We give bounds on the distribution and fragmentation functions that appear at leading order in deep inelastic one-particle inclusive leptoproduction or in Drell-Yan processes. These bounds simply follow from positivity of the defining matrix elements and are an important guidance in estimating the magnitude of the azimuthal and spin asymmetries in these processes.

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In deep-inelastic processes the transition from hadrons to quarks and gluons is described in terms of distribution and fragmentation functions. For instance, at leading order in the inverse hard scale $1/Q$, the cross section for inclusive electroproduction $e^-H \to e^-X$ is given as a charge squared weighted sum over quark distribution functions, which describe the probability of finding quarks in hadron $H$. In electron-positron annihilation, the one-particle inclusive cross section for $e^+e^- \to hX$ is given as a charge squared weighted sum over quark and antiquark fragmentation functions, describing the decay of the produced (anti)quarks into hadron $h$.

The distribution functions for a quark can be obtained from the light cone correlation function [1–3]

$$\Phi_{ij}(x) = \int \frac{d\xi^-}{2\pi}\, e^{ip\cdot\xi}\langle P,S|\overline{\psi}_j(0)\psi_i(\xi)|P,S\rangle \Big|_{\xi^+=\xi_T=0}, \tag{1}$$

depending on the light cone fraction $x = p^+/P^+$. To be precise, the lightlike directions $n_+$ and $n_-$, satisfying $n_+^2 = n_-^2 = 0$ and $n_+ \cdot n_- = 1$, define the light cone coordinates $a^\pm = a \cdot n_\mp$. The hadron momentum $P$ is chosen so that it has no components orthogonal to $n_+$ or $n_-$, thus the transverse hadron momentum $P_T = 0$. The correlator contains the soft parts appearing in hard scattering processes, and is related to the forward amplitude for antiquark-hadron scattering (see Fig. 1). The relevant part is $\Phi \not{n}_- = \Phi\gamma^+$. Inserting a complete set of intermediate states and generalizing to off-diagonal spin, one obtains

$$\begin{aligned}(\Phi\gamma^+)_{ij,s's} &= \int \frac{d\xi^-}{2\pi\sqrt{2}}\, e^{ip\cdot\xi}\langle P,s'|\psi^\dagger_{+j}(0)\psi_{+i}(\xi)|P,s\rangle \Big|_{\xi^+=\xi_T=0} \\ &= \frac{1}{\sqrt{2}}\sum_n \langle P_n|\psi_{+j}(0)|P,s'\rangle^* \langle P_n|\psi_{+i}(0)|P,s\rangle\,\delta[P_n^+ - (1-x)P^+],\end{aligned} \tag{2}$$

where $\psi_+ \equiv P_+\psi = \frac{1}{2}\gamma^-\gamma^+\psi$ is the good component of the quark field [4]. This representation shows that the correlation functions have a natural interpretation as light cone momentum densities.

In order to study the correlation function in a spin 1/2 target one introduces a spin vector $S$ that parametrizes the spin density matrix $\rho(P,S)$. It satisfies $P \cdot S = 0$ and $S^2 = -1$ (spacelike) for a pure state, $-1 < S^2 \leq 0$ for a mixed state. Using $\lambda \equiv MS^+/P^+$ and the transverse spin vector $S_T$, the condition becomes $\lambda^2 + \boldsymbol{S}_T^2 \leq 1$, as can be seen from the rest-frame expression $S = (0, \boldsymbol{S}_T, \lambda)$. The precise equivalence of a $2 \times 2$ matrix $\tilde{M}_{ss'}$ in the target spin space and the $S$-dependent function $M(S)$ is $M(S) = \mathrm{Tr}[\rho(S)\tilde{M}]$. Explicitly, the $S$-dependent function $M(S) = M_O + \lambda M_L + S_T^1 M_T^1 + S_T^2 M_T^2$ corresponds to a matrix, which in the target rest frame with as basis the spin 1/2 states with $\lambda = +1$ and $\lambda = -1$ becomes

$$\tilde{M}_{ss'} = \begin{pmatrix} M_O + M_L & M_T^1 - iM_T^2 \\ M_T^1 + iM_T^2 & M_O - M_L \end{pmatrix}. \tag{3}$$

From Eq. (2) follows that after transposing in Dirac space and subsequently extending the matrix $M(S) = (\Phi\gamma^+)^T$ to the target spin space gives a matrix in the combined Dirac ⊗ target spin space which satisfies $v^\dagger M v \geq 0$ for any vector $v$ in that combined space.

The most general form for the quantity $\Phi\gamma^+$ for a spin 1/2 target in terms of the spin vector is

$$\Phi(x)\gamma^+ = \{f_1(x) + \lambda g_1(x)\gamma_5 + h_1(x)\gamma_5 \not{S}_T\}P_+, \tag{4}$$

where the functions $f_1$, $g_1$, and $h_1$ are the leading order quark distribution functions [5]. By tracing over the Dirac indices one projects out $f_1$, which is the quark momentum density [see Eq. (2)]. By writing $\gamma_5$ as the difference of the chirality projectors $P_{R/L} = \frac{1}{2}(1 \pm \gamma_5)$ it follows that in a longitudinally polarized target ($\lambda \neq 0$)

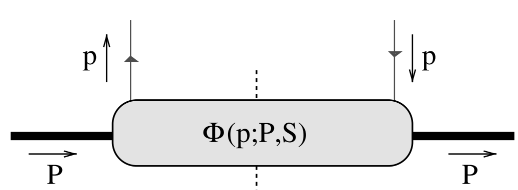


FIG. 1. Matrix element for distribution functions.

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$g_1$ is the difference of densities for right-handed and left-handed quarks. By writing $\gamma^i\gamma_5$ as the difference of the transverse spin projectors $P_{\uparrow/\downarrow} = \frac{1}{2}(1 \pm \gamma^i\gamma_5)$, it follows that in a transversely polarized target ($S_T \neq 0$) $h_1$ is the difference of quarks with transverse spin along and opposite the target spin [6–8]. Since $f_1(x)$ is the sum of the densities, it is positive and gives bounds $|g_1(x)| \leq f_1(x)$ and $|h_1(x)| \leq f_1(x)$.

By considering the combined Dirac $\otimes$ target spin space stricter bounds can be found. As mentioned above, we need to consider the function $M(S) = (\Phi\gamma^+)^T$ in Dirac space. For this we use a chiral representation. In that representation the good projector $P_+$ leaves only two (independent) dirac spinors, one right-handed ($R$) and one left-handed ($L$). On this (two-dimensional) basis of good $R$ and $L$ spinors the matrix $M = [\Phi(x)\gamma^+]^T$ obtained from Eq. (4) is given by

$$M_{ij} = \begin{pmatrix} f_1(x) + \lambda g_1(x) & (S_T^1 + iS_T^2)h_1(x) \\ (S_T^1 - iS_T^2)h_1(x) & f_1(x) - \lambda g_1(x) \end{pmatrix}. \tag{5}$$

Next we make the spin structure of the target explicit as outlined in Eq. (3), yielding on the basis $+R$, $-R$, $+L$, and $-L$

$$\tilde{M} = \begin{pmatrix} f_1 + g_1 & 0 & 0 & 2h_1 \\ 0 & f_1 - g_1 & 0 & 0 \\ 0 & 0 & f_1 - g_1 & 0 \\ 2h_1 & 0 & 0 & f_1 + g_1 \end{pmatrix}. \tag{6}$$

From the positivity of the diagonal elements one recovers the trivial bounds $f_1(x) \geq 0$ and $|g_1(x)| \leq f_1(x)$, but requiring the eigenvalues of the matrix to be positive gives the stricter Soffer bound [9]

$$|h_1(x)| \leq \frac{1}{2}[f_1(x) + g_1(x)]. \tag{7}$$

Analogously bounds can be obtained for transverse momentum dependent distribution and fragmentation functions. Transverse momenta of partons play an important role in hard processes with more than one hadron [10]. Examples are one-particle inclusive deep-inelastic electroproduction, $e^-H \to e^-hX$ [11], or Drell-Yan scattering, $H_1H_2 \to \mu^+\mu^-X$ [12].

The soft parts involving the distribution functions are contained in the light front correlation function

$$\Phi_{ij}(x, \boldsymbol{p}_T) = \int \frac{d\xi^- d^2\boldsymbol{\xi}_T}{(2\pi)^3}\, e^{ip\cdot\xi}\langle P, S|\overline{\psi}_j(0)\psi_i(\xi)|P, S\rangle\Big|_{\xi^+=0}, \tag{8}$$

depending on $x = p^+/P^+$ and the quark transverse momentum $\boldsymbol{p}_T$ in a target with $P_T = 0$. For the description of quark fragmentation one needs [13]

$$\Delta_{ij}(z, \boldsymbol{k}_T) = \sum_X \int \frac{d\xi^- d^2\boldsymbol{\xi}_T}{(2\pi)^3}\, e^{ik\cdot\xi}\, \mathrm{Tr}\langle 0|\psi_i(\xi)|P_h, X\rangle\langle P_h, X|\overline{\psi}_j(0)|0\rangle\Big|_{\xi^+=0}, \tag{9}$$

(see Fig. 2) depending on $z = P_h^+/k^+$ and the quark transverse momentum $\boldsymbol{k}_T$ leading to a hadron with $P_{hT} = 0$. A simple boost shows that this is equivalent to a quark producing a hadron with transverse momentum $P_{h\perp} = -zk_T$ with respect to the quark. Notice that the expressions given here are in a light cone gauge $A^+ = 0$. In a general gauge, a gauge link running along $n_-$ needs to be included. In the presence of transverse momentum dependence [14] and hence separation in $\xi_T$, the links run to light cone infinity $\xi^- = \pm\infty$.

Separating the terms corresponding to unpolarized ($O$), longitudinally polarized ($L$), and transversely polarized targets ($T$), the most general parametrizations *with $p_T$ dependence,* relevant at leading order, are

$$\Phi_O(x, \boldsymbol{p}_T)\gamma^+ = \left\{f_1(x, \boldsymbol{p}_T^2) + ih_1^\perp(x, \boldsymbol{p}_T^2)\frac{\not{p}_T}{M}\right\}P_+, \tag{10}$$

$$\Phi_L(x, \boldsymbol{p}_T)\gamma^+ = \left\{\lambda g_{1L}(x, \boldsymbol{p}_T^2)\gamma_5 + \lambda h_{1L}^\perp(x, \boldsymbol{p}_T^2)\gamma_5\frac{\not{p}_T}{M}\right\}P_+, \tag{11}$$

$$\begin{aligned}\Phi_T(x, \boldsymbol{p}_T)\gamma^+ = \Big\{& f_{1T}^\perp(x, \boldsymbol{p}_T^2)\frac{\epsilon_{T\rho\sigma}p_T^\rho S_T^\sigma}{M} + g_{1T}(x, \boldsymbol{p}_T^2)\frac{\boldsymbol{p}_T\cdot\boldsymbol{S}_T}{M}\gamma_5 \\ &+ h_{1T}(x, \boldsymbol{p}_T^2)\gamma_5\not{S}_T + h_{1T}^\perp(x, \boldsymbol{p}_T^2)\frac{\boldsymbol{p}_T\cdot\boldsymbol{S}_T}{M}\frac{\gamma_5\not{p}_T}{M}\Big\}P_+.\end{aligned} \tag{12}$$

As before, $f_{\ldots}$, $g_{\ldots}$, and $h_{\ldots}$ indicate unpolarized, chirality, and transverse spin distributions. The subscripts $L$ and $T$ indicate the target polarization, and the superscript $\perp$ signals explicit presence of transverse momentum of partons. Using the notation $f^{(1)}(x, \boldsymbol{p}_T^2) \equiv (\boldsymbol{p}_T^2/2M^2)f(x, \boldsymbol{p}_T^2)$, one sees that $f_1(x, \boldsymbol{p}_T^2)$, $g_1(x, \boldsymbol{p}_T^2) = g_{1L}(x, \boldsymbol{p}_T^2)$, and $h_1(x, \boldsymbol{p}_T^2) = h_{1T}(x, \boldsymbol{p}_T^2) + h_{1T}^{\perp(1)}(x, \boldsymbol{p}_T^2)$ are the functions surviving $p_T$ integration.

Analogously, $\Delta$ is parametrized in terms of unpolarized, chirality, and transverse-spin fragmentation functions [11], denoted by capital letters $D_{\ldots}$, $G_{\ldots}$, and $H_{\ldots}$, respectively.

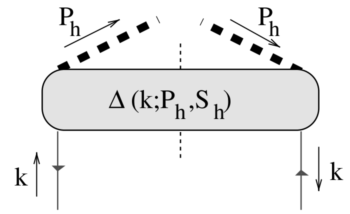


FIG. 2. Matrix element for fragmentation functions.

Time-reversal invariance has not been imposed in the above parametrization, allowing for nonvanishing $T$-odd functions $f_{1T}^{\perp}$ and $h_1^{\perp}$. Possible sources of $T$-odd effects in the initial state have been discussed in Refs. [15]. In the final state time-reversal invariance cannot be imposed [16–18], leading to nonvanishing fragmentation functions $D_{1T}^{\perp}$ [11] and $H_1^{\perp}$ [19].

To put bounds on the transverse momentum dependent functions, we again make the matrix structure explicit. One finds for $M = [\Phi(x, p_T)\gamma^+]^T$ the full spin matrix $\tilde{M}$ to be

$$\begin{pmatrix} f_1 + g_{1L} & \frac{|p_T|}{M}e^{i\phi}(g_{1T} + if_{1T}^{\perp}) & \frac{|p_T|}{M}e^{-i\phi}(h_{1L}^{\perp} + ih_1^{\perp}) & 2h_1 \\ \frac{|p_T|}{M}e^{-i\phi}(g_{1T} - if_{1T}^{\perp}) & f_1 - g_{1L} & \frac{|p_T|^2}{M^2}e^{-2i\phi}h_{1T}^{\perp} & -\frac{|p_T|}{M}e^{-i\phi}(h_{1L}^{\perp} - ih_1^{\perp}) \\ \frac{|p_T|}{M}e^{i\phi}(h_{1L}^{\perp} - ih_1^{\perp}) & \frac{|p_T|^2}{M^2}e^{2i\phi}h_{1T}^{\perp} & f_1 - g_{1L} & -\frac{|p_T|}{M}e^{i\phi}(g_{1T} - if_{1T}^{\perp}) \\ 2h_1 & -\frac{|p_T|}{M}e^{i\phi}(h_{1L}^{\perp} + ih_1^{\perp}) & -\frac{|p_T|}{M}e^{-i\phi}(g_{1T} + if_{1T}^{\perp}) & f_1 + g_{1L} \end{pmatrix},$$

where $\phi$ is the azimuthal angle of $\boldsymbol{p}_T$. First of all, this matrix is illustrative as it shows the full quark helicity structure accessible in a polarized nucleon [20], which is equivalent to the full helicity structure of the forward antiquark-nucleon scattering amplitude. Bounds to assure positivity of any matrix element can, for instance, be obtained by looking at the one-dimensional subspaces, giving the trivial bounds $f_1 \geq 0$ and $|g_{1L}| \leq f_1$. From the two-dimensional subspace one finds [omitting the $(x, \boldsymbol{p}_T^2)$ dependences]

$$|h_1| \leq \frac{1}{2}(f_1 + g_{1L}) \leq f_1\,, \tag{13}$$

$$|h_{1T}^{\perp(1)}| \leq \frac{1}{2}(f_1 - g_{1L}) \leq f_1\,, \tag{14}$$

$$\begin{aligned}(g_{1T}^{(1)})^2 + (f_{1T}^{\perp(1)})^2 &\leq \frac{\boldsymbol{p}_T^2}{4M^2}(f_1 + g_{1L})(f_1 - g_{1L}) \\ &\leq \frac{\boldsymbol{p}_T^2}{4M^2}f_1^2\,,\end{aligned} \tag{15}$$

$$\begin{aligned}(h_{1L}^{\perp(1)})^2 + (h_1^{\perp(1)})^2 &\leq \frac{\boldsymbol{p}_T^2}{4M^2}(f_1 + g_{1L})(f_1 - g_{1L}) \\ &\leq \frac{\boldsymbol{p}_T^2}{4M^2}f_1^2\,.\end{aligned} \tag{16}$$

Besides the Soffer bound, new bounds for the distribution functions are found. In particular, one sees that functions like $g_{1T}^{(1)}$ and $h_{1L}^{\perp(1)}$ appearing in azimuthal asymmetries in leptoproduction are proportional to $|\boldsymbol{p}_T|$ for small $p_T$. In the case of the $T$-odd fragmentation functions, the Collins function, $H_1^{\perp(1)}$, describing fragmentation of a transversely polarized quark into an unpolarized or spinless hadron, for instance, a pion, is bounded by $(|\boldsymbol{P}_{\pi\perp}|/2zM_\pi)D_1(z, \boldsymbol{P}_{\pi\perp}^2)$ while the other $T$-odd function $D_{1T}^{\perp(1)}$, describing fragmentation of an unpolarized quark into a polarized hadron such as a $\Lambda$, is given by $(|\boldsymbol{P}_{\Lambda\perp}|/2zM_\Lambda)D_1(z, \boldsymbol{P}_{\Lambda\perp}^2)$.

Before sharpening these bounds via eigenvalues, it is convenient to introduce two positive definite functions $F(x, \boldsymbol{p}_T^2)$ and $G(x, \boldsymbol{p}_T^2)$ such that $f_1 = F + G$ and $g_1 = F - G$ and define

$$h_1 = \alpha F\,, \tag{17}$$

$$h_{1T}^{\perp(1)} = \beta G\,, \tag{18}$$

$$g_{1T}^{(1)} + if_{1T}^{\perp(1)} = \gamma\frac{|p_T|}{M}\sqrt{FG}\,, \tag{19}$$

$$h_{1L}^{\perp(1)} + ih_1^{\perp(1)} = \delta\frac{|p_T|}{M}\sqrt{FG}\,, \tag{20}$$

where the $x$ and $\boldsymbol{p}_T^2$ dependent functions $\alpha$, $\beta$, $\gamma$, and $\delta$ have absolute values in the interval $[-1, 1]$. Note that $\alpha$ and $\beta$ are real valued but $\gamma$ and $\delta$ are complex valued, the imaginary part determining the strength of the $T$-odd functions. Actually, one sees that the $T$-odd functions $f_{1T}^{\perp}$ and $h_1^{\perp}$ could be considered as imaginary parts of $g_{1T}$ and $h_{1L}^{\perp}$, respectively.

Next we sharpen these bounds using the eigenvalues of the matrix, which are given by

$$e_{1,2} = (1 - \alpha)F + (1 + \beta)G \pm \sqrt{4FG|\gamma + \delta|^2 + [(1 - \alpha)F - (1 + \beta)G]^2}\,, \tag{21}$$

$$e_{3,4} = (1 + \alpha)F + (1 - \beta)G \pm \sqrt{4FG|\gamma - \delta|^2 + [(1 + \alpha)F - (1 - \beta)G]^2}\,. \tag{22}$$

Requiring them to be positive can be converted into the conditions

$$F + G \geq 0\,, \tag{23}$$

$$|\alpha F - \beta G| \leq F + G, \qquad \text{i.e., } |h_{1T}| \leq f_1\,, \tag{24}$$

$$|\gamma + \delta|^2 \leq (1 - \alpha)(1 + \beta)\,, \tag{25}$$

$$|\gamma - \delta|^2 \leq (1 + \alpha)(1 - \beta)\,. \tag{26}$$

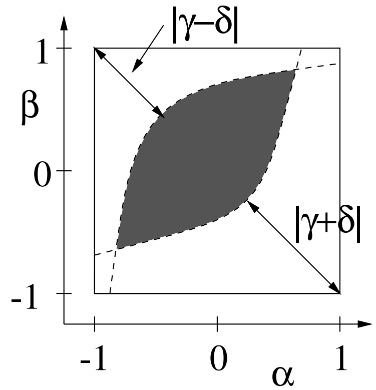


FIG. 3. Allowed region (shaded) for $\alpha$ and $\beta$ depending on $\gamma$ and $\delta$.

It is interesting for the phenomenology of deep inelastic processes that a bound for the transverse-spin distribution $h_1$ is provided not only by the inclusively measured functions $f_1$ and $g_1$, but also by the functions $g_{1T}$ and $h_{1L}^{\perp}$, responsible for specific azimuthal asymmetries [11,21]. This is illustrated in Fig. 3. The same goes for fragmentation functions, where, for instance, the magnitude of $H_1^{\perp}$ constrains the magnitude of $H_1$ [22]. Recently SMC [23], HERMES [24], and LEP [25] have reported preliminary results for azimuthal asymmetries. More results are likely to come in the next few years from HERMES, HERA, RHIC, and COMPASS experiments. Although much theoretical work is needed, for instance, on factorization, scheme ambiguities, and the stability of the bounds under evolution [26], these future experiments may provide us with the knowledge of the full helicity structure of quarks in a nucleon. The elementary bounds derived in this paper can serve as important guidance to estimate the magnitudes of asymmetries expected in the various processes.

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