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Multivariate Ornstein-Uhlenbeck processes with mean-field dependent coefficients: Application to postural sway

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We study the transient and stationary behavior of many-particle systems in terms of multivariate Ornstein-Uhlenbeck processes with friction and diffusion coefficients that depend nonlinearly on process mean fields. Mean-field approximations of this kind of system are derived in terms of Fokker-Planck equations. In such systems, multiple stationary solutions as well as bifurcations of stationary solutions may occur. In addition, strictly monotonically decreasing steady-state autocorrelation functions that decay faster than exponential functions are found, which are used to describe the erratic motion of the center of pressure during quiet standing.

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I. INTRODUCTION

The growing interest in the cooperative behavior of stochastic many-particle systems has amplified the need for suitable analytical tools for investigating their transient and stationary properties. Analytical treatments of overdamped stochastic systems consisting of a large number of mutually coupled subsystems often require analytical treatments in terms of mean-field approaches. Such approaches may then serve as entry points for investigations of complex systems exhibiting more general structures. Many authors have studied mean-field approximations of multivariate Langevin equations. The validity of these works will be taken for granted in this paper.

Kometani and Shimizu introduced a system involving mean-field coupling to describe biochemical reactions during muscular contraction [1]. This model was examined by Desai and Zwanzig in terms of a mean-field approximation [2,3], and a corresponding nonlinear Fokker-Planck equation (NLFPE) was derived [4]. Dawson [5] presented an alternative derivation of this NLFPE on the basis of studies by McKean [6]. Since then, many studies were devoted to the topic of such mean-field couplings. For instance, Shiino [7] derived an *H* theorem for the NLFPE proposed by Desai and Zwanzig. Several numerical techniques were proposed to deal with both the stochastic differential equations [8] and the NLFPE [9–11]. Mean-field approximations of models that describe spatially distributed systems and take multiplicative noise processes into account were extensively studied by van den Broeck and co-workers and others [12–16].

All the studies listed so far have in common that mean fields are established as linear superpositions of the states of the subsystems. In contrast, mean fields generated by nonlinear superpositions of the state variables of the subsystems were predominantly studied in the context of the coupled oscillator models proposed by Winfree [17] and Kuramoto [18,19], who found applications in neuroscience and artificial neural networks theory [20–23], with a specific emphasis on coupled oscillators with unequal eigenfrequencies [24–30]. Both Kuramoto [19] and Desai and Zwanzig [4] interpreted the respective mean fields as macroscopic variables that are generated by the subsystems and, at the same time, act on the subsystems. Irrespective of its generating structure (linear or nonlinear superposition) the dependency of the evolution equations of the subsystems on the mean fields is usually considered to be linear. In biology, however, we are often concerned with systems that depend in a nonlinear fashion on these macroscopic variables. For example, in neuroscience we may regard the soma membrane potential of a particular neuron of a spatially distributed population of neurons as a mean-field variable based on the superposition of the dendritic currents delivered by other members of the population in question. The pulse rate generated by this particular neuron may depend in a highly nonlinear fashion on the soma potential, that is, on the mean-field variable [20,31–36]. In studies of human movement, mean-field variables may express physiologically relevant quantities. For instance, the variance computed from the outputs of similar neurons of a single motor control unit may be seen as an accuracy measure for output signals of that unit. Beek and co-workers [37,38] and Bullock *et al.* [39], among others, emphasized the need for comprehensive models of human motor control that incorporate such physiologically meaningful variables (informational variables and variables of perception). Furthermore, it was argued that neurophysiological variables such as task-related neural processing times should be incorporated into models of human motor control [40,41]. This requirement is tantamount to taking stochastic properties of afferent and efferent signals into account because processing time is related to stochastic variables such as movement accuracy, variance, and entropy of motor performance (Ref. [42], Chaps. 6–9). At issue, therefore, is how to extend concepts derived for stochastic subsystems that depend linearly on their mean fields to more general, nonlinear cases. In other words, mean-field models such as proposed by Desai and Zwanzig and by Kuramoto depend linearly on mean fields composed of arbitrary interactions of their subsystems. Regardless of the explicit form of the interactions (linear or nonlinear), these models are, by definition, linear with respect to the mean fields. Both a phenomenological and a structural microscopic point of view, however, suggest that many systems depend nonlinearly on mean fields, which requires a generalization of the theory of mean-field coupled systems developed so far.

The present paper seeks to contribute to such a generalization by investigating appropriately defined classes of stochastic systems. NLFPE's are obtained from mean-field approximations of these systems, and their transient and stationary solutions are discussed. The paradigmatic case of mean fields established by the system's cumulants is studied in detail. In the context of pitchfork bifurcations, the emergence of multiple stationary solutions is discussed. Furthermore, stochastic systems featuring a general class of steady autocorrelations are proposed. These findings are finally applied in a physiologically motivated model of human motor control that can reproduce the experimentally observed stochastic properties of the random walk of the center of pressure during quiet standing of humans. In short, the present paper focuses on the derivation of NLFPE's for systems of large ensembles. The constituents arbitrarily interact with each other, and generate mean fields that affect the entire system in a nonlinear fashion. After investigating more general properties of such systems, we apply these forms to explain some key features of postural sway during quite stance.

We would like to emphasize that mean-field NLFPE's can be viewed as macroscopic descriptions of stochastic systems derived from microscopic descriptions of their constituents. Similar NLFPE's can be derived, however, on the basis of purely macroscopic considerations invoking variables such as the systems' entropy or energy (e.g., Refs. [43–49]).

II. SYSTEMS WITH MEAN-FIELD-DEPENDENT COEFFICIENTS

We consider a system that is composed of N individual subsystems. Each subsystem, indexed by $j=1,...,N$, is described by a real dimensionless stochastic variable ξ_j. For the entire set $\{\xi_j\}_{j=1,\ldots,N}$ of random variables we can formally define the corresponding multivariate probability density $\mathcal{P}(x_1,\ldots,x_N)$. However, we may also be interested in the one-variable probability densities achieved by integration, $\mathcal{W}_k(x_k):=\int\cdots\int\mathcal{P}(x_1\ldots,x_N)\prod_{l=1,l\neq k}^{N}dx_l$. Suppose that the subsystems are coupled by means of a superposition of their individual states ξ_j according to $s:=\{\sum_{j=1}^{N}f(\xi_j)\}/N$, with $f(z)=z$. In this particular case, we can interpret the variable s as a mean field generated by all subcomponents, so that for large populations ($N\gg 1$) the system as a whole becomes amenable to a mean-field approximation. Then s can be replaced by its expectation value $\langle s\rangle_{\mathcal{P}}$, where the functional $\langle\cdot\rangle_{\mathcal{P}}$ denotes the average with respect to the probability density $\mathcal{P}$ [2,3]. Since $\langle s\rangle_{\mathcal{P}}$ represents a scalar, this approximation usually yields a simplified description of the problem at hand, which can often be solved analytically in a self-consistent fashion. In a similar manner, nonlinear coupling functions of the form $\tilde{s}=\{\sum_{j=1}^{N}f(\xi_j)\}/N$ can be handled when f is an arbitrary infinitely differentiable function. In the present paper, we treat stochastic processes that arise from multivariate Ornstein-Uhlbenbeck processes by replacing the friction coefficient and the fluctuation strength by functions of mean field variables. In detail, the system of study reads, for $N\gg 1$,

$$\frac{d}{dt}\xi_j(t)=-\gamma(s_{\gamma,j})[\xi_j(t)-m]+\sqrt{Q(s_{Q,j})}\Gamma_j(t),$$
$$s_{\gamma,j}=\frac{1}{N-1}\sum_{k=1,k\neq j}^{N}f_\gamma(\xi_k),\quad s_{Q,j}=\frac{1}{N-1}\sum_{l=1,l\neq j}^{N}f_Q(\xi_l). \tag{1}$$

In the following, we restrict ourselves to the case of natural boundary conditions by assuming $|\vec{x}|\rightarrow\infty\Rightarrow\mathcal{P}(x_1,\ldots,x_N,t)\rightarrow 0$, with $\vec{x}=(x_1,\ldots,x_N)$. For the individual terms in Eq. (1) we further require $f_\gamma(\cdot),f_Q(\cdot),\gamma(\cdot),Q(\cdot)\in C^\infty(R)$ and we put $\gamma\geq 0$ and $f_v(cz)=c^{k_v}f_v(z)$, with $k_v>0$ for $c\in R$ and $v\in\{\gamma,Q\}$. Hence we utilize homogeneous functions of degree k_v larger than zero. As common, the Langevin forces $\Gamma_j(t)$ are assumed to be statistically independent white noise sources with $\langle\Gamma_i(t)\rangle=0$ and $\langle\Gamma_i(t)\Gamma_k(t')\rangle=\delta_{i,k}\delta(t-t')$, where $\delta(\cdot)$ is the δ distribution and $\delta_{i,k}$ denotes the Kronecker symbol. The fluctuation strength is measured by $Q>0$ and, in general, may depend on a mean field $s_{Q,j}$. Furthermore, we interpret the stochastic evolution equations as Ito-Langevin equations [50]. Note that in the deterministic case ($Q\rightarrow 0$), for $\gamma(z)=\text{const}>0$, the constant m describes the stable fixed point of the system, and we do not allow for self-interactions of the subsystems, cf. the definitions of $s_{\gamma,j}$ and $s_{Q,j}$ in Eqs. (1). In the following, we assume the existence of transient and stationary solutions of the system of equations (1), and attempt to determine these solutions, at least approximately, using mean-field approaches.

A. Steady-state solutions

We first study the stationary solutions of the mean field approximation of Eq. (1). To this end, we replace $s_{\gamma,j}$ and $s_{Q,j}$ by the respective expectation values, which yields

$$\frac{d}{dt}\xi_j(t)=-\gamma(\langle s_{\gamma,j}\rangle_{\mathcal{P}})[\xi_j(t)-m]+\sqrt{Q(\langle s_{Q,j}\rangle_{\mathcal{P}})}\Gamma_j(t),$$
$$\langle s_{\gamma,j}\rangle_{\mathcal{P}}=\frac{1}{N-1}\sum_{k\neq j}\langle f_\gamma(x_k)\rangle_{\mathcal{W}_k}, \tag{2}$$
$$\langle s_{Q,j}\rangle_{\mathcal{P}}=\frac{1}{N-1}\sum_{l\neq j}\langle f_Q(x_l)\rangle_{\mathcal{W}_l},$$

with $\langle f_v(x_r)\rangle_{\mathcal{W}_r}=\int f_v(x_r)\mathcal{W}_r(x_r,t)dx_r$ for $\nu\in\{\gamma,Q\}$. All the N subsystems in Eqs. (2) can be considered as formally equivalent random walks with similar stochastic properties. Consequently, we assume that for large N the corresponding probability densities are almost identical, that is, $\mathcal{W}_1(\cdot,t)\approx\mathcal{W}_2(\cdot,t)\approx\cdots\approx\mathcal{W}_N(\cdot,t)\approx\mathcal{R}(\cdot,t)$, so that Eqs. (2) becomes

$$\frac{d}{dt}\xi_j(t)=-\gamma(\langle f_\gamma(x_j)\rangle_{\mathcal{P}_{\rm st}})[\xi_j(t)-m]+\sqrt{Q[\langle f_Q(x_j)\rangle_{\mathcal{R}_{\rm st}}]}\Gamma_j(t), \tag{3}$$

which is a self-consistent stochastic differential equation. In the stationary case, the friction coefficient γ and the fluctuation strength Q correspond to the stationary values $\gamma_{\rm st}>0$ and $Q_{\rm st}\geqslant 0$, respectively. From Eq. (3) we can derive the stationary probability density $\mathcal{R}_{\rm st}(x)$ by computing the expectation values $\langle (x-m)^n\rangle_{\mathcal{R}_{\rm st}(x)}$ for all $n\geqslant 1$, cf., e.g., Ref. [50], Chap. 3, which leads to

$$\mathcal{R}_{\rm st}(x)=\left(\frac{\gamma_{\rm st}}{\pi Q_{\rm st}}\right)^{1/2}\exp\left\{-\frac{\gamma_{\rm st}}{Q_{\rm st}}(x-m)^2\right\} \tag{4}$$

and

$$\gamma_{\rm st}/Q_{\rm st}=\gamma[\langle f_\gamma(x)\rangle_{\mathcal{P}_{\rm st}}]/Q[\langle f_Q(x)\rangle_{\mathcal{P}_{\rm st}}]. \tag{5}$$

We dropped the index j to indicate that this result holds for any subsystem j. In order to determine the explicit form of $\mathcal{R}_{\rm st}(x)$, we first solve the transcendent equation (5) for the ratio $\gamma_{\rm st}/Q_{\rm st}$, which, when inserted into Eq. (4), yields the stationary solution.

B. Transient solutions—linear cases $\gamma(z)=\tilde{\gamma}z$ and $Q(z)=\tilde{Q}z$

We proceed by discussing mean-field approximations of the transient solutions of the general system given by Eq. (1). First, however, we review the linear case (also see Desai and Zwanzig [4]). The stochastic process [Eq. (1)] can equivalently be expressed by the multivariate Fokker-Planck equation (cf., e.g., Ref. [50])

$$\frac{\partial}{\partial t}\mathcal{P}(x_1,..,x_N,t)=\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\gamma(s_{\gamma,j})[x_j-m]\mathcal{P}(x_1,\dots,x_N,t)+\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2}Q(s_{Q,j})\mathcal{P}(x_1,\dots,x_N,t), \tag{6}$$

$$s_{\gamma,j}=\frac{1}{N-1}\sum_{k=1,k\neq j}^{N}f_\gamma(x_k),\quad s_{Q,j}=\frac{1}{N-1}\sum_{l=1,l\neq j}^{N}f_Q(x_l).$$

For linear forms $\gamma(z)=\tilde{\gamma}z$ and $Q(z)=\tilde{Q}z$, with $\tilde{\gamma}$, $\tilde{Q}>0$, one obtains

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{P}(x_1,\dots,x_N,t)=&\frac{\tilde{\gamma}}{N-1}\sum_{j=1}^{N}\left[\sum_{k=1,k\neq j}^{N}f_\gamma(x_k)\right]\times\frac{\partial}{\partial x_j}[x_j-m]\mathcal{P}(x_1,\dots,x_N,t)\\&+\frac{\tilde{Q}}{2(N-1)}\sum_{j=1}^{N}\left[\sum_{k=1,k\neq j}^{N}f_Q(x_k)\right]\times\frac{\partial^2}{\partial x_j^2}\mathcal{P}(x_1,\dots,x_N,t).\end{aligned} \tag{7}$$

We can then exploit the identity

$$\begin{aligned}&\left\{\int\cdots\int\tilde{h}(x_k)\frac{\partial}{\partial x_j}[x_j-m]\mathcal{P}(x_1,\dots,x_N,t)\prod_{l=1,l\neq r}^{N}dx_l\right\}_{j\neq k}\\&\equiv\left\{\delta_{j,r}\int\tilde{h}(x_k)\frac{\partial}{\partial x_r}[x_r-m]\rho_{r,k}(x_r,x_k,t)dx_k\right\}_{j\neq k},\end{aligned} \tag{8}$$

where $\tilde{h}(z)\in C^\infty$ denotes an arbitrary function, and $\rho_{r,k}(x_r,x_k,t)$ the joint probability density defined by $\rho_{r,k}(x_r,x_k,t):=\int\cdots\int\mathcal{P}(x_1,\dots,x_N)\Pi_{l=1,l\neq r,l\neq k}^{N}dx_l$. Accordingly, integrating Eq. (7) with respect to the variables x_j with $j\neq r$ yields the evolution equation of the one-variable probability densities $\mathcal{W}_r(x_r,t)$:

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{W}_r(x_r,t)=&\frac{\tilde{\gamma}}{N-1}\sum_{k=1,k\neq r}^{N}\frac{\partial}{\partial x_r}\times[x_r-m]\int f_\gamma(x_r)\rho_{r,k}(x_r,x_k,t)dx_k\\&+\frac{\tilde{Q}}{2(N-1)}\sum_{k=1,k\neq r}^{N}\frac{\partial^2}{\partial x_r^2}\times\int f_Q(x_r),\rho_{r,k}(x_r,x_k,t)dx_k.\end{aligned} \tag{9}$$

Following Desai and Zwanzig [4], we assume that for $N\gg 1$ the joint probability densities factorize according to $\rho_{r,k}(x_r,x_k,t)=\mathcal{W}_r(x_r,t)\mathcal{W}_k(x_k,t)$. Given that the one-variable probability densities again describe identical stochastic processes, that is, $\mathcal{W}_1(\cdot,t)\approx\mathcal{W}_2(\cdot,t)\approx\cdots\approx\mathcal{W}_N(\cdot,t)\approx\mathcal{R}(\cdot,t)$ [cf. Eq. (3)], the diffusion equation (9) becomes

$$\frac{\partial}{\partial t}\mathcal{R}(x,t)=\tilde{\gamma}\cdot\langle f_\gamma(x)\rangle_{\mathcal{R}}\frac{\partial}{\partial x}[x-m]\mathcal{R}(x,t)+\frac{\tilde{Q}}{2}\cdot\langle f_Q(x)\rangle_{\mathcal{R}}\frac{\partial^2}{\partial x_\xi^2}\mathcal{R}(x,t). \tag{10}$$

Note that we dropped the index r. Obviously, the stationary solution of the NLFPE (10) agrees with the stationary solution defined by Eqs. (4) and (5), and we can conclude that the NLFPE (10) describes a mean-field approximation of the transient and steady-state behavior of the system given by Eq. (1) for the linear case [$\gamma(z)=\tilde{\gamma}z$, $Q(z)=\tilde{Q}z$].

C. Transient solutions—nonlinear case and hierarchies of mean-field couplings

In analogy to the special case of a linear dependency of the friction coefficient and the fluctuation strength on mean fields, we now propose an NLFPE of the form

$$\frac{\partial}{\partial t}\mathcal{R}(x,t)=\gamma[\langle f_\gamma(x)\rangle_{\mathcal{R}}]\frac{\partial}{\partial x}[x-m]\mathcal{R}(x,t)+\frac{1}{2}Q[\langle f_Q(x)\rangle_{\mathcal{R}}]\frac{\partial^2}{\partial x^2}\mathcal{R}(x,t) \tag{11}$$

for the nonlinear case. Note that for $\gamma(z)=\tilde{\gamma}z$ and $Q(z)=\tilde{Q}z$ the NLFPE (11) recovers the linear case given by Eq. (10). Moreover, for arbitrary $\gamma(s_{\gamma,j})$ and $Q(s_{Q,j})$ the stationary solution of the NLFPE (11) agrees with the stationary solution given by Eqs. (4) and (5). Therefore, the question arises whether transient solutions of the NLFPE (11) can be viewed as mean-field approximations of the solutions of the original system defined by Eqs. (1). Unfortunately, we cannot answer this question in general but we can derive Eq. (11) using two approaches. We will first obtain Eq. (11) under quite restrictive conditions with regard to the stochastic processes being studied. This first derivation, however, appeals to our intuitive understanding of mean-field coupled systems. The second derivation of Eq. (11) is based on the central limit theorem, and is presented in Appendix A. Using this second derivation we can weaken the conditions imposed on the processes under considerations. In addition, we can discuss the accuracy of the mean-field approximation.

We confine ourselves now to an important special case in which the time-dependent joint probability density $\mathcal{P}(x_1,\ldots,x_N,t)$ has a single global maximum. Explicitly, we require

$$\forall t: \|\{\overline{x^*}:\mathcal{P}(x_1^*,\ldots,x_N^*,t)=\max\}\|=1, \tag{12}$$

where $\|\{\cdot\}\|$ denotes the number of elements of the set $\{\cdot\}$. Of course, whether stochastic processes determined by Eq. (1) obey condition (12) depends on the explicit forms of $\gamma(\cdot)$, $Q(\cdot)$, $f_\gamma(\cdot)$, and $f_Q(\cdot)$, as well as on the initial distributions. Nevertheless, the processes satisfying Eq. (12) represent a rather general class of stochastic processes. For the sake of simplicity, to derive Eq. (11) we first put $Q(z)=Q_0=\text{constant}$ with $Q_0>0$. To treat this case, we start with the evolution equation of the one-variable probability densities $\mathcal{W}_r(x_r,t)$ that can be obtained from the multivariate Fokker-Planck equation (6) according to

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{W}_r(x_r,t)=&\frac{\partial}{\partial x_r}[x_r-m]\\ &\times\int\cdots\int\gamma\left\{\frac{1}{N-1}\sum_{k\neq r}f_\gamma(x_r)\right\}\\ &\times\mathcal{P}(x_1,\ldots,x_N,t)\prod_{l\neq r}dx_l+\frac{Q_0}{2}\frac{\partial^2}{\partial x_r^2}\mathcal{W}_r(x_r,t).\end{aligned} \tag{13}$$

For $N\gg 1$ we assume that the random variables ξ_j become statistically independent. In this case, we can decompose $\mathcal{P}$ into the product of the one-variable probability density $\mathcal{W}_k$ of the random variable ξ_k and the joint probability density $\mathcal{M}_k$ of all other random variables ξ_j but ξ_k, that is, $\mathcal{P}(x_1,\ldots,x_N,t)=\mathcal{W}_k(x_k,t)\mathcal{M}_k(\ldots,x_{k-1},x_{k+1},\ldots,t)$. Then Eq. (13) can be transformed into

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{W}_r(x_r,t)=&\mathcal{Y}[\mathcal{M}_r]\frac{\partial}{\partial x_r}[x_r-m]\mathcal{W}_r(x_r,t)\\ &+\frac{Q_0}{2}\frac{\partial^2}{\partial x_r^2}\mathcal{W}_r(x_r,t),\\ \mathcal{Y}[\mathcal{M}_r]:=&\int\cdots\int\gamma\left\{\frac{1}{N-1}\sum_{k\neq r}f_\gamma(x_k)\right\}\\ &\times\mathcal{M}_r(\ldots,x_{r-1},x_{r+1},\ldots,t)\prod_{l\neq r}dx_l.\end{aligned} \tag{14}$$

Now we introduce new time-dependent variables $\vec{x}_r^{\max}(t)=[\ldots,x_{r-1}^{\max}(t),x_{r+1}^{\max}(t),\ldots]$ that describe the maximum of $\mathcal{M}_r$ at time t. Any point $\vec{x}_r$ of the respective subspace $R^{(N-1)}$ different from $\vec{x}_r^{\max}$ is thus assigned to a probability density $\mathcal{M}_r(\vec{x}_r,t)$ that is much smaller than $\mathcal{M}_r[\vec{x}_r^{\max}(t),t]$. That is, $N\gg 1\Rightarrow\mathcal{M}_r(\vec{x}_r,t)\ll\mathcal{M}_r[\vec{x}_r^{\max}(t),t]$ for $\vec{x}_r\neq\vec{x}_r^{\max}$. Consequently, for suitably chosen functions f_γ, for $N\gg 1$ the joint probability density $\mathcal{M}_r(\vec{x}_r,t)$ can act similarly to a δ distribution. Let us elucidate this point by substituting $y_i:=x_i/(N-1)^{1/k_\gamma}$, so that $\mathcal{Y}[\mathcal{M}_r]$ becomes

$$\begin{aligned}\mathcal{Y}[\mathcal{M}_r]=&\int\cdots\int\gamma\left\{\sum_{k\neq r}f_\gamma(y_k)\right\}\\ &\times\mathcal{M}_r'(\ldots,y_{r-1},y_{r+1},\ldots,t;N)\prod_{l\neq r}dy_l.\end{aligned} \tag{15}$$

$\mathcal{M}_r'(\ldots)$ is the normalized joint probability density

$$\mathcal{M}_r'(\vec{y}_r,t;N):=(N-1)^{(N-1)/k_\gamma}\mathcal{M}_r[(N-1)^{1/k_\gamma}\vec{y}_r,t]. \tag{16}$$

By definition, $\mathcal{M}_r(\vec{x}_r,t)$ has a unique maximum at $\vec{x}_r^{\max}(t)$ and, consequently, the rescaled joint probability density $\mathcal{M}_r'$ has a unique maximum at $\vec{y}_r^{\max}=\vec{x}_r^{\max}/(N-1)^{1/k_\gamma}$. For $N\gg 1$ we assume again that the individual random processes are statistically equivalent [i.e., $\mathcal{W}_1(\cdot,t)\approx\mathcal{W}_2(\cdot,t)\approx\cdots\approx\mathcal{W}_N(\cdot,t)\approx\mathcal{R}(\cdot,t)$]. Then, $\mathcal{M}_r'(\vec{y}_r(t),t;N)$ can be expressed as

$$\mathcal{M}_r'(\vec{y}_r,t;N):=(N-1)^{(N-1)/k_\gamma}\prod_{k=1,k\neq r}^{N}\mathcal{R}([N-1]^{1/k_\gamma}y_k,t) \tag{17}$$

and the ratio $\{\mathcal{M}_r'(\vec{y}_r,t;N)/\mathcal{M}_r'[\vec{y}_r^{\max}(t),t;N]\}$ vanishes for any $\vec{y}_r\neq\vec{y}_r^{\max}(t)$ for $N\gg 1$. In addition, the function $v_t(z):=\mathcal{R}[(N-1)^{1/k_\gamma}z,t]$ decays rapidly from its maximum value because $k_\gamma>0$. In sum, $\mathcal{M}_r'$ converges for $N\gg 1$ to a δ distribution: $\mathcal{M}_r'(\vec{y}_r,t;N\gg 1)\approx\delta(\vec{y}_r-\vec{y}_r^{\max}\{t\})$, allowing integral (15) to be written as

$$\begin{aligned}\mathcal{Y}[\mathcal{M}_r]=\gamma\Bigg(&\int\cdots\int\left[\sum_{k\neq r}f_\gamma(y_k)\right]\\ &\times\mathcal{M}_r'(\ldots,y_{r-1},y_{r+1},\ldots,t;N)\prod_{l\neq r}dy_l\Bigg)\end{aligned} \tag{18}$$

$$= \gamma\left(\sum_{k\neq r} \langle f_\gamma(x_k)\rangle_{\mathcal{W}_k(x_k,t)} \right)$$

$$= \gamma[\langle f_\gamma(x_r)\rangle_{\mathcal{R}(x_r,t)}]. \tag{19}$$

Inserting into Eq. (14) results in

$$\frac{\partial}{\partial t}\mathcal{R}(x,t) = \mathcal{Y}\frac{\partial}{\partial x}[x-m]\mathcal{R}(x,t) + \frac{Q_0}{2}\frac{\partial^2}{\partial x^2}\mathcal{R}(x,t). \tag{20}$$

The key property that we used in the derivation of Eq. (20) is the assumption that constraint (12) holds. By the same reasoning, we can deal with a fluctuation strength $Q=Q(z)$ depending on mean-field variables such as $z=s_{Q,j}$. In this case, Eq. (20) assumes the form of the NLFPE (11) proposed at the beginning of this section.

Note that the preceding findings can be even generalized further in terms of coupling functions $f_\gamma(z,u)$ and $f_Q(z,v)$ that depend on additional mean-field variables $u=s'_{\gamma,j}$ and $v=s'_{Q,j}$. The mean fields $s'_{\gamma,j}$ and $s'_{Q,j}$, in turn, may involve coupling functions $f'_\gamma(z,u)$ and $f'_Q(z,v)$ that depend on other mean-field variables $u=s''_{\gamma,j}$ and $v=s''_{Q,j}$. Let us denote $s_{\gamma,j}$ and $s_{Q,j}$ as $s^{(1)}_{\gamma,j}$ and $s^{(1)}_{Q,j}$, respectively; cf. Eq. (1). Then, we can ''recursively'' define a hierarchy of mean-field couplings, according to

$$s^{(n)}_{\nu,j} := \frac{1}{N-1}\sum_{k=1,k\neq j}^{N} f^{(n)}_\nu(\xi_k, s^{(n+1)}_{\nu,j}),$$
$$s^{(n_d)}_{\nu,j} := \frac{1}{N-1}\sum_{k=1,k\neq j}^{N} f^{(n_d)}_\nu(\xi_k), \tag{21}$$

for $n=1,2,3,...,(n_d-1)$ and $\nu\in\{\gamma,Q\}$. Analogously with previous considerations, we can then derive a mean-field approximation of the multivariate Langevin equation (1) in terms of the NLFPE,

$$\frac{\partial}{\partial t}\mathcal{R}(x,t) = \gamma\{{}^{\mathcal{R}}E^{(1)}_\gamma(t)\}\frac{\partial}{\partial x}[x-m]\mathcal{R}(x,t) + \frac{1}{2}Q\{{}^{\mathcal{R}}E^{(1)}_Q(t)\}\frac{\partial^2}{\partial x^2}\mathcal{R}(x,t),$$
$${}^{\mathcal{R}}E^{(n)}_\nu(t) := \langle f^{(n)}_\nu(x, {}^{\mathcal{R}}E^{(n+1)}_\nu\{t\})\rangle_{\mathcal{R}(x,t)},$$
$${}^{\mathcal{R}}E^{(n_d)}_\nu(t) := \langle f^{(n_d)}_\nu(x)\rangle_{\mathcal{R}(x,t)}, \tag{22}$$

for $n=1,2,3,...,(n_d-1)$ and $\nu\in\{\gamma,Q\}$. Based on the structure of the NLFPE (22), one can derive exact time-dependent solutions in terms of Gaussian probability densities. In detail, the one-point probability density $\mathcal{R}(x,t)$ is described by

$$\mathcal{W}[x;\varpi(t),\mu(t)] = \frac{1}{\sqrt{2\pi\varpi(t)}}\exp\left\{-\frac{[x-\mu(t)]^2}{2\varpi(t)}\right\}, \tag{23}$$

with time-dependent parameters $\varpi(t)$ and $\mu(t)$, that is, $\mathcal{R}(x,t)=\mathcal{W}[x;\varpi(t),\mu(t)]$. For the functions $\mu(t)$ and $\varpi(t)$, this leads to

$$\frac{d}{dt}\mu(t) = -\gamma\{{}^{\mathcal{W}}E^{(1)}_\gamma(t)\}[\mu(t)-m],$$
$$\frac{d}{dt}\varpi(t) = -2\gamma\{{}^{\mathcal{W}}E^{(1)}_\gamma(t)\}\varpi(t) + Q\{{}^{\mathcal{W}}E^{(1)}_Q(t)\}, \tag{24}$$

where ϖ corresponds to the variance of $\mathcal{W}$. For constant friction and diffusion coefficients, $\gamma=\gamma_0$ and $Q=Q_0$, Eqs. (23) and (24) constitute the exact time-dependent stochastic description of classical Ornstein-Uhlenbeck processes [50], while, in general, they portray the exact time-dependent solution of the NLFPE (22) in the case of an initial Gaussian probability density. It can be shown that any solution of the NLFPE (22) converges to these Gaussian solutions; see Appendix B.

III. CUMULANTS AS MEAN FIELD COUPLINGS

A. Pitchfork bifurcation

We examine system (1) specifically for $\gamma=\gamma_0>0$ and $m=0$ in the case of an arbitrary but symmetric coupling function $f_Q(z)=f_Q(-z)$ (here we dropped the superscript $f^{(1)}_Q\to f_Q$). The corresponding mean-field (MF) approximation [Eq. (22)] reads

$$\frac{\partial}{\partial t}\mathcal{R}(x,t) = \gamma_0\frac{\partial}{\partial x}x\mathcal{R}(x,t) + \frac{1}{2}Q[\langle f_Q(x)\rangle_{\mathcal{R}(x,t)}]\frac{\partial^2}{\partial x^2}\mathcal{R}(x,t), \tag{25}$$

which is solved by the Gaussian probability density $\mathcal{W}$; cf. Eq. (23). According to Eq. (24) the dynamics of mean $\mu(t)$ and variance $\varpi(t)$ of $\mathcal{W}$ then reads

$$\frac{d}{dt}\mu(t) = -\gamma_0\mu(t),$$
$$\frac{d}{dt}\varpi(t) = -2\gamma_0\varpi(t) + Q[\langle f_Q(x)\rangle_{\mathcal{W}[x;\varpi(t),\mu(t)]}]. \tag{26}$$

To discuss a stability criterion for this kind of stationary solutions, let u and v be small deviations from the stationary values $\mu_{\rm st}=0$ and $\varpi_{\rm st}=Q_{\rm st}/[2\gamma_0]$, respectively. Then, the linear stability analysis on the basis of Eq. (26) yields $du/dt=-\gamma_0 u$ and $dv/dt=\lambda v$, where the corresponding Lyapunov exponent λ reads

$$\lambda = -2\gamma_0\left[1-\frac{1}{2\gamma_0}\left.\frac{dQ'}{d\varpi}\right|_{\varpi_{\rm st}}\right],$$
$$Q'(\varpi) := Q[\langle f_Q(x)\rangle_{\mathcal{W}[x;\varpi(t),\mu(t)]}] = Q\left(\frac{1}{\sqrt{2\pi\varpi(t)}}\int f_Q(x)\exp\left\{-\frac{[x]^2}{2\varpi(t)}\right\}dx\right). \tag{27}$$

Because of the symmetry $f_Q(z)=f_Q(-z)$ the expression $\partial Q'/\partial u|_{\mu=\mu_{st}=0,\varpi=\varpi_{st}}$ vanishes. In fact, the transcendent equation $\varpi_{\rm st}=Q'(\varpi_{\rm st})/[2\gamma_0]$, in combination with Eq. (27), allows one to address the issue of stability in terms of geometric considerations. First we plot the functions $y_1(\varpi)=\varpi$ and $y_2(\varpi)=Q'(\varpi)/(2\gamma_0)$ in one diagram. Then the intersections of the curves yields the values $\varpi_{\rm st}$ at the stationary points. If the slope of $y_2(\varpi)$ at a particular point $\varpi^*_{\rm st}$ is larger than the slope of the diagonal, then the corresponding Lyapunov exponent is positive, that is, the stationary solution is unstable. Otherwise, the stationary solution is stable [51]—a similar geometric stability criterion was derived by Shiino for the mean-field model proposed by Desai and Zwanzig [7].

In the following we illustrate these results by a system whose diffusion coefficient Q is composed of two parts according to $Q=Q_0+Q_{\rm MF}(s_{Q,j})$. Q_0 corresponds to a constant fluctuation strength, whereas $Q_{\rm MF}$ couples the subsystems. For $Q_{\rm MF}\equiv 0$ the stationary second moment $M_{2,\rm st}$ is given by $M_{2,\rm st}=Q_0/(2\gamma_0)$. We assume that $Q_{\rm MF}$ measures the deviation of the actual empirical second moment $M_{2,j,\rm emp}(t):=\sum_{k\neq j}\xi_k^2(t)/(N-1)$ from $Q_0/(2\gamma_0)$. Note that in this section we consider systems with vanishing mean, so that second moments and second cumulants are identical. Specifically, we choose

$$Q(s_j):=Q_0+2\alpha\left(\left\{\frac{1}{N-1}\sum_{k\neq j}\xi_k^2\right\}-\frac{Q_0}{2\gamma_0}\right)-2\beta\left(\left\{\frac{1}{N-1}\sum_{k\neq j}\xi_k^2\right\}-\frac{Q_0}{2\gamma_0}\right)^3, \qquad (28)$$

with positive control parameters α and β. Comparing Eq. (28) with Eqs. (21) and (22), we find $f_Q^{(1)}(z)=z^2$ and $n_d=1$ and the mean-field approximation [Eq. (25)] with

$$Q(\langle x^2\rangle_{\mathcal{R}(x,t)})=Q_0+2\alpha\left(\langle x^2\rangle_{\mathcal{R}(x,t)}-\frac{Q_0}{2\gamma_0}\right)-2\beta\left(\langle x^2\rangle_{\mathcal{R}(x,t)}-\frac{Q_0}{2\gamma_0}\right)^3. \qquad (29)$$

More explicitly, inserting $Q(\langle x^2\rangle_{\mathcal{R}(x,t)})$ into Eq. (25) yields

$$\frac{\partial}{\partial t}\mathcal{R}(x,t)=\gamma_0\frac{\partial}{\partial x}x\mathcal{R}(x,t)+\left\{\frac{Q_0}{2}+\alpha\left(\langle x^2\rangle_{\mathcal{R}(x,t)}-\frac{Q_0}{2\gamma_0}\right)-\beta\left(\langle x^2\rangle_{\mathcal{R}(x,t)}-\frac{Q_0}{2\gamma_0}\right)^3\right\}\frac{\partial^2}{\partial x^2}\mathcal{R}(x,t). \qquad (30)$$

Since transient solutions converge to Gaussian solutions in the limit $t\to\infty$ (cf. Appendix B), we restrict the following stability analysis to a time-dependent Gaussian probability density $\mathcal{W}$ with vanishing mean defined by Eq. (23) with $\mu(t)\equiv 0$. In this case, we can identify $M_2(t):=\langle x^2\rangle_{\mathcal{R}(x,t)}$ with $\varpi(t)$. From Eq. (26) in combination with the definition for $Q(\cdot)$ [cf. Eq. (29)], we obtain the evolution equation of the second moment $M_2(t)$ according to

$$\frac{d}{dt}M_2(t)=-2(\gamma_0-\alpha)\left\{M_2(t)-\frac{Q_0}{2\gamma_0}\right\}-2\beta\left\{M_2(t)-\frac{Q_0}{2\gamma_0}\right\}^3, \qquad (31)$$

which reveals a pitchfork bifurcation of the variable $q(t)=M_2(t)-Q_0/(2\gamma_0)$ [52–54]. The stationary values become $M_{2,\rm st}^{(a)}=Q_0/(2\gamma_0)$ for $\alpha>0$ and $M_{2,\rm st}^{(b,\pm)}=Q_0/(2\gamma_0)\pm\sqrt{(\alpha-\gamma_0)/\beta}$ for $\alpha>\gamma_0$. A stability analysis based on Eq. (31) reveals that $M_{2,\rm st}^{(a)}$ represents a stable stationary solution (cf. Ref. [51]) for $\alpha<\gamma_0$, and an unstable one for $\alpha>\gamma_0$, whereas $M_{2,\rm st}^{(b,\pm)}$ describe stable stationary probability densities for $\alpha>\gamma_0$. In fact, we obtain identical results from the stability analysis based on Eq. (27) by substituting $Q(\cdot)$ defined by Eq. (29) into Eq. (27), and computing the Lyapunov exponent λ for the stationary solutions $M_{2,\rm st}^{(a)}$ and $M_{2,\rm st}^{(b,\pm)}$. In detail, for $M_{2,\rm st}^{(a)}$ we obtain $\lambda=2(\alpha-\gamma_0)$. For $M_{2,\rm st}^{(b,\pm)}$ we obtain $\lambda=-4(\alpha-\gamma_0)$.

To further examine the impacts of the statistically modulated diffusion coefficients [Eq. (28)] we finally analyze the system numerically. To this end, we rescale Eqs. (1) and (28) by means of $\tau=\gamma_0 t$, $\epsilon=\alpha/\gamma_0$, and $\beta'=\beta/\gamma_0$ and use the random variables $\xi_j'(\tau)=\xi_j(\tau/\gamma_0)$ for which the Langevin equations read

$$\frac{d}{dt}\xi_j'(\tau)=-\xi_j'(\tau)+\left[2C_0+2\epsilon\left(\left\{\frac{1}{N-1}\sum_{k\neq j}\xi_k^2\right\}-C_0\right)-2\beta'\left(\left\{\frac{1}{N-1}\sum_{k\neq j}\xi_k^2\right\}-C_0\right)^3\right]^{1/2}\Gamma(t), \qquad (32)$$

with $C_0:=Q_0/(2\gamma_0)$. Given Eq. (31), the pitchfork bifurcation occurs at the critical value $\epsilon=1$. Note that $Q(s_j)>0$ is always satisfied by the stationary values $M_{2,\rm st}^{(a)}$ and $M_{2,\rm st}^{(b,+)}$. However, the admissible range of ϵ is restricted in the case of $M_{2,\rm st}^{(b,-)}$ in terms of $\epsilon<C_0^2\beta'+1$ (cf. definitions of $M_{2,\rm st}^{(b,-)}$, β', and C_0). Figure 1 shows the bifurcation diagram in the stationary second moment $M_{2,\rm st}$ obtained by simulating Eq. (32) for $C_0=1$ and $\beta'=4$. The simulation of the lower branch ($M_{2,\rm st}^{(b,-)}$) indicates the convergence of the stationary probability density to a δ distribution when ϵ approaches its maximal admissible value $\epsilon_{\max}=5$. Figures 2 and 3 illustrate the stationary one-variable probability densities $\mathcal{W}_{\rm st}(x)$ for different values of ϵ corresponding to the upper branch (Fig. 2) and the lower branch (Fig. 3) of the diagram in Fig. 1.

B. Stationary autocorrelations

In anticipation of the application of mean-field models to human motor control that will be presented in Sec. IV, we now turn our attention to stochastic systems with nonvanishing means, and consider the empirical variance $\mathrm{s}^2_{\rm emp}$ in place of the second moment. In contrast to Sec. III A, we examine the solutions of Eq. (1) for a constant diffusion coefficient (i.e., $Q=Q_0=\text{const}$) but consider a statistically modulated drift term

$$\gamma:=\gamma\{\mathrm{s}^2_{i,\rm emp}(t)\}$$

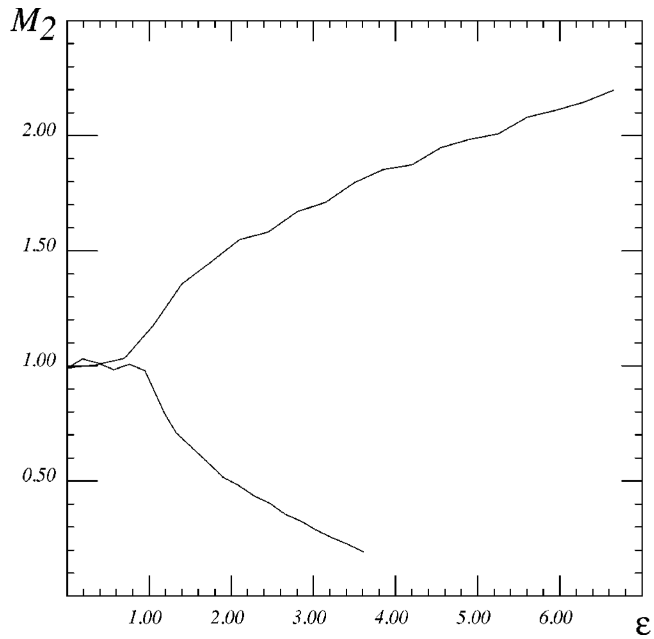

FIG. 1. Bifurcation diagram $M_{2,\mathrm{st}}$ as function of ϵ (stable solutions only). Both curves were computed from the discrete version (cf. Ref. [50]) of the Langevin equation (32): $N=10\,000$, single time step $\Delta t=0.01$, and 3000 iterations for every ϵ. A δ distribution at $x=0$ served as the initial distribution for the $\epsilon=0$ trial. ϵ was increased in steps of 0.35 (upper branch) and 0.19 (lower branch). The final distribution of every ϵ was used as initial distribution for the subsequent trial with increased ϵ. 20 trials were performed. Due to numerical constraints only one bifurcation branch was realized at the bifurcation point. To control the bifurcation of the solutions, at the bifurcation point the initial distribution was broadened by multiplying each representation ξ_j by 1.05 in the upper branch or squeezed by a factor of ξ_j by 0.95 in the lower branch.

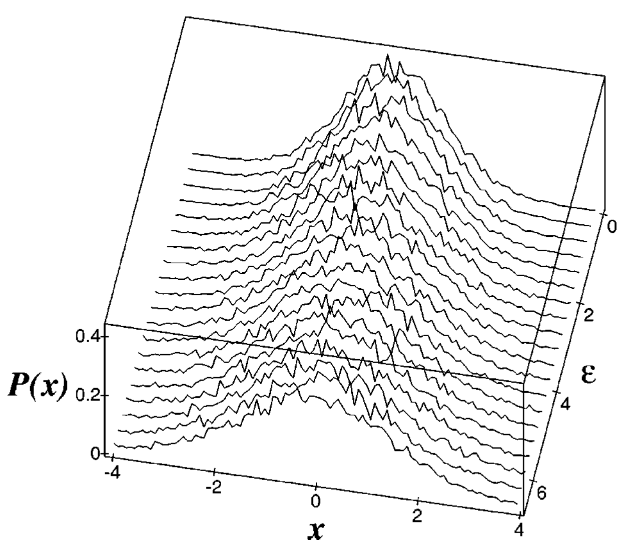

FIG. 2. Stationary stable probability densities corresponding to the upper branch in Fig. 1. Beyond the critical value $\epsilon=1$, the probability density evolves toward the uniform distribution when ϵ is increased.

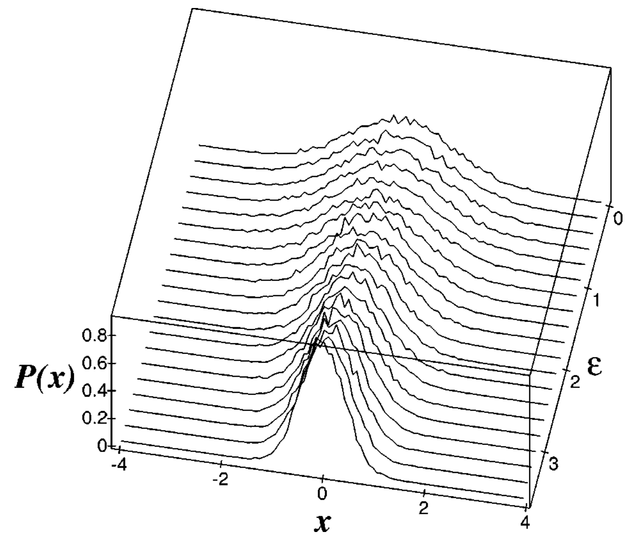

FIG. 3. Stationary stable probability densities corresponding to the lower branch in Fig. 1. Beyond the critical value $\epsilon=1$ the probability density evolves toward a δ distribution for increasing ϵ.

and

$$s_{i,\mathrm{emp}}^2(t) := \frac{1}{N-1}\sum_{k\neq i}\left(\xi_k(t)-\left\{\frac{1}{N-1}\sum_{l\neq i}\xi_l(t)\right\}\right)^2. \tag{33}$$

Comparing Eq. (33) with Eqs. (21) and (22), we find the identities $f_\gamma^{(1)}(z,s_{\gamma,j}^{(2)})=(z-s_{\gamma,j}^{(2)})^2$, $f_\gamma^{(2)}(z)=z$, and $n_d=2$. With respect to the upcoming application, we are interested in the steady-state autocorrelation function η_r of an individual subsystem r defined as

$$\eta_r(\Delta t;N) := \langle \xi_r(t+\Delta t)\xi_r(t)\rangle_{\mathcal{T}_{r,\mathrm{st}}^{(N)}}, \tag{34}$$

that can be computed from the stationary two-point probability density $\mathcal{T}_{r,\mathrm{st}}^{(N)}(y_r,t+\Delta t;x_r,t) := \int\cdots\int \mathcal{P}^{(N)}(\vec{y},t+\Delta t;\vec{x},t)\Pi_{i\neq r,k\neq r}dy_i dx_k$, when Δt denotes the time interval $\Delta t := t'-t \geqslant 0$. The key idea is to approximate $\eta_r(\Delta t;N)$ in terms of the steady-state autocorrelation function of the mean-field approximation η_{MF}; that is, we put

$$\eta_r(\Delta t;N) \approx \eta_{\mathrm{MF}}(\Delta t) := \langle \xi(t+\Delta t)\xi(t)\rangle_{\mathcal{T}_{\mathrm{st}}^{\mathrm{MF}}}. \tag{35}$$

$\mathcal{T}_{\mathrm{st}}^{(\mathrm{MF})}(y,t+\Delta t;x,t)$ is the stationary joint probability density of the NLFPE (22). Generally, the correlation function $\eta_{\mathrm{MF}}(\Delta t)$ can be computed as

$$\eta_{\mathrm{MF}}(\Delta t) = \int\int xy\mathcal{T}^{(\mathrm{MF})}(y,t';x,t)dy\,dx$$

$$= \int x\mathcal{R}(x,t)\underbrace{\left[\int y\mathcal{T}^{(\mathrm{MF})}(y,t'|x,t)dy\right]}_{=\langle \xi(t')\rangle_{\xi(t)=x}}dx. \tag{36}$$

Here $\mathcal{T}^{(\mathrm{MF})}(y,t'|x,t)$ represents the conditional probability density solving the NLFPE (22). Therefore, the inner integral in Eq. (36) can be viewed as the conditional mean of $\xi(t')$ that can be computed from the NLFPE (22) assuming the initial condition $\mathcal{R}(z,t)=\delta(z-x)$. Indeed, $\mathcal{T}^{(\mathrm{MF})}(y,t'|x,t)$ coincides with $\mathcal{W}[y;\varpi(t'),\mu(t')]$ defined by Eqs. (23) and (24) for $\mu(t'\to t)=x$ and $\varpi(t'\to t)=0$. Thus we can interpret $\varpi(t')$ as the conditional variance $\xi^2(t')_{\xi(t)=x}$ of $\mathcal{R}(x,t)$, and $\mu(t')$ as the conditional mean value $\langle\xi(t')\rangle_{\xi(t)=x}$, so that the first equation of Eqs. (24) becomes

$$\frac{d}{dt'}\langle\xi(t')\rangle_{\xi(t)=x}=-\gamma[\mathrm{s}^2(t')_{\xi(t)=x}]\{\langle\xi(t')\rangle_{\xi(t)=x}-m\}$$

$$\Rightarrow\langle\xi(t')\rangle_{\xi(t)=x}$$

$$=m+\{x-m\}\times\exp\left\{-\int_t^{t'}\gamma[\mathrm{s}^2(\tilde{t})_{\xi(t)=x}]d\tilde{t}\right\}. \quad (37)$$

Since we are interested in the stationary case, we replace the probability density in Eq. (36) by $\mathcal{R}(x,t)\to\mathcal{R}_{\mathrm{st}}(x)$. Inserting into Eq. (36) yields

$$\eta_{\mathrm{MF}}(\Delta t)=m\langle\xi\rangle_{\mathcal{R}_{\mathrm{st}}}+(\langle\xi^2\rangle_{\mathcal{R}_{\mathrm{st}}}-m\langle\xi\rangle_{\mathcal{R}_{\mathrm{st}}})\times\exp\left\{-\int_t^{t+\Delta t}\gamma[\mathrm{s}^2(\tilde{t})_{\xi(t)=x}]d\tilde{t}\right\}. \quad (38)$$

From the stationary solution [Eq. (4)] we can further read off that $\langle\xi\rangle_{\mathcal{R}_{st}}=m$ holds, which yields

$$\eta_{\mathrm{MF}}(\Delta t)=m^2+\mathrm{s}_{\mathrm{st}}^2\exp\left\{-\int_t^{t+\Delta t}\gamma[\mathrm{s}^2(\tilde{t})_{\xi(t)=x}]d\tilde{t}\right\}. \quad (39)$$

The conditional variance evolves like [cf. Eq. (24)]

$$\frac{d}{dt'}\mathrm{s}^2(t')_{\xi(t)=x}=-2\gamma[\mathrm{s}^2(t')_{\xi(t)=x}]\mathrm{s}^2(t')_{\xi(t)=x}+Q_0 \quad (40)$$

with the initial condition $\mathrm{s}^2(t)_{\xi(t)=x}=0$. When we require that $\gamma(0)\geqslant 0$ holds and that $\gamma(z)$ is a strictly monotonically increasing function, the structure of Eq. (40) already implies the existence of a stationary value of the variance, and indicates that for $t'\geqslant t$ the argument $z=\mathrm{s}^2(t')_{\xi(t)=x}$ increases strictly monotonically from $\mathrm{s}^2(t)_{\xi(t)=x}=0$ toward this steady state [55], $\mathrm{s}_{\mathrm{st}}^2$ is implicitly given via $\gamma[\mathrm{s}_{\mathrm{st}}^2]\mathrm{s}_{\mathrm{st}}^2=Q_0/2$, and for large $t'-t$ the integral in Eq. (37) converges to $Q_0(t'-t)/(2\mathrm{s}_{\mathrm{st}}^2)$. Consequently, we find that $t'-t\to\infty\Rightarrow\int\gamma(\cdot)d\tilde{t}\to\infty$, leading to $\lim_{t'-t\to\infty}\langle\xi(t')\rangle_{\xi(t)=x}=m$. Then the probability density $\mathcal{R}(x,t')=\mathcal{W}[x;\mathrm{s}^2(t'),\mu(t')]$ converges to a stationary solution.

Apart from these stationarity features, we can further unpack the explicit form $\gamma(z)$. For the sake of convenience, we denote the conditional variance by $\mathrm{s}^2(t)$ rather than by $\mathrm{s}^2(t)_{\xi(t)=x}$. We can express $\gamma(z)$ in terms of the steady-state autocorrelation function η_{MF}. For that purpose, we insert $\phi(t)=\gamma(\mathrm{s}^2\{t\})$ into Eq. (40),

$$\frac{d}{dt}\mathrm{s}^2(t)=-2\beta\phi(t)\mathrm{s}^2(t)+Q_0 \quad (41)$$

and solve Eq. (41) for $\mathrm{s}^2(t')$ with $t'\geqslant t$ and $\mathrm{s}^2(t)=0$. According to our previously derived results, there exists a unique time-dependent solution of Eq. (41) with $\mathrm{s}^2(t)=0$ that reads

$$\mathrm{s}_\phi^2(\Delta t)=Q\exp\left\{-2\int_0^{\Delta t}\phi(t')dt'\right\}\times\int_0^{\Delta t}\exp\left\{2\int_0^{t'}\phi(t'')dt''\right\}dt'. \quad (42)$$

Although this leads to $\phi(z)=\gamma(\mathrm{s}_\phi^2\{z\})$, the function $\gamma(z)$ remains unknown. On account of the monotony of the variance, however, $\mathrm{s}_\phi^2(z)$ is invertible. Let $[\mathrm{s}_\phi^2]^{-1}(\cdot)$ denote the inverse of $\mathrm{s}_\phi^2(\cdot)$. Then we can compute $\gamma(u)$ by means of $u=\mathrm{s}_\phi^2(z)$ and $z=[\mathrm{s}_\phi^2]^{-1}(u)$, which yields

$$\gamma(u)=\phi\{[\mathrm{s}_\phi^2]^{-1}(u)\}. \quad (43)$$

To further elaborate this form, we express $\phi(z)$ in terms of $\eta_{\mathrm{MF}}(z)$ by means of

$$\phi(z)=\left(\frac{1}{\eta_{\mathrm{MF}}(z)-m^2}\right)\frac{d}{dz}\eta_{\mathrm{MF}}(z) \quad (44)$$

cf. Eq. (39). In sum, from Eqs. (42)–(44) one can derive the explicit form of the friction coefficient $\gamma(u)$ for any steady-state autocorrelation function η_{MF}. The requirement that $\gamma(u)$ is a strictly monotonically increasing function, however, restricts η_{MF}. From Eq. (39) we can infer that η_{MF} always decays faster than an exponential function. Only in the trivial case, that is, for $\gamma=$const, does the autocorrelation function exhibit an exponential decay—as is known for conventional Ornstein-Uhlenbeck processes. Finally, we may express $\eta_{\mathrm{MF}}(z)$ by means of $\phi(z)$. Using Eq. (39), we readily obtain

$$\eta_{\mathrm{MF}}(\Delta t)=m^2+\mathrm{s}_{\mathrm{st}}^2\exp\left\{-\int_t^{t+\Delta t}\phi(z)dz\right\}. \quad (45)$$

IV. AN EXPLICIT APPLICATION—POSTURAL SWAY

A. Basic experimental findings

A classical paradigm showing the impact of fluctuations on motor control strategies is erratic motion of the center of pressure (COP) in upright stance. Corresponding stochastic aspects of quiet standing were frequently discussed in the literature [56–59]. In line with these studies, we interpret the random motion of the COP as a steady-state property of the postural control system [60–63]. Accordingly, the COP evolution (or, in general, the postural sway) may be considered

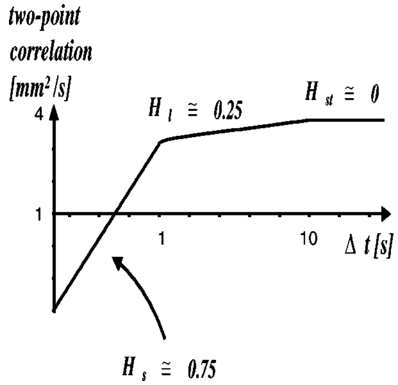

FIG. 4. Log-log plot illustrating the empirical correlation function $C^{T}_{\mathrm{emp}}(\Delta t)$ of the COP displacements as a function of time interval Δt. The graph was drawn from $C^{T}_{\mathrm{emp}}(\Delta t) = D_s(\Delta t)^{2H_s}\theta(1-t) + (D_s + D_l t^{2H_l})\theta(\Delta t - 1)\theta(10-\Delta t) + (D_s + D_l 10^{2H_l})\theta(\Delta t-10)$, where $\theta(x)$ denotes the Heaviside function and $D_s=2.7$, $D_l=0.45$, $H_s=0.73$, and $H_l=0.21$ (Ref. [56], Tables 1 and 2). Three regions can be distinguished as follows: $0s \leqslant \Delta t \leqslant 1s$, scaling faster than linear, $1s \leqslant \Delta t \leqslant 10$ s, scaling sublinear; and $\Delta t \geqslant 10$ s, saturation. The slopes in the different regimes represent the respective scaling exponents H_s, H_l, and H_{satu}; see the text.

as a stationary random walk whose statistical properties can be quantified by use of the two-point displacement function $C(t,t+\Delta t)$, defined as

$$C_{\xi}(t,t+\Delta t) := \langle [\xi(t) - \xi(t+\Delta t)]^2 \rangle. \tag{46}$$

The random variable $\xi(t)$ is the COP, trajectory and $\langle \cdot \rangle$ denotes the ensemble average—note that we always assume the time average and ensemble average to be interchangeable when discussing empirical data [56,57]. Dependent on the time lag Δt, the function C_{ξ} exhibits three qualitatively different regimes. In the short-term regime, covering time intervals Δt from zero to about 1 s, the correlation function scales faster than Δt, that is, $\Delta t \in (0\ \mathrm{s}, 1\ \mathrm{s}]: C_{\xi} \propto (\Delta t)^{2H_s}$ with a characteristic scaling exponent $H_s > 0.5$. In the long-term regime, ranging from about 1 s to about 10 s, the correlation function increases sublinear, that is, $\Delta t \in (1\ \mathrm{s}, 10\ \mathrm{s}], C_{\xi} < A_1 + A_2(\Delta t - A_3)$, where $A_1,\dots,A_3$ s represent positive constants, and $A_3 \approx 1$ s. Alternatively, the experimental results regarding the long-term regime may be written as $C_{\xi} = A_1 + A_2(\Delta t - A_3)^{2H_l}$, with $H_l < 0.5$ for $\Delta t \in (1\ \mathrm{s}, 10\ \mathrm{s}]$. Finally, for time lags longer than 10 s the displacement function typically attains its saturation value by means of $\Delta t > 10\ \mathrm{s}: C_{\xi} \rightarrow \mathrm{const}$ or $C_{\xi} \propto \Delta t^{H_{\mathrm{satu}}}$, with $H_{\mathrm{satu}} \approx 0$. The scaling exponents H_s, H_l, and H_{satu} can be read off from the logarithmic representation of the correlation function C_{ξ} shown in Fig. 4. They represent the slopes of the graph in the three regimes.

In line with the preceding sections, we now interpret the COP random walk as a phenomenon generated by a stochastic mean-field model as described by Eq. (1). To this end, we rewrite the mean squared displacement $C_{\xi}(t,t+\Delta t)$ according to

$$\begin{aligned} C_{\xi}(t,t+\Delta t) &= 2\{\langle \xi^2 \rangle_{\mathrm{st}} - \langle \xi(t)\xi(t+\Delta t)\rangle\} \\ &= 2\{\langle \xi^2 \rangle_{\mathrm{st}} - \eta(t+\Delta t,t)\}. \end{aligned} \tag{47}$$

For the special case discussed in Sec. III B, we immediately obtain

$$C_{\xi}(t,t+\Delta t) = 2\varsigma^2_{\mathrm{st}}\left(1-\exp\left\{-\int_t^{t+\Delta t}\phi(z)\,dz\right\}\right). \tag{48}$$

Inspired by this simple structure, we could, of course, fit the function $\phi(z)$ to the experimental findings. The characteristic kink in the graph of the correlation function C^{T}_{emp} (see Fig. 4) would then correspond to a kink in the function $\phi(z)$ which determines the stochastic mean-field process in question. The origin of this discontinuity in the derivative of $C_{\xi}(\Delta t)$, however, would remain obscure. Put differently, a plain data fit does not really deepen our understanding of the underlying postural control mechanisms. To achieve such an understanding, we have to incorporate recent findings in the study of human motor control, which basically hint at the presence of, at least, two control processes, even in the case of very simple movements (see, e.g., Refs. [64–66]). Such findings also reflect an earlier suggestion of Collins and De Luca to the effect that quiet standing is characterized by the presence of two different processes [56]. Recently, Dijkstra attempted to identify these two processes with the stabilization of a set point and the dynamics of the set-point itself [67].

B. Stochastic VITE model

To specify an explicit model structure we adopt the so-called vector-integration-to-end-point (VITE) model to describe global movements of the body during quiet standing and, in particular, the observed erratic motion of the COP. The VITE model was originally proposed by Bullock and Grossberg to explain the emergence of typical properties of reaching such as a speed-accuracy trade-off and a bell-shaped velocity profile (e.g., Refs. [39,68]). Its central elements are three-dimensional vectors denoted as $\vec{D}$ and $\vec{V}$ and the scalar g. The vector $\vec{D}$ is called the difference vector, and is a measure of the distance between the limb position and the target position, and $\vec{V}$ assumes time-averaged weighted values of $\vec{D}$, and is referred to as the averaged difference vector. The scalar g represents a time-dependent gain signal (comparable to Bullock and Grossberg's GO signal) which controls the rate of change of the vectors $\vec{D}$ and $\vec{V}$. The scalar g does not depend on $\vec{D}$ and $\vec{V}$. We restrict ourselves to an investigation of a random walk in one dimension, and consider the components D and V related to either the mediolateral or the anteroposterior direction. Furthermore, we collapse the neural motor control units related to agonist and antagonist muscles into a single control unit, while being fully aware that a more sophisticated model should reflect on the brain level the reciprocal organization of the muscular level (cf. Ref. [68]). In line with this simplification, D and V can take positive and negative values: positive D values would, for instance, correspond to the activity of agonist muscles, and negative D values to the activity of antagonist muscles. Following Bullock and Grossberg, we study three kinds of interacting neural populations [68]. In particular, we study the set of deterministic evolution equations

$$\frac{d}{dt}V=a(V-D), \tag{49}$$

$$\frac{d}{dt}g=-b(g-m), \tag{50}$$

$$\frac{d}{dt}D=-cD-G_0 gV, \tag{51}$$

in which the variables a, b, c, m, and G_0 are positive constants (see below). At movement onset t_{start}, the difference D is initialized by the distance between the limb and target position. Furthermore, we assume $g(t_{\mathrm{start}})=0$ and $V(t_{\mathrm{start}})=0$ (cf. Ref. [68] for alternative scenarios). D is viewed as an index of the efferent output signal of a particular motor control unit, and serves two purposes: first, so-called efferent copies of D are weighted and integrated, giving rise to the averaged difference vector V [cf. Eq. (49)]; second, D signals lead to motor commands, and thus give rise to limb movements. By means of a time-varying gain signal g and a time-independent amplitude G_0 the averaged difference V is amplified, and impinges on the neural D population of the motor control unit according to Eq. (51). The gain g is determined by the autonomous differential equation [cf. Eq. (50)] and increases monotonically toward the stationary value m. The system of equations (49)–(51) has a unique and globally stable fixed point at $V_{\mathrm{st}}=0$, $g_{\mathrm{st}}=m$, and $D_{\mathrm{st}}=0$—note that $D(t)=0$ reflects the case in which limb and target position coincide. Three subtleties are worth mentioning. First, according to the VITE model, efferent (output) signals that are sufficient to execute successfully goal-directed movements can be produced solely on a neural level irrespective of any afferent information provided by the sensorymotor system. This is, in fact, in agreement with various experimental findings showing that goal-directed movements can be performed in the absence of afferent signals. Second, for the sake of convenience, in the place of the so-called present position vector of the original VITE model of Bullock and Grossberg, we use the variable $D(t)$. Both variables agree except for a shift by the target position. Finally, the model given by Eqs. (49)–(51) provides a consistent integrate-to-end-point model because both V and D can be conceived of as quadratures involving exponential memory by means of $V(t)=\int^t \exp\{-a[t-\tau]\}D(\tau)d\tau$ and $D(t)=G_0\int^t \exp\{-c[t-\tau]\}g(\tau)V(\tau)d\tau$, respectively. Regarding the latter, we note that Bullock and Grossberg studied the case of vanishing c [68], whereas we will always consider c values that are large compared to the parameter a which determines the time scale of V. Recall that D signals are conveyed to muscles leading to limb movements and, hence, $D=0$ corresponds to the absence of a movement. Here, however, we deal with postural control rather than the control of individual limb movements. Thus we interpret the evolution of D as measure for global body movement, and identify the random walk of $D(t)$ with the COP trajectory.

However, a system given by Eqs. (49)–(51) is mute to fundamental properties of postural and motor control systems. Taking a neurophysiological point of view, neural control systems putatively consist of many interacting subsystems, which act together in generating a motor command. Evidence for such a collective behavior was gathered in several studies that focused on intracranial brain activity and encephalographic signals related to motor performance; see, e.g., Refs. [69–73]. Therefore, we study an ensemble of mutually coupled neural control systems rather than a single postural or motor control system. Here we implement this idea by investigating mean-field coupled systems. Moreover the VITE model given by Eqs. (49)–(51) discards the relevance of informational variables. Postural control and motor performance, however, may evaluate temporal and positional information such as the expected execution time or performance accuracy. In fact, such information can be gained via statistical quantities. For example, process execution time may be related to the entropy of neural motor command signals (Fitts law [74]) or accuracy may be measured by means of the output variance of neural control units. Statistical quantities can be computed from ensembles of similar subsystems, and may assume the form of mean-field variables. In sum, from a neurophysiological point of view as well as from a phenomenological point of view it seems plausible to incorporate mean-field variables in postural and motor control systems.

In line with the preceding observation, we extend the system of equations (49)–(51) in terms of an interaction through a mean field. In addition, we take stochastic forces into account. In detail, we study a stochastic system defined by

$$\frac{d}{dt}\xi_V=-(\xi_V-\xi_D)+\Gamma_V, \tag{52}$$

$$\frac{d}{dt}\xi_{g,j}=-\gamma(\varsigma^2_{j,\mathrm{emp}})(\xi_{g,j}-m)+\Gamma_{g,j}, \tag{53}$$

$$\frac{d}{dt}\xi_D=-c\xi_D-G_0\xi_g\xi_V, \tag{54}$$

where Γ_V and $\Gamma_{g,j}$ are statistically independent Langevin forces, and only Eq. (53) is regarded as a particular subsystem j of a population of subsystems that are described by Eqs. (2) and (33). For the sake of simplicity, we neglect explicit fluctuations acting on the difference vector and confine ourselves to a mean-field interaction for the gain signal g because this population might be particularly sensitive to perceptual and informational influences [37]. In this case we can interpret the empirical variance $\varsigma^2_{j,\mathrm{emp}}$ as an accuracy measure or as a measure of temporal aspects of postural control [75]. Alternatively, $\varsigma^2_{j,\mathrm{emp}}$ may account for interactions between subsystems of the g population which scale with the square of the subsystems' states. Since $\gamma(\varsigma^2_{j,\mathrm{emp}})$ does not depend on $\xi_{g,j}$, a linear stability analysis of Eqs. (52)–(54) can be carried out by conventional techniques; see e.g., Refs. [52–54]. In detail, we linearize the system in the vicinity of its fixed point, yielding a set of eigenvalues

$$\lambda^{(\pm)} := -\frac{1+c}{2} \pm Z \quad \text{with} \quad Z := \sqrt{\left[\left(\frac{1-c}{2}\right)^2 - G_0 m\right]^{1/2}}, \tag{55}$$

and the eigenvectors $\vec{\omega}^{(\pm)}$ and dual vectors $\vec{\Omega}^{(\pm)}$;

$$\vec{\omega}^{(\pm)} = \begin{pmatrix} \omega_1^{(\pm)} \\ \omega_2^{(\pm)} \end{pmatrix} := \begin{pmatrix} 1 \\ (1-c)/2 \pm Z \end{pmatrix},$$
$$\vec{\Omega}^{(\pm)} = \begin{pmatrix} \Omega_1^{(\pm)} \\ \Omega_2^{(\pm)} \end{pmatrix} := \frac{1}{2Z} \begin{pmatrix} Z \mp (1-c)/2 \\ \pm 1 \end{pmatrix}, \tag{56}$$

where $\vec{\omega}^{(+)}\vec{\Omega}^{(+)}=1$, $\vec{\omega}^{(+)}\vec{\Omega}^{(-)}=0$, $\vec{\omega}^{(-)}\vec{\Omega}^{(-)}=1$, and $\vec{\omega}^{(-)}\vec{\Omega}^{(+)}=0$. Subsequently, we transform the system of equations (52)–(54) to

$$\frac{d}{dt}\tilde{\xi}_V(t) = \lambda^{(+)}\tilde{\xi}_V(t) + \vec{\Omega}^{(+)} \times \begin{pmatrix} \Gamma_V \\ -G_0\tilde{\xi}_g(t)[\omega_1^{(+)}\tilde{\xi}_V(t) + \omega_1^{(-)}\tilde{\xi}_D(t)] \end{pmatrix}, \tag{57}$$

$$\frac{d}{dt}\tilde{\xi}_g(t) = -\gamma(\cdot)\tilde{\xi}_g(t) + \Gamma_g, \tag{58}$$

$$\frac{d}{dt}\tilde{\xi}_D(t) = \lambda^{(-)}\tilde{\xi}_D(t) + \vec{\Omega}^{(-)} \times \begin{pmatrix} \Gamma_V \\ -G_0\tilde{\xi}_g(t)[\omega_1^{(+)}\tilde{\xi}_V(t) + \omega_1^{(-)}\tilde{\xi}_D(t)] \end{pmatrix}, \tag{59}$$

where the variables $\tilde{\xi}_V$, $\tilde{\xi}_d$, and $\tilde{\xi}_g$ are defined as

$$\tilde{\xi}_V(t) := \vec{\Omega}^{(+)} \begin{pmatrix} \xi_v(t) \\ \xi_D(t) \end{pmatrix}, \quad \tilde{\xi}_D(t) := \vec{\Omega}^{(-)} \begin{pmatrix} \tau_v(t) \\ \xi_D(t) \end{pmatrix},$$
$$\tilde{\xi}_g(t) := \xi_{g,j}(t) - m. \tag{60}$$

To gain further insight into the model properties, we now reduce the number of independent variables. We look for cases in which a distinction can be made between order parameters and enslaved variables [53,76]. It has been argued that neural systems typically allow for such a partitioning (Ref. [77], Sec. 20.1). Accordingly, we consider the aforementioned case in which c is much larger than 1 (note that we rescaled time to eliminate a). In addition, we assume that the product $G_0 m$ is bounded, so that for $c \gg 1$ the conditions $4G_0 m < (1-c)^2$, $|\lambda^{(-)}| \gg |\lambda^{(+)}|$, and $|\lambda^{(-)}| \gg 1$ hold. Then both eigenvalues are real and negative, and we can identify $\tilde{\xi}_V(t)$ and $\tilde{\xi}_g(t)$ as the order parameters, while $\tilde{\xi}_D(t)$ can be viewed as the enslaved mode. Accordingly, adiabatic elimination yields $\tilde{\xi}_D(t)$ as a function of $\tilde{\xi}_V(t)$, $\tilde{\xi}_g(t)$, and of the time-averaged noise source $\int^t \exp\{-|\lambda^{(-)}|(t-\tau)\}\Gamma_V(\tau)d\tau$. Neglecting this noise source and using the inverse transformation $\xi_D(t) = \omega_2^{+}\tilde{\xi}_V(t) + \omega_2^{(-)}\tilde{\xi}_D(t)$, we can explicitly express $\xi_D(t)$ in terms of $\tilde{\xi}_V(t)$ and $\tilde{\xi}_g$ as

$$\xi_D(t) = \tilde{\xi}_V(t)\left(\omega_2^{(+)} + \frac{\omega_2^{(-)}\omega_1^{(+)}\Omega_2^{(-)}G_0}{\lambda^{(-)}}\tilde{\xi}_g(t)\right). \tag{61}$$

To cast the evolution equations for $\tilde{\xi}_V(t)$, $\tilde{\xi}_g$, and $\tilde{\xi}_D(t)$ [cf. Eqs. (57), (58), and (61)] into forms similar to Eqs. (52)–(54), we define the constants $\tilde{c} := (\omega_2^{(-)}\omega_1^{(+)}\Omega_2^{(-)}G_0)/\lambda^{(-)} > 0$ and $\tilde{m} := \omega_2^{(+)}/\tilde{c}$. Further, we introduce a shifted gain signal $\xi_{g'}(t) := \tilde{\xi}_g(t) + \tilde{m} = \xi_g(t) + (\tilde{m} - m)$, approximate Eq. (57) by its linear parts, and ignore any multiplicative noise sources. In sum, we obtain

$$\frac{d}{dt}\tilde{\xi}_V(t) = -|\lambda^{(+)}|\tilde{\xi}_V(t) + \Omega_1^{(+)}\Gamma_V, \tag{62}$$

$$\frac{d}{dt}\xi_{g'}(t) = -\gamma(s^2_{\text{emp.}})[\xi_{g'}(t) - \tilde{m}] + \Gamma_g, \tag{63}$$

$$\xi_D(t) = -\tilde{c}\tilde{\xi}_V(t)\xi_{g'}(t). \tag{64}$$

Given Eq. (64) the stationary correlation function C_D of the random variable ξ_D consists of autocorrelation functions related to both $\tilde{\xi}_V(t)$ and $\xi_{g'}(t)$. In line with the arguments advanced in Sec. III, we therefore approximate the autocorrelation function of $\xi_{g'}(t)$ by the autocorrelation function of the mean-field NLFPE [cf. Eq. (35)]. Thus we obtain

$$C_D(\Delta t) \approx 2\tilde{c}^2\{\langle\tilde{\xi}_V^2\rangle_{\text{st}}\langle\xi_{g'}^2\rangle_{\text{st}} - \eta_{\tilde{V}}(\Delta t)\,\eta_{\text{MF};g'}(\Delta t)\}, \tag{65}$$

where $\eta_{\tilde{V}}$ is the well-known stationary autocorrelation function of an Ornstein-Uhlenbeck process with vanishing mean, that is,

$$\xi_{\tilde{V}}(\Delta t) = \langle\tilde{\xi}_V^2\rangle_{\text{st}}\exp\{-|\lambda^{(+)}|\Delta t\}. \tag{66}$$

By means of Eqs. (45), (65), and (66) we can now explain the characteristic features of the mean squared displacement $C(\Delta t)$ of the COP motion (cf. Fig. 4).

The correlation function $C(\Delta t)$ exhibits two different regimes because of the two different time scales of the V and g dynamics that are already present in the conventional VITE model; cf. Eqs. (49)–(51). Moreover, the correlation function $C(\Delta t)$ scales faster than linearity in the short-term regime due to the mean field affecting the g-dynamics of the extended version; cf. Eqs. (52)–(54). In short, the interplay of the mean-field coupling and the VITE model allows for an interesting interpretation of postural data. In detail, we assume that in the short-term regime (i.e., $\Delta t \in (0s, 1s]$) the autocorrelation function $\eta_{\tilde{V}}$ varies slowly, whereas $\eta_{MF;g'}(\Delta t)$ decays rapidly to its saturation value $\tilde{m}^2$. Choosing $\phi(z)$ appropriately, we can model the characteristic scaling behavior $C(\Delta t) \propto (\Delta t)^{H_s}$, with $H_s > 0.5$. For example, introducing a characteristic time scale $\tau_{g'}$ and utilizing $\phi(z) = (2\nu z^{2\nu-1})/\tau_{g'}$ for $|\lambda^{(+)}|\Delta t \approx 0$, Eq. (65) can be approximated by

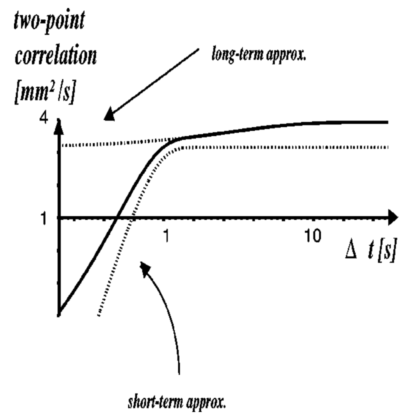

FIG. 5. Illustration of the correlation function C_D of the multiplicative compound process and its two constituents that are related to a ''fast'' Ornstein-Uhlenbeck process with mean field coupling and to a ''slow'' and ordinary Ornstein-Uhlenbeck process. The graph of C_D (solid line) was computed from Eq. (65). The graph of the short-term approximation (lower dashed line) was computed from Eq. (67). The graph of the long-term approximation (upper dashed line) varies only slightly in the short-term regime ($\Delta t < 1$ s), and merges with the graph of C_D in the long-term regime ($\Delta t > 1$ s). The long-term approximation was calculated from Eq. (68), and reflects the contribution of the ''slow'' ordinary Ornstein-Uhlenbeck process to the behavior of C_D. $\nu=3$, $|\lambda^{(+)}|^{-1}=0.5$, $\tau_{g'}=3.3$, $\langle \xi_{g'}^2\rangle=3.85$, $\tilde{m}^2=1.1$, and $\tilde{c}^2\langle\tilde{\xi}_V^2\rangle=0.5$.

$$C_D(\Delta t)\approx 2\tilde{c}^2\langle\tilde{\xi}_V^2\rangle_{\rm st} s_{{\rm st},g'}^2\left(1-\exp\left\{-\frac{(\Delta t)^{2\nu}}{\tau_{g'}}\right\}\right) \overset{(\Delta t)^{2\nu}\ll\tau_{g'}}{\Rightarrow} C_D(\Delta t) \propto (\Delta t)^{2\nu}. \tag{67}$$

After this rapid saturation of $\eta_{{\rm MF};g'}(\Delta t)$, there might still be some significant exponential decay of the autocorrelation function $\eta_{\tilde{\xi}_V}$. In this case, the mean squared displacement decreases sublinearly (concavity of exponential functions) in the long-term regime, and we can approximate $C_D(\Delta t)$ by

$$C_D(\Delta t)\approx 2\tilde{c}^2\langle\tilde{\xi}_V^2\rangle_{\rm st}(\langle\xi_{g'}^2\rangle_{\rm st}-\tilde{m}^2\exp\{-|\lambda^{(+)}|\Delta t\}). \tag{68}$$

The qualitative reproduction of the aforementioned experimental findings can be readily achieved by adjusting the free parameters: $\tilde{c}^2\langle\tilde{\xi}_V^2\rangle_{\rm st}$, $\langle\xi_{g'}^2\rangle$, $\tilde{m}$, $\tau_{g'}$, and $\lambda^{(+)}$. Figure 5 shows a rough fit of $C_D(\Delta t)$ defined by Eq. (65) to the empirical correlation function $C_{\rm emp}^T(\Delta t)$ displayed in Fig. 4. In Fig. 5 a more general case is shown, in which the time scales of the autocorrelation function $\eta_{\tilde{V}}$ and $\eta_{{\rm MF};g'}$ are not markedly different. There, $\tau_{g'}<1/|\lambda^{(+)}|$ holds rather than $\tau_{g'}\ll 1/|\lambda^{(+)}|$, so that the random process ξ_ϑ also affects the short-term regime. This may result in a scaling exponent ν that is significantly larger than the one of the short-term regime H_s (cf. Fig. 5). The main features, however, of the mean squared displacement [Eq. (65)], as discussed under the condition $\tau_{g'}\ll 1/|\lambda^{(+)}|$, can also be observed for the weaker condition $\tau_{g'}<1/|\lambda^{(+)}|$.

V. CONCLUSIONS

In the present paper we studied interactions between subsystems that depend nonlinearly on mean fields which, in turn, are generated by these subsystems. In line with conventional mean-field approaches, we derived mean-field NLFPE's which can approximate the overall stochastic behavior of such systems. We illustrated our results by two examples: a formal dynamical system showing a pitchfork bifurcation in the second moment, and a neurophysiologically motivated model for postural control.

In closing, we would like to highlight two specific but theoretically important features of the stochastic concept presented in this paper by comparing it with the theoretical approach to the problem of quiet standing adopted by Chow and co-workers: the pinned polymer model [61,62]. This model is based on the assumption that postural control is achieved in a spatially distributed physical body (such as the human body) whose degrees of freedom are restricted to two dimensions due to a pointlike rigid connection to the environment (pinning). In contrast, the description of a COP random walk via zero-dimensional stochastic processes does not require the physical body under consideration to be vastly extended in space. Consequently, the present paper raises the question of whether correlated random walks with the texture of fractional Brownian motions and stationary two-point correlation functions displaying kinks can also be found, for example, during postural control of limbs with comparatively small masses and extensions such as fingers or the human hand (hand tremor). In addition, the above-mentioned features of postural sway may further be found, for instance, in the stationary performance of rhythmic movements because—as the analyses in this paper clearly suggest—they do not necessitate a rigid link of the limb with the environment. In contrast, our main finding was that two-point correlation functions that resemble correlation functions of fractional Brownian walks, and are interspersed by kinks, can be induced by cooperative effects of distinct neural motor control units being subjected to noise and composed of a large number of mean-field coupled subsystems. This result can provide a sound basis for future experimental investigations designed to study stochastic phenomena of this kind. The second important issue that we want to emphasize here concerns the applicability of the concept of stochastic processes defined by mean-field nonlinear Fokker-Planck equations. Unlike the pinned polymer model, the concepts of stochastic processes described by mean-field nonlinear Fokker-Planck equations can be applied to various deterministic motor control theories, provided that they are formulated in terms of ordinary differential equations. The latter have to be replaced by a set of identical ordinary differential equations with additional (white) noise forces and appropriately defined mean-field couplings. Such applications, however, should not be viewed as mere supplements of deterministic theories aiming at a stochastic description of phenomena that can already be explained by deterministic models. On the contrary, the concepts of stochastic processes determined by nonlinear Fokker-Planck equations predict and explain phenomena that can hardly be handled by traditional stochastic theories, and that are impossible to handle by deterministic models.

APPENDIX A: SECOND DERIVATION OF THE MEAN-FIELD NLFPE (11)

The crucial step in the derivation of the mean-field NLFPE (11) is to show that $\mathcal{Y}[\mathcal{M}_r]$, as defined by Eq. (14), can be approximated by Eq. (19), provided that for $N \gg 1$ the joint probability density $\mathcal{M}_r$ factorizes into $N-1$ copies of a one-variable limiting case probability density $\mathcal{R}$. Now, we dispense with condition (12), and assume that mean and variance of $\mathcal{R}(x,t)$ are finite. If the transformed probability density $\mathcal{R}'(x',t)$, defined by $\mathcal{R}'(x',t)dx' := \mathcal{R}(x,t)dx$ with $x' = f_\gamma(x)$, satisfies the Lindeberg condition [78,79], then, as shown below, we find that

$$\mathcal{Y}[\mathcal{M}_r] = \tilde{g}\left(N, K_1 := \langle f_\gamma(x) \rangle_{\mathcal{R}(x,t)}, K_2 := \int_{-\infty}^{\infty} [f_\gamma(x) - K_1]^2 \mathcal{R}(x,t)dx \right) + O\left(\frac{1}{\sqrt{N}} \right), \tag{A1}$$

$$\tilde{g}(N,K_1,K_2) := \sqrt{\left(\frac{N}{2\pi K_2} \right)^{1/2}} \int_{-\infty}^{\infty} \gamma(z) \exp\left\{ \frac{N(z-K_1)^2}{2\pi K_2} \right\} dz = \int_{-\infty}^{\infty} \gamma(z) \mathcal{G}(z; K_1, K_2/N) dz,$$

where the Gaussian probability density $\mathcal{G}(z;K_1,K_2/N)$ has mean K_1 and variance K_2/N, and tends to a δ distribution for $N \gg 1$; that is, we have $\lim_{N\to\infty} \tilde{g}(N,K_1,K_2) = \gamma(K_1)$. Inserting this limit into Eq. (A1) yields

$$\mathcal{Y}[\mathcal{M}_r] = \gamma(\langle f_\gamma(x) \rangle_{\mathcal{R}(x,t)}), \tag{A2}$$

which coincides with Eq. (19). However, by inserting Eq. (A1) instead of Eq. (A2) into Eq. (20), we obtain a mean-field NLFPE which can be considered as a higher order approximation. According to Eq. (A1) the error of these estimate is of the order $1/\sqrt{N}$, whereas for Eq. (A2) an additional error $e(N) = |\gamma(K_1) - \int \gamma(z) \mathcal{G}(z;K_1,K_2/N)dz|$ occurs. For very large N, however, both the term proportional to $1/\sqrt{N}$ and the term $e(N)$ become arbitrarily small and negligible. Note that, by analogy, we can derive a result similar to Eq. (A1) for the diffusion term of the NLFPE (20).

To outline the derivation of Eq. (A1), we first rewrite the integral $\mathcal{Y}[\mathcal{M}_r]$ in Eq. (14) in terms of $\mathcal{R}(x_r,t)$ as

$$\mathcal{Y}[\mathcal{M}_r] := \int \cdots \int \gamma\left(\frac{1}{N-1} \sum_{k \neq r} f_\gamma\{x_k\} \right) \prod_{l \neq r} \mathcal{R}(x_l,t) dx_l. \tag{A3}$$

We can express the probability $\mathcal{D}(\Phi,t)d\Phi$ that we find the mean field $s_{\gamma,r} = \Sigma_{k\neq r} f_\gamma(\xi_k)/(N-1)$ [cf. Eq. (1)], in $[\Phi, \Phi + d\Phi]$ by

$$\mathcal{D}(\Phi,t) := \int \cdots \int \delta\left(\Phi - \frac{1}{N-1} \sum_{k \neq r} f_\gamma\{x_k\} \right) \times \prod_{l \neq r} \mathcal{R}(x_l,t) dx_l. \tag{A4}$$

Then Eq. (A3) can be written

$$\mathcal{Y}[\mathcal{M}_r] = \int_{-\infty}^{\infty} \gamma(\Phi) \mathcal{D}(\Phi,t) d\Phi. \tag{A5}$$

$\mathcal{D}(\Phi,t)$ can be determined by use of $\mathcal{R}'(z',t)$ defined on $A' := \{z' : z' = f_\gamma(z) \wedge z \in (-\infty,\infty)\}$ according to

$$\mathcal{D}(\Phi,t) = \int_{A'} \cdots \int_{A'} \delta\left(\Phi - \frac{1}{N-1} \sum_{k \neq r} x'_k \right) \prod_{l \neq r} \mathcal{R}'(x'_l,t) dx'_l. \tag{A6}$$

If $\overline{x}(t)$ and $\overline{x'}(t)$ $[\sigma^2(t)$ and $\sigma'^2(t)]$ denote the means (variances) of $\mathcal{R}(x,t)$ and $\mathcal{R}'(x',t)$, then we find

$$\overline{x'}(t) := \int_{A'} z' \mathcal{R}'(z',t) dz' = \int_{-\infty}^{\infty} f_\gamma(z) \mathcal{R}(z,t) dz, \tag{A7}$$

$$\sigma'^2(t) := \int_{A'} [z - \overline{x'}]^2 \mathcal{R}'(z',t) dz' = \int_{-\infty}^{\infty} [f_\gamma(z) - \overline{x'}]^2 \mathcal{R}(z,t) dz.$$

We assume that $\overline{x}(t)$ and $\sigma^2(t)$ are finite, which implies for $k_\gamma > 1$ that $\overline{x'}(t)$ and $\sigma'^2(t)$ are also finite [for $k_\gamma \in (0,1]$ we have to guarantee that $\overline{x'}(t)$ and $\sigma'^2(t)$ are finite]. When we use the parameter-dependent scaling $z''(N) = z'/(N-1)$, so that the corresponding variance, mean value, and probability density read $\sigma''^2(t,N) := \sigma'^2(t)/(N-1)^2$, $\overline{x''}(t,N) := \overline{x'}(t)/(N-1)$, and $\mathcal{R}''(z'',t,N)dz''(N) := \mathcal{R}'(z',t)dz'$, then, Eq. (A6) becomes

$$\mathcal{D}(\Phi,t) = \int \cdots \int \delta\left(\Phi - \sum_{k \neq r} x''_k \right) \prod_{l \neq r} \mathcal{R}''(x''_l,t) dx''_l. \tag{A8}$$

Now $\mathcal{D}(\Phi,t)$ is the probability density of a random variable which can be computed from the sum of $N-1$ random variables. As stated earlier, we require that $\mathcal{R}'(z',t)$ obeys the Lindeberg condition [78,79], that is, $\forall \lambda > 0: \lim_{N\to\infty} \int_{B'} [z']^2 \mathcal{R}'[z' - \overline{x'}(t),t] dz' = 0$, where the set B' is defined by $B' := A'/\{z' : |z'| < \lambda \sqrt{N}\}$. Note that the Lindeberg condition states that the asymptotic parts of probability densities (i.e., the tails) are negligible. Since, by assumption, the Lindeberg condition is satisfied for $\mathcal{R}'(z',t)$, the Lindeberg condition is also satisfied for $\mathcal{R}''(z'',t)$ [80]. Consequently, the central limit theorem [2,78,79,81] applies to Eq. (A8), and for $N \gg 1$ we obtain

$$\mathcal{D}(\Phi,t)=\sqrt{\frac{1}{2\pi N\sigma''^2(t,N)}}\exp\left\{-\frac{(\Phi-N\overline{x}''(t,N))^2}{2\pi N\sigma''^2(t,N)}\right\}+O\left(\frac{1}{\sqrt{N}}\right) \tag{A9}$$

$$=\underbrace{\sqrt{\frac{N}{2\pi\sigma'^2(t)}}\exp\left\{-\frac{N(\Phi-\bar{x}'(t))^2}{2\pi\sigma'^2(t)}\right\}}_{\mathcal{G}(\Phi;\bar{x}'(t),\sigma'^2(t)/N}+O\left(\frac{1}{\sqrt{N}}\right). \tag{A10}$$

This result, in combination with Eqs. (A5) and (A7), corresponds to proposition (A1). To estimate the error in Eq. (A9), we can derive Eq. (A9) via Taylor expansion of probability operators (Ref. [78], Chap. 8), and find that the range in which the Taylor expansion is evaluated scales with $1/\sqrt{N}$ (i.e., ϵ defined in Ref. [78] is proportional to $1/\sqrt{N}$). Hence $\forall X:\|\int\chi(X-z)[\mathcal{D}(z,t)-\mathcal{G}(z,t)]dz\|\propto O(1/\sqrt{N})$ (theorem 1 in Ref. [78], Sec. 8.4), where $\chi(z)\in\mathcal{C}[-\infty,\infty]$ denotes a test function and $\|\cdot\|$ corresponds to the supremum norm. Consequently, with $X=0$ and $\chi(z)=\underline{\gamma}(-z)$ we find that $\|\int\gamma(z)[\mathcal{D}(z,t)-\mathcal{G}(z,t)]dz\|\propto O(1/\sqrt{N})$, which is consistent with Eq. (A10). For an alternative derivation of the $O(1/\sqrt{N})$ term, we refer to Ref. [81], Chap. 1.

APPENDIX B: ASYMPTOTIC SOLUTIONS—*H* THEOREM

We show that any probability density $\mathcal{R}(x,t_0)$ given at the initial time t_0 converges in the limit $t-t_0\to\infty$ to a corresponding Gaussian probability density $\mathcal{W}[x,\varpi(t),\mu(t)]$ as defined by Eqs. (23) and (24). To prove this assertion, let us consider an arbitrary probability density $\mathcal{R}(x,t)$ solving Eq. (22), and a function $\mathcal{U}(x,t)$ which satisfies

$$\frac{\partial}{\partial t}\mathcal{U}(x,t)=\frac{\partial}{\partial x}\gamma\{{}^{\mathcal{R}}E_{\gamma}^{(1)}(t)\}x\mathcal{U}(xt)+\frac{1}{2}Q\{{}^{\mathcal{R}}E_{Q}^{(1)}(t)\}\frac{\partial^2}{\partial x^2}\mathcal{U}(x,t). \tag{B1}$$

Using the ansatz

$$\mathcal{U}(x,t)=\frac{1}{\sqrt{2\pi\varpi(t)}}\exp\left\{-\frac{[x-\mu(t)]^2}{2\varpi(t)}\right\} \tag{B2}$$

leads, in analogy to Eqs. (23) and (24), to the set of ordinary differential equations

$$\frac{d}{dt}\mu(t)-\gamma\{{}^{\mathcal{R}}E_{\gamma}^{(1)}(t)\}[\mu(t)-m],$$

$$\frac{d}{dt}\varpi(t)=-2\gamma\{{}^{\mathcal{R}}E_{\gamma}^{(1)}(t)\}\varpi(t)+Q\{{}^{\mathcal{R}}E_{Q}^{(1)}(t)\}, \tag{B3}$$

$$\mu(t_0)=\langle x\rangle_{\mathcal{R}(x,t_0)},\quad \varpi(t_0)=\langle[x-\mu(t_0)]^2\rangle_{\mathcal{R}(x,t_0)}.$$

Note that on account of Eq. (B2), the function $\mathcal{U}$ is normalized, and can be seen as a probability density. As indicated by the preceding upper index $\mathcal{R}$, the expectation values ${}^{\mathcal{R}}E_{\gamma}^{(n)}(t)$ and ${}^{\mathcal{R}}E_{Q}^{(n)}(t)$ for $n=1,\dots,n_d$ in Eqs. (B1)–(B3) are computed from $\mathcal{R}(x,t)$ rather than $\mathcal{U}(x,t)$. Hence $\mathcal{U}(x,t)$ does not coincide with $W[x;a(t),\mu(t)]$, and Eq. (B1) remains linear with respect to $\mathcal{U}$ but contains the time-dependent drift and diffusion coefficients $\tilde{\gamma}(t):=\gamma\{{}^{\mathcal{R}}E_{\gamma}^{(1)}(t)\}$ and $\tilde{Q}(t):=Q\{{}^{\mathcal{R}}E_{Q}^{(1)}(t)\}$. We can therefore adopt the H theorem of the theory of linear Fokker-Planck equations by introducing the functional

$$H(t):=\int_{-\infty}^{\infty}\mathcal{R}(x,t)\ln\left[\frac{\mathcal{R}(x,t)}{\mathcal{U}(x,t)}\right]dx\geqslant 0. \tag{B4}$$

Since the Fokker-Planck equations for $\mathcal{R}$ and $\mathcal{U}$ have common drift and diffusion coefficients $\tilde{\gamma}(t)$ and $\tilde{Q}(t)$, we can directly calculate the derivative of H, and obtain

$$\frac{d}{dt}H(t)=-\tilde{Q}(t)\int_{-\infty}^{\infty}\mathcal{R}(x,t)\left\{\frac{\partial}{\partial x}\ln\left[\frac{\mathcal{R}(x,t)}{\mathcal{U}(x,t)}\right]\right\}dx\leqslant 0. \tag{B5}$$

From Eqs. (B4) and (B5), it follows that in the limit $t-t_0\to\infty$ the derivative of H vanishes. Given the positivity of the fluctuation strength $\tilde{Q}$, $\lim_{t-t_0\to\infty}dH(t)/dt=0$ implies

$$\lim_{t-t_0\to\infty}[\mathcal{R}(x,t)-\mathcal{U}(x,t)]=0. \tag{B6}$$

For further details the reader is referred to Ref. [50]. In the limit $t-t_0\to\infty$, we can now replace the expectation values ${}^{\mathcal{R}}E_{\gamma}^{(n)}(t)$ and ${}^{\mathcal{R}}E_{Q}^{(n)}(t)$ computed from $\mathcal{R}(x,t)$ by ${}^{\mathcal{U}}E_{\gamma}^{(n)}(t)$ and ${}^{\mathcal{U}}E_{Q}^{(n)}(t)$ computed from $\mathcal{U}(x,t)$, so that Eqs. (B2) and (B3) agree with Eqs. (23) and (24), and $U(x,t)$ coincides with $\mathcal{W}[x;\varpi(t),\mu(t)]$. By virtue of Eq. (B6), we can thus conclude that

$$\lim_{t-t_0\to\infty}\mathcal{R}(x,t)=\lim_{t-t_0\to\infty}\mathcal{U}(x,t)=\mathcal{W}[x;\varpi(t),\mu(t)], \tag{B7}$$

with $\mathcal{W}[x;\varpi(t),\mu(t)]$ described by Eqs. (23) and (24), and the initial conditions formulated in Eq. (B3).

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