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Chapter 1

Introduction

This manuscript is a collection of the author's results on realization theory of hybrid systems and some other related issues. This work is based on a number of papers by the author. Each chapter roughly corresponds to a paper. The author tried to avoid unnecessary repetition of concepts and definitions, thus achieving a somewhat more concise presentation.

The current chapter is intended to serve as a short informal introduction to the contents of this work. In order to improve readability, we will split this chapter into several sections. Section 1.1 deals with the notion of hybrid systems and describes informally the major classes of *hybrid systems* we will be dealing with in this chapter. Section 1.2 describes the problem known under the name of the *realization problem*. Section 1.3 give a short informal description of the main results presented in this work. Finally, Section 1.4 outlines the structure of this work and gives a brief presentation of the contents of individual chapters and their interdependency.

1.1 Hybrid Systems

The field of hybrid systems emerged more than a decade ago. There is a vast literature on the subject, see [44, 79]. The field of hybrid systems became popular in the beginning of 1990's, but isolated papers on hybrid systems appeared already earlier. One of the first papers addressing hybrid systems is [15]. There a class of control systems was introduced, which is closely related to the class of piecewise-affine hybrid systems. The relationship between the class of systems introduced in [15] and more modern definitions of hybrid systems was discussed in [62]. The term "hybrid systems" is used to denote a broad range of control systems. We will only give an informal description of the class of hybrid systems which will be the subject of this thesis. There are many books and papers discussing hybrid systems in detail, the interested reader is referred to the literature [44, 79, 68] for more details.

A hybrid system is a control system of the following form. The discrete dynamics is specified by a finite-state automaton. The states of the automaton are called *discrete modes*. The elements of the input alphabet of the automaton will be called *discrete events*. We associate a time-invariant nonlinear control system with each discrete mode. All the nonlinear control systems associated with the discrete modes are assumed to have the same input and output spaces but their state spaces may be different. The state evolution of each nonlinear system is assumed to be determined by a differential or difference equation. Most of hybrid systems considered in this thesis will be such that the contnious control systems will be defined by *differential equations*.

The finite state automaton is assumed to be endowed with a discrete output space. With each discrete mode and discrete event we associate a map on the continuous state spaces. We will refer to these maps as the *reset maps*.

The state evolution takes place as follows. The system is started in a certain discrete mode. The continuous state changes according to the differential/difference equation associated with the current discrete mode. The evolution of the continuous state stops if a discrete event occurs. In this case the discrete mode is changed according to the finite-state automaton. Then the continuous state is changed by applying the reset map corresponding to the discrete event and the discrete mode. After that the evolution of the continuous state is resumed according to the differential/difference equation associated with the new discrete mode. The new contnious state which is obtained as a result of application of the reset maps serves as the new initial state for the differential/difference equation associated with the new discrete state.

Discrete events can be triggered either externally or internally. In the latter case a discrete event arises if the continuous state variable reaches a designated subset of the state-space. Such a subset is called a guard. If we allow discrete events which can be triggered internally, then we say that the hybrid system admits *autonomous switching*. The discrete events which are triggered externally can be considered as discrete inputs. The inputs of a hybrid system consist of the continuous inputs of the nonlinear systems associated with each discrete mode and the externally triggered discrete events. The outputs of a hybrid system consist of the continuous outputs of the control systems associated with each discrete mode and the discrete outputs of the finite-state automaton.

The main motivation for studying hybrid systems is the increasing significance of digital control in engineering systems. As the role of computers and the sheer amount of functionalities of engineering systems which are subject to digital control increases, it becomes more and more difficult to ignore the effect of discrete behaviour of the controllers on the overall behaviour of the system. By considering the plant and the controllers realized by digital controllers as one single system we naturally arrive to hybrid systems. Of course, systems exhibiting hybrid behaviour can arise from purely physical considerations too.

The problems which are normally studied for hybrid systems can be divided into two types. The first type of problems are essentially classical control theoretic problems such as observability, stability, reachability, existence of a controller, optimal control. The second type of questions comes from computer science and it is mostly concerned with verification of hybrid systems, that is, proving that some property holds for the system. Very often the property is that a certain set of states is never reached, or the negation of it, that is certain set of states can be reached. The presence of the two approaches reflects the involvement of both control theory and computer science communities in the field.

The main topic of this thesis is realization theory of hybrid systems. Realization theory is concerned with finding a (preferably minimal) hybrid system which exhibits the specified input-output behaviour. We will discuss realization theory in more detail in the next section, the definition given above should be enough for the objectives of this section.

Realization theory is mostly viewed as a classical control system theoretic question, however, its version for finite state automata play a significant role in computer science. The significance of realization theory for control theory will be elaborated on in the next section. That is why we consider realization theory an important development for the field of hybrid systems. We hope that our results on realization theory will be useful for both control theorists and computer scientists working in the field of hybrid systems.

1.2 Realization Problem and Realization Theory

Realization theory is one of the central problems of systems theory. Historically it is also one of the oldest, in fact, to some extend, the whole modern control systems theory started with realization theory.

So what is realization theory about ? There are several ways to define a control systems. One of the most widespread methods is to give a description in terms of differential and/or difference equation. Representations of this kind we will call *state-space representations*. In fact, this is not a really adequate definition of the concept known as *state*, but let's accept this definition for the time being.

Such a representation has a number of attractive properties. First of all, such models are usually easy to derive using first principles, i.e. from laws of physics or chemistry. Probably the most spectacular case of deriving the differential equation from the laws of physics is the derivation of models for mechanical systems. In fact, most of us had to do it for simple toy systems in high-school physics classes. Another attractive property of such systems is that sometimes it is not that difficult to develop control laws for such a system. So what is the drawback of such models ? The problem with such a model is that it is not necessarily a verifiable model. That is, the description of the model might contain components which cannot be tested by experiments. Typically not all relevant variables can be measured, which means that there are variables about which we might be unable to say anything based on our measurements. In fact, we might have several models, which behave in the same way in the measurable variables. That is, by looking at the experiments we can not distinguish these models.

It is quite a grim news. Not only does it mean that engineers might be unable to derive an appropriate model, it also makes the concept of a control system itself unclear. After all, if the differential equation does not determine the external behaviour of the system uniquely, then we can not identify the system with the differential equation describing it.

Of course, as an alternative, we could try to look at the behaviour of the measurable variables of the system, instead of the differential/difference equations describing the system. In this way we avoid all the ambiguity about the concept of a system which arose before. The idea is not new and it too has a serious deficiency. It turned out extremely difficult to design a control law for a system for which only the behaviour of the measurable variables is known.

In fact, historically the latter approach to control systems was the first one. Control theory was started by electrical engineers who had to ensure reliable communication of telecommunication devices (telephones, more precisely) in the thirties of the twentieth century. The systems were described by transfer functions, which is in fact a description of the input-output behaviour of the system.

Intuitively the reason for the failure of designing control laws for systems in input-output description is the following. In order to determine the control at some point of time, one naturally needs the values of the past control actions. Generally, this means that as we advance in time, more and more data points are needed to design the next input action. Obviously this is not a realistic option. What we would like to have is a *finite* description of some sort encoding the information about the past inputs. Such a description is what is usually called a *state* of the system. For example, if the system is determined by a differential equation, then the vector of variables on which the equation is defined will play the role of the state. Indeed, the knowledge of the vector of these variable gives us enough information to determine the future behaviour of the system.

Therefore, it becomes important to represent systems in a state-space form and to relate the input-output behaviour and state space representations. This is precisely the topic of realization theory.

Realization theory is concerned with the following questions.

- When does a specified input-output behaviour have a state-space representation of certain class. What are the necessary and sufficient conditions for existence of a state-space representation of a certain form.
- What is the "smallest" or "minimal" state-space representation of a given class for a specified input-output behaviour. Does such a representation exists and if it does is it unique in any way. What are the system theoretic characteristics of such a representation.

The two problems above together are called the *realization problem*. As the reader might have noticed, we are looking for state-space representations of a certain class. Typical classes of state-space representations researchers looked at before are state-space representations in terms of linear, polynomial, analytic or rational differential, or difference equations. The reason for not just looking for a state-space representation is that the form of the state-space representation has huge implication of what kind of problems can be solved in it. Obviously, a problem which is relatively easy to solve for state-space representations of one form, might be quite difficult to solve for state-space representations of another form. Besides, physical models are usually obtained as state-space models of certain form and the input-output behaviour of the system very often reflects a physical model.

Of course, a certain input-output behaviour can have state-space representations of different types and a "minimal" state-space representation of one type might be a non "minimal" state-space representation of another type. Thus, speaking of "minimal" state-space representations makes sense only with respect to a certain class of state-space representations.

As the reader might have already realized, realization theory plays an important role in identification of systems. As it was mentioned earlier, many mathematical models of real-life systems are derived using the laws of physics or chemistry, or other science depending on domain of application. The models obtained in this way are in state-space form but usually they are only partially known. For example, the equations of the model might contain parameters which are not known. In such cases the need arises to further determine the model by means of experiments of the physical system. Of course, only the measurable variables can be studied by experiments. That is, from experiments one can obtain information only on the input-output behaviour of the system. Based on this input-output behaviour one would like to determine a good approximation of the state-space model of the system. This range of problems is called *system identification*. It is quite easy to see that realization theory and system identification are closely related. Of course, in system identification one would like to actually *compute* the state-space realization, that is one would like to use *finite data*. Another important point is that in the identification problem one cannot assume that the data is exact. That is, one has to take noise and other types of uncertainty into account. In some sense the realization problem is an idealised version of the identification problem. Therefore, it is highly unlikely that one can find a satisfactory solution to the identification problem for a certain class of systems before developing the realization theory for that class of systems first.

Another important aspect of realization theory is characterisation of minimal state-space representations of a certain class. Intuitively it is easy to see why minimal representations are important. Any output feedback control law in fact operates on the minimal sub-representation of the state-space representation. Not surprisingly, in most cases minimal representations have turned out to have such important system-theoretic properties as observability and reachability. They are also unique up to isomorphism for most cases.

Realization theory was developed for several classes of control systems. The first class of control systems for which realization theory was developed is the class of linear systems [39, 40, 41, 38, 31]. Later on realization theory was developed for bilinear systems [33, 30, 64, 73, 75, 66, 65], analytic nonlinear systems [21, 36, 34, 35, 6, 10, 67], polynomial [64, 84, 2] and rational control systems [84] both in discrete- and continuous time. There are also some results on general nonlinear systems, mostly concerning characterisations of minimal realizations [34, 72] Linear systems have by far the most complete realization theory. It is also the most known to the wider audience. Realization theory of nonlinear systems is much less well-known.

The most active research in realization theory was carried out in the two decades between the sixties and the middle of eighties of the last century. After that the topic did not attract substantial interest any more. There were some works in the nineties on realization theory, mostly on positive [77, 76], max-plus algebraic [12, 13, 14] and multidimensional systems [1]. There were some early attempts to develop realization theory for hybrid systems in [28], but the actual development of the theory was never done.

1.3 Main Results

The goal of this section is to present an informal overview of the main results of the thesis. As it was already noted at the beginning on the chapter the main topic of the thesis is realization theory of various classes of hybrid systems. Sure, as matter of course, reachability and observability properties of hybrid systems are studied too, but only to the extend which is necessary for realization theory. Most of the classes of hybrid systems discussed in this thesis are hybrid systems *without guards* defined in *continuous time*. The only exception is the short chapter on piecewise-affine hybrid systems. A great part of the thesis is devoted to study of switched systems. Another major class of hybrid systems extensively studied in this thesis is the class of linear and bilinear hybrid systems without guards. Hybrid systems with completely nonlinear continuous dynamics are studied much less, but there is a chapter devoted to them which lays down the basics of the theory. There is a short chapter on piecewise-affine discrete-time systems, describing preliminary results.

Hybrid systems without guards are a quite restricted subclass of hybrid systems. Their practical relevance is unclear. The main motivation for studying such hybrid systems is that they might help to understand other, more general hybrid system classes better. In some sense they form an extreme case of hybrid systems with guards. Indeed, if we assume that our hybrid system is such that by a suitable choice of input we can steer the system to a guard arbitrary fast and thus trigger a discrete-state transition at any time we wish, then we in fact have a hybrid system without guards. That is why we think that understanding realization theory of hybrid systems without guards is necessary for developing realization theory for hybrid systems with guards. An other way to look at hybrid systems is to think of them as interconnection of three systems: a finite state automaton, a collection of classical control systems and a control system defined on continuous state-space but with discrete output space. The output space of the last system is in fact the set of input symbols of the finite-state automaton. The role of each of the systems is quite clear. The automaton is responsible for the discrete dynamics, the classical control systems are responsible for the continuous dynamics, and the control system with discrete outputs is responsible for generating discrete-state transitions which depend on the continuous states.

The different weights with which the different classes of systems are represented in the thesis by no means reflects their relative importance. On the contrary, the author considers the class of piecewise-affine continuous-time hybrid systems to be the most important class of hybrid systems and this class is not even mentioned in the thesis. The reason for that is that these systems are very difficult to study and so far they have withstood all the attempts of the author to develop realization theory for them. In fact, all the other systems presented in this thesis were studied in the hope that the results obtained for them will provide a clue for solution of the realization problem for piecewise-affine hybrid systems in continuous-time.

In the remaining part of the section we will go through each class of systems discussed in the thesis and we will present a short description of the main results.

1.3.1 Switched Systems

Switched systems is the class of hybrid systems which is probably the closest one to classical control theory. A switched system is nothing else but a collection of classical continuous control systems defined on the same state-space and having the same input and output spaces. In the setting of this thesis the sequence of discrete modes (the switching sequence) is considered to be part of the input. That is, one can choose when to switch and to which discrete mode. Consequently, the input-output maps for switched systems are defined both on the space of piecewise-contnious input functions and switching sequences. The outputs live in the shared output space of the control systems comprising the switched system. That is, the input-output maps map piecewise-continuous inputs and switching sequences to continuous outputs. Two important versions of the realization problem can be distinguished. In the first case the input-output maps are defined on *all the possible switching sequences*. In the second case only a *subset of the possible switching sequences is allowed* and the input-output maps are defined only with respect to these switching sequences. We will consider only those restrictions of the set of admissible switching sequences where the switching times are arbitrary and the sequence of discrete modes is required to belong to a certain set (or language, in the terminology of formal language theory).

In this thesis we consider two particular types of switched systems: *linear switched systems* and *bilinear switched systems*. Linear switched systems are switched systems which consist of continuous-time linear control systems. Bilinear switched systems are switched systems which consist of bilinear control systems.

We develop a full realization theory for both classes of switched systems.

For linear or bilinear switched systems such that arbitrary switching sequences are allowed, we will present the following results. We will formulate necessary and sufficient conditions in terms of the finiteness of the rank of the Hankel-matrix. We will characterise minimality in terms of observability and span-reachability and show that minimal realizations are unique up to isomorphism. Partial realization theory will be developed too and algorithms will be formulated to compute a minimal (partial) realization and to check minimality.

For linear or bilinear switched systems such that not all switching sequences are allowed the results presented in this thesis are more modest if compared to the case of arbitrary switching. First of all, we treat only the case when arbitrary switching times are allowed and the only restriction is on the relative order of the discrete modes. Moreover, we assume that the set of admissible sequences of discrete modes form a regular language, i.e. it can be decided by a finite state automaton whether a particular finite sequence of discrete modes belongs to the admissible set or not. For such classes of linear or bilinear switched systems we will formulate necessary and sufficient conditions for existence of a realization in terms of finiteness of the rank of the Hankel-matrix. Unfortunately we did not succeed in characterising minimality for such systems. However, instead we can consider realizations which are observable and semi-reachable and behave almost like minimal realizations. More precisely, for any other realization the quotient of the dimension of the "almost" minimal realization and the dimension of the specified realization are bounded from above by a constant (in the ideal case, for a true minimal realization this constant equals 1). The reason for that is the following. In case of restricted switching there are sequences of discrete modes, for which we have no information on the input-output behaviour. On the other hand, any switched system realization of the input-output behaviour does imply certain information on the behaviour for forbidden switchings. That is, if we find a switched system realization, then at the same time we impose a certain behaviour on the system with respect to the forbidden switching sequences. Thus behaviour is in some sense arbitrary but different choices of this hidden behaviour may result in systems having different state space dimensions. On the other hand, if the set of admissible sequences of discrete modes is regular (which means that the set of forbidden sequences of the discrete modes is a regular language too), then the choice of the hidden behaviour can change the dimension only by at most a certain constant. By the way, the choice of the "almost" minimal realization amounts to choosing the hidden behaviour to be zero.

As it is almost always the case with realization theory, the main tools are algebraic in nature. The main tool for realization theory of both linear and bilinear switched systems is the theory of rational formal power series. The theory of rational formal power series has a rich history going back to the 1960's. The concept itself was rediscovered several times and was applied successfully to bilinear and multidimensional control systems. In this thesis we will use a slight extension of the classical theory which enables us to deal with families of formal power series instead of one single formal power series. In fact, E.Sontag and Y.Wang have already looked at families of formal power series before, see [84]. But they were looking at problems which were a bit different and their paper does not contain the formulation of all the results we need for realization theory. That is why we felt compelled to present the theory completely again. Interestingly, Gohberg, Kaashoek and Lerer in [37] also looked at algebraic objects, the so called *nodes*, which were very similar to rational formal power series representations. They studied the properties of those nodes which were minimal in a certain sense. This notion of a minimal node presented in [37] was applied to a number of control systems. In terms of classical formal power series theory, the notion of minimality investigated in [37] corresponds to minimality of partial representations of formal power series. By a partial representation of a formal power series we mean a representation which generates some (not necessarily all) of the coefficients of the formal power series. Hence, the results of [37] could be potentially useful for studying the realization problem of switched systems with constrained switching. However, in this thesis we will not use any of the results of [37].

When arbitrary switching sequences are allowed, then it turns out that both for linear and bilinear switched systems we can associate a suitable family of formal power series with each family of input-output maps. There is a one-to-one correspondence between switched linear or bilinear system realizations of a set of input-output maps and rational representations of the associated family of formal power series. Moreover, this correspondence maps minimal switched systems realizations to minimal rational representations, in case of arbitrary switchings. Thus, we can just use the classical theory of formal power series to derive the results on realization theory of linear and bilinear switched systems.

The case of restricted switching is a bit more involved. We can still associate a family of formal power series with each family of input-output maps. But in contrast to the case of arbitrary switching, we have some freedom in choosing such a family of formal power series. This freedom of choice stems from the fact that the behaviour of the input-output maps is not known for those switching sequences which are not admissible. Since the associated family of formal power series has to capture the behaviour for all the switching sequences, we are compelled to "make up" some behaviour for the disallowed switching sequences. As a result, although there is still a correspondence between switched systems and rational representations, this correspondence fails to be one-to-one and it does not map minimal switched systems to minimal rational representations. Nevertheless, as was already mentioned above, there is still a possibility to define "almost minimal" switched system realizations, which posses quite useful properties. Such "almost minimal" realizations arise from minimal rational representations of the family of formal power series. Although these switched system realizations are not minimal, they are not "too big", in a sense that they cannot exceed the dimension of any other realization by more than a constant factor. This constant factor depends only on the nature of admissible sequences of discrete modes, i.e. it is independent of the family of input-output maps considered.

1.3.2 Hybrid Systems Without Guards

Hybrid systems without guards is probably the next simplest class of hybrid systems after switched systems. A hybrid system without guards consists of an automaton and a finite collection of classical control systems. With each state of the automaton we associate a classical control system. For each discrete-state transition we define maps which map the continuous states of the control system associated with the old discrete state to the continuous states of the control system associated with the new discrete state. We will call the states of the automaton discrete modes and the input symbols of the automaton discrete events. We will assume that all the control systems associated with the discrete modes are continuous-time systems endowed with the same input- and output-spaces.

The state evolution takes place as follows. The system is started in a certain discrete mode. The continuous state changes according to the differential equation associated with the current discrete mode. During the evolution of the continuous state the discrete mode remains unchanged. The evolution of the continuous state stops if a discrete event occurs. In this case the discrete mode is changed according to the finite-state automaton. The continuous state is changed by applying the reset map corresponding to the discrete event and the discrete mode. After that the evolution of the continuous state is resumed according to the differential equation associated with the new discrete mode.

The expression "a discrete event takes place" means the following. We assume that discrete events act as discrete inputs. That is, we can initiate any discrete event at any time we like. The reader might think of discrete events as pressing buttons on the control board of a machine. If a particular button is pressed, then a discrete event takes place. Of course, one can press any button at any time one likes.

The inputs of a hybrid system without guards consist of the continuous inputs of the control systems associated with each discrete mode and the discrete events. The outputs of a hybrid system without guards consist of the continuous outputs of the control system associated with each discrete mode and the discrete outputs of the finite-state automaton.

Of course, switched systems are a particular subclass of hybrid systems without guards. In the case of switched systems the set of discrete modes and the set of discrete events coincide. That is, with each discrete state we associate a discrete event and this correspondence is onto and one-to-one. The state-transition map of the automaton is trivial, i.e. if a discrete event takes place, then the new discrete mode is the discrete mode which corresponds to the discrete event which took place. That is, the automaton does not have a memory, in the sense that the new discrete state depends only on the discrete event but not on the previous discrete mode.

As the reader might have noticed, in this thesis we will be primarily concerned with hybrid systems without guards.

Apart from switched systems, the following three subclasses of hybrid systems without guards will be studied: linear hybrid systems, bilinear hybrid systems and analytic nonlinear hybrid systems.

Linear and Bilinear Hybrid Systems

Linear hybrid systems are hybrid systems without guards such that the control systems associated with discrete modes are linear control systems and the reset maps are linear. Bilinear hybrid systems are hybrid systems without guards such that the control systems associated with discrete modes are bilinear control systems and the reset maps are linear.

Due to the linear structure, a fairly complete theory can be derived for the realization problem of linear and bilinear hybrid systems. We will be able to give necessary and sufficient conditions for existence of a (bi)linear hybrid system realization of a family of input-output maps. These conditions involve finiteness of the rank of the Hankel matrix. We will also present a procedure for constructing a (bi)linear hybrid system realization from the columns of the Hankel-matrix. We will give a characterisation of minimal (bi)linear hybrid system realizations in terms of observability and semi-reachability. Semi-reachability means that the continuous state-spaces are linearly spanned by reachable continuous states. We will also present rank conditions for observability and reachability of (bi)linear hybrid system. These conditions will enable us to check observability and semi-reachability of (bi)linear hybrid systems by algorithms. Partial realization theory for (bi)linear hybrid systems will be formulated too, along with algorithms for computing a minimal (bi)linear hybrid realization from finite number of input-output data points.

A necessary condition for existence of a realization by a linear (bilinear) hybrid system for a family of input-output maps is that the family admits a so called hybrid kernel representation (hybrid Fliess-series expansion respectively). The requirement that a family of input-output maps has a hybrid kernel representation roughly means that the continuous valued part of each input-output map depends linearly and continuously on the continuous input, the continuous output is analytic in time for constant continuous inputs and the discrete-valued part depends only on sequences of discrete events. The requirement that a family of input-output maps has a hybrid

Fliess-series expansion is more or less equivalent to requiring that the discrete-valued parts of the input-output maps should depend only on the sequences of discrete input events and that the continuous-valued parts should be representable as infinite series of iterated integrals of the continuous inputs. Thus, hybrid Fliess-series expansion can be viewed as a generalisation of the classical notion of Fliess-series expansion from nonlinear systems theory [83, 32]. Similarly, hybrid kernel representation is a generalisation of the classical condition that the outputs of linear control systems can be represented as the convolution of the inputs with an analytic convolution kernel.

The main tool for developing realization theory for linear and bilinear hybrid systems is the theory of so called rational hybrid formal power series. A hybrid formal power series is a pair consisting of a classical formal power series in non-commuting variables and a discrete-valued input-output map. That is, the first component of the pair is a formal power series defined over an alphabet and having real vector valued coefficients. Recall that such a formal power series can be viewed as a function mapping the words over the alphabet to p tuples of real numbers for some p. The second component is a function, mapping words over a finite alphabet to elements of some finite set. In case of a hybrid formal power series we assume that the input alphabet of the discrete-valued input-output map (the second component of the pair) is a subset of the alphabet, over which the formal power series (the first component of the pair) is defined. Thus, a hybrid formal series can be viewed as an input-output map, mapping words over the bigger alphabet of the formal power series to pairs consisting of real vectors and elements of a finite set. The real vectors are the values (coefficients) of the formal power series for the given word. The element of the finite set arises by applying the discrete-valued input-output map to the word obtained from the specified one by forgetting all the letters which do not belong to the input alphabet of the discrete-valued input-output map.

We will be interested in families of hybrid formal power series which admit a rational hybrid representation. A rational hybrid representation is roughly speaking an interconnection of a finite state Moore-automaton with a number of rational formal power series representations. A family of hybrid formal power series admits a rational hybrid representation if the hybrid representation, viewed as a Moore-automaton, realizes the family of hybrid formal power series, viewed as a family of input-output maps. Recall that a rational formal power series representation can be thought of as an automaton, the state space of which is a finite dimensional vector space and the readout and state-transition maps are linear. Thus, rational hybrid formal power series can be thought of as input-output maps of machines, which are interconnections of finite state Moore-automata and finite-dimensional linear Moore-automata, i.e. rational representations.

It turns out that there is a one-to-one correspondence between rational hybrid representations and (bi)linear hybrid systems. One can easily associate with each family of input-output maps which admits a hybrid kernel representation or a hybrid Fliess series expansion, a family of hybrid formal power series. It turns out that a (bi)linear hybrid system is a realization of the family of input-output maps if and only if the rational hybrid representation associated with the (bi)linear hybrid system is a representation of the family of hybrid formal power series associated with the family of input-output maps. In particular, a family of input-output maps has a realization by a linear (bilinear) hybrid system, if and only if it admits a hybrid kernel representation (hybrid Fliess-series expansion) and the associated family of hybrid formal power series is rational. There is one-to-one correspondence between minimal rational hybrid representations and minimal (bi)linear hybrid systems. System theoretic properties of (bi)linear hybrid systems such as semi-reachability and observability can be characterised through reachability and respectively observability of rational hybrid representations.

Thus, the realization problem for (bi)linear hybrid systems is equivalent to the problem of finding (a preferably minimal) rational hybrid representation for a suitable family of hybrid formal power series. Moreover, characterisation of minimality of hybrid representations immediately yields a characterisation of minimality of (bi)linear hybrid systems. That is, instead of investigating (bi)linear hybrid systems it is sufficient to study hybrid representations.

Realization theory for hybrid formal power series can be developed by combining results of automata theory and theory of rational formal power series. It turns out that we can associate a family of classical formal power series and a family of discrete-valued input-output maps with each family of hybrid formal power series. The family of hybrid formal power series is rational if and only if the associated family of formal power series is rational and the associate family of discrete-valued input-output maps admit a realization by a finite Moore-automaton. Moreover, it can be shown that one can construct a rational hybrid representation of the family of hybrid formal power series from a minimal rational representation of the associated family of formal power series and a minimal Moore-automaton realization of the associated family of discrete-valued input-output maps. Moreover, this construction yields a minimal rational hybrid representation. Observability and reachability of a hybrid representation can also be translated into observability and reachability of suitable rational formal power series representations and Moore-automata. In fact, a hybrid representation is minimal if and only if it is reachable and observable.

Thus, the more or less classical results of automata theory and formal power series theory yield sufficient and necessary conditions for rationality of hybrid formal power series, along with characterisation of minimality of hybrid representations in terms of reachability and observability. They also enable us to define the Hankel-matrix of a family of hybrid formal power series and a procedure for constructing a hybrid representation of the family from the columns of the Hankel-matrix. In fact, we can construct such a hybrid representation from the columns of a suitably big finite sub-matrix of the Hankel-matrix. The algorithm for computing a minimal rational representation and a minimal automaton realization yields an algorithm for computing a minimal hybrid representation. Observability, reachability and minimality of a hybrid representation can be checked by checking observability, reachability of suitable a rational formal power series representation and a suitable Moore-automaton. Thus, by using algorithms for checking observability and reachability of rational formal power series representations and Moore-automata one can formulate algorithms for checking reachability, observability and minimality of hybrid representations.

The algorithms in turn can be applied to (bi)linear hybrid systems. Thus, we are able to formulate algorithms for checking observability, semi-reachability and minimality of (bi)linear hybrid systems and for constructing minimal (bi)linear hybrid system realizations. The algorithm for computing a hybrid representation from a finite sub-matrix of a Hankel-matrix enables us to formulate partial realization theory for (bi)linear hybrid systems. It also gives a procedure for computing a (bi)linear hybrid system realization from finitely many input-output data.

Nonlinear Hybrid Systems Without Guards

In this thesis we will also present results on realization theory of hybrid systems without guards which have a bit more general structure than linear and bilinear hybrid systems. We will tentatively call them nonlinear hybrid systems in the subsequent text (in the corresponding chapter they will be called *nicely analytic nonlinear hybrid systems*). As the name suggests nonlinear hybrid systems are hybrid systems without guards such that the control system at each discrete mode is an analytic input-affine nonlinear control system and the reset maps are analytic such that the following condition holds. For each discrete mode there exists a distinguished point in the state-space of the underlying analytic input-affine control system such that the reset maps map these points into each other. That is, the value of a reset map at a distinguished point is a distinguished point itself. We will be looking at realizations of a single input-output map by a nonlinear hybrid system such that the continuous component of the initial state from which the input-output map is realized is a distinguished point.

The assumptions that the underlying nonlinear control systems are analytic and that the reset maps are analytic enable us to translate the global realization problem to a local one. That is, instead of trying to find a hybrid system which realizes a certain input-output map we will aim at finding a hybrid system which realizes the specified input-output map *locally, i.e. for small enough times and small enough continuous inputs*. That is, we will be looking for a hybrid system and an initial state, such that for small enough times and continuous inputs, the input-output map induced by the initial states coincides with the specified input-output map. Due to analyticity of input-output maps existence of such a hybrid system realization will also imply that the input-output map induced by the hybrid system and the specified input-output map, realization of which is wanted, will coincide on the intersection of their domains of definition. That is, if the found hybrid system induces an input-output map which is defined for all times and continuous inputs, then the hybrid system will be a realization of the specified input-output map.

The reason why we prefer to deal with the local rather than the global realization problem is that the local realization problem can be translated to a purely algebraic problem. Thus, the local realization problem is somewhat easier and its solution might give important insight into the solution of the global problem. Moreover, the conditions one gets for existence of a local realization might be easier to handle algorithmically.

The way we translate the local realization problem to an algebraic problem resembles the formal power series approach, classical in realization theory of nonlinear systems [36, 21].

The classical solution to local nonlinear realization problem starts with associating with each nonlinear system a formal system defined as a follows. We associate with each vector field of the nonlinear systems a derivation on the ring of formal power series. The derivations are obtained by taking the Taylor-series expansion of each vector field around the initial point. The solution to the local realization problem is reduced to finding a formal system realization for a map, which maps sequences of input symbols to continuous outputs.

In order to repeat the procedure above for hybrid systems we will need the fact that distinguished points are mapped to distinguished points. This will enable us to look at Taylor series expansions of vector field, readout maps and reset maps around the distinguished points. By viewing the continuous state-spaces as formal power series rings and viewing the vector fields, resets maps and readout maps as derivations, homomorphisms on formal power series and formal power series respectively, we will be able to associate a *formal hybrid system* with each hybrid system. Conversely, if we have a formal hybrid system and the corresponding vector fields, homomorphism and formal power series are convergent, then we can associate with the formal hybrid system a hybrid system, continuous state spaces of which are open neighbourhoods of $\mathbb{R}^n$, where n depends on the discrete-state. It turns out that a hybrid system is a local realization of an input-output map if and only if the corresponding formal hybrid system is a realization of a map obtained from the input-output map as follows. The map maps sequences of discrete inputs and indices indicating the directions of continuous inputs to continuous and discrete outputs. The continuous valued part is obtained by taking high-order derivatives of the input-output map with respect to inputs and arrival times of discrete-inputs. The discrete valued part simply coincides with the discrete-valued part of the original input-output map. In fact, in this thesis we will pursue a seemingly different manner of obtaining this map, by defining the concept of hybrid Fliess-series expansions and hybrid convergent generating series. We will define the map which should be realized by the formal hybrid system by using hybrid convergent generating series, but it is easy to see that the values of hybrid convergent generating series can be obtained by taking high-order derivatives of the input-output map.

Thus a necessary condition for existence of a local realization by a hybrid system is that the map obtained from the input-output map has a realization by a formal hybrid system.

Unfortunately, even this formal version of the realization problem is quite difficult and we did not manage to find a satisfactory solution to it. In this thesis we will present necessary conditions for existence of a formal hybrid realization and we will present conditions which are "almost" sufficient. By "almost" sufficient we mean that if the conditions are satisfied, then there exists a realization by an abstract hybrid system, which is slightly more general than formal hybrid systems. Both the necessary and the sufficient conditions involve two types of conditions. The first type essentially requires that a certain discrete input-output map should have a realization by a Moore-automaton. The second type requires that a certain vector space should be finite-dimensional. Conditions of the second type are analogous to the classical finite Lie-rank condition for classical nonlinear systems. In fact, they imply the Lie-rank condition if applied to the special case of classical nonlinear systems.

As the reader could see, even the formal realization problem for nonlinear hybrid systems is more difficult than the corresponding problem for simple nonlinear systems. There are many ways to solve the problem of existence of a formal realization for classical nonlinear system. One of them is to use the theory of Sweedler-type coalgebras and bialgebras [29, 27]. The other one gives a direct construction of a realization, using theory of Lie-algebras [36, 21]. In this thesis we will use the theory of Sweedler-type coalgebras and bialgebras for studying the formal realization problem for hybrid systems. Note that this thesis is not the first attempt to use Sweedler-type coalgebra theory for hybrid systems, a similar approach was proposed in [28], but there only some elements of the framework were sketched and no new result proven.

We believe that coalgebra theory is the natural framework for a range of problems, including realization theory of hybrid and nonlinear systems. It also presents a framework, which connects well to the notion of costate, introduced by E.Sontag [64] for realization of polynomial and rational discrete-time systems. The two approaches are dual to each other and connect roughly as follows. The state-space representation of a system, where the state-space is a manifold, finite-dimensional linear space or an algebraic variety corresponds to a coalgebra system. The costate-space representation of a system, where the costate (or the space of observables) is an algebra of certain class corresponds to an algebra systems. In fact, this duality between the two types of representations was noticed already by Sontag in his work on polynomial systems [64], but there he stated only the duality between finitely generated algebras or algebras with finite transcendence degree and varieties. This duality is a special case of duality between algebra and coalgebra systems. We hope that recognising this duality and using the representation which suits the particular problem better might be a useful problem solving technique.

One of the technical obstacles we encountered while trying to solve the formal realization problem is the presence of non-invertible reset maps. It is due to the presence of non-invertible reset maps that we failed to find necessary and sufficient conditions for existence of a formal hybrid system realization.

1.3.3 Piecewise-affine Discrete-time Hybrid Systems

In this thesis we will also discuss realization theory for discrete-time piecewise-affine hybrid systems. A discrete-time piecewise-affine hybrid system is essentially a PL-system according to the terminology by E.Sontag [15]. That is, it is a discrete-time system, such that the state-transition and readout maps are piecewise-affine. By a piecewise-affine map we mean a map such that there exists a partitioning of its domain into polyhedra such that the restriction of the map to each such polyhedron is a linear map. We will study only autonomous systems, that is, systems without inputs. We will also assume that there exists a partition of the state-space into polyhedra, such that on each polyhedra the state-transition and readout maps are linear. We will assume that each such polyhedron is indexed by an element of a finite set. We will call this finite set the *set of discrete modes*. For autonomous systems, if we start in a particular state of the system, then the sequence of indices of polyhedra which the state-trajectory started in this particular states visits, is completely determined by the structure of the system and by the particular state. We will refer to this sequence of indices (discrete modes) as the *switching sequence* induced by the particular state.

For the sake of simplicity in the subsequent text we will refer to discrete-time autonomous piecewise-affine hybrid systems simply as hybrid systems.

We will be interested in finding necessary and sufficient conditions for existence of an autonomous piecewise-affine hybrid system realization of an output trajectory. Since we are looking at discrete-time systems, the output trajectory is simply an infinite sequence of output values. We will distinguish two cases. In the first case the set of discrete modes of the sought hybrid systems is fixed, moreover, the sequence of discrete modes which should be visited by the state-trajectory generating the output trajectory is fixed too. That is, the output trajectory can be viewed as a map from finite subwords of an infinite word over the fixed set of discrete modes to output values. The problem of finding a realization with a specified set of discrete modes and with a specified switching sequence will be called the *weak realization problem*. In contrast, in the second case we do not assume any a priori knowledge on the set of discrete modes or switching sequence. That is, in this case the output map is just a sequence of output values and the desired hybrid realization can have any set of discrete modes and it can generate any switching sequence from its initial state. We will refer to the problem described as the second case as the *strong realization problem*.

The results presented in this thesis are quite elementary, they represent the first step towards realization theory of piecewise-affine hybrid systems.

We will give necessary and sufficient conditions for existence of a realization by an autonomous discrete-time piecewise-affine hybrid system. The conditions are of two type. Conditions of the first type are conditions which are necessary and sufficient for existence of a hybrid system realization such that the switching sequence induced by the initial state is *almost-periodic*. The second type of conditions are conditions for existence of a hybrid system without any further restriction on the switching sequence induced by the initial state.

In the first case, i.e. when the induced switching sequence is required to be almost-periodic, the sufficient and necessary condition is finiteness of an infinite matrix, reminiscent of the Hankel-matrix. For the weak realization problem the Hankel-matrix is very similar to the Hankel matrix of a formal power series. That is, the output trajectory is viewed as a formal power series, which maps each finite subword of the desired switching sequence to the value of the output trajectory at natural number which is equal to the length of the subword. It maps each word which is not a subword of the desired output trajectory to zero. In plain English, for all those sequences of discrete modes for which we have no information, we assume that the output is zero. Notice that this approach is similar to what was done for (bi)linear switched systems with constrained switching. In fact, the construction of a realization in both cases relies on the very same properties of formal power series. For the strong realization problem the Hankel-matrix is simply the classical Hankel-matrix. In fact, if the desired switching sequence is almost-periodic, then existence of a hybrid system realization in the strong sense is equivalent to existence of a realization by a linear system.

The second case, i.e. when there is no restriction on the desired switching sequence induced by the initial state is a bit more involved. We used ideas very similar to those which appeared in theory of time-varying systems and in theory of systems over abstract rings. If we adopt the operations of point-wise addition and multiplication then the set of all infinite sequences of real numbers becomes a ring. Consider the set of all infinite sequences of real number such that each sequence from the set takes finitely many values, i.e., if it is viewed as a map from natural numbers to reals, then its range if finite. The set of sequences with finite range forms a sub-ring of the ring of sequences. The output trajectory can be viewed as a collection of p sequences of real numbers, where p is the dimension of the output space. Both for the weak and strong realization problems we define a number of sub-rings of the ring of sequences with finite range. It is easy to see that the ring of sequences is a module over the ring of sequences with finite range. It turns out that a necessary and sufficient condition for existence of a hybrid system realization is that the output trajectories are contained in a finitely generated shift invariant submodule of the module of all sequences, where the space of all sequences is viewed as a module over a suitable sub-ring of the ring of sequences with finite range. The choice of the sub-ring depends on whether we consider the weak or the strong realization problems. For the strong realization problem we use the whole ring of sequences with finite range. The choice of the sub-ring for the weak realization problem is a bit more involved.

The necessary and sufficient condition discussed above yields a sufficient condition. Namely, if the set of shifts of the output trajectories generates a finitely generated module over a suitable sub-ring of the ring of sequences with finite range, then the output trajectory has a realization by a hybrid system. Notice that the set of shifts of the output trajectories is simply the set of columns of the Hankel-matrix, thus the sufficient condition above simply says that if the module spanned by the columns of the Hankel-matrix is finitely generated, then the output trajectory has a realization by a hybrid system.

At this stage the reader might be puzzled as to how we reconstruct the switching mechanism, i.e., how we find a suitable partitioning of the state-space into polyhedra. The answer is quite simple, and yet, in the author's opinion, it is one of the most interesting observations of the thesis on the theory of piecewise-affine discrete-time hybrid systems. As we mentioned earlier, in the autonomous case the switching sequence induced by the initial state depends only on the initial state and the structure of the system. Conversely, given any switching sequence over a suitable set of discrete modes, we can find a suitable initial state and a piecewise-affine discrete-time hybrid system, such that the sequence of outputs of this hybrid system is the desired switching sequence. The construction of such a system is in fact known, see [8]. Thus, existence of a realization by a discrete-time piecewise-affine hybrid system is equivalent to existence of a realization by a linear switched system realization with a specified switching signal. Or, in other words, existence of a hybrid system realization is equivalent to existence of a linear time-varying realization of a very special structure. Hence, we can use ideas from realization theory of switched systems and time-varying systems to develop realization theory of autonomous discrete-time piecewise-affine hybrid systems.

1.4 Structure of the Manuscript

In this section we will give a brief outline of the structure of the thesis.

Chapter 2 This chapter describes some notation and terminology which will be use throughout the thesis. It also presents the formal definitions of the classes of hybrid system which are discussed in the thesis. The last section of this chapter presents the concept of abstract generating series, which will be used only in the sections dealing with bilinear switched and hybrid systems. The only sections the reader is strongly advised to read before going further are Section 2.1, Section 2.2 and Section 2.3. All the other sections can be read later, when the reader arrives to the corresponding chapters which refer to them.

Chapter 3 This chapter discusses the theory of classical and hybrid formal power series. This is one of the most important chapters of the thesis, most of the other chapters rely on this one. The only chapters which are independent of this one are Chapter 5 and Chapter 6. However, for Chapter 4 and Section 10.1, Section 10.4 of Chapter 10 it is enough to read Section 3.1 of Chapter 3. The material of this chapter can be found in [55, 51].

Chapter 4 The chapter presents realization theory of linear and bilinear switched systems. The approach to realization theory pursued in this chapter relies on theory of formal power series. Thus, Section 2.3, 2.4 and Section 3.1 are prerequisites for this chapter. Section 2.6 is a prerequisite for Section 4.2. The material of this chapter was published in [55, 51, 53].

Chapter 5 This chapter deals with the structure of the reachable set of linear switched systems. The only prerequisite for this chapter is Section 2.4 and Subsection 4.1.1. The results presented in this chapter were published in [56]. The material of this chapter is in some sense a detour from the main topic of the thesis. It presents an investigation of the structure of the reachable set for linear switched systems using methods of nonlinear systems theory. It does not touch the issue of realization theory.

Chapter 6 This chapter presents an alternative approach to realization theory of linear switched systems. Instead of using formal power series, it discusses a direct construction of a minimal realization. This chapter is based on [50], which was the earliest publication on realization theory of linear switched systems. Although all the results of this section are implied by results of Chapter 4, this chapter still provides an interesting alternative view of realization theory of linear switched systems. The only prerequisite for this chapter are Subsection 4.1.1 and Section 2.4.

Chapter 7 This chapter presents realization theory for linear and bilinear hybrid systems. Perhaps this is one of the most interesting chapters of the thesis. The approach to realization theory relies heavily on theory of hybrid formal power series. Prerequisites for this chapter are Chapter 3 and Section 2.3. Section 7.2 of this chapter relies on Section 2.6. The material of this chapter is partially based on [48, 54, 47].

Chapter 8 This chapter deals with realization theory of nonlinear analytic systems without guards. The chapter is based on the conference paper [57]. The only prerequisite to this chapter is Section 2.3 and Section 2.6. This chapter only sketches the main constructions and states the main results. It does not provide detailed proofs of the stated results.

Chapter 9 This chapter discusses realization theory of discrete-time autonomous piecewise-affine hybrid systems. The prerequisites to this chapter are Section 2.5 and Section 3.1. The chapter is based on [49]. The chapter merely sketches the main constructions, without providing too much details and omitting a number of proofs.‘;

Chapter 10 This chapter discusses partial realization theory for linear and bilinear switched and hybrid systems. It also presents algorithms for checking observability, semi-reachability and minimality of hybrid systems and computing a minimal hybrid system realization. Prerequisites for this chapter are Chapter

2, Chapter 3, Chapter 4 and Chapter 7. However, Section 10.1 requires only the results of Section 3.1, Section 10.2 requires only Section 3.2. Section 10.4 relies only on Section 10.1 and Chapter 4. Section 10.3 uses results from Section 3.3 and Section 10.2., Finally, Section 10.5 uses results from Chapter 3, Chapter 7, Section 10.1 and Section 10.3.

There already exists a preliminary software implementation of the algorithms presented in this chapter.

Chapter 2

Preliminaries

The goal of this chapter is twofold. First, we will set up notation and terminology, which will be used in the rest of the thesis. Second, we will define those classes of control systems which will be the object of the study in this thesis. We will start by describing in Section 2.1 some general notation and terminology, which will be used in the thesis without any further reference. Then we will proceed by defining the concept of Moore-automata in Section 2.2. Section 2.3 presents the definition and basic properties of hybrid systems without guards. Section 2.4 presents the definition and elementary properties of switched systems. Section 2.6 presents the framework of abstract generating convergent series. Abstract generating convergent series are a generalisation of generating convergent series from nonlinear systems theory, see [32, 84, 83]. Abstract generating series will be used in Section 4.2 and in Section 7.2 for defining the concepts of generalised generating convergent series and hybrid generating convergent series respectively. The latter two notions play a central role in realization theory of bilinear switched and hybrid systems.

2.1 Notation and Terminology

For suitable sets S, B, $S \subseteq \mathbb{R}$ denote by $PC(S, B)$ the class of piecewise-continuous maps from S to B. That is, $f \in PC(S, B)$ if f has finitely many points of discontinuity on each finite interval and at each point of discontinuity the right- and left-hand side limits exist and they are finite. For a set Σ denote by Σ^* the set of finite strings of elements of Σ. For $w = a_1 a_2 \cdots a_k \in \Sigma^*$ the length of w is denoted by $|w|$, i.e. $|w| = k$. The empty sequence is denoted by ϵ. The length of ϵ is zero: $|\epsilon| = 0$. Let $\Sigma^+ = \Sigma^* \setminus \{\epsilon\}$. The concatenation of two strings $v = v_1 \cdots v_k, w = w_1 \cdots w_m \in \Sigma^*$

is the string $vw = v_1 \cdots v_k w_1 \cdots w_m$. If $w \in \Sigma^+$ then w^k denotes the word $\underbrace{ww \cdots w}_{k-times}$. The word w^0 is just the empty word ϵ. Denote by T the set $[0, +\infty) \subseteq \mathbb{R}$. Denote by $\mathbb{N}$ the set of natural numbers including 0. Denote by $F(A, B)$ the set of all functions from the set A to the set B. By abuse of notation we will denote any constant function $f : T \to A$ by its value. That is, if $f(t) = a \in A$ for all $t \in T$, then f will be denoted by a. For any function f the range of f will be denoted by $\mathrm{Im} f$. If A, B are two sets, then the set $(A \times B)^*$ will be identified with the set $\{(u, w) \in A^* \times B^* \mid |u| = |w|\}$. For any two sets J, X the surjective function $A : J \to X$ is called an *indexed subset* of X or simply and *indexed set*. It will be denoted by $A = \{a_j \in X \mid j \in J\}$. The set J will be called the *index set* of A. The indexed subset $A = \{a_j \in X \mid j \in J\}$ is said to be a *subset* of the indexed subset $B = \{b_i \in X \mid i \in I\}$ if there exists $g : J \to I$ such that $a_j = b_{g(j)}$. The fact that A is a subset of B will be denoted by $A \subseteq B$.

Let $f : A \times (B \times C)^+ \to D$. Then for each $a \in A$, $w \in B^+$ we define the function $f(a, w, .) : C^{|w|} \to D$ by $f(a, w, .)(v) = f(a, (w, v)), v \in C^{|w|}$. By abuse of notation we denote $f(a, w, .)(v)$ by $f(a, w, v)$.

Let $\phi : \mathbb{R}^k \to \mathbb{R}^p$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$. We define $D^\alpha \phi$ as the partial derivative

$$D^\alpha \phi = \frac{d^{\alpha_1}}{dt_1^{\alpha_1}} \frac{d^{\alpha_2}}{dt_2^{\alpha_2}} \cdots \frac{d^{\alpha_k}}{dt_k^{\alpha_k}} \phi(t_1, t_2, \ldots, t_k)|_{t_1 = t_2 = \cdots = t_k = 0}.$$

Let $f, g \in PC(T, A)$ for some suitable set A. Define for any $\tau \in T$ the concatenation $f \#_\tau g \in PC(T, A)$ of f and g by

$$f \#_\tau g(t) = \begin{cases} f(t) & \text{if } t \le \tau \\ g(t) & \text{if } t > \tau \end{cases}$$

If $f : T \to A$, then for each $\tau \in T$ define $\mathrm{Shift}_\tau(f) : T \to A$ by $\mathrm{Shift}_\tau(f)(t) = f(t + \tau)$. If $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are vector spaces over $\mathbb{R}$, and $F_1 : \mathcal{X} \to \mathcal{Y}$, $F_2 : \mathcal{Y} \to \mathcal{Z}$ are linear maps, then $F_1 F_2$ denotes the composition $F_1 \circ F_2$ of F_1 and F_2. If $x \in \mathcal{X}$, then $F_1 x$ denote the value $F_1(x)$ of F_1 at x.

2.2 Moore automata

A *finite Moore-automaton* is a tuple $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ where Q, Γ are finite sets, $\delta : Q \times \Gamma \to Q$, $\lambda : Q \to O$. The set Q is called the *state-space*, O is called the *output space* and Γ is called the *input space*. The function δ is called the *state-transition map* and the function λ is called the *readout map*. Denote by $card(\mathcal{A})$ the cardinality of the state-space Q of $\mathcal{A}$, i.e. $card(\mathcal{A}) = card(Q)$.

Define the functions $\widetilde{\delta} : Q \times \Gamma^* \to Q$ and $\widetilde{\lambda} : Q \times \Gamma^* \to O$ as follows. Let $\widetilde{\delta}(q, \epsilon) = q$ and

$$\widetilde{\delta}(q, w\gamma) = \delta(\widetilde{\delta}(q, w), \gamma), w \in \Gamma^*, \gamma \in \Gamma$$

Let $\widetilde{\lambda}(q, w) = \lambda(\widetilde{\delta}(q, w)), w \in \Gamma^*$. By abuse of notation we will denote $\widetilde{\delta}$ and $\widetilde{\lambda}$ simply by δ and λ respectively.

Let $\mathcal{D} = \{\phi_j \in F(\Gamma^*, O) \mid j \in J\}$ be an indexed set of functions. A pair $(\mathcal{A}, \zeta)$ is said to be *an automaton realization* of $\mathcal{D}$ if $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$, $\zeta : J \to Q$ and

$$\lambda(\zeta(j), w) = \phi_j(w), \forall w \in \Gamma^*, j \in J$$

An automaton $\mathcal{A}$ is said to be a realization of $\mathcal{D}$ if there exists a $\zeta : J \to Q$ such that $(\mathcal{A}, \zeta)$ is a realization of $\mathcal{D}$.

Let $(\mathcal{A}, \zeta)$ and $(\mathcal{A}^{'}, \zeta^{'})$ be two automaton realizations. Assume that

$$\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$$

and

$$\mathcal{A}^{'} = (Q^{'}, \Gamma, O, \delta^{'}, \lambda^{'})$$

. A map $\phi : Q \to Q^{'}$ is said to be an *automaton morphism* from $(\mathcal{A}, \zeta)$ to $(\mathcal{A}^{'}, \zeta^{'})$, denoted by $\phi : (\mathcal{A}, \zeta) \to (\mathcal{A}^{'}, \zeta^{'})$ if $\phi(\delta(q, \gamma)) = \delta^{'}(\phi(q), \gamma), \forall q \in Q, \gamma \in \Gamma$, $\lambda(q) = \lambda^{'}(\phi(q)), \forall q \in Q$, $\phi(\zeta(j)) = \zeta^{'}(j), j \in J$. It is easy to see that composition of two automaton morphisms is again an automaton morphism. The automaton morphism ϕ is called injective (surjective) if the map ϕ is injective (surjective). If ϕ is a bijection, then $\phi^{-1} : (\mathcal{A}^{'}, \zeta^{'}) \to (\mathcal{A}, \zeta)$ is an automaton morphism too. An automaton realization $(\mathcal{A}, \zeta)$ of $\mathcal{D}$ is called *minimal* if for each automaton realization $(\mathcal{A}^{'}, \zeta^{'})$ of $\mathcal{D}$ $card(\mathcal{A}) \leq card(\mathcal{A}^{'})$. Let $\phi : \Gamma^* \to O$. For every $w \in \Gamma^*$ define $w \circ \phi : \Gamma^* \to O$*–the left shift of* ϕ *by* w as $w \circ \phi(v) = \phi(wv)$. For $\mathcal{D} = \{\phi_j \in F(\Gamma^*, O) \mid j \in J\}$ define the set $W_{\mathcal{D}} \subseteq F(\Gamma^*, O)$ by

$$W_{\mathcal{D}} = \{w \circ \phi_j : \Gamma^* \to O \mid w \in \Gamma^*, j \in J\}$$

An automaton $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ is called *reachable from* $Q_0 \subseteq Q$, if

$$\forall q \in Q : \exists w \in \Gamma^*, q_0 \in Q_0 : q = \delta(q_0, w)$$

A realization $(\mathcal{A}, \zeta)$ is called reachable if $\mathcal{A}$ is reachable from $\mathrm{Im}\zeta$. A realization $(\mathcal{A}, \zeta)$ is called *observable* or *reduced*, if

$$\forall q_1, q_2 \in Q : [\forall w \in \Gamma^* : \lambda(q_1, w) = \lambda(q_2, w)] \implies q_1 = q_2$$

2.3 Hybrid Systems Without Guards

In this subsection we will present a formal definition of hybrid systems without guards. As the name indicates, a hybrid system without guards is a hybrid system where all the discrete events are externally triggered. More precisely, one could describe a hybrid system without guards as follows. The system consists of a finite state Moore-automaton, a finite collection of control systems and a collection of *reset maps*. We associate a control system with each state of the Moore-automaton. The states of the Moore-automaton are referred to as *discrete states*. The control systems are assumed to be determined by differential equations. Thus, in general, we consider *nonlinear control systems*, state-space of which, generally speaking is a *manifold*. We associate a reset map with each discrete state transitions. Reset maps are assumed to be maps between state-spaces of the control systems comprising the hybrid system. The control systems associated with the discrete states are assumed to be endowed with the input and output spaces but the state-spaces are allowed to vary with the discrete states. The state evolution of such a hybrid system takes place as follows. One starts in a certain discrete state with a certain continuous initial state. The state trajectory evolves according to the differential equation of the control system associated with the current discrete mode, until a discrete even arrives. When a discrete even arrives, the evolution of the continuous state stops and the discrete state of the hybrid system changes according to the state transition rule of the Moore-automaton. The new continuous state is obtained by applying the reset map associated with the current discrete state transition to the continuous state where the evolution of the control stopped. All these transitions are assumed to take place instantaneously, in zero time. After the discrete state transition and reseting of the continuous state the state evolution proceeds according to the differential equation of the new discrete state,by applying the flow of the differential equation to the new continuous state. The continuous input is fed to the control system associated with the current discrete mode. The continuous output trajectory is obtained by concatenating the continuous state trajectories. The contnious output trajectory is piecewise-constant, it is formed by the outputs associated with the discrete states of the Moore-automaton visited during the state-space evolution.

We assume that the discrete events and their arrival is subject to control. In other words, we assume that the discrete events are inputs and any specific discrete event can be triggered at any time. Thus, timed sequences of discrete events play the role of inputs, just as sequences of input symbols play the role of inputs for finite-state automata.

After having described in an informal way the concept of hybrid systems without guards we proceed with giving a formal definition.

Definition 1. *A hybrid systems without guards (HSWG) is a tuple*

$$H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, f_q, h_q)_{q \in Q}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\})$$

where

- $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ *is a finite-state Moore-automaton,*
- $\mathcal{X}_q$ *is a manifold for each* $q \in Q$*,*
- $\mathcal{U}$ *is the set of continuous input values, it is assumed to be a manifold.*
- $\mathcal{Y}$ *is the set of continuous output values,* $\mathcal{Y}$ *is assumed to be a manifold.*
- $h_q : \mathcal{X}_q \to \mathcal{Y}$ *is a smooth map*
- $f_q : \mathcal{X}_q \times \mathcal{U} \to TX_q$ *is a smooth map, such that for each* $u \in \mathcal{U}$ *the map* $x \mapsto f_q(x, u)$ *defines a vector field.*

The set Q of states of $\mathcal{A}$ is called the *set discrete modes*, the input alphabet Γ of $\mathcal{A}$ is called the *set of discrete events*. The tuple $(\mathcal{X}_q, f_q, h_q)$ can be viewed as the contnious control system associated with the discrete state $q \in Q$. The map h_q is called the *readout map.* We will assume that f_q, is *globally Lipschitz*, or more precisely, the coordinate functions are globally Lipschitz, so that the solution of the differential equation

$$\frac{d}{dt}x(t) = f_q(x(t), u(t))$$

is well-defined for all $t \in \mathbb{R}$ and u piecewise-continuous functions, i.e., $u \in PC(\mathbb{R}, \mathcal{U})$. In the rest of the section we will refer to hybrid systems without guards simply as hybrid systems.

Let $\mathcal{H} = \bigcup_{q \in Q} \{q\} \times \mathcal{X}_q$. Let $\mathcal{X} = \bigcup_{q \in Q} \mathcal{X}_q$, $\mathcal{A}_H = \mathcal{A}$. As we already indicated at the beginning of the section, hybrid systems without guards admit two types of inputs. The inputs of the hybrid system H are functions from $PC(T, \mathcal{U})$ and sequences from $(\Gamma \times T)^*$.

The interpretation of a sequence $(\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*$ is the following. The event γ_i took place *after* the event γ_{i-1} and t_{i-1} is the *elapsed time between the arrival of* γ_{i-1} *and the arrival of* γ_i. That is, t_i is the difference of the arrival times of γ_i and γ_{i-1}. Consequently, $t_i \geq 0$ but we allow $t_i = 0$, that is, we allow γ_i to arrive instantly after γ_{i-1}. If $i = 1$, then t_1 is simply the time when the event γ_1 arrived.

The *state trajectory* of the system H is a map

$$\xi_H : \mathcal{H} \times PC(T, \mathcal{U}) \times (\Gamma \times T)^* \times T \to \mathcal{H}$$

of the following form. For each $u \in PC(T,\mathcal{U})$, $w = (\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*$, $t_{k+1} \in T$, $h_0 = (q_0, x_0) \in \mathcal{H}$ it holds that

$$\xi_H(h_0, u, w, t_{k+1}) = (\delta(q_0, \gamma_1 \cdots \gamma_k), x_H(h_0, u, w, t_{k+1}))$$

where the map $x : T \ni t \mapsto x_H(h_0, u, w, t) \in \mathcal{X}$ is the solution of the differential equation

$$\frac{d}{dt}x(t) = f_{q_k}(x(t), u(t + \sum_{1}^{k} t_j)$$

where $q_i = \delta(q_0, \gamma_1 \cdots \gamma_i)$, $i = 1, \ldots, k$ and

$$x(0) = x_H(h_0, u, w, 0) = R_{q_k, \gamma_k, q_{k-1}} x_H(x_0, u, (\gamma_1, t_1) \ldots (\gamma_{k-1}, t_{k-1}), t_k)$$

if $k > 0$ and $x(0) = x_0$ if $k = 0$.

In fact, one can define a map $x_H : \mathcal{H} \times PC(T,\mathcal{U}) \times (T \times \Gamma)^* \times T \to \bigcup_{q \in Q} \mathcal{X}_q$, by $(h, u, s, t) \mapsto x_H(h, u, s, t)$. It is easy to see that $\Pi_{\bigcup_{q \in Q} \mathcal{X}_q} \circ \xi_H = x_H$. Define the set of reachable states from a subset $\mathcal{H}_0 \subseteq \mathcal{H}$ in an obvious way as follows.

$$R(H, \mathcal{H}_0) = \{\xi_H(h, u, w, t) \mid h \in \mathcal{H}_0, u \in PC(T,\mathcal{U}), w \in (\Gamma \times T)^*, t \in T\}$$

We will say that the hybrid system H is reachable from $\mathcal{H}_0$ if $R(H, \mathcal{H}_0) = \mathcal{H}$).

One could give an alternative definition of reachability. Define the set of *continuous states* reachable from $\mathcal{H}_0$ by

$$Reach(H, \mathcal{H}_0) = \{x_H(h_0, u, w, t) \in \mathcal{X} \mid u \in PC(T,\mathcal{U}), w \in (\Gamma \times T)^*, t \in T, h_0 \in \mathcal{H}_0\}$$

Then H is reachable from $\mathcal{H}_0$ if $Reach(H, \mathcal{H}_0) = \mathcal{X}$ and the automaton A_H is reachable from $\Pi_Q(\mathcal{H}_0)$.

Define the function $\upsilon_H : \mathcal{H} \times PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T \to O \times \mathcal{Y}$ by

$$\upsilon_H((q_0, x_0), u, (w, \tau), t) = (\lambda(q_0, w), h_q(x_H((q_0, x_0), u, (w, \tau), t)))$$

where $q = \delta(q_0, w)$. For each $h \in \mathcal{H}$ the *input-output map of the system H induced by h* is the function

$$\upsilon_H(h, .) : PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T \ni (u, (w, \tau), t) \mapsto \upsilon_H(h, u, (w, \tau), t) \in O \times \mathcal{Y}$$

We will denote the map $(u, s, t) \mapsto \Pi_{\mathcal{Y}} \circ \upsilon_H(h, u, s, t) \in \mathcal{Y}$ by $y_H(h, .)$ and we will denote $y_H(h, .)(u, s, t)$ simply by $y_H(h, u, s, t)$. Two states $h_1 \neq h_2 \in \mathcal{H}$ of the linear hybrid system H are *indistinguishable* if $\upsilon_H(h_1, .) = \upsilon_H(h_2, .)$. H is called *observable* if it has no pair of indistinguishable states.

Throughout the thesis we will mostly be concerned with realization of a *set of input-output maps*. It means that we will have to look at systems which have not one, but several initial states. We will use the following formalism to deal with the issue. Let H be a hybrid system and let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ be a subset of the set of input-output maps. Let $\mu : \Phi \to \mathcal{H}$ be any map. We will call the pair (H, μ) a *realization* . The map μ just specifies a way to associate an initial state to each element of Φ. The statement that (H, μ) is a realization *does not imply* that H is realized Φ from the set of initial states $\Im\mu$. The set $\Phi \subseteq F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ is said to be *realized* by a hybrid realization (H, μ) where $\mu : \Phi \to \mathcal{H}$, if

$$\forall f \in \Phi : \ \upsilon_H(\mu(f), .) = f$$

We will say that H *realizes* Φ if there exists a map $\mu : \Phi \to \mathcal{H}$ such that (H, μ) realizes Φ. With slight abuse of terminology, sometimes we will call both H and (H, μ) a *realization* of Φ. Thus, H realizes Φ if and only if for each $f \in \Phi$ there exists a state $h \in \mathcal{H}$ such that $\upsilon_H(h, .) = f$. We say that a realization (H, μ) is observable if H is observable and we say that (H, μ) is reachable if H is reachable from $\mathrm{Im}\mu$. We will denote by μ_D the map $\Phi \ni f \mapsto \Pi_Q(\mu(f)) \in Q$, where Q is the discrete-state space of H. The map μ can be thought of as a map which assigns to each input-output map f an initial state of the system H. It is just an alternative way to fix a set of initial states. If we speak of a realization (H, μ) it will always imply that $dom(\mu)$ is a subset of $F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$, i.e. it is a set of input-output maps, and $\mu : dom(\mu) \to \mathcal{H}$.

For a hybrid system H the dimension of H is defined as

$$\dim H = (card(Q), \sum_{q \in Q} \dim \mathcal{X}_q) \in \mathbb{N} \times \mathbb{N}$$

The first component of $\dim H$ is the cardinality of the discrete state-space, the second component is the sum of dimensions of the continuous state-spaces. For each $(m, n), (p, q) \in \mathbb{N} \times \mathbb{N}$ define the partial order relation $(m, n) \leq (p, q)$, if $m \leq p$ and $n \leq q$. A realization H of Φ is called a *minimal realization* of Φ, if for any realization $H^{'}$ of Φ:

$$\dim H \leq \dim H^{'}$$

The partial order relation on the dimensions of hybrid systems realizations induces a partial order on the set of all hybrid realizations. If the set of all realizations of Φ is considered as a partially ordered set, then a minimal realization defines a minimal element of this set. Notice however, that our definition of a minimal realization is quite different from the usual definition of a minimal element of a partially ordered set. The definition of a minimal element of a partially ordered set does not imply that the minimal element is comparable (in relation) with other elements of the set. Our definition of a minimal realization explicitly requires that the minimal realization should have dimension which is smaller than the dimension of any other realization, thus, in particular, it has to be comparable with all the realizations. That is, it is not necessarily true that any minimal element of the partially ordered set of realizations yields a minimal realization.

The reason for defining the dimension of a hybrid system as above is that there is a trade-off between the number of discrete states and dimensionality of each continuous state-space component. That is, one can have two realizations of the same input/output maps, such that one of the realizations has more discrete states than the other, but its continuous state components are of smaller dimension than those of the other system.

Let (H, μ) and $(H^{'}, \mu^{'})$ be two realizations such that $dom(\mu) = dom(\mu^{'})$ and

$$\begin{aligned} H &= (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, f_q, h_q)_{q \in Q}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}) \\ H^{'} &= (\mathcal{A}^{'}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}^{'}_q, f^{'}_q, h^{'}_q)_{q \in Q'}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q^{'}, \gamma \in \Gamma\}) \end{aligned}$$

where $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ and $\mathcal{A}^{'} = (Q^{'}, \Gamma, O, \delta^{'}, \lambda^{'})$. A pair $T = (T_D, T_C)$ is called a *hybrid system morphism* from (H, μ) to $(H^{'}, \mu^{'})$, denoted by $T : (H, \mu) \to (H^{'}, \mu^{'})$, if the the following holds. The map $T_D : (\mathcal{A}, \mu_D) \to (\mathcal{A}^{'}, \mu^{'}_D)$, where $\mu_D(f) = \Pi_Q(\mu_D(f))$, $\mu^{'}_D(f) = \Pi_{Q'}(\mu^{'}_D(f))$, is an automaton morphism and $T_C : \bigcup_{q \in Q} \mathcal{X}_q \to \bigcup_{q \in Q'} \mathcal{X}^{'}_q$ is a map such that

- $\forall q \in Q : T_C(\mathcal{X}_q) \subseteq \mathcal{X}^{'}_{T_D(q)}$,
- For each $q \in Q$, the restriction $T_C|_{\mathcal{X}_q} : \mathcal{X}_q \to \mathcal{X}_{T_D(q)}$ is a smooth map
- For all $q \in Q$, $x \in \mathcal{X}_q$, $u \in \mathcal{U}$

  $$D(T_C|_{\mathcal{X}_q})(x) f_q(x, u) = f^{'}_{T_D}(T_C(x), u) \text{ and } h_q(x) = h^{'}_{T_D(q)}(T_C(x))$$

  where $D(T_C|_{\mathcal{X}_q})(x)$ denotes the Jacobian of the smooth map $T_C|_{\mathcal{X}_q}$ at x.
- For all $q_1, q_2 \in Q, \gamma \in \Gamma, \delta(q_2, \gamma) = q_1$, $T_C R_{q_1,\gamma,q_2} = R^{'}_{T_D(q_1),\gamma,T_D(q_2)} T_C$
- $T_C(\Pi_{\mathcal{X}_q}(\mu(f))) = \Pi_{\mathcal{X}^{'}_{T_D(q)}}(\mu^{'}(f))$ for each $q = \mu_D(f)$, $f \in \Phi$.

The hybrid morphism T is called a hybrid isomorphism if T_D is a bijective map and for each $q \in Q$ the map $T_C|_{\mathcal{X}_q}$ is a diffeomorphism. Two hybrid system realizations are *isomorphic* if there exists a hybrid isomorphisms between them. Notice that a hybrid morphism can be defined only between hybrid system realizations (H, μ) and

$(H^{'}, \mu^{'})$ such that the domains of μ and $\mu^{'}$ coincide. Denote the state space of H_1 by $\mathcal{H}_1 = \bigcup_{q\in Q}\{q\} \times \mathcal{X}_q$ and denote the state-space of H_2 by $\mathcal{H}_2 = \bigcup_{q\in Q'}\{q\} \times \mathcal{X}^{'}_q$. It is easy to see that $T = (T_C, T_D)$ defines a map $\phi_T : \mathcal{H}_1 \ni (q, x) \mapsto (T_D(q), T_C x) \in \mathcal{H}_2$. By abuse of notation we will denote ϕ_T simply by T.

Proposition 1. *If T is a hybrid isomorphism then the map $\phi(T)$ is a bijective as a map from $\mathcal{H}_1$ to $\mathcal{H}_2$.*

Proof. Indeed, for each $(q, x) \in \mathcal{H}_1$, $\phi(T)((q, x)) = (T_D(q), T_C(x))$. Thus, if

$$\phi(T)((q^{'}, x^{'})) = \phi(T)((q, x))$$

then $T_D(q) = T_D(q^{'})$ and $T_C(x) = T_C(x^{'})$. By injectivity of T_D we get $q = q^{'}$ and thus $x, x^{'} \in \mathcal{X}_q$. Then from the assumption that $T_C|_{\mathcal{X}_q}$ is a diffeomorphism we get that $x = x^{'}$. Thus, $\phi(T)$ is injective. If $(s, z) \in \{s\} \times \mathcal{X}^{'}_s$, then by surjectivity of T_D there exists a $q \in Q$ such that $T_D(q) = s$. Since $T_C|_{\mathcal{X}_q} : \mathcal{X}_q \to \mathcal{X}^{'}_s$ is a diffeomorphism, there exists a $x \in \mathcal{X}_q$ such that $T_C(x) = z$. Thus, $\phi(T)((q, x)) = (z, s)$.

□

The following proposition gives an important system theoretic characterisation of hybrid morphisms.

Proposition 2. *Let (H_i, μ_i), $i = 1, 2$ be two hybrid systems and let $T : (H_1, \mu_1) \to (H_2, \mu_2)$ be a hybrid morphism. Then the following holds.*

$$T \circ \xi_{H_1}(h, .) = \xi_{H_2}(T(h), .) \text{ and } \upsilon_{H_1}(h, .) = \upsilon_{H_2}(T(h), .), \forall h \in \mathcal{H}_1 \tag{2.1}$$

If T is an hybrid isomorphism, then (H_1, μ_1) is reachable if and only if (H_2, μ_2) is and (H_1, μ_1) is observable if and only if (H_2, μ_2) is observable.

Proof. It is easy to see that $T_D(\delta^1(q, w)) = \delta^2(T_D(q), w)$ for all $q \in Q^1$ and $w \in \Gamma^*$. We will first show that if $x : T \to \mathcal{X}^1_q$ is a solution of the differential equation

$$\frac{d}{dt}x(t) = f^1_q(x(t), u(t))$$

then $\phi : T \ni t \mapsto T_C(x(t)) \in \mathcal{X}^2_{T_D(q)}$ is a solution to the differential equation

$$\frac{d}{dt}\phi(t) = f^2_{T_D(q)}(\phi(t), u(t))$$

Indeed,

$$\frac{d}{dt}\phi(t) = DT_C|_{\mathcal{X}^1_q}(x(t))\frac{d}{dt}x(t) = DT_C|_{\mathcal{X}^1_q}(\phi(t))f^1_q(x(t), u(t)) =$$
$$= f^2_{T_D(q)}(T_C(x(t)), u(t)) = f^2_{T_D(q)}(\phi(t), u(t))$$

Thus, $\phi(t)$ is indeed the solution of the differential equation $\frac{d}{dt}\phi(t) = f^2_{T_D(q)}(\phi(t), u(t))$. Next, we will show that for any $(q, x) \in \mathcal{H}_1$, $u \in PC(T, \mathcal{U})$, $w = (\gamma_1, t_1)(\gamma_2, t_2)\cdots(\gamma_k, t_k) \in (\Gamma \times T)^*$, $k \geq 0$,$t_{k+1} \in T$

$$T_C(x_{H_1}((q, x), u, w, t_{k+1})) = x_{H_2}((q, x), u, w, t_{k+1}) \tag{2.2}$$

We proceed by induction on k. If $k = 0$, then the map $T \ni t \mapsto x_{H_1}((q, x), u, \epsilon, t) \in \mathcal{X}^1_q$ is the solution to the differential equation $\frac{d}{dt}x(t) = f^1_q(x(t), u(t))$ with initial condition $x(0) = x$. Thus, the map $\phi : T \ni t \mapsto T_C(x_{H_1}((q, x), u, \epsilon, t))$ is the solution to the differential equation $\frac{d}{dt}x(t) = f^2_{T_D(q)}(x(t), u(t))$ with the initial condition $x(0) = T_C(x)$. But the map $t \mapsto x_{H_2}((T_D(q), T_C(x)), u, \epsilon, t)$ is also a solution the differential equation above with the same initial condition $T_C(x)$. Thus,

$$T_C(x_{H_1}((q, x), u, \epsilon, t)) = \phi(t) = x_{H_2}((T_D(q), T_C(x)), u, \epsilon, t)$$

for all $t \in T$.

Assume that (2.2) is true for all $k \leq n$. Let $w_n = (\gamma_1, t_1)\cdots(\gamma_n, t_n)$, $w_{n+1} = (\gamma_1, t_1)\cdots(\gamma_{n+1}, t_{n+1})$. By induction hypothesis,

$$T_C(x_{H_1}((q, x), u, w_n, t_{n+1})) = x_{H_2}((T_D(q), T_C(x)), u, w_n, t_{n+1})$$

From the definition of hybrid morphisms and we get that

$$\begin{aligned} T_C(R^1_{q_{n+1},\gamma_{n+1},q_n}(x_{H_1}((q, x), u, w_n, t_{n+1})) &= \\ = R^2_{T_D(q_{n+1}),\gamma_{n+1},T_D(q_n)} T_C(x_{H_1}((q, x), u, w_n, t_{n+1})) &= \\ = R^2_{T_D(q_{n+1}),\gamma_{n+1},q_n}(x_{H_2}((T_D(q), T_C(x)), u, w_n, t_{n+1}) \end{aligned}$$

where $q_i = \delta(q, \gamma_1\cdots\gamma_i)$, $i = n, n+1$. From the definition of state trajectories of hybrid systems it follows that

$$R^1_{q_{n+1},\gamma_{n+1},q_n}(x_{H_1}((q, x), u, w_n, t_{n+1})) = x_{H_1}((q, x), u, w_{n+1}, 0)$$

and

$$\begin{aligned} R^2_{T_D(q_{n+1}),\gamma_{n+1},T_D(q_n)}(x_{H_2}((T_D(q), T_C(x)), u, w_n, t_{n+1})) &= \\ x_{H_2}((T_D(q), T_C(x)), u, w_{n+1}, 0) \end{aligned}$$

Thus,

$$T_C(x_{H_1}((q, x), u, w_{n+1}, 0)) = x_{H_2}((T_D(q), T_C(x)), u, w_{n+1}, 0)$$

Notice that the map $t \mapsto x_{H_1}((q, x), u, w_{n+1}, t)$ is solution to the differential equation

$$\frac{d}{dt}x(t) = f^1_{q_{n+1}}(x(t), u(\sum_{j=1}^{n} t_j + t))$$

Thus, the map $\phi : t \mapsto T_C((x_{H_1}((q,x),u,w_{n+1},t))$ is the solution to the differential equation $\frac{d}{dt}x(t) = f^2_{T_D(q)}(x(t), u(\sum_{j=1}^{n} t_j + t))$. Notice that by the definition of hybrid state trajectories the map

$$t \mapsto x_{H_2}((T_D(q), T_C(x)), u, w_{n+1}, t)$$

is also a solution of the differential equation above. We just showed that $\phi(0) = T_C(x_{H_1}((q,x),u,w_{n+1},0)) = x_{H_2}((T_D(q),T_C(x)),u,w_{n+1},0)$. Therefore, by uniqueness of solutions of differential equations

$$T_C(x_{H_1}((q,x),u,w_{n+1},t)) = \phi(t) = x_{H_2}((T_D(q),T_C(x)),u,w_{n+1},t))$$

for all $t \in T$. That is, we have just proven (2.2) for the case $k = n+1$.

But equation (2.2) implies that for any $(q,x) \in \mathcal{H}_1$, $u \in PC(T,\mathcal{U})$, $w = (\gamma_1,t_1)\cdots(\gamma_k,t_k) \in (\Gamma \times T)^*$,$t_{k+1} \in T$, $k \geq 0$

$$\begin{aligned}
&T \circ \xi_{H_1}((q,x),.)(u,w,t_{k+1}) = T(\xi_{H_1}((q,x),u,w,t_{k+1})) = \\
&= (T_D(\delta^1(q,\gamma_1\cdots\gamma_k)), T_C((x_{H_1}((q,x),u,w,t_{k+1}))) = \\
&(\delta^2(T_D(q),\gamma_1\cdots\gamma_k), x_{H_2}((T_D(q),T_C(x)),u,w,t_{k+1})) = \\
&= \xi_{H_2}(T((q,x)),u,w,t_{k+1}) = \xi_{H_2}(T((q,x)),.)(u,w,t_{k+1})
\end{aligned}$$

Thus, we have shown the first part of (2.1). The second part follows from the following observation. $h^1_q = h^2_{T_D(q)} \circ T_C$ and $\lambda^1(q) = \lambda^2(T_D(q))$ for all $q \in Q^1$. Thus, we get that

$$\begin{aligned}
&\upsilon_{H_1}((q,x),u,w,t_{k+1}) = (\lambda^1(q_k), h^1_{q_k}(x_{H_1}((q,x),u,w,t_{k+1}))) = \\
&= (\lambda^2(T_D(q_k)), h^2_{T_D(q_k)}(T_C(x_{H_1}((q,x),u,w,t_{k+1}))) = \\
&= (\lambda^2(T_D(q_k)), h^2_{T_D(q_k)}(x_{H_2}((T_D(q),T_C(x),u,w,t_{k+1}))) = \\
&= \upsilon_{H_2}((T_D(q),T_C(x)),u,w,t_{k+1})
\end{aligned}$$

where $q_k = \delta^1(q,\gamma_1\cdots\gamma_k)$ and thus $T_D(q_k) = \delta^2(T_D(q),\gamma_1\cdots\gamma_k)$. Thus, we proved (2.1).

Assume that T is a hybrid isomorphism. We will proceed with the proof of the remaining part of the proposition.

Notice that $T(\mathrm{Im}\mu_1) = \mathrm{Im}\mu_2$, thus from (2.2) we get that $T_C(R(H_1,\mathrm{Im}\mu_1)) = R(H_2,\mathrm{Im}\mu_2)$. If T is a hybrid isomorphism, then by Proposition 1 T is bijective as a map from $\mathcal{H}_1$ to $\mathcal{H}_2$ and thus $T(\mathcal{H}_\infty) = \mathcal{H}_2$. Moreover, is $S \subseteq \mathcal{H}_1$ and $S \neq \mathcal{H}_1$, then $T(S) \neq \mathcal{H}_2$. That is, $R(H_1,\mathrm{Im}\mu_1) = \mathcal{H}_1$ if and only if $T(R(H_1,\mathrm{Im}\mu_1)) = R(H_2,\mathrm{Im}\mu_2) = \mathcal{H}_1$.

Assume that H_1 is not observable. Then there exists two states $h,h^{'} \in \mathcal{H}_1$ such that $h \neq h^{'}$ and $\upsilon_{H_1}(h,.) = \upsilon_{H_2}(h^{'},.)$. It implies that $\upsilon_{H_2}(T(h),.) = \upsilon_{H_1}(h,.) =$

$\upsilon_{H_1}(h^{'},.) = \upsilon_{H_2}(T(h^{'}),.)$. Since $h \neq h^{'}$, it follows that $T(h) \neq T(h^{'})$ thus H_2 is not observable. Conversely, assume that H_2 is not observable. Then there exists two states $s, s^{'} \in \mathcal{H}_2$ such that $s \neq s^{'}$ and $\upsilon_{H_2}(s,.) = \upsilon_{H_2}(s^{'},.)$. Since T is bijective, there exists $h, h^{'} \in \mathcal{H}_1$ such that $h \neq h^{'}$ and $T(h) = s$, $T(h^{'}) = s^{'}$. But then it follows that $\upsilon_{H_1}(h,.) = \upsilon_{H_2}(s,.) = \upsilon_{H_2}(s^{'},.) = \upsilon_{H_1}(h^{'},.)$. That is, H_1 is not observable. □

In this thesis we will mostly deal with hybrid systems without guards. One particular class of hybrid systems without guards is the class of switched systems. The class of switched systems has a structure, quite different from the general case, therefore we will discuss switched systems in a separate section. Two important subclasses of hybrid systems without guards are *linear hybrid systems* and *bilinear hybrid systems*. They both merit separate treatment, and we will discuss their properties extensively in the corresponding sections. At this stage we will just give their definition, without delving too much into details.

Definition 2. *A (time-invariant) linear hybrid system (abbreviated as* LHS *) is hybrid system*

$$H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, f_q, h_q)_{q \in Q}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\})$$

such that

- *For each $q \in Q$ $\mathcal{X}_q = \mathbb{R}^{n_q}$, i.e. $\mathcal{X}_q$ has the structure of the linear space R^{n_q} for some $n_q > 0$,*
- *$\mathcal{U} = \mathbb{R}^m$ and $\mathcal{Y} = \mathbb{R}^p$, i.e the input and output spaces have the structure of the linear spaces $\mathbb{R}^m$ and $\mathbb{R}^p$, $p, m \in \mathbb{N}, n, m > 0$.*
- *For each $q \in Q$ there exist linear maps $A_q : \mathcal{X}_q \to \mathcal{X}_q$, $B_q : \mathcal{U} \to \mathcal{X}_q$, such that with the usual identification on $\mathbb{R}^{n_q}$ of the tangent vectors with elements of $\mathbb{R}^{n_q}$ the following holds*

$$\forall x \in \mathcal{X}_q, u \in \mathcal{U} = \mathbb{R}^m : f_q(x,u) = A_q x + B_q u$$

- *For each $q \in Q$ there exists a linear map $C_q : \mathcal{X}_q \to \mathcal{Y}$ such that*

$$\forall x \in \mathcal{X}_q : h_q(x) = C_q x$$

- *The reset maps are linear, i.e., for each $q_1, q_2 \in Q$, $\gamma \in \Gamma$, $\delta(q_2, \gamma) = q_1$ there exists a linear map $M_{q_1,\gamma,q_2} : \mathcal{X}_{q_2} \to \mathcal{X}_{q_1}$ such that*

$$\forall x \in \mathcal{X}_q : R_{q_1,\gamma,q_2}(x) = M_{q_1,\gamma,q_2} x$$

We will use the following shorthand notation for linear hybrid systems

$$H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, B_q, C_q)_{q \in Q}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2, \gamma)\})$$

Definition 3. *A bilinear hybrid system (abbreviated as* BHS *) is hybrid system*

$$H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, f_q, h_q)_{q \in Q}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\})$$

such that

- *For each* $q \in Q$ $\mathcal{X}_q = \mathbb{R}^{n_q}$*, i.e.* $\mathcal{X}_q$ *has the structure of the linear space* R^{n_q} *for some* $n_q > 0$*,*
- $\mathcal{U} = \mathbb{R}^m$ *and* $\mathcal{Y} = \mathbb{R}^p$*, i.e the input and output spaces have the structure of the linear spaces* $\mathbb{R}^m$ *and* $\mathbb{R}^p$ $p, m \in \mathbb{N}, n, m > 0$.
- *For each* $q \in Q$ *there exist linear maps* $A_q : \mathcal{X}_q \to \mathcal{X}_q$, $B_q : \mathcal{X}_q \to \mathcal{X}_q$*, such that with the usual identification on* $\mathbb{R}^{n_q}$ *of the tangent vectors with elements of* $\mathbb{R}^{n_q}$ *the following holds*

$$\forall x \in \mathcal{X}_q, u = (u_1, \ldots, u_m)^T \in \mathcal{U} = \mathbb{R}^m, \quad f_q(x, u) = A_q x + \sum_{j=1}^{m} (B_{q,j} x) u_j$$

- *For each* $q \in Q$ *there exists a linear map* $C_q : \mathcal{X}_q \to \mathcal{Y}$ *such that*

$$\forall x \in \mathcal{X}_q : h_q(x) = C_q x$$

- *The reset maps are linear, i.e., for each* $q_1, q_2 \in Q$, $\gamma \in \Gamma$, $\delta(q_2, \gamma) = q_1$ *there exists a linear map* $M_{q_1,\gamma,q_2} : \mathcal{X}_{q_2} \to \mathcal{X}_{q_1}$ *such that*

$$\forall x \in \mathcal{X}_q : R_{q_1,\gamma,q_2}(x) = M_{q_1,\gamma,q_2} x$$

We will use the following shorthand notation for bilinear hybrid systems

$$H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, \{B_{q,j}\}_{j=1,\ldots,m}, C_q)_{q \in Q}, \{M_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\})$$

2.4 Switched Systems

This section contains the definition and elementary properties of switched systems.

Definition 4. *A switched (control) system is a tuple*

$$\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\})$$

where

- $\mathcal{X} = \mathbb{R}^n$ *is the state-space*
- $\mathcal{Y} = \mathbb{R}^p$ *is the output-space*
- $\mathcal{U} = \mathbb{R}^m$ *is the input-space*
- Q *is the finite set of discrete modes*
- $f_q : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$*, is a function smooth in both variables* x *and* u*, and globally Lipschitz in* x
- $h_q : \mathcal{X} \to \mathcal{Y}$ *is smooth map for each* $q \in Q$

Elements of the set $(Q \times T)^+$ are called *switching sequences*. The inputs of the switched system Σ are functions from $PC(T, \mathcal{U})$ and sequences from $(Q \times T)^+$. That is, the switching sequences are part of the input, they are specified externally and we allow any switching sequence to occur. In fact, the switching sequences can be considered as discrete inputs.

In the hybrid systems literature the discrete modes are usually viewed as part of the state. One can think of switched systems as hybrid systems without guards, such that the discrete state transitions are triggered by discrete inputs and the discrete state transition rules are trivial. More precisely, there is one-to-one correspondence between discrete states and discrete inputs, and a discrete input changes the discrete state to the discrete state which corresponds to this particular discrete input. That is, the new discrete state of the system *depends only on the discrete input, but not on the previous discrete state.* The continuous state-space does not depend on discrete modes, i.e. all the continuous state-spaces are the same for all discrete modes. The reset maps are assumed to be the identity maps.

Let $u \in PC(T, \mathcal{U})$ and $w = (q_1, t_2)(q_2, t_2) \cdots (q_k, t_l) \in (Q \times T)^+$. The inputs u and w steer the system Σ from state x_0 to the state $x_\Sigma(x_0, u, w)$ given by

$$x_\Sigma(x_0, u, w) = F(q_k, \mathrm{Shift}_{\sum_1^{k-1} t_i}(u), t_k) \circ F(q_{k-1}, \mathrm{Shift}_{\sum_1^{k-2} t_i}(u), t_{k-1}) \circ \cdots$$
$$\cdots \circ F(q_1, u, t_1)(x_0)$$

where $F(q, u, t) : \mathcal{X} \to \mathcal{X}$ and for each $x \in \mathcal{X}$ the function $F(q, u, t, x) : t \mapsto F(q, u, t)(x)$ is the solution of the differential equation

$$\frac{d}{dt} F(q, u, t, x) = f_q(F(q, u, t, x), u(t)),\ F(q, u, 0, x) = x$$

The empty sequence $\epsilon \in (Q \times T)^*$ leaves the state intact: $x_\Sigma(x_0, u, \epsilon) = x_0$.

The *reachable set* of the system Σ from a set of initial states $\mathcal{X}_0 \subseteq \mathcal{X}$ is defined by

$$\mathit{Reach}(\Sigma, \mathcal{X}_0) \quad = \quad \{x_\Sigma(x_0, u, w) \in \mathcal{X} \mid u \in PC(T, \mathcal{U}), w \in (Q \times T)^*, x_0 \in \mathcal{X}_0\}$$

Σ is said to be *reachable* from $\mathcal{X}_0$ if $Reach(\Sigma, \mathcal{X}_0) = \mathcal{X}$ holds. Σ is *semi-reachable* from $\mathcal{X}_0$ if $\mathcal{X}$ is the smallest vector space containing $Reach(\Sigma, \mathcal{X}_0)$. In other words, Σ is semi-reachable from $\mathcal{X}_0$ if

$$\mathcal{X} = \mathrm{Span}\{x \in \mathcal{X} \mid x \in Reach(\Sigma, \mathcal{X}_0)\}$$

Define the function $y_\Sigma : \mathcal{X} \times PC(T,\mathcal{U}) \times (Q \times T)^+ \to \mathcal{Y}$ by

$$\forall x \in \mathcal{X}, u \in PC(T, U), w = (q_1, t_1)(q_2, t_2) \cdots (q_k, t_k) \in (Q \times T)^+ : \\ y_\Sigma(x, u, w) = h_{q_k}(x_\Sigma(x, u, w))$$

By abuse of notation, for each $x \in \mathcal{X}$ define the input-output map $y_\Sigma(x, ., .) : PC(T,\mathcal{U}) \times (Q \times T)^+ \to \mathcal{Y}$ by

$$y_\Sigma(x, ., .)(u, w) = y_\Sigma(x, u, w)$$

The map $y_\Sigma(x, ., .)$ is called the *input-output map of the system* Σ *induced by the state* x. By abuse of notation we will use $y_\Sigma(x, u, w)$ for $y_\Sigma(x, ., .)(u, w)$.

Two states $x_1 \neq x_2 \in \mathcal{X}$ of the switched system Σ are *indistinguishable* if

$$\forall w \in (Q \times T)^+, u \in PC(T,\mathcal{U}) : \qquad y_\Sigma(x_1, u, w) = y_\Sigma(x_2, u, w)$$

Σ is called *observable* if it has no pair of indistinguishable states.

Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$ be a subset of the set of input-output maps. Let Σ be a switched system and let $\mu : \Phi \to \mathcal{X}$ be a map. Just as it was the case for general hybrid systems without guards, we will call the pair (Σ, μ) a *realization* . As for the case of hybrid systems without guards, μ associates an initial state to each element of Φ. Again, (Σ, μ) *need not be* a realization of Φ. The pair (Σ, μ) is a *realization of* Φ, if

$$\forall f \in \Phi\colon \quad y_\Sigma(\mu(f), ., .) = f$$

or, in other words,

$$\forall f \in \Phi, u \in PC(T,\mathcal{U}), w \in (Q \times T)^+\colon\ y_\Sigma(\mu(f), u, w) = f(u, w)$$

We will say that Σ is a *realization* of Φ, if there exists a map $\mu : \Phi \to \mathcal{X}$ such that (Σ, μ) is a realization of Φ in the above sense. By abuse of terminology, both Σ and (Σ, μ) will be called a *realization* of Φ. One can think of the map μ as a way to determine the corresponding initial condition for each element of Φ. That is, Σ realizes Φ if and only if for each $f \in \Phi$ there exists a state $x \in \mathcal{X}$ such that $y_\Sigma(x, ., .) = f$. Denote by $\dim \Sigma := \dim \mathcal{X}$ the dimension of the state space of the switched system Σ.

A switched system Σ is a *minimal realization of* Φ if Σ is a realization of Φ and for each switched system Σ_1 such that Σ_1 is a realization of Φ it holds that

$$\dim \Sigma \leq \dim \Sigma_1$$

For any $L \subseteq Q^+$ define the subset of admissible switching sequences $TL \subseteq (Q \times T)^+$ by

$$TL := \{(w, \tau) \in (Q \times T)^+ \mid w \in L\}$$

That is, TL is the set of all those switching sequences, for which the sequence of discrete modes belongs to L and the sequence of times is arbitrary. Notice that if $L = Q^+$ then $TL = (Q \times T)^+$. Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$ be a set of input-output maps defined only on switching sequences belonging to TL. The system $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\})$ *realizes* Φ *with constraint* L if there exists $\mu : \Phi \to \mathcal{X}$ such that

$$\forall f \in \Phi\colon\ y_{\Sigma}(\mu(f), ., .)|_{PC(T,\mathcal{U}) \times TL} = f$$

or, in other words,

$$\forall w \in \Phi, u \in PC(T,\mathcal{U}), w \in TL\colon\ \ y_{\Sigma}(\mu(f), u, w) = f(u, w)$$

We will call both (Σ, μ) and Σ a *realization* of Φ. Notice that if $L = Q^+$ then Σ realizes Φ with constraint L if and only if Σ realizes Φ. If Σ is a switched system, then we say that the realization (Σ, μ) is *semi-reachable* , if Σ is semi-reachable from $\mathrm{Im}\mu$.

In this work we will especially be interested in the following two classes of switched systems: *linear switched systems* and *bilinear switched systems*. We will postpone describing them in more detail until Chapter 4. Here we will restrict ourselves to giving the definition of these classes of switched systems.

Definition 5 (Linear switched systems). *A switched system*

$$\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\})$$

is called linear, if for each $q \in Q$ *there exist linear mappings* $A_q : \mathcal{X} \to \mathcal{X}$, $B_q : \mathcal{U} \to \mathcal{X}$ *and* $C_q : \mathcal{X} \to \mathcal{Y}$ *such that*

- $\forall u \in \mathcal{U}, \forall x \in \mathcal{X} : f_q(x, u) = A_q x + B_q u$
- $\forall x \in \mathcal{X} : h_q(x) = C_q x$

To make the notation simpler, linear switched systems will be denoted by

$$\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$$

The term linear switched system will be abbreviated by LSS.

Definition 6 (Bilinear switched systems). *A switched system*

$$\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\})$$

is called bilinear if for each $q \in Q$ there exist linear mappings $A_q : \mathcal{X} \to \mathcal{X}$, $B_{q,j} : \mathcal{X} \to \mathcal{X}$, $j = 1, 2, \ldots, m$, $C_q : \mathcal{X} \to \mathcal{Y}$ such that

- $\forall x \in \mathcal{X}, u = (u_1, \ldots, u_m)^T \in \mathcal{U} = \mathbb{R}^m : f_q(x, u) = A_q x + \sum_{j=1}^{m} u_j B_{q,j} x$
- $\forall x \in \mathcal{X} : h_q = C_q x$.

We will use the notation $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\ldots,m}, C_q) \mid q \in Q\})$ to denote bilinear switched systems.

2.5 Piecewise-affine Discrete-time Hybrid Systems

In this section definition and some elementary properties of piecewise-affine systems will be presented. Recall the a subset $H \subseteq \mathbb{R}^n$ is a polyhedral set if it is of the form $H = \{x \in \mathbb{R}^n \mid Ax \leq b, Fx < d\}$ for some $A \in \mathbb{R}^{p \times n}$, $b \in \mathbb{R}^p$, $F \in \mathbb{R}^{d \times n}$, $d \in \mathbb{R}^d$, $p, d \in \mathbb{N}$, $p, d > 0$.

Definition 7 (Piecewise-affine hybrid systems). *A time invariant discrete-time autonomous piecewise-affine hybrid system (abbreviated DTAPA) is a tuple*

$$\Sigma = (\mathcal{X}, \mathcal{Y}, Q, (\mathcal{X}_q, A_q, a_q, C_q, c_q)_{q \in Q}, (q_0, x_0))$$

where

- *Q is a finite set, called the set of discrete modes*
- *$\mathcal{X} = \bigcup_{q \in Q} \mathcal{X}_q$, $\mathcal{X} \subseteq \mathbb{R}^n$. The set $\mathcal{X}$ is called the state-space.*
- *For each $q \in Q$ the set $\mathcal{X}_q$ is polyhedral and $\mathcal{X}_{q_1} \cap \mathcal{X}_{q_2} = \emptyset$, for each $q_1, q_2 \in Q$, $q_1 \neq q_2$.*
- *$\mathcal{Y} = \mathbb{R}^p$. The space $\mathcal{Y}$ is called the output space.*
- *For each $q \in Q$, $c_q \in \mathbb{R}^p$, $C_q \in \mathbb{R}^{p \times n}$,*

- *For each* $q \in Q$*,* $a_q \in \mathbb{R}^n$*,* $A_q \in \mathbb{R}^{n\times n}$ *and for each* $x \in \mathcal{X}_q$*,* $A_q x + a_q \in \mathcal{X}$*.*
- (q_0, x_0) *is the initial state, where* $q_0 \in Q$ *and* $x_0 \in \mathcal{X}_{q_0}$

Define the following maps. Define the map $h_\Sigma : \mathcal{X} \to \mathcal{Y}$ by $h(x) = C_q x + c_q$ for all $q \in Q$, $x \in \mathcal{X}_q$. Define $f_\Sigma : \mathcal{X} \to \mathcal{X}$ by $f(x) = A_q x + a_q$ for all $x \in \mathcal{X}_q$, $q \in Q$. It is clear that the maps f_Σ and h_Σ are well-defined maps. If it does not create confusion we will drop the subscript Σ and will write simply f and h instead of f_Σ and h_Σ.

Define $f^k : \mathcal{X} \to \mathcal{X}$ by $f^0(x) = x$ and $f^{k+1}(x) = f(f^k(x))$ for all $k \geq 0, x \in \mathcal{X}$. The *state-trajectory* of the system Σ is the map $x_\Sigma : \mathcal{X} \times \mathbb{N} \to \mathcal{X}$ such that $x_\Sigma(x, k) = f^k(x)$. The *output-trajectory* of the system Σ is the map $y_\Sigma : \mathcal{X} \times \mathbb{N} \to \mathcal{Y}$ such that $y_\Sigma(x, k) = h(x_\Sigma(x, k)) = h(f^k(x))$.

That is, a DTAPA system Σ can be thought of as a discrete-time system of the form

$$x_{k+1} = f_\Sigma(x_k), \quad y_k = h_\Sigma(x_k)$$

Denote by Q^ω denotes the set of all infinite sequences of elements of Q. Define the map $\phi : \mathcal{X} \to Q^\omega$ by $\phi(x) = q_0 q_1 q_2 \cdots q_k \cdots$ if and only if $f^k(x) \in \mathcal{X}_{q_k}$ for all $k \geq 0$. It is easy to see that ϕ is well-defined.

We will say that the DTAPA system Σ has *almost-periodic dynamics* if the set $\{S^k(\phi(x_0)) \mid k \geq 0\} \subseteq Q^\omega$ is finite, where $S : Q^\omega \to Q^\omega$ is the shift map $S(w_0 w_1 w_2 \cdots) = w_1 w_2 w_3 \cdots$.

A map $y : \mathbb{N} \to \mathcal{Y}$ is said to be realized by a DTAPA $\Sigma = (\mathcal{X}, \mathcal{Y}, f, h, x_0)$, if

$$\forall k \in \mathbb{N} : y(k) = y_\Sigma(x, k) = h(f^k(x_0))$$

Two DTAPA systems are said to be *equivalent* if they realize the same output map. In this paper we will try to solve the following two problems.

Weak realization problem for DTAPA systems For a specified set of discrete modes $\widetilde{Q}$, for a specified sequence $w \in \widetilde{Q}^\omega$ and output trajectory $y : \mathbb{N} \to \mathcal{Y}$ find a DTAPA system $\Sigma = (\mathcal{X}, \mathcal{Y}, Q, (\mathcal{X}_q, A_q, a_q, C_q, c_q)_{q\in Q}, (q_0, x_0))$ such that Σ realizes y, $\widetilde{Q} \subseteq Q$, and $\phi(x_0) = w$.

Strong realization problem for DTAPA systems For any specified $y : \mathbb{N} \to \mathcal{Y}$ find a DTAPA system Σ such that Σ realizes y. That is, in the case of strong realization problem we also have to reconstruct the set of discrete modes.

Let $\Sigma_i = (\mathcal{X}_i, \mathcal{Y}, Q_i, (\mathcal{X}_{q,i}, A_{q,i}, a_{q,i}, C_{q,i}, c_{q,i})_{q\in Q_i}, (q_{0,i}, x_{0,i}))$, $i = 1, 2$ be two DTAPA systems. A map $T : \mathcal{X}_1 \to \mathcal{X}_2$ is called a *DTAPA morphism* if

$$T \circ f_{\Sigma_1} = f_{\Sigma_2} \circ T,\ h_{\Sigma_1} = h_{\Sigma_2} \circ T \text{ and } T(x_{0,1}) = x_{0,2}$$

The DTAPA morphism T will be called injective, surjective, an isomorphism if the corresponding map T is injective, surjective, bijective respectively. It is easy to see that if $T : \Sigma_1 \to \Sigma_2$ is a DTAPA morphism then $T(x_{\Sigma_1}(x,k)) = x_{\Sigma_2}(T(x),k)$ and $y_{\Sigma_1}(x,k) = y_{\Sigma_2}(T(x),k)$ for all $k \geq 0$. In particular, $y_{\Sigma_1}(x_{0,1},k) = y_{\Sigma_2}(x_{0,2},k)$ for all $k \geq 0$. Thus, Σ_1 realizes a map $y : \mathbb{N} \to \mathcal{Y}$ if and only if Σ_2 realizes y.

Thus, in particular, if for some discrete mode the underlying polyhedral set $\mathcal{X}_q$ is shifted or rotated, then the values c_q and a_q is changed accordingly.

Let $\Sigma = (\mathcal{X}, \mathcal{Y}, Q, (\mathcal{X}_q, A_q, a_q, C_q, c_q)_{q \in Q}, (q_0, x_0))$ be a DTAPA system. Notice that without loss of generality f and h can be assumed being piecewise-linear, that is, we can assume that $a_q = 0$ and $c_q = 0$ for all $q \in Q$. Indeed, define the DTAPA system

$$\Sigma_l = (\widetilde{\mathcal{X}}, (\widetilde{\mathcal{X}}_q, \widetilde{A}_q, 0, \widetilde{C}_q), (q_0, \widetilde{x}_0))$$

where $\widetilde{\mathcal{X}}_q = \{(x^T, 1)^T \mid x \in \mathcal{X}_q\} \subseteq \mathbb{R}^{n+1}$, $\widetilde{X} = \bigcup_{q \in Q} \widetilde{X}_q$ and

$\widetilde{A}_q = \begin{bmatrix} A_q & a_q \\ 0 & 1 \end{bmatrix}$, $\widetilde{C}_q = \begin{bmatrix} C_q & c_q \end{bmatrix}$, $\widetilde{x}_0 = (x_0^T, 1)^T$. Define the map $S : \mathcal{X} \to \widetilde{\mathcal{X}}$ by $S(x) = (x^T, 1)^T$. It is easy to see that $S : \Sigma \to \Sigma_l$ is a DTAPA isomorphism Hence, $y_\Sigma(x_0, k) = y_{\Sigma_l}(\widetilde{x}_0, k)$ and thus Σ is equivalent to Σ_l. We will call DTAPA systems for which $a_q = 0$ and $c_q = 0$ for all $q \in Q$, *linearised DTAPA systems* and we will use the following notation for them.

$$(\mathcal{X}, \mathcal{Y}, Q, (\mathcal{X}_q, A_q, C_q)_{q \in Q}, (q_0, x_0))$$

Notice that for any DTAPA system Σ the DTAPA system Σ_l is a linearised DTAPA and we will call Σ_l the *linearised DTAPA associated with* Σ.

Notice that DTAPA systems and PL systems from [15] are essentially the same objects. In fact, any DTAPA system can be transformed to a PL systems generating the same output map and conversely, any autonomous PL system can be written as a DTAPA systems generating the same output map.

2.6 Abstract Generating Series

The aim of this section is to present some simple results on objects, which are best thought of as a generalisation of generating convergent series. In order to formulate the results notation has to be set up. For each $u = (u_1, \ldots, u_k) \in \mathcal{U}$ denote

$$d\zeta_j[u] = u_j, j = 1, 2, \ldots, m, \quad d\zeta_0[u] = 1$$

Denote the set $\{0,1,\ldots,m\}$ by Z_m. For each $j_1,\cdots,j_k \in \mathrm{Z}_m$, $k \geq 0, t \in T$, $u \in PC(T,\mathcal{U})$ define $V_{j_1\cdots j_k}[u](t) \in \mathbb{R}$ as

$$V_{j_1\cdots j_k}[u](t) = \begin{cases} 1 & \text{if } k=0 \\ \int_0^t d\zeta_{j_k}[u(\tau)]V_{j_1,\ldots,j_{k-1}}[u](\tau)d\tau & \text{if } k>1 \end{cases}$$

For each $w_1,\ldots,w_k \in \mathrm{Z}_m^*$, $(t_1,\cdots,t_k) \in T^k$, $u \in PC(T,\mathcal{U})$ define

$$V_{w_1,\ldots,w_k}[u](t_1,\ldots,t_k) \in \mathbb{R}$$

by

$$V_{w_1,\ldots,w_k}[u](t_1,\ldots,t_k) = V_{w_1}(t_1)[u]V_{w_2}(t_2)[\mathrm{Shift}_1(u)]\cdots V(w_k)[\mathrm{Shift}_{k-1}(u)](t_k)$$

where $\mathrm{Shift}_i(u) = \mathrm{Shift}_{\sum_1^i t_i}(u)$, $i = 1,2,\ldots,k-1$. We will call $V_{w_1,\ldots,w_k}[u](t_1,\ldots,t_k)$ the *iterated integral of u at $t_1,\ldots,t_k$ with respect to $w_1,\ldots,w_k$*.

Let I_k, $k \in \mathbb{N}$ be a family of sets. Let $p \in \mathbb{N}$. Define the set $I = \bigcup_{k=1}^{\infty} I_k \times (\mathrm{Z}_m^*)^k$. That is, elements of I are of the form $(i,(w_1,\ldots,w_k))$, where $k \geq 0$, $i \in I_k$ and $w_1,\ldots,w_k \in Z_m^*$.

Definition 8 (Abstract convergent generating series). *A map $c : I \to \mathbb{R}^p$ is called an abstract generating convergent series on $\{I_k\}_{k\geq 0}$ with values in $\mathbb{R}^p$ if There exists $M > 0$ and a collection $K_i > 0, i \in I_k, k \geq 1$, such that for each $k \geq 1$, $(i,(w_1,\ldots,w_k)) \in I_k \times (Z_m^*)^k$*

$$||c((i,(w_1,\ldots,w_k)))|| < |w_1|!\cdots|w_k|!K_iM^{|w_1|}\cdots M^{|w_k|}$$

The map c is called an abstract globally convergent generating series, if for all $k \geq 1$, $(i,(w_1,\ldots,w_k) \in I_k \times (\mathrm{Z}_m^*)^k$, there exists $M \geq 0$, $K_i > 0, i \in I_k, k \geq 1$, such that

$$||c((i,(w_1,\ldots,w_k)))|| < K_iM^{|w_1|}\cdots M^{|w_k|}$$

The notion of generating convergent series is an extension of the notion of convergent power series from [67, 32]. If $I_k = \emptyset$, $k > 1$ and I_1 is a singleton set, then a generating convergent series in the sense of Definition 8 can be viewed as a convergent generating series in the sense of [67, 32, 82]. Convergent generating series in the latter sense play an important role in the theory of nonlinear control systems. The paper by Wang and Sontag [82] offers an excellent exposition of the topic and it contains many useful results which cannot be found elsewhere.

Let $c : I \to \mathbb{R}^p$ be a generating convergent series. Define the set $I^T = \bigcup_{k=1}^{\infty} I_k \times T^k$. For each $u \in PC(T,\mathcal{U})$ and $s = (i,(t_1,\ldots,t_k) \in I^T$ define the series

$$F_c(u,s) = \sum_{w_1,\ldots,w_k \in \mathrm{Z}_m^*} c((i,(w_1,\ldots,w_k))V_{w_1,\ldots,w_k}[u](t_1,\ldots,t_k) \tag{2.3}$$

We will prove that if either $\sum_{j=1}^{k} t_i$ is small enough and u restricted to $\sum_{j=1}^{k} t_i$ is small enough, or u, t_i are arbitrary and c is globally convergent generating series, then $F_c(u, s)$ is absolutely convergent.

Consider a map $u \in PC(T, \mathcal{U})$, let $S \in T$. Denote by $||u||_{S,\infty}$ the supremum of the restriction of u to $[0, S]$, that is

$$||u||_{S,\infty} = sup_{t\in[0,S]} ||u(t)||$$

where $||.||$ is the Euclidean norm on $\mathcal{U} = \mathbb{R}^m$. Since u is piecewise-continuous, and it has finite left- and right-hand side limits at points of discontinuity, we get that $||u||_{S,\infty}$ is finite for all $S \in T$.

Lemma 1. *Let $c : I \to \mathbb{R}^p$ is an abstract convergent generating series. Consider arbitrary $u \in PC(T, \mathcal{U})$, $s \in I^T$. If one of the following conditions hold, then $F_c(u, s)$ is absolutely convergent*

(a) The abstract convergent generating series c is an abstract globally convergent generating series.

(b) Assume $s = (i, (t_1, \ldots, t_k))$ and $T_s = \sum_{j=1}^{k} t_i$. Then with the notation of formula (2.3)

$$T_s \cdot ||u||_{T_s,\infty} < \frac{1}{2M(1+m)}$$

Proof. Assume that $s = (i, (t_1, \ldots, t_k)) \in I^T$. Since u is piecewise-continuous, there exists $R > 1$ such that
$sup\{|u_j(t)| \mid j = 1, 2, \ldots, m, t \in [0, \sum_1^k t_i]\} < R$. Then by induction it is easy to see that for all $w \in Z_m$ it holds that $|V_w[u](t_i)| \leq \frac{R^{|w|} t^{|w|}}{|w|!}$, consequently

$$|V_{w_1,\ldots,w_k}[u](t_1, \ldots, t_k)| = \Pi_{i=1}^{k} |V_{w_i}[u](t_i)| \leq \frac{t_1^{|w_1|}}{|w_1|!} \cdots \frac{t_k^{|w_k|}}{|w_k|!} R^{|w_1|+\cdots+|w_k|}$$

Assume that condition (a) of the statement of the Lemma holds. Then with the notation of Definition 8, for all $w_1, \ldots, w_k \in Z_m$.

$$||c((i, (w_1, \ldots, w_k)))|| < K_i M^{|w_1|} M^{|w_2|} \cdots M^{|w_k|}$$

We get that

$$\sum_{w_1,\ldots,w_k \in Z_m^*, |w_1|+\ldots+|w_k| \leq N} ||c((i, (w_1, \ldots, w_k)) V_{w_1,\ldots,w_k}[u](t_1, \ldots, t_k)|| \leq$$

$$\leq \sum_{l_1+\cdots+l_k \leq N} K_i (MR(m+1))^{l_1+\cdots+l_k} \frac{t_1^{l_1}}{l_1!} \cdots \frac{t_k^{l_k}}{l_k!} \leq \sum_{l=0}^{N} K_i (MRk(m+1))^l \frac{T^l}{l!} \leq$$
$$\leq K_i \exp(MRk(m+1)T)$$

where $T = \sum_1^k t_i$. That is, each finite sum of absolute values of coefficients of $F_c(u,s)$ is bounded by $K_i \exp(MRk(m+1)T)$, thus the series $F_c(u,s)$ is absolutely convergent.

Assume that condition (b) of the statement of the lemma holds. Then, R can be chosen such that $T_s R < \frac{1}{2M(m+1)}$. Moreover, for any $w_1, \ldots, w_k \in \mathrm{Z}_m^*$,

$$||c((i,(w_1,\ldots,w_k)))|| < |w_1|!|w_2|!\cdots|w_k|!K_i M^{|w_1|}M^{|w_2|}\cdots M^{|w_k|}$$

Thus,

$$\begin{aligned}
&\sum_{w_1,\ldots,w_k \le N} ||c((i,(w_1,\ldots,w_k)))V_{w_1,\ldots,w_k}[u](t_1,\ldots,t_k)|| < \\
&< \sum_{w_1,\ldots,w_k \le N} ||c((i,(w_1,\ldots,w_k)))|| \cdot |V_{w_1,\ldots,w_k}[u](t_1,\ldots,t_k)| < \\
&< \sum_{l_1+\cdots+l_k \le N} K_i M^{l_1}\cdots M^{l_k}(1+m)^{\sum_{j=1}^k l_j} l_1!\cdots l_k! R^{\sum_{j=1}^k l_i} \frac{t_1^{l_1}}{l_1!}\frac{t_2^{l_2}}{l_2!}\cdots\frac{t_k^{l_k}}{l_k!} = \\
&= \sum_{l_1+\cdots+l_k \le N} K_i (MR(1+m)T_s)^{l_1+\cdots+l_k} = \sum_{l=0}^{N}(\sum_{j=1}^{k-1}\binom{l}{j})K_i(MR(1+m)T_s)^l < \\
&< \sum_{l=0}^{N} 2^l K_i (MR(1+m)T_s)^l < \sum_{l=0}^{\infty} K_i(2MR(1+m)T_s)^l
\end{aligned}$$

In the last step we used the fact that $(2MR(1+m)T_s) = (T_s \cdot R)(2M(1+m)) < 1$. Thus, each finite sum of absolute values of elements of $F_c(u,s)$ is bounded by $\sum_{l=1}^{\infty} K_i(2MR(1+m)T_s)^l < +\infty$, hence we get that the series $F_c(u,s)$ is absolutely convergent. □

Let's introduce the following notation. If c is an abstract *globally* convergent generating series, then let $\mathrm{dom}(F_c) = PC(T,\mathcal{U}) \times I^T$. Otherwise, let

$$\mathrm{dom}(F_c) = \{(u,s) \in PC(T,\mathcal{U}) \times I^T \mid ||u||_{T_s,\infty} \cdot T_s < \frac{1}{2M(1+m)}\}$$

In fact we can define a function F_c

$$F_c : \mathrm{dom}(F_c) \ni (u,s) \mapsto F_c(u,s) \in \mathbb{R}^p$$

Notice that $F_c(u,s)$ depends only on the restriction of u to $[0,T_s]$.

Lemma 2. *Let $c : I \to \mathbb{R}^p$ be an abstract generating convergent series. Then the following holds. For each $s = (i,(t_1,\ldots,t_k)) \in I^T$, $u,v \in PC(T,\mathcal{U})$, $(u,s),(v,s) \in$ dom(F_c),*

$$(\forall t \in [0,T_s] : u(t) = v(t)) \implies F_c(u,s) = F_c(v,s)$$

It is a natural to ask whether c determines F_c uniquely. The following result answers this question.

Lemma 3. *let $d, c : I \to \mathbb{R}^p$ be two convergent generating series. If $F_c = F_d$, then $c = d$.*

In order to prove the lemma above, we will need the following result.

Lemma 4. *For each $w \in \mathrm{Z}_m^*$:*

$$V_w[u](t_1 + t_2) = \sum_{s,z \in \mathrm{Z}_m^*, sz=w} V_s[u](t_1) V_z[\mathrm{Shift}_{t_1}(u)](t_2)$$

Proof of Lemma 4. We proceed by induction on $|w|$. Assume that $|w| = 1$, that is, $w = j \in \mathrm{Z}_m$. Then

$$V_w[u](t_1 + t_2) = \int_0^{t_1+t_2} d\zeta_j(\tau) d\tau = \int_0^{t_1} d\zeta_j(\tau) d\tau + \\ \int_0^{t_2} d\zeta_j(t_1 + \tau) d\tau = V_j[u](t_1) + V_j[\mathrm{Shift}_{t_1}(u)](t_2)$$

Assume that $w = vj$. Then

$$V_w[u](t_1 + t_2) = \int_0^{t_1+t_2} d\zeta_j(\tau) V_v[u](\tau) d\tau = \\ = \int_0^{t_1} d\zeta_j(\tau) V_v[u](\tau) d\tau + \int_0^{t_2} d\zeta_j(t_1 + \tau) = \\ = V_v[u](t_1 + \tau) d\tau V_w[u](t_1) + \int_0^{t_2} d\zeta_j(t_1 + \tau) V_v[u](t_1 + \tau) d\tau$$

By induction hypothesis we get that

$$\int_0^{t_2} d\zeta_j(t_1 + \tau) V_v[u](t_1 + \tau) d\tau = \sum_{sz=v, s,z \in \mathrm{Z}_m^*} V_s[u](t_1) \times \\ \times \int_0^{t_2} d\zeta_j(t_1 + \tau) V_z[\mathrm{Shift}_{t_1}(u)](\tau) d\tau = \sum_{sz=v, s,z \in \mathrm{Z}_m^*} V_s[u](t_1) V_{zj}[\mathrm{Shift}_{t_1}(u)](t_2)$$

That is, we get that

$$V_w[u](t_1 + t_2) = V_w[u](t_1) + \sum_{sz=v, s,z \in \mathrm{Z}_m^*} V_s[u](t_1) V_{zj}[\mathrm{Shift}_{t_1}(u)](t_2) = \\ \sum_{sz=w, s,z, \in \mathrm{Z}_m^*} V_s[u](t_1) V_z[\mathrm{Shift}_{t_1}(u)](t_2)$$

$\square$

Proof of Lemma 3. We will use the same method as in [83]. In fact, our proof is an easy generalisation of the proof presented in [83].

Assume that $F_d = F_c$. It is equivalent to $F_{d-c} = 0$. That is, it is enough to show that if $F_c = 0$ then $c = 0$.

Assume that $F_c(u, s) = 0$ for all $(u, s) \in dom(F_c)$.

Assume that $w_i = w_{i,1} \cdots w_{i,k_i}$, $w_{i,1}, \dots, w_{i,k_i} \in \mathrm{Z}_m$, $k_i \geq 0$, $i = 1, \dots, k$. Let $u_{i,1}, \dots, u_{i,k_i} \in \mathcal{U}$ and $\tau_{i,1}, \dots, \tau_{i,k_i} \in T$, $u_{k_i}(t) = u_{i,j}$ for all $t \in [\sum_{z=1}^{j-1} \tau_{i,z}, \sum_{z=1}^{j} \tau_{i,z})$, $j = 1, \dots, k_i$. Then it follows from Lemma 4 that for all $w \in \mathrm{Z}_m^*$,

$$V_w[u_{k_i}](\sum_{z=1}^{k_i} \tau_{i,z}) = \sum_{v_1,\dots,v_{k_i} \in \mathrm{Z}_m^*, v_1 \cdots v_{k_i} = w} V_{v_1}[u_{i,1}](\tau_{i,1}) \cdots V_{v_{k_i}}[u_{i,k_i}](\tau_{i,k_i})$$
$$= \sum_{v_1,\dots,v_{k_i}, v_1 \cdots v_{k_i} = w_i} u_{i,1}^{v_1} u_{i,2}^{v_2} \cdots u_{i,k_i}^{v_{k_i}} \tau_{i,1}^{|v_1|} \cdots \tau_{i,k_i}^{|v_{k_i}|} \frac{1}{|v_1|!|v_2|! \cdots |v_{k_i}|!}$$

where we used the following notation. If $u = (u_1, \dots, u_m)^T \in \mathcal{U} = \mathbb{R}^m$, then $u^{j_1 \cdots j_d} = u_{j_1} u_{j_2} \cdots u_{j_d}$, where $u_0 = 1$ is assumed.

Thus, the following equality holds

$$\frac{d}{d\tau_{i,1} d\tau_{i,2} \cdots d\tau_{i,k_i}} V_v[u_{k_i}](\sum_{j=1}^{k_i} \tau_{i,j})|_{\tau_{i,j}=0, j=1,\dots,k_i} =$$
$$\begin{cases} u_{i,1}^{v_1} \cdots u_{i,k_i}^{v_{k_i}} & \text{if there exists } v_1 v_2 \cdots v_{k_i} = v \text{ and } v_1, \dots, v_{k_i} \in \mathrm{Z}_m \\ 0 & \text{otherwise} \end{cases}$$

That is,

$$\frac{d}{du_{i,1}^{w_{i,1}} \cdots du_{i,k_i}^{w_{i,k_i}}} \frac{d}{d\tau_{i,1} d\tau_{i,2} \cdots d\tau_{i,k_i}} V_v[u_{w_i}](\sum_{j=1}^{k_i} \tau_{i,j})|_{\tau_{i,j}=0, j=1,\dots,k_i} = \begin{cases} 1 & \text{if } v = w_i \\ 0 & \text{otherwise} \end{cases}$$

Let $\xi \in I_k$, $\tau_{i,1}, \dots, \tau_{i,k_i} \in T$, $i = 1, \dots, k$. Define the map

$$g_\xi : W \times V \ni (\tau_{1,1}, \dots, \tau_{1,k_1}, \dots, \tau_{k,1},$$
$$\dots, \tau_{k,k_k}, u_{1,1}, \dots, u_{i,k_1}, \dots, u_{k,1}, \dots, u_{k,k_k}) \mapsto F_c(\widetilde{u}, \widetilde{s})$$

where $\widetilde{u}(t) = u_{i,j}$ if $t \in [\sum_{z=1}^{i} \sum_{l=1}^{j-1} \tau_{z,l}, \sum_{z=1}^{i} \sum_{l=1}^{j} \tau_{z,l})$ for some $i = 1, \dots, k$, $j = 1, \dots, k_i$, and $s = (\xi, (\sum_{j=1}^{k_1} \tau_{1,j}, \sum_{j=1}^{k_2} \tau_{2,j}, \dots, \sum_{j=1}^{k_k} \tau_{k,j}))$, and $W \subseteq T^{\sum_{i=1}^{k} k_i}$, $V \subseteq \mathcal{U}^{\sum_{i=1}^{k} k_i}$ are suitably small neighbourhoods such that $(\widetilde{u}, \widetilde{s}) \in \mathrm{dom}(F_c)$. It is easy to see that $F_c = 0$ implies that $g_\xi = 0$ for all $\xi = 0$ and g_ξ is an analytic mapping.

Notice that

$$g_\xi(\tau_{1,1},\ldots,\tau_{k,kk},u_{1,1},\ldots,u_{k,k_k}) =$$

$$\sum_{v_1,\ldots,v_k\in \mathrm{Z}_m^*} c(\xi,(v_1,\ldots,v_k))V_{v_1}[u_{k_1}](\sum_{j=1}^{k_1}\tau_{1,j})V_{v_2}[u_{k_2}](\sum_{j=1}^{k_2}\tau_{2,j},\ldots,\sum_{j=1}^{k_k}\tau_{k,j}) =$$

$$= \sum_{v_{1,1},\ldots,v_{1,k_1},\ldots,v_{k,1}\ldots v_{k,k_k}\in \mathrm{Z}_m^*} c((\xi,(v_{1,1}\cdots v_{1,k_1},v_{2,1}\cdots v_{2,k_2},\ldots,v_{k,1}\cdots v_{k,k_k})))\times$$

$$\times\, \Pi_{i=1}^{k}\Pi_{j=1}^{k_i}u_{i,j}^{v_{i,j}}\frac{\tau_{i,j}^{|v_{i,j}|}}{|v_{i,j}|!}$$

Denote by D_i the operator $D_{w_i} = \frac{d}{du_{i,1}^{w_{i,1}}\cdots du_{i,k_i}^{w_{i,k_i}}}\frac{d}{d\tau_{i,1}\cdots d\tau_{i,k_i}}$. Thus, for each $i = 1,\ldots,k$,

$$D_{w_1}D_{w_2}\cdots D_{w_k}g_\xi|_{\tau_{i,j}=0,u_{i,j}=0,i=1,\ldots,k,j=1,\ldots,k_i} = c((\xi,(w_1,\ldots,w_k)))$$

But if $g_\xi = 0$, then for all $w_1,\ldots,w_k \in \mathrm{Z}_m^*$, $D_{w_1}\cdots D_{w_k}g_\xi = 0$, i.e.

$$c((\xi,(w_1,\ldots,w_k))) = 0$$

□

Chapter 3

Hybrid Formal Power Series

The aim of this chapter is to present the necessary "abstract nonsense" which will be used for developing realization theory for a number of classes of hybrid systems. As the title of the chapter indicates, we will be mostly concerned with formal power series-like objects in this chapter.

Although the output trajectories of hybrid systems are functions of time, for hybrid systems without guards they are piecewise-analytic maps. Moreover, they are such that for any switching sequence the dependence of the continuous output on the times between consecutive switches is analytic. Thus, for small enough switching times the behaviour of the input-output maps is determined by their high-order derivatives at zero with respect to the relative switching times. Moreover, if the continuous valued parts of the input-output maps are *entire* analytic functions of the switching times, the high-order derivatives determine the *whole global behaviour* of the input-output maps. Besides, if the input-output maps admit a hybrid system realization, then the high-order derivatives can be expressed as composition of the vector fields of the system with the reset and readout maps evaluated at the initial state.

Thus, for a number classes of hybrid systems, the realization problem turns out to be equivalent to the existence of a particular representation of the sequence of high-order derivatives of the input-output maps. For example, for hybrid systems with linear or affine vector fields with linear reset maps and linear readout maps, the existence of a realization yields that there exists a finite collection of matrices such that each high-order derivative can be expressed as a product of those matrices taken in a particular order. With some extra condition the existence of such a representation of the high-order derivatives is also sufficient for existence of a hybrid realization.

In this way, we arrive to variations of the following problem. Given a sequence of real numbers, indexed by words over a certain alphabet, when does this sequence admit a representation of the following form. There exists finitely many matrices of suitable dimensions indexed by the elements of the alphabet, such that any element of the sequence indexed by some word w can be represented as a product of the matrices (possibly multiplied from left- and right- by suitable vectors) above taken in the order prescribed by a subword of w chosen in a particular way.

In the simplest case the problem above amounts to the classical problem of rationality of formal power series in several non-commuting indeterminates (indeterminates correspond to the letters of the alphabet and multiplication of indeterminates correspond to concatenation of words, hence non-commutativity). The theory of rational formal power series is a classical topic, it has been around in different forms for over forty years. It has been successfully applied to realization problem of several classes of control systems, the most well-known one is the class of bilinear systems.

Unfortunately, for hybrid systems the framework of rational formal power series is no longer sufficient (although it is still suitable for handling switched systems). The reason for that is that we have to take into account the discrete output and the dependence of the value of the high-order derivatives on the change of the discrete state. In order to capture these specific features of hybrid systems, we will have to introduce a new formal framework, the framework of what we will call *hybrid formal power series*

One can think of formal power series in non-commuting indeterminates as maps assigning each to sequence of indeterminates a real vector in some vector space $\mathbb{R}^p$. Hybrid formal power series are pairs of consisting of a discrete-valued input-output map and a classical formal power series. We will be interested in *families of hybrid formal power series*. We will then try to find conditions for existence of a hybrid representation of such a family of hybrid formal power series. The notion of hybrid representation is analogous to the notion of rational formal power series representation. Roughly speaking, a hybrid representation is a composition of several rational formal power series representations with a finite Moore-automaton.

Within such a family, certain hybrid formal power series will be grouped together, such that such the members of each group will have the same discrete-valued parts but different continuous valued parts. The idea is that the common discrete value part should be realized by one of the state of the Moore-automaton of the hybrid representation and the different classical formal power series should be represented by different states of the representations belonging to discrete states reachable from the discrete state realizing the discrete valued part. If it seems a bit confusing to the reader, then we suggest to wait until the formal definition is presented.

The theory of hybrid formal power series presented in this paper relies very much on the classical theory of rational formal power series [64, 65, 4] and automata theory [17, 24]. In fact, it combines the two theories. The main questions will be the following.

Existence of a hybrid representation When does such a collection of hybrid power series admit a *hybrid representation* ?

Minimality of hybrid representation What is the smallest possible hybrid representation of a family of hybrid formal power series ? How can such hybrid representations be characterised ? Is there always a smallest possible hybrid representation of a family of hybrid formal power series ? Is such a minimal hybrid representation unique ?

Partial realization theory How to construct a hybrid representation for a family of hybrid formal power series using only finite number of data ?

The results obtained for rational hybrid representations are very similar to those of rational formal power series and finite automata. In fact, we will proceed as follows. We will associate with each family of hybrid formal power series a family of classical formal power series and a family of discrete input-output maps. It turns out that there is a correspondence between rational representations of this family of formal power series and automaton realizations of the family of discrete input-output maps on the one hand and hybrid representations of the original family of hybrid formal power series on the other hand. Let us formulate the main results on hybrid formal power series in an informal way.

Existence of a hybrid representation A family of hybrid formal power series has a hybrid representation, i.e., it is rational if and only if the corresponding family of classical formal power series has a rational representation, i.e, it is rational and the corresponding family of discrete input-output maps has a realization by finite a Moore-automaton.

Minimality of hybrid representations If a family of hybrid formal power series has a hybrid representation, then it has a minimal hybrid representation. A hybrid representation is minimal if and only if it is reachable and observable. Any two minimal hybrid representations of the same family of hybrid formal power series are isomorphic. Minimality, observability and reachability can be checked algorithmically. Any hybrid representation can be transformed to a minimal one and the transformation can be done by an algorithm.

Partial realization theory If the number of available data points is big enough and the family of hybrid formal power series is finite, then it is possible to construct a minimal hybrid representation of the family of hybrid formal power series from finitely many data points. The precise conditions for the number of data points are similar to the conditions in partial realization theory of linear and bilinear systems.

All the results announced above will be discussed in this chapter with the exception of partial realization theory, presentation of which will be postponed until Section 10.3 of Chapter 10.

The structure of the chapter is the following. Section 3.1 presents a concise treatment of the results on classical formal power series and their rational representations. In this thesis we will need the theory of *rational family of formal power series*, that is, we will be looking at rational representations of a *family* of formal power series. Most of the existing literature deals with rational representations of a *single* formal power series. The only exception the author is aware of is [84], but unfortunately the results we need in this thesis are not explicitly stated there. However, the existing theory can be easily (almost trivially) extended to deal with families of formal power series and we will present this extension in Section 3.1.

Section 3.2 presents realization theory of Moore-automata. Again, we will need theory for realization by Moore-automata of a *family of input-output maps*. The theory of realization of a *single input-output map* by a Moore-automaton is a classical topic, and we will need a simple extension of the existing theory, which will be discussed in Section 3.2. Finally, Section 3.3 deals with the main topic of the chapter, the theory of rational families of hybrid formal power series.

3.1 Theory of Formal Power Series

The section presents results on formal power series. The material of this section is based on the classical theory of formal power series, see [4, 43]. However, a number of concepts and results are extensions of the standard ones. In particular, the definition of the rationality is more general than that one occurring in the literature. Consequently, the theorems characterising minimality are extensions of the well-known results. These generalisations and extensions are rather straightforward and can be easily derived in a manner similar to the classical case. In order to keep the exposition self-contained and complete, the proofs of those theorems which are not part of the classical theory, will be presented too.

Let X be a finite alphabet. A *formal power series* S with coefficients in $\mathbb{R}^p$ is a map

$$S : X^* \to \mathbb{R}^p$$

We denote by $\mathbb{R}^p \ll X^* \gg$ the set of all formal power series with coefficients in $\mathbb{R}^p$. Let $S \in \mathbb{R}^p \ll X^* \gg$. For each $i = 1, \ldots, p$ define the formal power series $S_i \in \mathbb{R} \ll X^* \gg$ by the following equation

$$S_i(w) \quad = \quad (S(w))_i = e_i^T S(w)$$

where e_i is the ith unit vector of $\mathbb{R}^p$. Let J be an arbitrary (possibly infinite) set. An indexed set of formal power series $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ with the index set J is called *rational* if there exists a vector space $\mathcal{X}$ over $\mathbb{R}$, $\dim \mathcal{X} < +\infty$ and linear maps

$$C : \mathcal{X} \to \mathbb{R}^p, \quad A_\sigma \in \mathcal{X} \to \mathcal{X} \quad , \sigma \in X$$

and an indexed set with the index set J

$$B = \{B_j \in \mathcal{X} \mid j \in J\}$$

such that for all $j \in J$, $\sigma_1, \ldots, \sigma_k \in X$, $k \geq 0$

$$S_j(\sigma_1 \sigma_2 \cdots \sigma_k) = C A_{\sigma_k} A_{\sigma_{k-1}} \cdots A_{\sigma_1} B_j.$$

The 4-tuple $R = (\mathcal{X}, \{A_x\}_{x \in X}, B, C)$ is called a *representation* of S. The number $\dim \mathcal{X}$ is called the *dimension* of the representation R and it is denoted by $\dim R$. We will refer to $\mathcal{X}$ as the *state-space* of the representation R. A formal power series $S \in \mathbb{R}^p \ll X^* \gg$ is called rational if the indexed set $\{S_j \mid j \in \{\emptyset\}\}$, $S_\emptyset = S$, with the singleton index $\{\emptyset\}$, is rational. That is, S is rational is the above sense if and only if it is rational in the classical sense.

In fact, a representation can be viewed as a Moore-automaton with the state-space $\mathcal{X}$, with input space X^*, with output space $\mathbb{R}^p$. The state transition function $\delta : \mathcal{X} \times X \to \mathcal{X}$ is given by the linear map $\delta(x, \sigma) = A_\sigma x$. The output map $\mu : \mathcal{X} \to \mathbb{R}^p$ is given by $\mu(x) := Cx$. The set of initial conditions is given by $\{B_j \mid j \in J\}$. The problem of finding a representation for a set of formal power series Ψ is equivalent to finding a realization of Ψ by a Moore-automaton of the form described above. That is, finding a representation is equivalent to finding a realization by a special class of Moore-automaton. We will not pursue the analogy with automaton theory in this paper. Instead, to keep the presentation self-contained, we will built the theory directly.

A representation R_{min} of Ψ is called *minimal* if for each representation R of Ψ

$$\dim R_{min} \leq \dim R$$

In the sequel the following short-hand notation will be used. Let $A_\sigma : \mathcal{X} \rightarrow \mathcal{X}, \sigma \in X$ be linear maps. Then

$$A_w := A_{w_k} A_{w_{k-1}} \cdots A_{w_1} \ , \ w = w_1 w_2 \cdots w_k \in X^*, w_1, \dots, w_k \in X$$

Let $R = (\mathcal{X}, \{A_z\}_{z \in X}, B, C)$, $\widetilde{R} = (\widetilde{\mathcal{X}}, \{\widetilde{A}_z\}_{z \in X}, \widetilde{B}, \widetilde{C})$ be two representations. A linear map $T : \mathcal{X} \rightarrow \widetilde{\mathcal{X}}$ is called a *representation morphism* from R to $\widetilde{R}$ and is denoted by $T : R \rightarrow \widetilde{R}$ if the following equalities hold

$$TA_z = \widetilde{A}_z T, \forall z \in X, \quad TB_j = \widetilde{B}_j, \forall j \in J, \quad C = \widetilde{C} T$$

Using the automaton-theoretic interpretation discussed one can think of representation morphisms as Moore-automaton morphisms which are linear morphisms between the state-spaces. The representation morphism T is called surjective, injective, isomorphism if T is a surjective, injective or isomorphism respectively if viewed as a linear vector space morphism.

Let $L \subseteq X^*$. If L is a regular language then, by the classical result [4], the power series $\bar{L} \in \mathbb{R} \ll X^* \gg$, $\bar{L}(w) = \begin{cases} 1 & \text{if } w \in L \\ 0 & \text{otherwise} \end{cases}$ is a rational power series. Consider two power series $S, T \in \mathbb{R}^p \ll X^* \gg$. Define the *Hadamard product* $S \odot T \in \mathbb{R}^p \ll X^* \gg$ by

$$(S \odot T)_i(w) = S_i(w) T_i(w), , \ i = 1, \dots, p$$

Let $w \in X^*$ and $S \in \mathbb{R}^p \ll X^* \gg$. Define $w \circ S \in \mathbb{R}^p \ll X^* \gg$ – the *left shift of S by w* by

$$\forall v \in X^* : w \circ S(v) = S(wv)$$

The following statements are generalisations of the results on rational power series from [4, 65]. Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$. be an indexed set of formal power series with the index set J. Define the set W_Ψ by

$$W_\Psi = \mathrm{Span}\{w \circ S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J, w \in X^*\}$$

Define the Hankel-matrix H_Ψ of Ψ as the infinite matrix $H_\Psi \in \mathbb{R}^{(X^* \times I) \times (X^* \times J)}$, $I = \{1, 2, \dots, p\}$ and $(H_\Psi)_{(u,i)(v,j)} = (S_j)_i(vu)$.

Theorem 1. *Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$.*

(i) Assume that $\dim W_\Psi < +\infty$ holds. Then a representation R_Ψ of Ψ is given by

$$R_\Psi = (W_\Psi, \{A_\sigma\}_{\sigma \in X}, B, C)$$

– $A_\sigma : W_\Psi \to W_\Psi$, $\forall T \in W_\Psi$: $A_\sigma(T) = \sigma \circ T$, $\sigma \in X$.

– $B = \{B_j \in W_\Psi \mid j \in J\}$, $B_j = S_j$ *for each* $j \in J$.

– $C : W_\Psi \to \mathbb{R}^p$, $C(T) = T(\epsilon)$.

(ii) The following equivalences hold

$$\Psi \textit{ is rational} \iff \dim W_\Psi < +\infty \iff \text{rank } H_\Psi < +\infty$$

Moreover, $\dim W_\Psi = \text{rank } H_\Psi$ *holds.*

Proof. `Part (i)`
Notice that for any $w \in X^*$, $w = w_1 \cdots w_k$, $w_1, \ldots, w_k \in X$ and for any $T \in \mathbb{R}^p \ll X^* \gg$

$$w \circ T = w_k \circ (w_{k-1} \circ (\cdots (w_1 \circ T) \cdots)))$$

Since $B_j = S_j$, and $A_\sigma T = \sigma \circ T$, we get that for all $w \in X^*$

$$w \circ S_j = A_w S_j = A_w B_j$$

But $S_j(w) = w \circ S_j(\epsilon) = C(w \circ S_j)$, so we get that $S_j(w) = CA_w B_j$, i.e., R_Ψ is indeed a representation of Ψ.

`Part (ii)`
The statement

$$\dim W_\Psi < +\infty \implies \Psi \text{ is rational}$$

follows from part (i) of the theorem. We will prove that Ψ rational $\implies \dim W_\Psi < +\infty$. Assume $R = (\mathcal{X}, A_{\sigma\sigma\in X}, B, C)$ is a representation of Ψ. Let $\dim \mathcal{X} = n$ and let $e_l \in \mathcal{X}, l = 1, 2, \ldots, n$ be a basis of $\mathcal{X}$. Define $Z_l \in K^p \ll X^* \gg$ by $Z_l(w) = CA_w e_l$, $w \in X^*$. For each $j \in J$ there exist $\alpha_{j,1}, \ldots, \alpha_{j,n} \in \mathbb{R}$ such that $B_j = \sum_{l=1}^n \alpha_{j,l} e_l$. We get that

$$S_j(w) = CA_w B = \sum_{l=1}^n \alpha_{j,l} CA_w e_l = \sum_{l=1} \alpha_{j,l} Z_l(w)$$

On the other hand

$$w \circ Z_l(v) = Z_l(wv) = CA_v A_w e_l = \sum_{k=1}^n \beta_{k,l} CA_v e_k = \sum_{k=1}^n \beta_{k,l} Z_k$$

where $\mathcal{X} \ni A_w e_l = \sum_k^n \beta_{k,l} e_k$. Thus, $w \circ S_j, S_j \in \text{Span}\{Z_i \mid i = 1, \ldots, n\}$ holds, which implies that $W_\Psi \subseteq \text{Span}\{Z_i \mid i = 1, \ldots, n\}$. That is, $\dim W_\Psi < +\infty$.

Finally, we show that $\dim W_\Psi < +\infty \iff \text{rank } H_\Psi < +\infty$. In fact, $\dim W_\Psi = \text{rank } H_\Psi$ and W_Ψ is naturally isomorphic to the span of column vectors of H_Ψ. Indeed, it easy to see that $w \circ S_j$ corresponds to $(H_\Psi)_{.,(w,j)}$ and the rest of the statement follows easily from this observation. $\square$

The representation R_Ψ is called *free*. Using the theorem above we can easily show that

Lemma 5. *The indexed set formal power series $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ is rational if and only if the indexed set of formal power series $\Xi = \{S_{(i,j)} \in \mathbb{R}^p \mid (i,j) \in \{1,\dots,p\} \times J\}$ is rational, where $S_{(i,j)} = (S_j)_i$, $j \in J, i = 1,\dots,p$.*

Proof. Indeed, define $pr_i : \mathbb{R}^p \to \mathbb{R}$ by $pr_i(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_p) = x_i$ for $i = 1,\dots,p$. It is easy to see that pr_i is linear and $S_{i,j} = pr_i \circ S_j$. Define the linear maps $P_i : W_\Psi \ni T \mapsto pr_i \circ T$, $i = 1,\dots,p$. Notice that $\bigcap_{i=1}^p \ker P_i = \{0\}$. It is easy to see that $W_\Xi = \sum_{i=1}^p P_i(W_\Psi)$. That is, $\dim W_\Psi < +\infty \implies \dim W_\Xi < +\infty$. Conversely, assume that $\dim W_\Xi < +\infty$. Define $P : W_\Psi \to \bigoplus_{i=1}^p Z_i$, $Z_i = W_\Xi$, $P(T) = \sum_{i=1}^p z_i$, $\forall i = 1,\dots,p : z_i = P_i(T) \in Z_i$. Then $\ker P = \bigcap_{i=1}^p \ker P_i = \{0\}$, thus $\dim W_\Psi < p \cdot \dim W_\Xi < +\infty$. $\square$

Theorem 1 implies the following lemma.

Lemma 6. *Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ and $\Theta = \{T_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ be rational indexed sets. Then $\Psi \odot \Theta := \{S_j \odot T_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ is a rational set. Moreover,* $\operatorname{rank} H_{\Psi\odot\Theta} \le \operatorname{rank} H_\Psi \cdot \operatorname{rank} H_\Theta$.

Proof. By Theorem 1 it is enough to show that
$\dim W_{\Psi\odot\Theta} < +\infty$. First, notice that for any $T_1, T_2 \in \mathbb{R}^p \ll X^* \gg$ it holds that $w \circ (T_1 \odot T_2) = (w \circ T_1) \odot (w \circ T_2)$. Indeed, $w \circ (T_1 \odot T_2)_l(v) = (T_1)_l \odot (T_2)_l(wv) = (T_1(wv))_l(T_2(wv))_l = (w \circ T_1)_l(v)(w \circ T_2)_l(v) = ((w \circ T_1) \odot (w \circ T_2))_l(v)$. Then we get that

$$\begin{aligned} W_{\Psi\odot\Theta} &= \operatorname{Span}\{(w \circ S_j) \odot (w \circ T_j) \mid j \in J, w \in X^*\} \\ &\subseteq \operatorname{Span}\{(w \circ S_j) \odot (v \circ T_z) \mid z, j \in J, w, v \in X^*\} \end{aligned}$$

Let $w_l \circ T_{z_l}, l = 1,2,\dots m, z_l \in J, w_l \in X^*$be a basis of W_Θ. Let $v_k \circ S_{j_k}, v_k \in X^*, k = 1,2,\dots n, j_k \in J$ be a basis of W_Ψ. Then it is easy to see that
$\operatorname{Span}\{(w \circ S_j) \odot (v \circ T_z) \mid z, j \in J, w, v \in X^*\}$ is spanned by $w_k \circ S_{j_k} \odot v_l \circ T_{z_l}, l = 1,2,\dots,m, k = 1,2,\dots n, j_k, z_l \in J$. That is, $\dim W_{\Psi\odot\Theta} \le \dim W_\Psi \cdot \dim W_\Theta$. $\square$

The classical version of the lemma above can be found in [4].

Let $R = (\mathcal{X}, \{A_\sigma\}_{\sigma\in X}, B, C)$ be a representation of $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$. Define the subspaces W_R and O_R of $\mathcal{X}$ by

$$\begin{aligned} W_R &= \operatorname{Span}\{A_w B_j \mid w \in X^*, j \in J\} \\ O_R &= \bigcap_{w \in X^*} \ker C A_w \end{aligned}$$

The sets above have the following automaton-theoretic interpretation. The subspace W_R is the span of states reachable by a $w \in X^*$ from an initial state B_j. Two states x_1, x_2 are indistinguishable, i.e.

$$CA_w x_1 = CA_w x_2 \text{ for all } w \in X^*$$

if and only if $x_1 - x_2 \in O_R$. That is, the automaton corresponding to R is reduced if and only if $O_R = \{0\}$. We will say that the representation R is *reachable* if $\dim W_R = \dim R$, and we will say that R is *observable* if $O_R = \{0\}$.

Lemma 7. *Let $R = (\mathcal{X}, \{A_\sigma\}_{\sigma \in X}, B, C)$ be a representation of Ψ. Then there exists a representation*

$$R_{can} = (\mathcal{X}_{can}, \{A^{can}_\sigma\}_{\sigma \in X}, B^{can}, C^{can})$$

of Ψ such that R_{can} is reachable and observable, and $\mathcal{X}_{can}$ is isomorphic to the quotient $W_R/(O_R \cap W_R)$.

The word *can* in R_{can} stands for canonical. A system which is both reachable and observable is often called canonical and R_{can} is reachable and observable, hence the notation.

Proof of Lemma 7. Let $R = (\mathcal{X}, \{A_\sigma\}_{\sigma \in X}, B, C)$ be a representation of Ψ. Define $R_r = (W_R, \{A^r_\sigma\}_{\sigma \in X}, B^r, C^r)$ by $A^r_\sigma = A_\sigma|_{W_R}$, $B^r_j = B_j \in W_R$ and $C^r = C|_{W_R}$. Since W_R is invariant w.r.t A_σ, the representation R_r is well defined. It is easy to see that $C^r A^r_w B^r_j = CA_w B_j$, so R_r is a representation of Ψ. It is easy to see that $W_{R_r} = W_R$ and $O_{R_r} = O_R \cap W_R$. Define $R_o = (W_R/O_{R_r}, \{\widetilde{A}_\sigma\}_{\sigma \in X}, \widetilde{B}, \widetilde{C})$ by $\widetilde{A}_\sigma[x] = [A^r_\sigma x]$, $\widetilde{B}_j = [B^r_j]$ and $\widetilde{C}[x] = C^r x$, for each $x \in W_R$. Here $[x]$ denotes the equivalence class of W_R/O_{R_r} represented by $x \in W_R$. The representation R_o is well defined. Indeed, if $x_1 - x_2 \in O_{R_r}$, then $\forall w \in X^* : C^r A^r_w (x_1 - x_2) = 0$, so we get that $\forall w \in X^* : C^r A^r_w A^r_\sigma (x_1 - x_2) = 0$. That is $A^r_\sigma x_1 - A^r_\sigma x_2 \in O_{R_r}$. It implies that $\widetilde{A}_\sigma$ is well defined. It is straightforward to see that $\widetilde{B}_j$ is well defined. Since $x_1 - x_2 \in O_{R_r}$ implies that $x_1 - x_2 \in \ker C^r$, we get that $\widetilde{C}$ is well defined too. Moreover $\widetilde{C}\widetilde{A}_w\widetilde{B}_j = CA_w B_j$, so R_o is a representation of Ψ. It is easy to see that $O_{R_o} = \{0\}$. That is, R_o is observable. Moreover, R_o is reachable, since $\mathrm{Span}\{\widetilde{A}_w \widetilde{B_j} \mid w \in X^*, j \in J\} = \mathrm{Span}\{[A^r_w B^r_j] \mid j \in J, w \in X^*\} = W_R/O_{R_r}$. $\square$

Theorem 2 (Minimal representation). *Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$. The following are equivalent.*

(i) $R_{min} = (\mathcal{X}, \{A^{min}_\sigma\}_{\sigma \in X}, B^{min}, C^{min})$ is a minimal representation of Ψ.

(ii) R_{min} is reachable and observable.

(iii) If R is a reachable representation of Ψ then there exists a surjective representation morphism $T : R \to R_{min}$.

(iv) rank $H_\Psi = \dim W_\Psi = \dim R_{min}$

Proof. *(i) $\implies$ (ii)*
Assume that $W_{R_{min}} \neq \mathcal{X}$ or $O_{R_{min}} \neq \{0\}$. Then by Lemma 7 there exists $R_{can} = (R_{min})_{can}$ representing Ψ such that

$$\dim R_{can} = \dim W_{R_{min}}/(O_{R_{min}} \cap W_{R_{min}}) < \dim R_{min}$$

which implies that R_{min} is not a minimal representation.

(ii) $\implies$ (iii)
Let $R = (\mathcal{X}, \{A_z\}_{z\in X}, B, C)$ be a reachable representation of Ψ. Notice that $CA_wB_j = S_j(w) = C^{min}A_w^{min}B_j^{min}$. Define T by $T(A_wB_j) = A_w^{min}B_j^{min}$. We will show that T is well-defined. Assume that $A_uB_j = \sum_{k=1}^{l} \alpha_k A_{w_k} B_{j_k}$ holds for some $u, w_1, \dots w_l \in X^*, j_1, \dots, j_l \in J$, $\alpha_1, \dots, \alpha_l \in \mathbb{R}$. Then for each $v \in X^*$ it holds that $CA_vA_uB_j = \sum_{k=1}^{l} \alpha_k CA_vA_{w_k}B_{j_l}$ which implies

$$C^{min}A_v^{min}A_u^{min}B_j^{min} = \sum_{k=1}^{l} \alpha_k C^{min}A_v^{min}A_{w_k}^{min}B_{j_l}^{min}$$

Thus, $A_u^{min}B_j^{min} - \sum_{k=1}^{l} \alpha_k A_{w_k}^{min}B_{j_k}^{min} \in O_{R_{min}} = \{0\}$ which means that

$$A_u^{min}B_j^{min} = \sum_{k=1}^{l} \alpha_k A_{w_k}^{min}B_{j_k}^{min}$$

. That is $T(A_uB_j) = \sum_{k=1}^{l} \alpha_k T(A_{w_k}B_{j_k})$. Thus, T is indeed well-defined and linear. The mapping T is surjective, since the following holds.

$$\mathcal{X}_{min} = \mathrm{Span}\{A_w^{min}B_j^{min} \mid j \in J\} = \mathrm{Span}\{T(A_wB_j) \mid j \in J\} = T(\mathcal{X})$$

We will show that T defines a representation morphism. Equality $TA_\sigma = A_\sigma^{min}T$ holds since
$T(A_\sigma A_wB_j) = A_\sigma^{min}A_w^{min}B_j^{min} = A_\sigma^{min}T(A_wB_j)$. Equality $B_j^{min} = TB_j$ holds by definition of T. Equality $C_{min}T = C$ holds because of the fact that $C_{min}A_w^{min}B_j^{min} = CA_wB_j = C_{min}T(A_wB_j)$.

(iii) $\implies$ (i)
Indeed, if R is a representation of Ψ, then it follows from the proof of Lemma 7 that $R_r = (W_R, \{A_z|_{W_R}\}_{z\in X}, B, C|_{W_R})$ is a reachable representation of Φ and $\dim R_r \leq \dim R$. By part (iii) there exists a surjective map $T : R_r \to R_{min}$. But $\dim R \geq \dim R_r \geq \dim T(W_R) = \dim R_{min}$, so R_{min} is indeed a minimal representation of Ψ.

(iv) $\iff$ *(i)*
The proof of Corollary 1 doesn't depend on the equivalence to be proved, so we can use it. By Corollary 1 R_Ψ is a minimal representation of Ψ. By construction $\dim R_\Psi = \dim W_\Psi = \text{rank } H_\Psi$. A representation is minimal whenever it has the same dimension as another minimal representation. Thus we get that R_{min} is minimal if and only if $\dim R_{min} = \dim R_\Psi = \text{rank } H_\Psi = \dim W_\Psi$. □

Corollary 1. *(a) All minimal representations of* Ψ *are isomorphic.*

(b) The free representation *from Theorem 1 is a minimal representation.*

Proof of Corollary 1. `Part (a)`
Let $R_{min} = (\mathcal{X}_{min}, \{A_\sigma^{min}\}_{\sigma\in X}, B^{min}, C^{min})$ be a minimal representation of Ψ. Let $R = (\mathcal{X}, \{A_\sigma\}_{\sigma\in X}, B, C)$ be another minimal representation of Ψ. Then R is reachable and there exists a surjective representation morphism $T : R \to R_{min}$. Since $\dim R \leq \dim R_{min}$ and $\dim R_{min} \leq \dim R$, we get that $\dim R = \dim R_{min}$, which implies that $\dim \mathcal{X}_{min} = \dim \mathcal{X} = \dim T(\mathcal{X})$, which implies that T is a linear isomorphism, that is, T is a representation isomorphism.

`Part (b)`
The equality $W_\Psi = \text{Span}\{w \circ S_j \mid j \in J, w \in X^*\} = \text{Span}\{A_w B_j \mid j \in J, w \in X^*\}$ implies that $W_{R_\Psi} = W_\Psi$. If $T \in W_\Psi$ has the property that for all $w \in X^* : CA_wT = 0$ then it means that for all $w \in X^*$ it holds that $C(w \circ T) = w \circ T(\epsilon) = T(w) = 0$, i.e T=0. So we get that $O_{R_\Psi} = \{0\}$. By Theorem 2 we get that R_Ψ is a minimal representation of Ψ. □

Lemma 8. *Let* $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ *and* $\Psi^{'} = \{T_{j^{'}} \in \mathbb{R}^p \ll X^* \gg \mid j^{'} \in J^{'}\}$ *be two indexed sets of formal power series with index sets* J *and* $J^{'}$ *respectively. Assume that there exists a map* $f : J^{'} \to J$*, such that* $\forall j^{'} \in J^{'} : S_{f(j^{'})} = T_{j^{'}}$*. Then, if* Ψ *is rational, then* $\Psi^{'}$ *is also rational and* $\text{rank } H_{\Psi^{'}} \leq \text{rank } H_\Psi$*. If* f *is surjective, then* $\text{rank } H_{\Psi^{'}} = \text{rank } H_\Psi$.

Proof. Indeed, let $R = (\mathcal{X}, \{A_x\}_{x\in X}, B, C)$ be a minimal representation of Ψ. Then it is easy to see that $R^{'} = (\mathcal{X}, \{A_x\}_{x\in X}, B^{'}, C)$ is a representation of $\Psi^{'}$, where $B^{'}_{j^{'}} = B_{f(j^{'})}, j^{'} \in J^{'}$. That is, if Ψ is rational, then $\Psi^{'}$ is rational too. By Lemma 7 there exists a reachable and observable representation $R^{'}_{can}$ such that $\dim R^{'}_{can} \leq \dim R^{'} = \dim R$. But $R^{'}_{can}$ is a minimal representation of $\Psi^{'}$. Thus, $\text{rank } H_{\Psi^{'}} = \dim R_{can} \leq \dim R = \text{rank } H_\Psi$. The representation R is reachable and observable. It is also easy to see that $O_R = O_{R^{'}} = \{0\}$, thus $R^{'}$ is observable too. It is also easy to see that if f is surjective, then $W_{R^{'}} = W_R = \mathcal{X}$, that is, $R^{'}$ is reachable. Thus, if f is surjective, then $R^{'}$ is a minimal representation of $\Psi^{'}$ and $\text{rank } H_\Psi = \dim R = \dim R^{'} = \text{rank } H_{\Psi^{'}}$. □

Lemma 9. *Let $J_1, \dots, J_n$ be disjoint sets. Let $\Psi_i = \{S_j \in \mathbb{R}^p \ll Q^* \gg \mid j \in J_i\}$, $i = 1, \dots, n$ be indexed sets of formal power series. Let $J = J_1 \cup J_2 \cup \dots \cup J_n$ and let $\Psi = \{S_j \in \mathbb{R}^p \ll Q^* \gg \mid j \in J\}$. Then Ψ is rational if and only if each Ψ_i, $i = 1, \dots n$ is rational.*

Proof. It is easy to see that $W_\Psi = \mathrm{Span}\{S_j \mid j \in J_1 \cup \dots \cup J_n\} = \sum_{i=1}^{n} \mathrm{Span}\{S_j \mid j \in J_i\} = W_{\Psi_1} + \dots + W_{\Psi_n}$. For each $i = 1, \dots, n$, W_{Ψ_i} is a subspace of W_Ψ. If Ψ is rational, then by Theorem 1 $\dim W_\Psi < +\infty$ and thus $\dim W_{\Psi_i} < +\infty$ for all $i = 1, \dots, n$. That is, each Ψ_i, $i = 1, \dots n$ is rational. Conversely, if each Ψ_i, $i = 1, \dots, n$ is rational, then by Theorem 1, for each $i = 1, \dots, n$, $\dim W_{\Psi_i} < +\infty$ holds. Thus, $\dim W_\Psi = \dim(W_{\Psi_1} + \dots + W_{\Psi_n}) < +\infty$, that is, Ψ is rational ☐

Corollary 2. *Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ be an indexed set of formal power series with the index set J. Assume that J is finite. Then Ψ is rational if and only if $S_j \in \mathbb{R}^p \ll X^* \gg$ is rational for each $j \in J$*

Proof. Let $J = \{j_1, \dots, j_n\}$. Let $\Psi_i = \{S_j \mid j \in \{j_i\}\}$, $i = 1, \dots, n$. Then $\Psi = \{S_j \mid j \in \{j_1\} \cup \dots \cup \{j_n\}\}$. Thus, by Lemma 9 Ψ is rational if and only if each Ψ_i, $i = 1, \dots, n$ is rational. Let $f_i : \{j_i\} \ni j_i \mapsto \emptyset \in \{\emptyset\}$, $i = 1, \dots, n$. Each f_i is a bijection. For each $i = 1, \dots, n$ let $Q_i = \{T_j \mid j \in \{\emptyset\}\}$, $T_\emptyset = S_{j_i}$. Applying Lemma 8 to Ψ_i, Q_i, f_i and f_i^{-1} we get that Q_i is rational if and only if Ψ_i is rational. Thus, Ψ_i is rational $\iff$ S_{j_i} is rational, for each $i = 1, \dots, n$. Therefore, Ψ is rational $\iff$ for each $j \in J$, S_j is rational. ☐

In the classical literature one often finds a procedure for constructing a representation of a rational formal power series from the columns of its Hankel-matrix. A similar construction can be carried out in the setting of this chapter too. Indeed, let $\mathrm{Im}H_\Psi = \mathrm{Span}\{(H_\Psi)_{.,(v,j)} \in \mathbb{R}^{X^* \times I} \mid (v,j) \in X^* \times J\}$. Then the map $T : W_\Psi \to \mathrm{Im}H_\Psi$ defined by $T(w \circ S_j) = (H_\Psi)_{.,(w,j)}$ is a well defined vector space isomorphism. Moreover, if $R_f = (W_\Psi, \{A_\sigma\}_{\sigma \in X}, B, C)$ is the free representation of Ψ, then $TB_j = (H_\Psi)_{.,(\epsilon,j)}$, $CT^{-1}(H_\Psi)_{.,(v,j)} = \begin{bmatrix} (H_\Psi)_{(\epsilon,1),(v,j)} & \cdots & (H_\Psi)_{(\epsilon,p),(v,j)} \end{bmatrix}^T$ and $TA_\sigma T^{-1}(H_\Psi)_{.,(v,j)} = (H_\Psi)_{(.,(v\sigma,j)}$ for each $\sigma \in X$. Define the representation

$$R_{H,\Psi} = (\mathrm{Im}H_\Psi, \{TA_\sigma T^{-1}\}_{\sigma \in X}, TB, CT^{-1})$$

Then it is easy to see that $T : R_f \to R_{H,\Psi}$ is a representation isomorphism and $R_{H,\Psi}$ is a representation of Ψ. It is also straightforward to see that the definition of $R_{H,\Psi}$ corresponds to the definition of the representation on the columns of the Hankel-matrix as it is described in the classical literature.

If $R = (\mathcal{X}, \{A_\sigma\}_{\sigma\in\Sigma}, B, C)$ is a representation of Ψ, then for any vector space isomorphism $T : \mathcal{X} \to \mathbb{R}^n$, $n = \dim R$, the tuple

$$TR = (\mathbb{R}^n, \{TA_\sigma T^{-1}\}_{\sigma\in\Sigma}, TB, CT^{-1})$$

is also a representation of Ψ. It is easy to see that R is minimal if and only if TR is minimal. Moreover, $T : R \to TR$ is a representation isomorphism. That is, when dealing with representations, we can assume without loss of generality that $\mathcal{X} = \mathbb{R}^n$. From now on, we will silently assume that $\mathcal{X} = \mathbb{R}^n$ holds for any representation considered.

3.2 Realization Theory of Moore-automata

Recall from Section 2.2 the concept of Moore-automata and realization by a Moore-automaton. In this section we will review the main results on realization theory of Moore-automata. The results are classical, in fact, they are the oldest results on realization theory. For more on the topic see [17, 24].

Let $\mathcal{D} = \{\phi_j : \Gamma^* \to O \mid j \in J\}$ be an indexed set of input-output maps. Let $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ a Moore automaton, $\zeta : J \to Q$ and assume that $(\mathcal{A}, \zeta)$ is a realization of $\mathcal{D}$. Define the realization $(\mathcal{A}_r, \zeta_r)$ by $\mathcal{A}_r = (Q_r, \Gamma, O, \delta_r, \lambda)$, $Q_r = \{q \in Q \mid \exists j \in J, w \in \Gamma^* : \delta(\zeta_j, w) = q\}$, $\delta_r(q, \gamma) = \delta(q, \gamma)$, $q \in Q_r, \gamma \in \Gamma$, $\zeta_r(j) = \zeta(j)$. It is easy to see that $(\mathcal{A}_r, \zeta_r)$ is well-defined, it is reachable and $card(\mathcal{A}_r) \leq card(\mathcal{A})$. Moreover, $card(\mathcal{A}_r) < card(\mathcal{A})$ if and only if $\mathcal{A}$ is not reachable. Thus, all minimal realizations are reachable. Indeed, if $(\mathcal{A}, \zeta)$ is a minimal realization of $\mathcal{D}$ and it is not reachable, then $(\mathcal{A}_r, \zeta_r)$ is a realization of $\mathcal{D}$ such that $card(\mathcal{A}_r) < card(\mathcal{A})$. But this contradicts to minimality of $(\mathcal{A}, \zeta)$. The following result is a simple reformulation of the well-known properties of realizations by automaton. For references see [17, 24].

Theorem 3. *Let $\mathcal{D} = \{\phi_j \in F(\Gamma^*, O) \mid j \in J\}$. $\mathcal{D}$ has a realization by a finite Moore-automaton if and only if $W_\mathcal{D}$ is finite. In this case a realization of $\mathcal{D}$ is given by $(\mathcal{A}_{can}, \zeta_{can})$ where $\mathcal{A}_{can} = (W_\mathcal{D}, \Gamma, O, L, T)$, $\zeta_{can}(j) = \phi_j$ and*

$$L(\phi, \gamma) = \gamma \circ \phi,\ T(\phi) = \phi(\epsilon),\ \phi \in W_\mathcal{D}, \gamma \in \Gamma$$

The realization $(\mathcal{A}_{can}, \zeta_{can})$ is reachable and observable.

Proof. Assume that $(\mathcal{A}, \zeta)$, $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ is a realization of $\mathcal{D}$. For any $w \in \Gamma^*, j \in J$, $w \circ \phi_j(v) = \lambda(\zeta(j), wv)$. Define the map $F : Q \to F(\Gamma^*, O)$, such that $F(q)(v) = \lambda(q, v), v \in \Gamma^*$. Then it is easy to see that $W_\mathcal{D} \subseteq F(Q)$. Since $card(Q) < +\infty$, we get that $card(W_\mathcal{D}) \leq card(F(Q)) < +\infty$. It is easy to see that L, ζ_{can}

and T are well-defined maps and thus $(\mathcal{A}_{can}, \zeta_{can})$ is a well-defined finite Moore-automaton. It is left to show that $(\mathcal{A}_{can}, \zeta_{can})$ is a realization of $\mathcal{D}$. The crucial observation is that $L(\phi, w_1 \dots w_k) = L(L(\cdots (L(\phi, w_1) \cdots), w_{k-1}), w_k) = w_k \circ (w_{k-1} \circ \cdots (w_1 \circ \phi) \cdots)) = w_1 \cdots w_k \circ \phi$ for each $\phi \in W_{\mathcal{D}}$ and $w_1, \dots, w_k \in \Gamma, k \geq 0$. For each $j \in J$, $T(\zeta_{can}(j), w) = T(L(\zeta_{can}(j), w)) = T(w \circ \phi_j) = \phi_j(w\epsilon) = \phi_j(w)$ for each $w \in \Gamma^*, k \geq 0$. Thus $(\mathcal{A}_{can}, \zeta_{can})$ is a realization of $\mathcal{D}$. It is easy to see that $(\mathcal{A}_{can}, \zeta_{can})$ is reachable and observable. Indeed, for each $w \in \phi_j \in W_{\mathcal{D}}$, $L(\zeta_{can}(j), w) = L(\phi_j, w) = w \circ \phi_j$, thus $\mathcal{A}_{can}$ is reachable. If $f, g \in W_{\mathcal{D}}$ are such that $T(f, w) = T(g, w)$ for each $w \in \Gamma^*$, then we get that $g(w) = T(w \circ g) = T(g, w) = T(f, w) = T(w \circ f) = f(w)$ for all $w \in \Gamma^*$, i.e., $f = g$ and thus $\mathcal{A}_{can}$ is observable. $\square$

The realization $(\mathcal{A}_{can}, \zeta_{can})$ is called the *free realization*. The following theorem gives equivalent conditions for minimality of a realization.

Theorem 4. *Let $(\mathcal{A}, \zeta)$ be a finite Moore-automaton realization of $\mathcal{D} = \{\phi_j \in F(\Gamma^*, O) \mid j \in J\}$. The following are equivalent:*

(i) *$(\mathcal{A}, \zeta)$ is minimal,*

(ii) *$(\mathcal{A}, \zeta)$ is reachable and observable,*

(iii) *$card(\mathcal{A}) = card(W_{\mathcal{D}})$,*

(iv) *For each reachable realization $(\mathcal{A}^{'}, \zeta^{'})$ of $\mathcal{D}$ there exists a surjective automaton morphism $T : (\mathcal{A}^{'}, \zeta^{'}) \to (\mathcal{A}, \zeta)$. In particular, all minimal realizations of $\mathcal{D}$ are isomorphic*

Proof. Consider the free realization $(\mathcal{A}_{can}, \zeta_{can})$ of $\mathcal{D}$ described in Theorem 3. We will show that (iv) holds for $(\mathcal{A}_{can}, \zeta_{can})$. Let $(\mathcal{A}, \zeta)$ be a reachable realization of $\mathcal{D}$. Assume that $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$. Define the map $F : Q \to F(\Gamma^*, O)$ by $F(q)(w) = \lambda(q, w)$, $w \in \Gamma^*$. We claim that F is an automaton morphism and $F(Q) = W_{\mathcal{D}}$. It is easy to see that $F(\delta(q, v))(w) = \lambda(\delta(q, v), w) = \lambda(q, vw) = F(q)(vw)$, thus $F(\delta(q, v)) = v \circ F(q)$ for all $q \in Q$, $v \in \Gamma^*$. Notice that $\lambda(q) = F(q)(\epsilon) = T(F(q))$. Thus, F is indeed an automaton morphism. It is again easy to see that $W_{\mathcal{D}} \subseteq F(Q)$, since $(\mathcal{A}, \zeta)$ is a realization of $\mathcal{D}$. On the other hand, if $(\mathcal{A}, \zeta)$ is reachable, then for any $q \in Q$ there exists $j \in J$, $w \in \Gamma^*$, such that $\delta(\zeta(j), w) = q$. Thus, $F(q) = w \circ \phi_j \in W_{\mathcal{D}}$, i.e. $F(q) \subseteq W_{\mathcal{D}}$. Thus, $F : (\mathcal{A}, \zeta) \to (\mathcal{A}_{can}, \zeta_{can})$ is a surjective automaton morphism.

Assume that $(\mathcal{A}, \zeta)$ above is observable. Then the map F is injective. Indeed, $\lambda(q_1, w) = F(q_1)(w) = F(q_2)(w) = \lambda(q_2, w), \forall w \in \Gamma^*$ implies that $q_1 = q_2$. Thus, if $(\mathcal{A}, \zeta)$ is observable and reachable, then it is isomorphic to $(\mathcal{A}_{can}, z_{can})$.

Notice that if $F : (\bar{\mathcal{A}}, \bar{\zeta}) \to (\mathcal{A}_{can}, \zeta_{can})$ is an automaton isomorphism, then F^{-1} defines an automaton isomorphism $F^{-1} : (\mathcal{A}_{can}, \zeta_{can}) \to (\bar{\mathcal{A}}, \bar{\zeta})$. If $(\mathcal{A}^{'}, \zeta^{'})$ is a reachable realization of $\mathcal{D}$, then there exists a surjective morphism $T : (\mathcal{A}^{'}, \zeta^{'}) \to (\mathcal{A}_{can}, \zeta_{can})$. Thus, $F^{-1} \circ T : (\mathcal{A}^{'}, \zeta^{'}) \to (\bar{\mathcal{A}}, \bar{\zeta})$ is a surjective automaton morphism.

Applying the remark above to a reachable and observable realization $(\mathcal{A}, \zeta)$ of $\mathcal{D}$, we get that there exists an isomorphism $F : (\mathcal{A}, \zeta) \to (\mathcal{A}_{can}, \zeta_{can})$ and thus $(\mathcal{A}, \zeta)$ satisfies (iv). Thus (ii) implies (iv).

Next we will show that any minimal realization $(\mathcal{A}_{min}, \zeta_{min})$ is isomorphic to $(\mathcal{A}_{can}, \zeta_{can})$. Indeed, if $(\mathcal{A}_{min}, \zeta_{min})$ is minimal, then $(\mathcal{A}_{min}, \zeta_{min})$ has to be reachable. But then there exists a surjective automaton morphism $T : (\mathcal{A}_{min}, \zeta_{min}) \to (\mathcal{A}_{can}, \zeta_{can})$. Thus, $card(W_{\mathcal{D}}) \leq card(\mathcal{A}_{min})$. By minimality of $(\mathcal{A}_{min}, \zeta_{min})$ we get that $card(W_{\mathcal{D}}) = card(\mathcal{A}_{min})$. Thus, $(\mathcal{A}_{can}, \zeta_{can})$ is minimal and all minimal realization of $\mathcal{D}$ are isomorphic. Thus, (i) is equivalent to (iii) and (i) implies (iv) and (i) is equivalent to (ii).

Finally, we will show that (iv) implies (i). Indeed, assume that $(\mathcal{A}, \zeta)$ satisfies (iv). Then there exists a surjective morphism $T : (\mathcal{A}_{can}, \zeta_{can}) \to (\mathcal{A}, \zeta)$. Thus, $card(W_{\mathcal{D}}) \geq card(\mathcal{A})$. But this is impossible unless $card(W_{\mathcal{D}}) = card(\mathcal{A})$, and thus $(\mathcal{A}, \zeta)$ is minimal.

We get that (i) $\iff$ (iii), (i) $\iff$ (ii), (i) $\iff$ (iv). □

The realization $(\mathcal{A}_{can}, \zeta_{can})$ is minimal.

3.3 Hybrid Formal Power Series

The section introduces the concept of hybrid power series and hybrid power series representation. This section contains the main contribution of the chapter. Subsection 3.3.1 contains the definition and basic properties of hybrid formal power series and hybrid representations. Subsection 3.3.2 discusses the problem of existence of hybrid representations. It gives necessary and sufficient conditions for a family of hybrid formal power series to admit a hybrid representation. Subsection 3.3.3 characterises minimal hybrid representations. Throughout the section the notation of Section 3.1 will be used.

3.3.1 Definitions and Basic Properties

Let X be an alphabet, i.e. a finite set and let O be an arbitrary finite set. Assume that $X = X_1 \cup X_2$ such that $X_1 \cap X_2 = \emptyset$. We allow X_1 or X_2 to be the empty set.

Let J be any set of the following form.

$$\begin{aligned} J = J_1 \cup (J_1 \times J_2) \\ J_2 \text{ is a finite set}, J_2 \cap J_1 = \emptyset \end{aligned} \tag{3.1}$$

Sets with the property (3.1) above will be called *hybrid power series index sets*. Notice that we allow J_2 to be the empty set.

A *hybrid formal power series* over X_1, X_2 with coefficients in $\mathbb{R}^p \times O$ is a pair

$$S = (S_C, S_D) \in \mathbb{R}^p \ll X^* \gg \times F(X_2^*, O)$$

That is, a hybrid formal power series S is a pair of functions. The first component of the pair is a map $S_C : X^* \to \mathbb{R}^p$, the second component is a map $S_D : X_2^* \to O$. We will denote the set of all hybrid formal power series over X_1, X_2 with coefficients in $\mathbb{R}^p \times O$ by $\mathbb{R}^p \ll X^* \gg \times F(X_2^*, O)$. If the space of coefficients and the alphabets X_1, X_2 are clear from the context we will simply speak of hybrid formal power series. If $S \in \mathbb{R}^p \ll X^* \gg \times F(X_2^*, O)$ is a hybrid formal power series, then define the formal power series $S_C \in \mathbb{R}^p \ll X^* \gg$ and the map $S_D : X_2^* \to O$ in such a way that $S = (S_C, S_D)$. That is, S_D denotes the discrete valued (O valued) component of S and S_C denotes the continuous ($\mathbb{R}^p$) valued component of S.

Assume that J is a hybrid formal power series index set. Let $\Omega = \{Z_j \in \mathbb{R}^p \ll X^* \gg \times F(X_2^*, O) \mid j \in J\}$ be an indexed set of hybrid formal power series indexed by J such that

$$\forall k \in J_1, j \in J_2 : (Z_{k,j})_D = (Z_k)_D \text{ and } (Z_{k,j})_C(w) = 0, \forall w \in X_2^* \tag{3.2}$$

Indexed sets of hybrid formal power series with the property (3.2) above will be called *well-posed indexed sets of hybrid power series* . The intuition behind the definition of well-posed indexed sets of hybrid power series is the following. We can think of the indexed set Ω as an encoding of the indexed set $\Psi = \{f_j \mid j \in J_1\}$, where $f_j : X \ni w \mapsto ((Z_j)_C(w), (Z_j)_D(v), ((Z_{j,k})_C)(w))_{k \in J_2})$, where $v = \gamma_1 \cdots \gamma_k \in X_2^*$ and w is assumed to be of the form $w = z_1\gamma_1 z_2 \cdots \gamma_k z_{k+1}$, $z_1, \ldots, z_{k+1} \in X_1^*$, $\gamma_1, \ldots, \gamma_k \in X_2$. The indexed set Ψ is supposed to contain input-output maps of a system which is an interconnection of a special form of a finite Moore-automaton and formal power series representations. The requirement $(Z_{j,k})_C(w) = 0$ for all $w \in X_2^*$ reflects the special structure of this interconnection. The motivation of the definition of a well-posed indexed set of hybrid power series should become clear to the reader after seeing the definition of a hybrid power series representation. A hybrid formal power series representation defines exactly an interconnection of a Moore-automaton and formal power series representations such that the input-output maps of the interconnection can be encoded by a well-defined indexed set of hybrid formal power series.

In the sequel, we will mostly work with well-posed indexed sets of hybrid formal power series. In the rest of the paper, unless stated otherwise, we will always mean a well posed indexed set of hybrid formal power series whenever we speak of indexed sets of hybrid formal power series.

Definition 9. *A* hybrid representation (abbreviated by HR) over J *is a tuple*

$$HR = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j \in J_2, z \in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y \in X_2})_{q \in Q}, J, \mu)$$

where

$\mathcal{A} = (Q, X_2, O, \delta, \lambda)$ *is a Moore-automaton*

$\mathcal{X}_q$ *is a finite-dimensional vector space for all* $q \in Q$*. Without loss of generality we can assume that* $\mathcal{X}_q = \mathbb{R}^{n_q}$ *for some* $n_q > 0$*.*

$\mathcal{Y}$ *is a finite-dimensional vector space and* $\mathcal{Y} = \mathbb{R}^p$ *for some* $p \in \mathbb{N}, p > 0$*.*

$M_{q_1,x,q_2} : \mathcal{X}_{q_2} \to \mathcal{X}_{q_1}$ *is a linear map, for each* $q_1, q_2 \in Q$*,* $x \in X_2$ *such that* $\delta(q_2, x) = q_2$*.*

$A_{q,x} : \mathcal{X}_q \to \mathcal{X}_q$ *is a linear map for each* $x \in X_1$ *and* $q \in Q$*.*

$C_q : X_q \to \mathcal{Y}$ *is a linear map for each* $q \in Q$*.*

For each $q \in Q$*,* $j \in J_2$*,* $x \in X_1$*, the vector* $B_{q,x,j}$ *belongs to* $\mathcal{X}_q$*, i.e.* $B_{q,x,j} \in \mathcal{X}_q$*.*

$\mu : J_1 \to \bigcup_{q \in Q} \{q\} \times \mathcal{X}_q$ *is a map*

Define $\mu_D : J_1 \to Q$ and $\mu_C : J_1 \to \bigcup_{q \in Q} \mathcal{X}_q$ by

$$\forall j \in J_1 : \mu(j) = (q, x) \Leftrightarrow \mu_D(j) = q \text{ and } \mu_C(j) = x$$

If $J_2 = \emptyset$, then we will use the following short-hand notation for the hybrid representation HR

$$(\mathcal{A}, (\mathcal{X}_q, \{A_{q,z}\}_{z \in X_1}, C_q)_{q \in Q}, \{M_{\delta(q,y),y,q} \mid q \in Q, y \in X_2\}, J, \mu)$$

In fact, a hybrid representation can be viewed as a some sort of cascade interconnection of a Moore-automaton and a formal power series representations. Recall from Section 3.1 that a formal powers series representation can be thought of as a Moore-automaton, state-space of which is a vector space (thus, not necessarily finite). One could define a suitable notion of cascade interconnection for Moore-automata, see for example [17] and view a hybrid representation as an interconnection of a finite

Moore-automaton with a number of Moore-automata which are in fact formal power series representations.

A hybrid representation can be itself viewed as a Moore-automaton. Before we can explain how to view a hybrid representation as a Moore-automata, we will need some additional definitions and notation.

Define the set

$$\bar{O} = \prod_{j \in J_2} \mathbb{R}^p \ll X^* \gg$$

An element of the set $\bar{O}$ is a tuple $(S_j)_{j \in J_2}$ such that $S_j \in \mathbb{R}^p \ll X^* \gg$ for all $j \in J_2$. If $J_2 = \emptyset$ then $\bar{O}$ will be viewed as the singleton set $\{\emptyset\}$.

Denote by $\mathcal{H}_{HR}$ the set $\mathcal{H}_{HR} = \bigcup_{q \in Q} \{q\} \times \mathcal{X}_q$. Define the maps $\Pi_Q : \mathcal{H}_{HR} \ni (q,x) \mapsto q \in Q$ and $\Pi_X : \mathcal{H}_{HR} \ni (q,x) \mapsto x \in \bigcup_{q \in Q} \mathcal{X}_q$.

Consider any $w \in X^*$. It is easy to see that w can be represented as $w = x_1 y_1 x_2 y_2 \cdots x_k y_k x_{k+1}$, for some $x_1, x_2, \ldots, x_{k+1} \in X_1^*$, $y_1, y_2, \ldots, y_k \in X_2$ and $k \geq 0$. It is easy to see that the representation above is unique. Such a representation can be easily obtained by grouping together those letters of w which belong to X_1.

The reader who wishes to see a formal proof, will find one below. The proof goes by induction. If $|w| = 1$, then $w = w_1$ and either $w_1 \in X_1$ or $w_1 \in X_2$. If $w_1 \in X_1$ then set $k = 0$ and $x_1 = w_1$. If $w_1 \in X_2$, then set $k = 1$, $y_1 = w_1$ and $x_1 = x_2 = \epsilon$. In both cases $w = x_1 y_1 \cdots y_k x_{k+1}$. Assume that a representation of the above form exists for all words $w \in X^*$, $|w| \leq n$. Assume that $w = w_1 \cdots w_{n+1}$, $w_1, \ldots, w_{n+1} \in X$. For each $i = 1, \ldots, n+1$ either $w_i \in X_1$ or $w_i \in X_2$. Assume that $w_1, w_2, \ldots, w_j \in X_1$ and $w_{j+1} \in X_2$. Let $x_1 = w_1 \cdots w_j \in X_1^*$ and $y_1 = w_{j+1} \in X_2$. If $w_1 \in X_2$ then $j = 0$ and $x_1 = \epsilon$. Consider the representation of $v = w_{j+2} \cdots w_{n+1}$, i.e assume that $v = x_2 y_2 \cdots y_k x_{k+1}$, $x_2, \ldots, x_{k+1} \in X_1^*$, $y_2, \ldots, y_k \in X_2$. Such a representation of v exists by the induction hypothesis. Then $w = x_1 y_1 v = x_1 y_1 x_2 \cdots y_k x_{k+1}$, that is, $x_1, \ldots, x_{k+1} \in X_1^*$, $y_1, \ldots, y_k \in X_2$.

For each $q \in Q$, $w = x_1 \cdots x_k \in X_1^*$, $x_1, \ldots, x_k \in X_1$ denote by $A_{q,w}$ the composition of linear maps $A_{q,x_k} A_{q,x_{k-1}} \cdots A_{q,x_1}$. If $k = 0$, i.e. $w = \epsilon$ then let $A_{q,w} = A_{q,\epsilon}$ be the identity map on $\mathcal{X}_q$.

Define the map $\xi_{HR} : \mathcal{H}_{HR} \times X^* \to \mathcal{H}_{HR}$ by

$$\begin{aligned}\xi_{HR}((q,x), z_1 w_1 \cdots z_k w_k z_{k+1}) = (\delta(q, w_1 \cdots w_k), A_{q_k, z_{k+1}} M_{q_k, w_k, q_{k-1}} A_{q_{k-1}, z_k} \\ \cdots\cdots A_{q_1, z_2} M_{q_1, w_1, q_0} A_{q_0, z_1} x)\end{aligned}$$

for all $z_1, \ldots, z_{k+1} \in X_1^*$, $w_1, \ldots, w_k \in X_2$, $k \geq 0$, where $q_i = \delta(q, w_1 \cdots w_i)$ for all $i = 0, \ldots, k$ (i.e. $q_0 = q$).

For each $q \in Q$, $j \in J_2$ define the power series $T_{q,j} \in \mathbb{R}^p \ll X^* \gg$ as follows. Recall that each $w \in X^*$ can be uniquely written as $w = x_1 y_1 x_2 \cdots y_k x_{k+1}$, for some

$y_1, \ldots, y_k \in X_2$, $x_1, \ldots, x_{k+1} \in X_1^*$ and $k \geq 0$. Then for each $w \in X^*$ define $T_{q,j}(w)$ as

$$T_{q,j}(w) = T_{q,j}(x_1 y_1 \cdots x_k y_k x_{k+1}) = C_{q_k} A_{q_k, x_{k+1}} M_{q_k, y_k, q_{k-1}} \cdots \\ \cdots M_{q_l, y_l, q_{l-1}} A_{q_{l-1}, z_l} B_{q_{l-1}, s_l, j}$$

where $1 \leq l \leq k+1$, $x_1 = x_2 = \cdots = x_{l-1} = \epsilon$, $x_l = s_l z_l$, $s_l \in X_1$, $z_l \in X_1^*$, $q_i = \delta(q, y_1 \cdots y_i)$ for all $i = 0, \ldots, k$.

The tuple $(T_{q,j})_{j \in J_2} \in \bar{O}$ will serve as the output of the hybrid representation HR. Define the map $\upsilon_{HR} : \mathcal{H}_{HR} \times X^* \to \mathbb{R}^p \times O \times \bar{O}$ as follows

$$\forall w \in X^* : \upsilon_{HR}((q, x), w) = (C_s z, \lambda(s), (T_{s,j})_{j \in J_2}) \text{ where } (s, z) = \xi_{HR}((q, x), w)$$

The map ξ_{HR} plays the role of state-trajectories and υ_{HR} plays the role of output-trajectories of the automaton associated with the hybrid representation HR

Now we are in position to explain the analogy between hybrid representations and Moore-automata. A hybrid representation HR can be viewed as an infinite-state Moore-automata, which is defined as follows. Its state space is the set $\mathcal{H}_{HR}$. Each state is a pair (q, x), consisting of a discrete component q and a continuous component $x \in \mathcal{X}_q$ The input alphabet of a hybrid representation viewed as a Moore-automaton is X. The output alphabet is the set $O \times \mathbb{R}^p \times \bar{O}$. The state-space evolution of a hybrid representation can be viewed as follows. If the hybrid representation receives a symbol $z \in X_1$, then the state changes as follows. If the current state is of the form $(q, x) \in \{q\} \times \mathcal{X}_q$, then the current state changes to $(q, A_{q,z} x)$. If the hybrid representation receives a symbol $y \in X_2$ then the state of the hybrid representation changes as follows. If the current state is of the form $(q, x) \in \{q\} \times \mathcal{X}_q$, then the current state changes to $(\delta(q, y), M_{\delta(q,y), y, q} x) \in \{\delta(q, y)\} \times \mathcal{X}_{\delta(q,y)}$. If the current state is of the form $(q, x) \in \{q\} \times \mathcal{X}_q$, then the output of the hybrid representation is $(C_q x, \lambda(q), (T_{q,j})_{j \in J_2})$. The tuple $(T_{q,j})_{j \in J_2}$ can be thought as an analog of impulse response for linear systems. The map μ can be thought of as a way to define the set of initial states of the Moore-automaton interpretation of the hybrid representation. Namely, the set of initial states is made up by the states $\mu(j) \in \mathcal{H}_{HR}$, $j \in J_2$.

We will not use the interpretation of a hybrid power series representation as a Moore-automaton presented above to prove mathematical properties of hybrid representations. However, we will frequently refer to this interpretation in order to give an intuitive description of results and concepts.

We define the *dimension of the hybrid representation HR* as the pair

$$(card(Q), \sum_{q \in Q} \dim \mathcal{X}_q)$$

and it is denoted by $\dim HR$. We will use the following partial order relation on $\mathbb{N} \times \mathbb{N}$. We will say that $(p, q) \in \mathbb{N}$ is smaller than or equal $(r, s) \in \mathbb{N}$ if $p \leq r$ and $q \leq s$. We will denote the fact that (p, q) is smaller than or equal (r, s) by $(p, q) \leq (r, s)$. Note the the order relation $\leq$ in $\mathbb{N} \times \mathbb{N}$ is indeed a partial order, it is not possible to compare all elements of $\mathbb{N} \times \mathbb{N}$.

Consider an indexed set of hybrid formal power series $\Omega = \{X_j \in \mathbb{R}^p \ll X^* \gg \times F(X_2^*, O) \mid j \in J\}$ with $J = J_1 \cup J_1 \times J_2$. The hybrid representation HR is said to be a *hybrid representation of* Ω if for all $w = x_1 y_1 \cdots x_k y_k x_{k+1} \in X^*, x_i \in X_1^*, y_j \in X_2, i = 1, 2, \ldots, k+1, j = 1, 2, \ldots, k, k \geq 0$ the following holds

$$
\begin{aligned}
&\forall j \in J_1 : \\
&(Z_j)_C(w) = C_{q_k} A_{q_k, x_{k+1}} M_{q_k, y_k, q_{k-1}} A_{q_{k-1}, x_k} \cdots M_{q_1, y_1, q_0} A_{q_0, x_1} \mu_C(j) \\
&\forall j \in J_1 : \\
&(Z_j)_D(y_1 \cdots y_k) = \lambda(\mu_D(j), y_1 \cdots y_k) \\
&\forall (j_1, j_2) \in J_1 \times J_2 : \\
&(Z_{j_1, j_2})_C(w) = C_{q_k} A_{q_k, x_{k+1}} M_{q_k, y_k, q_{k-1}} A_{q_{k-1}, x_k} \cdots \\
&\cdots M_{q_l, y_l, q_{l-1}} A_{q_{l-1}, z_l} B_{q_{l-1}, s_l, j_1} \\
&\ \text{where } x_l \in X_1^*, x_l = s_l z_l, s_l \in X_1, z_l \in X_1^* \\
&\ \text{and } x_1 = x_2 = \cdots = x_{l-1} = \epsilon, l > 0 \text{ and} \\
&\forall w \in X_2^* : (Z_{j_1, j_2})_C(w) = 0
\end{aligned}
\tag{3.3}
$$

where $q_0 = \mu_D(j), q_l = \delta(q_0, y_1 \cdots y_l), 1 \leq l \leq k$. Can can think of $(Z_j)_C$ as continuous output, $(Z_j)_D$ as discrete-output and $(Z_{k,j})_C$ as continuous output corresponding to impulse response. This is of course only an analogy, there is no formal correspondence between the objects mentioned above. An indexed set of hybrid formal power series is called *rational* if it has a hybrid representation. Note that the framework above resembles very much the concept of rational representations described in [64]. In fact, when $Q = \{q\}$ is a singleton set, the notion of hybrid representation and the notion of rational representation coincides. We say that the hybrid representation HR is a *minimal hybrid representation of* Ω if HR is a hybrid representation of Ω and for any hybrid representation $HR^{'}$ of Ω

$$\dim HR \leq \dim HR^{'}$$

Recall the interpretation of a hybrid representation as a Moore-automaton. Then the statement that HR is a hybrid representation of Ω simply says that for each $j_1 \in J_1$ the Moore-automaton interpretation of the hybrid representation HR realizes the map:

$$T_{j_1} : X^* \ni w \mapsto ((Z_{j_1})_C(w), (Z_{j_1})_D(\Pi_{X_2}(w)), ((Z_{j_1,j_2})_C(w))_{j_2 \in J_2}$$

from the initial states $\mu(j_1)$. Here $\Pi_{X_2} : X^* \to X_2^*$ is a map which erases all the letters not in X_2, i.e., $\Pi_{X_2}(x_1 y_1 \cdots x_k y_k x_{k+1}) = y_1 \cdots y_k$ for each $x_1, \ldots, x_{k+1} \in X_1^*$, $y_1, \ldots, y_k \in X_2$, $k \geq 0$.

Thus HR is a representation of Ω if and only if

$$\begin{aligned} &\forall j \in J_1, \forall w \in X^* : \\ &((Z_j)_C(w), (Z_j)_D(\Pi_{X_2}(w)), (Z_{j,j_2}(w))_{j_2 \in J_2}) = \upsilon_{HR}(\mu(j), w) \end{aligned} \tag{3.4}$$

Let $HR = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j \in J_2, z \in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y \in X_2})_{q \in Q}, J, \mu)$ be a hybrid representation. Let

$$HR' = (\mathcal{A}', \mathcal{Y}, (\mathcal{X}'_q, \{A'_{q,z}, B'_{q,z,j_2}\}_{j \in J_2, z \in X_1}, C'_q, \{M'_{\delta'(q,y),y,q}\}_{y \in X_2})_{q \in Q'}, J, \mu')$$

be another hybrid representation. A pair $T = (T_D, T_C)$ is a *HR-morphism* from HR to HR' denoted by $T : HR \to HR'$ if $T_D : (\mathcal{A}, \mu_D) \to (\mathcal{A}', \mu'_D)$ is an automaton realization morphism, $T_C : \bigoplus_{q \in Q} \mathcal{X}_q \to \bigoplus_{q \in Q'} \mathcal{X}'_q$ is a linear map such that

$T_C(\mathcal{X}_q) \subseteq \mathcal{X}'_{T_D(q)}$ for all $q \in Q$,

$T_C M_{q_1,x,q_2} = M'_{T_D(q_1),x,T_D(q_2)} T_C$ for all $q_1, q_2 \in Q, x \in X_2$ such that $\delta(q_2, x) = q_1$,

$T_C A_{q,z} = A'_{T_D(q),z} C(T)$ for all $q \in Q, z \in X_1$,

For all $q \in Q$, $j \in J_2$, $z \in X_1$, $T_C B_{q,z,j} = B'_{T_D(q),z,j}$

$C_q = C'_{T_D(q)} T_C$ for each $q \in Q$,

$T_C \mu_C(j) = \mu'_C(j)$ for all $j \in J_1$

It is easy to see that the pair $T = (T_D, T_C)$ defines a map

$$\phi(T) : \mathcal{H}_{HR} \ni (q, x) \to (T_D(q), T_C(x)) \in \mathcal{H}_{HR'}$$

The intuition behind the definition of T is the following. Notice that T can be extended to act on the state-spaces of the Moore-automaton interpretations of HR and HR' by defining $T((q,j)) = (T_D(q), j)$ for all $q \in Q$, $j \in J_2$. This extension of T becomes a Moore-automaton morphism, if T is a hybrid representation morphism.

We will call HR *observable* if for each $h_1, h_2 \in \mathcal{H}_{HR}$

$$(\forall w \in X^* : \upsilon_{HR}(h_1, w) = \upsilon_{HR}(h_2, w)) \implies h_1 = h_2$$

Define the set

$$\begin{aligned}\mathcal{H}_{0,HR} = \{(q,x) \mid (\exists j \in J_1 : \mu(j) = (q,x)) \text{ or } (q = \delta(\mu_D(j), v), \\ x = B_{q,z,j}, \text{ for some } v \in X_2^*, z \in X_1, j \in J_2)\}\end{aligned}$$

Define the set

$$\begin{aligned}Reach(HR) = \{(q,x) \mid \exists w_1,\dots,w_k \in X^*, \alpha_1,\dots,\alpha_k \in \mathbb{R}, \\ h_1,\dots,h_k \in \mathcal{H}_{0,HR}, k \geq 0, \\ x = \sum_{j=1}^{k} \alpha_j \Pi_X(\xi_{HR}(h_i,w_i)) \\ \text{and } q = \Pi_Q(\xi_{HR}(h_i,w_i)), i = 1,\dots,k\}\end{aligned}$$

We will call HR *reachable* if $\mathcal{H}_{HR} = Reach(HR)$.

Below we will give a reformulation of observability and reachability of hybrid representations. For the HR HR define the following spaces

$$\begin{aligned}W_{HR} = &\mathrm{Span}(\\ &\{A_{q_k,x_{k+1}} M_{q_k,y_k,q_{k-1}} A_{q_{k-1},x_k} M_{q_{k-1},y_{k-1},q_{k-2}} \cdots M_{q_1,y_1,q_0} A_{q_0,x_1} \mu_C(j) \mid \\ &j \in J_1, x_1,\dots,x_{k+1} \in X_1^*, y_1,\dots,y_k \in X_2, \\ &q_0 = \mu_D(j), q_l = \delta(q_0, y_1 \cdots y_l), 1 \leq l \leq k, k \geq 0\} \cup \\ &\cup \{A_{q_k,x_{k+1}} M_{q_k,y_k,q_{k-1}} A_{q_{k-1},x_k} M_{q_{k-1},y_{k-1},q_{k-2}} \cdots \\ &\cdots M_{q_l,y_l,q_{l-1}} A_{q_{l-1},z_l} B_{q_{l-1},s_l,j} \mid \\ &j \in J_2,, j \in J_1, x_1,\dots,x_{k+1} \in X_1^*, x_l \in X_1, x_1 = x_2 = \cdots = x_{l-1} = \epsilon, \\ &x_l = s_l z_l, s_l \in X_1, z_l \in X_1^*, 1 \leq l \leq k+1, y_1,\dots,y_k \in X_2, \\ &q_0 = \mu_D(j), q_i = \delta(q_0, y_1 \cdots y_i), 1 \leq i \leq k, k \geq 0\}) \subseteq \bigoplus_{q \in Q} \mathcal{X}_q\end{aligned}$$

The following statement is an easy consequence of the definition.

Proposition 3. *The hybrid representation HR is reachable, if and only if $(\mathcal{A}, \mu_D)$ is reachable and $W_{HR} = \bigoplus_{q \in Q} \mathcal{X}_q$.*

Again, if we look at the Moore-automaton interpretation of HR, then W_{HR} is precisely the linear span of the continuous components of the states which belong to $\bigcup_{q \in Q} \{q\} \times \mathcal{X}_q$ and can be reached from some initial state.

Below we will give a characterisation of observability of hybrid representations. For each $q \in Q$, define

$$O_{HR,q} = \bigcap_{q \in Q, w \in X^*} O_{q,w}$$

where for all $w = x_1 y_1 \cdots y_k x_{k+1} \in X^*, k \geq 0, x_1, \cdots, x_{k+1} \in X_1^*, y_1, \cdots, y_k \in X_2$

$$O_{q,w} = \ker C_{q_k} A_{q_k,x_{k+1}} M_{q_k,y_k,q_{k-1}} A_{q_{k-1},x_k} M_{q_{k-2},y_{k-1},q_{k-1}} \cdots M_{q_1,y_1,q_0} A_{q_0,x_1}$$

where $q = q_0 \in Q, q_l = \delta(q, y_1 \cdots y_l), 0 \leq l \leq k$. The space $O_{HR,q}$ is analogous to the observability kernel of linear (bilinear) systems and plays a very similar role. Unfortunately, the spaces $O_{HR,q}$ are not sufficient to characterise observability for hybrid representations. The following proposition characterises observability of hybrid representations.

Proposition 4. *The hybrid representation HR is observable, if and only if the following two conditions hold*

(i) For each $q_1, q_2 \in Q$, if for all $w \in X_2^*$, $j \in J_2$

$$\lambda(q_1, w) = \lambda(q_2, w) \text{ and } T_{q_1,j} = T_{q_2,j}$$

then $q_1 = q_2$.

(ii) For each $q \in Q$, $O_{HR,q} = \{0\}$

Proof. First we will show that

$$\forall w \in X_2^* : \lambda(q_1, w) = \lambda(q_2, w) \text{ and } T_{q_1,j} = T_{q_2,j} \text{ for all } j \in J_2$$

is equivalent to

$$\upsilon_{HR}((q_1, 0), v) = \upsilon_{HR}((q_2, 0), v), \forall v \in X^*$$

Indeed, let $q_1, q_2 \in Q$ such that for all $w \in X_2^*$, $\lambda(q_1, w) = \lambda(q_2, w)$ and $T_{q_1,j} = T_{q_2,j}$ for all $j \in J_2$. Then it follows that for all $v \in X^*$, such that $s = \Pi_{X_2}(v)$, $\upsilon_{HR}((q_i, 0), v) = (0, \lambda(q_i, w), (T_{\delta(q_i,s),j})_{j \in J_2})$.

It is easy to see from the definition of $T_{q,j}$ that $T_{\delta(q,w),j} w \circ T_{q,j}$ for all $q \in Q$, $j \in J_2$, $w \in X_2^*$. Indeed,

$$T_{\delta(q,w),j}(y_1 \cdots y_{l-1} x z_l y_l \cdots z_k y_k z_{k+1}) = C_{s_k} A_{s_k,z_{k+1}} M_{s_k,y_k,s_{k-1}} \cdots$$
$$\cdots M_{s_l,y_l,s_{l-1}} A_{s_{l-1},z_l} B_{s_{l-1},x,j} = T_{q,j}(w y_1 \cdots y_{l-1} x z_{l-1} y_l \cdots z_k y_k z_{k+1})$$

where $s_i = \delta(\delta(q, w), y_1 \cdots y_i) = \delta(q, w y_1 \cdots y_i)$, $i = 0, \ldots, k$, $y_1, \ldots, y_k \in X_2^*$, $z_l, \ldots, z_{k+1} \in X_1^*$, $x \in X_1$, $k \geq 0$. $1 \leq l \leq k+1$. Thus, we get that

$$\forall q \in Q, j \in J_2, w \in X_2^* : T_{\delta(q,w),j} = w \circ T_{q,j} \tag{3.5}$$

Since we assumed that $T_{q_1,j} = T_{q_2,j}$, $j \in J_2$ and $\lambda(q_1, w) = \lambda(q_2, w)$, $\forall w \in X_2^*$ it follows that $(w \circ T_{q_1,j})_{j \in J_2} = (w \circ T_{q_2,j})_{j \in J_2}$ and $\lambda(q_1, w) = \lambda(q_2, w)$ for all $w \in X_2^*$. Thus, we get that $\upsilon_{HR}((q_1, 0), v) = \upsilon_{HR}((q_2, 0), v)$.

Next, we show that for all $q \in Q$, $x_1, x_2 \in \mathcal{X}_q$

$$x_1 - x_2 \in O_{HR,q}$$

is equivalent to

$$\forall v \in X^* : \upsilon_{HR}((q, x_1), v)) = \upsilon_{HR}((q, x_2), v)$$

Indeed, assume that $v = z_1 w_1 \cdots w_k z_{k+1}$, $z_1, \ldots, z_{k+1} \in X_1^*$, $w_1, \ldots, w_k \in X_2$. Then

$$\upsilon_{HR}((q, x_i), v) = (C_{q_k} h_i, \lambda(q_k), (T_{q_k,j})_{j \in J_2})$$

where $(q_k, h_i) = \xi_{HR}((q, x_i), v)$, $i = 1, 2$. Thus, $\upsilon_{HR}((q, x_1), v) = \upsilon_{HR}((q, x_2), v)$ if and only if $C_{q_k} h_1 = C_{q_k} h_2$. From definition of ξ_{HR} it follows that for $i = 1, 2$,

$$h_i = A_{q_k, z_{k+1}} M_{q_k, w_k, q_{k-1}} A_{q_{k-1}, z_k} \cdots M_{q_1, w_1, q_0} A_{q_0, z_1} x_i$$

Thus, $C_{q_k} h_1 = C_{q_k} h_2$ if and only if

$$x_1 - x_2 \in \ker C_{q_k} A_{q_k, z_{k+1}} M_{q_k, w_k, q_{k-1}} \cdots M_{q_1, w_1, q_0} A_{q_0, z_1}$$

Since v runs through all elements of X^*, i.e. through all $w_1, \ldots, w_{k+1} \in X_2$, $z_1, \ldots, z_{k+1} \in X_1^*$, $k \geq 0$, we get the desired equivalence.

Now we are ready to prove the statement of the proposition. Assume that HR is observable. Assume there exists $q_1, q_2 \in Q$ such that $\lambda(q_1, w) = \lambda(q_2, w)$, $w \in X_2^*$ and $T_{q_1,j} = T_{q_2,j}$, $j \in J_2$. Then we get that $\upsilon_{HR}((q_1, 0), v) = \upsilon_{HR}((q_2, 0), v)$ for all $v \in X^*$. By observability of HR it implies $q_1 = q_2$. Thus, condition (i) of the proposition holds. Assume there exists $x = x - 0 \in O_{HR,q}$ for some $q \in Q$. Then we get that $\upsilon_{HR}((q, x), v) = \upsilon_{HR}((q, 0), v)$ for all $v \in X^*$. By observability of HR it implies $x = 0$, i.e. $O_{HR,q} = \{0\}$, that is, condition (ii) of the proposition holds.

Assume now that condition (i) and (ii) of the proposition hold. We will show that HR is observable. Assume that there exists $(q_i, x_i) \in \mathcal{H}_{HR}$, $i = 1, 2$ such that $\upsilon_{HR}((q_1, x_1), v) = \upsilon_{HR}((q_2, x_2), v)$ for all $v \in X^*$. Assume that $q_1 \neq q_2$. But $\upsilon_{HR}((q_1, x_1), v) = \upsilon_{HR}((q_2, x_2), v)$ implies that $\lambda(q_1, \Pi_{X_2}(v)) = \lambda(q_2, \Pi_{X_2}(v))$ and $\Pi_{X_2}(v) \circ T_{q_1,j} = \Pi_{X_2}(v) \circ T_{q_2,j}$ for all $j \in J_2$. Since v runs through all elements of X^* we get that $\lambda(q_1, w) = \lambda(q_2, w)$ $\forall w \in X_2^*$ and $T_{q_1,j} = T_{q_2,j}$ for all $j \in J_2$. Then by condition (i) we get that $q = q_1 = q_2$. But $\upsilon_{HR}((q, x_1), v) = \upsilon_{HR}((q, x_2), v)$ for all $v \in X^*$ is equivalent to $x_1 - x_2 \in O_{HR,q}$, thus by condition (ii) we get that $x_1 = x_2$. That is $(q_1, x_1) = (q_2, x_2)$. Thus,

$$(\forall v \in X^* : \upsilon_{HR}(h_1, v) = \upsilon_{HR}(h_2, v)) \implies h_1 = h_2$$

That is, we get that HR is observable. $\square$

Notice that if $J_2 = \emptyset$ then the first condition in the definition of observability is equivalent to $\mathcal{A}$ being observable.

If we look at the Moore-automaton interpretation of hybrid representations, then a hybrid representation is observable if and only if the Moore-automaton interpretation of the hybrid representation is observable.

Formula (3.5) is worth remembering. It will play an important role in the later subsections. It essentially says that $\Pi_{\bar{O}} \circ \upsilon_{HR}((q,x),v)) = ((\Pi_{X_2}(v) \circ T_{q,j})_{j \in J_2}$ for all $v \in X^*$, that is, the $\bar{O}$-valued component of the output trajectory induced by (q,x) is uniquely determined by $T_{q,j}, j \in J_2$ and it is independent of x.

Next we will discuss certain elementary properties of hybrid representation morphisms. Recall that any hybrid representation morphism $T : HR \rightarrow HR^{'}$ induces a map $\phi(T) : \mathcal{H}_{HR} \rightarrow \mathcal{H}_{HR'}$.

Proposition 5. *A hybrid representation morphism T is a hybrid representation isomorphism if and only if $\phi(T)$ is a bijective map.*

Proof. Indeed, assume that $\phi(T) : \mathcal{H}_{HR} \rightarrow \mathcal{H}_{HR'}$ is bijective. Then for all $q \in Q^{'}$ there exists uniquely a $q \in Q$ such that $T((q,0)) = (T_D(q), T_C(0)) = (q^{'},0)$, i.e., $T_D(q) = q^{'}$. Thus, T_D is bijective. For any $x \in \mathcal{X}^{'}_{q^{'}}$ there exists a unique $z \in \mathcal{X}_q$ such that $T((q,z)) = (T_D(q), T_C z) = (q,x)$, i.e., $T_C z = x$. Thus, T_C is surjective. We will show that T_C is injective. Indeed, assume that $T_C y = x$. Then $y = y_{q_1} + \cdots + y_{q_{|Q|}}$, where $y_{q_i} \in \mathcal{X}_{q_i}$, $i = 1, \ldots, |Q|$. But $T_C(y_{q_i}) \in \mathcal{X}^{'}_{T_D(q_i)}$, thus $T_C(y_{q_i}) = 0$ for all $i = 1, \ldots, |Q|, q_i \neq q$. Thus, $y \in \mathcal{X}_q$, and thus $y = z$. Conversely, assume that T is a hybrid representation isomorphism. Then for any $(q^{'},x) \in \mathcal{H}_{HR'}$ there exists a unique $q \in Q$, $y \in \bigoplus_{q \in Q} \mathcal{X}_q$, such that $T_D(q) = q^{'}$ and $x = T_C y$. But $T_C^{-1}(\mathcal{X}^{'}_{q^{'}}) = \bigoplus_{q \in Q, T_D(q) = q'} \mathcal{X}_q = \mathcal{X}_q$, thus $y \in \mathcal{X}_q$. That is, $(q,y) \in \mathcal{H}_{HR}$, i.e., T is bijective map from $\mathcal{H}_1$ to $\mathcal{H}_2$. ☐

Proposition 6. *Let HR_1 and HR_2 be two hybrid representations. Assume that $T : HR_1 \rightarrow HR_2$ is a hybrid representation morphism. Then the following holds.*

- *If T is injective, then* $\dim HR_1 \leq \dim HR_2$.
- *If T is surjective, then* $\dim HR_2 \leq \dim HR_1$.
- *If T is either injective or surjective and* $\dim HR1 = \dim HR_2$, *then T is an hybrid representation isomorphism.*

Proof. Let $HR_1 = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j \in J_2, z \in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y \in X_2})_{q \in Q}, J, \mu)$ and $HB_2 = (\mathcal{A}^{'}, \mathcal{Y}, (\mathcal{X}^{'}_q, \{A^{'}_{q,z}, B^{'}_{q,z,j_2}\}_{j \in J_2, z \in X_1}, C^{'}_q, \{M^{'}_{\delta^{'}(q,y),y,q}\}_{y \in X_2})_{q \in Q'}, J, \mu^{'})$.

Then $T_C : \bigoplus_{q\in Q} \mathcal{X}_q \to \bigoplus_{q\in Q'} \mathcal{X}^{'}_q$ is a linear morphism. Assume that T is injective. Then T_C and T_D are injective. Then $card(Q) = card(T_D(Q)) \le card(Q^{'})$ and

$$\text{rank } T_C = \sum_{q\in Q} \dim \mathcal{X}_q = \dim \bigoplus_{q\in Q} \mathcal{X}_q \le \sum_{q\in Q'} \dim \mathcal{X}^{'}_q$$

Thus

$$\dim HR_1 = (card(Q), \sum_{q\in Q} \dim \mathcal{X}_q) \le (card(Q^{'}), \sum_{q\in Q'} \dim \mathcal{X}^{'}_q)$$

Similarly, if T is surjective, then T_C and T_D are surjective. Thus,

$$\sum_{q\in Q} \dim \mathcal{X}_q \ge \text{rank } T_C = \sum_{q\in Q'} \dim \mathcal{X}^{'}_q$$

and $card(Q) \ge card(T_D(Q)) = card(Q^{'})$. Thus, $\dim HR_1 \ge \dim HR_2$. Assume that T is injective and $\dim HR_1 = \dim HR_2$. Then

$$\text{rank } T_C = \sum_{q\in Q} \dim \mathcal{X}_q = \sum_{q\in Q'} \dim \mathcal{X}^{'}_q \text{ and } card(T_D(Q)) = card(Q) = card(Q^{'})$$

Similarly, if T is surjective and $\dim HR_1 = \dim HR_2$, then

$$\text{rank } T_C = \sum_{q\in Q'} \dim \mathcal{X}^{'}_q = \sum_{q\in Q} \dim \mathcal{X}_q \text{ and } card(T_D(Q)) = card(Q^{'}) = card(Q)$$

Thus, if T is injective or surjective and $\dim HR_1 = \dim HR_2$, then T_C and T_D are bijections, and thus T is a hybrid representation isomorphism. □

The following proposition gives an important system theoretic characterisation of hybrid representation morphisms.

Proposition 7. *Let HR_i, $i = 1, 2$ be two hybrid representations and let $T : HR_1 \to HR_2$ be a hybrid representation morphism. Then the following holds.*

$$\phi(T)(\xi_{HR_1}(h, v)) = \xi_{HR_2}(\phi(T)(h), v) \text{ and } \upsilon_{HR_1}(h, v) = \upsilon_{HR_2}(\phi(T)(h), v)$$

for all $h \in \mathcal{H}_{HR_1}, v \in X^*$. If T is a hybrid representation isomorphism, then HR_1 is reachable if and only if HR_2 is reachable and HR_1 is observable if and only if HR_2 is observable.

Proof. Let

$$HR_1 = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j\in J_2, z\in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y\in X_2})_{q\in Q}, J, \mu)$$
$$HR_2 = (\mathcal{A}^{'}, \mathcal{Y}, (\mathcal{X}^{'}_q, \{A^{'}_{q,z}, B^{'}_{q,z,j_2}\}_{j\in J_2, z\in X_1}, C^{'}_q, \{M^{'}_{\delta^{'}(q,y),y,q}\}_{y\in X_2})_{q\in Q'}, J, \mu^{'})$$

Let $\mathcal{A} = (Q, X_2, O, \delta, \lambda)$ and $\mathcal{A}^{'} = (Q^{'}, X_2, O, \delta^{'}, \lambda^{'})$. It is easy to see that

$$T_D(\delta(q,w)) = \delta^{'}(T_D(q), w)$$

for all $q \in Q, w \in X_2^*$. For all $q \in Q$, $x \in \mathcal{X}_q$, $T_C A_{q,z} x = A^{'}_{T_D(q),z} T_C x$, $T_C M_{\delta(q,y),y,q} x = M^{'}_{\delta^{'}(T_D(q),y),y,T_D(q)} T_C x$. Thus, by induction we get that for all $z_1, \ldots, z_{k+1} \in X_1^*$, , $w_1, \ldots, w_k \in X_2$, $k \geq 0$

$$\begin{aligned} T_C(A_{q_k,z_{k+1}} M_{q_k,w_k,q_{k-1}} \cdots M_{q_1,w_1,q_0} A_{q_0,z_1} x) = \\ A^{'}_{d_k,z_{k+1}} M^{'}_{d_k,w_k,d_{k-1}} \cdots M^{'}_{d_1,w_1,d_0} A^{'}_{d_0,z_1} T_C x \end{aligned} \tag{3.6}$$

where $q_i = \delta(q, w_1 \cdots w_i)$, $T_D(q_i) = d_i$, $i = 0, \ldots, k$. Thus, we get that

$$\begin{aligned} &\phi(T)(\xi_{HR_1}((q,x), z_1 w_1 \cdots z_k w_k z_{k+1})) = \\ &= (T_D(\delta(q, w_1 \cdots w_k)), T_C A_{q_k,z_{k+1}} M_{q_k,w_k,q_{k-1}} \cdots M_{q_1,w_1,q_0} A_{q_0,z_1} x) = \\ &(\delta^{'}(T_D(q), w_1 \cdots w_k), A^{'}_{d_k,z_{k+1}} M^{'}_{d_k,w_k,d_{k-1}} \cdots M_{d_1,w_1,d_0} A^{'}_{d_0,z_1} T_C x) = \\ &= \xi_{HR_2}((T_D(q), T_C x), z_1 w_1 \cdots w_k z_{k+1}) \end{aligned}$$

Thus we get that

$$\phi(T)(\xi_{HR_1}(h,v)) = \xi_{HR_2}(\phi(T)(h), v) \tag{3.7}$$

for all $v \in X^*$.

We will proceed with proving that for all $h \in HR_1, v \in X^*$,

$$\upsilon_{HR_1}(h,v) = \upsilon_{HR_2}(\phi(T)(h), v) \tag{3.8}$$

As the first step we will show that if $(q_e, x_e) = \xi_{HR_1}((q,x),v)$ and $(q_e^{'}, x_e^{'}) = \xi_{HR_2}(\phi(T)((q,x)),v)$ then $C^{'}_{q_e^{'}} x_e^{'} = C_{q_e} x_e$. Notice $C_q x = C^{'}_{T_D(q)} T_C x$ for all $q \in Q$, $z \in X_1$, $x \in \mathcal{X}_q$. Since $\phi(T)((q_e, x_e)) = (T_D(q_e), T_C x_e) = (q_e^{'}, x_e^{'})$ by formula (3.7) we get the required equality. Notice that $\lambda(q_e) = \lambda^{'}(T_D(q_e)) = \lambda^{'}(q_e^{'})$. That is, we get that

$$\Pi_{\mathbb{R}^p \times O} \circ \upsilon_{HR_1}((q,x),v) = \Pi_{\mathbb{R}^p \times O} \circ \upsilon_{HR_2}(\phi(T)((q,x)),v) \tag{3.9}$$

Thus, in order to prove (3.8) it is left to show that $\Pi_{\bar{O}} \circ \upsilon_{HR_1}((q,x),v) = (T_{q_e,j})_{j \in J_2} = (T_{q_e^{'},j})_{j \in J_2} = \Pi_{\bar{O}} \circ \upsilon_{HR_2}(\phi(T)((q,x)),v)$ where as before $(q_e, x_e) = \xi_{HR_1}((q,x),v)$ and $(q_e^{'}, x_e^{'}) = \xi_{HR_2}(\phi(T)((q,x)),v)$. Since $T_D(q_e) = q_e^{'}$ it is enough to show that for all $j \in J_2$, $q \in Q$, $T_{q,j} = T_{T_D(q),j}$.

Notice that $T_C B_{q,z,j} = B^{'}_{T_D(q),z,j}$. $j \in J_2$, It is also easy to see that $T_{q,j}(zv) = \Pi_{\mathbb{R}^p} \circ \upsilon_{HR_1}((q, B_{q,z,j}), v))$ for all $v \in X^*$ and $z \in X_1$. Recall that $w \circ T_{q,j} = T_{\delta(q,w),j}$. It is easy to see that for all $s \in X^*$, $s = wzv$ for some $z \in X_1, w \in X_2^*$, $v \in X^*$. Thus, we get that

$$T_{q,j}(s) = T_{q,j}(wzv) = T_{\delta(q,w),j}(zv) = \Pi_{\mathbb{R}^p} \circ \upsilon_{HR_1}((\delta(q,w), B_{\delta(q,w),z,j}), v)$$

Since $\phi(T)((\delta(q,w), B_{\delta(q,w),z,j})) = (\delta^{'}(T_D(q),w)), B^{'}_{\delta^{'}(T_D(q),w),z,j})$ by formula (3.9) we get that

$$T_{q,j}(s) = \Pi_{\mathbb{R}^p} \circ \upsilon_{HR_1}((\delta(q,w), B_{\delta(q,w),z,j}), v) =$$
$$= \Pi_{\mathbb{R}^p} \circ \upsilon_{HR_2}((\delta^{'}(T_D(q),w), B^{'}_{\delta^{'}(T_D(q),w),z,j}), v) = T_{T_D(q),j}(s)$$

That is, $T_{q,j} = T_{T_D(q),j}$ for all $j \in J_2$. Thus we have shown that for all $h \in HR_1, v \in X^*$ (3.8) holds.

Assume that T is an hybrid representation isomorphism. Then T_C and T_D are bijective maps. It is easy to see that

$$\phi(T)(Reach(HR_1)) = \{(T_D(q), T_C(x)) \mid \exists k \geq 0, h_1, \dots, h_k \in \mathcal{H}_{0,HR_1},$$
$$w_1, \dots, w_k \in X^*, \alpha_1, \dots, \alpha_k \in \mathbb{R} :$$
$$(q, x_i) = \xi_{HR_1}(h_i, w_i), i = 1, \dots, k \text{ and } x = \sum_{j=1}^{k} \alpha_i x_i\} =$$
$$= \{(q^{'}, x^{'}) \mid \exists k \geq 0, h_1, \dots, h_k \in \phi(T)(\mathcal{H}_{0,HR_1}), w_1, \dots, w_k \in X^*, \alpha_1, \dots, \alpha_k \in \mathbb{R} :$$
$$(q^{'}, x_i^{'}) = \xi_{HR_2}(h_i, w_i), i = 1, \dots, k \text{ and } x^{'} = \sum_{j=1}^{k} \alpha_i x_i^{'}\}$$

It is easy to see that $\phi(T)(\mathcal{H}_{0,HR_1}) = \mathcal{H}_{0,HR_2}$. Indeed, $\phi(T)(\mu(j)) = \mu^{'}(j)$ and for all $q^{'} = \mu_D^{'}(j)$, $w \in X_2^*$, if $q = \delta(\mu_D(j), w)$, then $\phi(T)((q, B_{q,z,j})) = (T_D(q), B^{'}_{T_D(q),z,j}) = (\delta^{'}(q^{'}, w), B^{'}_{\delta^{'}(q,w),z,j})$ for all $z \in X_1, j \in J_2$. Thus,

$$\phi(T)(Reach(HR_1)) = Reach(HR_2)$$

Notice that HR_1 is reachable if and only if $Reach(HR_1) = \mathcal{H}_{HR_1}$. Since $\phi(T)$ is a bijection, the latter condition is equivalent to $Reach(HR_2) = \phi(T)(Reach(HR_1)) = \phi(T)(\mathcal{H}_{HR_1}) = \mathcal{H}_{HR_2}$, i.e. it is equivalent to HR_2 being reachable.

Similarly, HR_1 is observable if and only if for each $h_1, h_2 \in \mathcal{H}_{HR_1}$,

$$(\forall v \in X^* : \upsilon_{HR_1}(h_1, v) = \upsilon_{HR_1}(h_2, v)) \implies h_1 = h_2$$

But this is equivalent to the following. For any $h_1^{'}, h_2^{'} \in \mathcal{H}_{HR_2}$,

$$(\forall v \in X^* : \upsilon_{HR_1}(\phi(T)^{-1}(h_1^{'}), v) = \upsilon_{HR_2}(h_1^{'}, v) =$$
$$\upsilon_{HR_2}(h_2^{'}, v) = \upsilon_{HR_1}(\phi(T)^{-1}(h_2^{'}), v))$$
$$\implies \phi(T)^{-1}(h_1^{'}) = \phi(T)^{-1}(h_2^{'})$$

Since $\phi(T)$ is bijective, it implies that $\phi(T)^{-1}(h_1^{'}) = \phi(T)^{-1}(h_2^{'})$ if and only if $h_1^{'} = h_2^{'}$. Thus we get that $(\forall v \in X^* : \upsilon_{HR_1}(h_1^{'}, .) = \upsilon_{HR_2}(h_2^{'}, .)) \implies h_1 = h_2$. That is, observability of HR_1 is equivalent to observability of HR_2. $\square$

Corollary 3. *Let HR_1, HR_2 be hybrid representations and let $T : HR_1 \to HR_2$ be a hybrid representation morphism. Then HR_1 is a representation of an indexed set of hybrid power series Ω if and only if HR_2 is a representation of Ω.*

Proof. Assume that

$$HR_i = (\mathcal{A}^i, \mathcal{Y}, (\mathcal{X}^i_q, \{A^i_{q,z}, B^i_{q,z,j_2}\}_{j\in J_2, z\in X_1}, C^i_q, \{M^i_{\delta^i(q,y),y,q}\}_{y\in X_2})_{q\in Q^i}, J, \mu^i)$$

for $i = 1, 2$. Notice that for any $j \in J_1$, $\phi(T)(\mu^1(j)) = \mu^2(j)$, thus by Proposition 7 for any $j \in J_1$

$$\forall v \in X^* : \upsilon_{HR_1}(\mu^1(j), v) = \upsilon_{HR_2}(\mu^2(j))$$

Recall form (3.4) that HR_1 is a representation of $\Omega = \{Z_j \mid j \in J\}$ if and only if for all $j \in J_1$, $v \in X^*$

$$\upsilon_{HR_2}(\mu^2(j), v) = \upsilon_{HR_1}(\mu^1(j), v) = ((Z_j)_C(w), (Z_j)_D(\Pi_{X_2}(w)), ((Z_{j,j_2})_C)_{j_2\in J_2})$$

By (3.4) the latter equality is equivalent to HR_2 being a representation of Ω. $\square$

3.3.2 Existence of Hybrid Representation

In this subsection we will give necessary and sufficient conditions for existence of a hybrid representation for a family of hybrid formal power series. Recall that hybrid representations can be viewed as an interconnection of Moore-automata and rational representations. In the light of this remark it should not be surprising that finding a hybrid representation for an indexed set of hybrid power series can be reduced to finding a rational representation for a indexed set of formal power series and finding a finite Moore-automaton realization for an indexed set of discrete input-output maps.

We will proceed as follows. We will associate with each family of hybrid formal power series a family of classical formal power series and a family of discrete input-output maps. It turns out that there is a correspondence between rational representations of this family of formal power series and automaton realizations of the family of discrete input-output maps on the one hand and hybrid representations of the original family of hybrid formal power series on the other hand.

Let $HR = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j\in J_2, z\in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y\in X_2})_{q\in Q}, J, \mu)$ be a hybrid representation. Assume that $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$, $Q = \{q_1, \dots, q_N\}$ and $card(J_2) = m$. Fix a basis $\{e_{q,j} \mid q \in Q, j \in J_2\}$ in $\mathbb{R}^{Nm}$. Define the *representation associated with HR* by

$$R_{HR} = (\mathcal{X}, \{M_z\}_{z\in X}, \widetilde{B}, \widetilde{C})$$

where

- $\mathcal{X} = (\bigoplus_{q\in Q} \mathcal{X}_q) \oplus \mathbb{R}^{Nm}$, if $m > 0$ and $\mathcal{X} = \bigoplus_{q\in Q} \mathcal{X}_q$ if $m = 0$.
- $\widetilde{C} : \mathcal{X} \to \mathbb{R}^p$, is a linear map such that $\widetilde{C}x = C_q x$ if $x \in \mathcal{X}_q$ and $\widetilde{C}e_{q,j} = 0$ for each $q \in Q$,$j \in J_2$,
- $\widetilde{B} = \{\widetilde{B}_j \in \mathcal{X} \mid j \in J\}$ is defined by $\widetilde{B}_j = x_j \in \mathcal{X}_{q_j}$ and $\widetilde{B}_{(j,l)} = e_{q_j,l}$, for each $j \in J_1$, $l \in J_2$ such that $\mu(j) = (q_j, x_j)$
- For each $z \in X_1$, $M_z : \mathcal{X} \to \mathcal{X}$ is a linear map, such that for each $q \in Q$, $\forall x \in \mathcal{X}_q : M_z x = A_{q,z}x$ and for each $q \in Q, j \in J_2$, $M_z e_{q,j} = B_{q,z,j} \in \mathcal{X}_q$.
- For each $y \in X_2$, $M_y : \mathcal{X} \to \mathcal{X}$ is a linear map such that $\forall x \in \mathcal{X}_q : M_y x = M_{\delta(q,y),y,q}x$ and $M_y e_{q,j} = e_{\delta(q,z),j}$, for all $q \in Q, j \in J_2$.

Note that R_{HR} depends on the structure of the finite Moore-automaton $\mathcal{A}$ too.

The idea behind the choice of R_{HR} is the following. Consider the Moore-automaton interpretation of HR. The representation R_{HR} can be also viewed as a Moore-automaton. We would like R_{HR} to be a realization of the continuous, i.e. $\mathbb{R}^p$ valued part of the input-output behaviour of HR. That is, if HR is a representation of some family of hybrid formal power series $\Omega = \{Z_j \mid j \in J\}$, then we would like R_{HR} to be a representation of $\{(Z_j)_C \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$. By "stacking up" the matrices $A_{q,z}, M_{q_1,y,q_2}$ and taking the "state-space" $\bigoplus_{q\in Q} \mathcal{X}_q$, we encoded most of the information on the discrete-state dynamics which has effect on the continuous valued part of the input-output behaviour of the hybrid representation. But we still need to keep track of the elements $B_{q,z,j}$, and for that we need to simulate the discrete-state transitions. This is done by introducing the vectors $e_{q,j}$ and defining the action of M_y on these vectors accordingly. Of course, if $J_2 = \emptyset$, we have no vectors $B_{q,z,j}$ and there is no need to include $e_{q,j}$ into the state-space of the representation R_{HR}.

Recall the definition of the set $\bar{O}$

$$\bar{O} = \prod_{j\in J_2} \mathbb{R}^p \ll X^* \gg$$

Consider a hybrid representation of the form

$$HR = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j\in J_2, z\in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y\in X_2})_{q\in Q}, J, \mu)$$

and assume that $\mathcal{A} = (Q, X_2, O, \delta, \lambda)$. Define

$$\bar{\mathcal{A}}_{HR} = (Q, \Gamma, O \times \bar{O}, \delta, \bar{\lambda}) \tag{3.10}$$

where $\bar{\lambda}(q) = (\lambda(q), (T_{q,j})_{j\in J_2})$ if $J_2 \neq \emptyset$ and $\bar{\lambda}(q) = (\lambda(q), \emptyset)$ if $J_2 = \emptyset$. The realization $(\bar{\mathcal{A}}_{HR}, \mu_D)$ will be called the *finite Moore-automaton realization associated with HR.*

Let $\Omega = \{Z_j \in \mathbb{R}^p \ll X^* \gg \times F(X_2^*, O) \mid\in j \in J\}$ be an indexed set of formal power series. Then define the *indexed set of formal power series* Ψ_Ω *associated with* Ω by

$$\Psi_\Omega = \{(Z_j)_C \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$$

Define the *Hankel-matrix* H_Ω of Ω to be the Hankel-matrix H_{Ψ_Ω} of Ψ_Ω, i.e. $H_\Omega = H_{\Psi_\Omega}$. Define the *indexed set of discrete input-output maps associated with* Ω by

$$\mathcal{D}_\Omega = \{\kappa_j : X_2^* \to O \times \bar{O} \mid j \in J_1\}$$

where the maps κ_j are defined as follows

$$\kappa_j : X_2^* \ni w \mapsto ((Z_j)_D(w), (w \circ (Z_{j,l})_C)_{l \in J_2}) \in O \times \bar{O}$$

The following theorem describes the relationship between rationality of Ω and rationality of Ψ_Ω and realisability of $\mathcal{D}_\Omega$ by a finite Moore-automaton.

Theorem 5. *The hybrid representation HR is a hybrid representation of the indexed set of hybrid formal power series Ω if and only if R_{HR} is a representation of the indexed set of formal power series Ψ_Ω and $(\bar{A}_{HR}, \mu_D)$ is a finite Moore-automaton realization of $\mathcal{D}_\Omega$.*

Proof. Notice that for each $z_1, \ldots, z_{k+1} \in X_1^*$, $\gamma_1, \ldots, \gamma_k \in X_2$, $k \geq 0$, $q_0 \in Q$,

$$\begin{aligned}
&A_{q_k,z_{k+1}} M_{q_k,\gamma_k,q_{k-1}} A_{q_{k-1},z_k} \cdots M_{q_1,\gamma_1,q_0} A_{q_0,z_1} x = \\
&= M_{z_{k+1}} M_{\gamma_k} M_{z_k} \cdots M_{\gamma_1} M_{z_1} x \in \mathcal{X}_{q_k} \\
&C_{q_k} A_{q_k,z_{k+1}} M_{q_k,\gamma_k,q_{k-1}} A_{q_{k-1},z_k} \cdots M_{q_1,\gamma_1,q_0} A_{q_0,z_1} x = \\
&= C M_{z_{k+1}} M_{\gamma_k} M_{z_k} \cdots M_{\gamma_1} M_{z_1} x \\
&B_{q_i,z,j} = M_{\gamma_i} M_{\gamma_{i-1}} \cdots M_{\gamma_1} e_{q_0,z,j} \in \mathcal{X}_{q_k} \text{ for all } z \in X_1, j \in J_2
\end{aligned} \tag{3.11}$$

where $q_i = \delta(q_0, \gamma_1 \cdots \gamma_i)$, $i = 0, \ldots, k$. From definition we get that HR is a representation of Ω if and only if the following holds. For all $j \in J_1$, $w \in X_2^*$, $w = w_1, \ldots, w_k, w_1, \ldots, w_k \in X_2$, $z_1, \ldots, z_{k+1} \in X_1^*$, $k \geq 0$ and $j_2 \in J_2$ it holds that $\lambda(\mu_D(f), w) = (Z_j)_D(w)$, and

$$\begin{aligned}
&(Z_{j,j_1})_C(w_1 w_2 \cdots w_{l-1} z_l w_l z_{l+1} \cdots w_k z_{k+1}) = \\
&= C_{q_k} A_{q_k,z_{k+1}} M_{q_k,w_k,q_{k-1}} \cdots M_{q_l,w_l,q_{l-1}} A_{q_{l-1},v} B_{q_{l-1},s,j_1} \\
&(Z_j)_C(z_1 w_1 z_2 \cdots w_k z_{k+1}) = \\
&= C_{q_k} A_{q_k,z_{k+1}} M_{q_k,w_k,q_{k-1}} A_{q_{k-1},z_k} \cdots M_{q_1,w_1,q_0} A_{q_0,z_1} \mu_C(j)
\end{aligned} \tag{3.12}$$

$l = 1, \ldots k$, $z_l = sv$, $s \in X_1$, $q_i = \delta(\mu_D(j), w_1 \cdots w_i)$, $i = 0, \ldots, k$.

That is, by (3.11) we get that for all $z_1, \dots, z_{k+1} \in X_2^*$, $z_l = sv$, $s \in X_1$, $w_1, \dots, w_k \in X_1$, $j, j_1 \in J_1$, $j_2 \in J_2$,

$$\begin{aligned}(Z_{(j_1,j_2)})_C(w_1 w_2 \cdots w_{l-1} z_l w_l z_{l+1} \cdots w_k z_{k+1}) = \\ = C M_{z_{k+1}} M_{w_k} M_{z_k} \cdots M_{w_l} M_v M_s M_{w_{l-1}} \cdots M_{w_1} e_{q,s,j_2}\end{aligned} \tag{3.13}$$

$$(Z_{j_1})_C(z_1 w_1 z_2 \cdots w_k z_{k+1}) = C M_{z_{k+1}} M_{w_k} \cdots M_{w_1} M_{z_1} \widetilde{B}_{j_1} \tag{3.14}$$

The equations above are equivalent to R_{HR} being a representation of Ψ_Ω. On the other hand,

$$\lambda(\mu_D(j), w) = (Z_j)_D(w), w \in X_2^*$$

is equivalent to $(\mathcal{A}, \mu_D)$ being a realization of $\Omega_D = \{(Z_j)_D \in F(X_2^*, O) \mid j \in J_1\}$.

Assume now that R_{HR} is a representation of Ψ_Ω and $(\bar{\mathcal{A}}_{HR}, \mu_D)$ is a realization of $\mathcal{D}_\Omega$. The fact that R_{HR} is a representation of Ψ_Ω implies (3.12). If $(\bar{\mathcal{A}}_{HR}, \mu_D)$ is a realization of $\mathcal{D}_\Omega$, then for each $j \in J_1$

$$\Pi_O \circ \bar{\lambda}(\mu_D(j), w) = \Pi_O \circ (\kappa_j)(w) = (Z_j)_D(w),\ w \in X_2^*$$

Thus, $(\mathcal{A}, \mu_D)$ is a realization of Ω_D. That is, from the discussion above we get that HR is a representation of Ω.

Conversely, assume that HR is a representation of Ω. Then (3.12) holds, which implies that R_{HR} is a representation of Ψ_Ω. Formula (3.12) also implies that for all $j \in J_1$, $q = \mu_D(j) \in Q$, $(Z_{j,j_2})_C = T_{q,j_2}$ for all $j_2 \in J_2$. Thus, $w \circ (Z_{j,j_2})_C = w \circ T_{q,j_2} = T_{\delta(q,w),j_2}$. Since $\lambda(q,w) = (Z_j)_D(w)$, we get that

$$\bar{\lambda}(q,w) = (\lambda(q,w), (T_{\delta(q,w),j_2})_{j_2 \in J_2}) = ((Z_j)_D(w), (w \circ (Z_{j_2,j_1})_C)_{j_2 \in J_2})$$

Thus, $(\bar{\mathcal{A}}, \mu_D)$ is a realization of $\mathcal{D}_\Phi$. □

Consider the following set of discrete input-output maps.

$$\Omega_D = \{(Z_j)_D : X_2^* \to O \mid j \in J_1\}$$

It is easy to see that if $(\bar{\mathcal{A}}_{HR}, \mu_D)$ is a realization of $\mathcal{D}_\Omega$, then $(\mathcal{A}, \mu_D)$ is a realization of Ω_D. It is also easy to see that if $J_2 = \emptyset$ then $(\bar{\mathcal{A}}_{HR}, \mu_D)$ is a realization of $\mathcal{D}_\Omega$ whenever $(\mathcal{A}, \mu_D)$ is a realization of Ω_D. Thus, we get the following corollary.

Corollary 4. *Assume that $J_2 = \emptyset$. Then HR is a hybrid representation of Ω if and only if R_{HR} is a representation of Ω_Ψ and $(\mathcal{A}, \mu_D)$ is a realization of Ω_D.*

Above we associated with each hybrid representation HR a representation and a finite Moore-automaton realization. Below we will present the converse of it. That is, we will associate a hybrid representation with any suitable representation and suitable finite Moore-automaton realization. The construction goes as follows.

Let $R = (\mathcal{X}, \{M_z\}_{z\in X}, \widetilde{B}, \widetilde{C})$ be an observable representation of Ψ_Ω and let $(\bar{\mathcal{A}}, \zeta), \bar{\mathcal{A}} = (Q, X_2, O \times \bar{O}, \delta, \bar{\lambda})$ be a reachable Moore-automaton realization of $\mathcal{D}_\Omega$. Then define $(HR_{R,\bar{\mathcal{A}},\zeta}$ – *the hybrid representation associated with* R *and* $(\bar{\mathcal{A}}, \zeta)$ as

$$HR_{R,\bar{\mathcal{A}},\zeta} = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j\in J_2, z\in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y\in X_2})_{q\in Q}, J, \mu)$$

where

- $\mathcal{A} = (Q, X_2, O, \delta, \Pi_O \circ \bar{\lambda})$,
- For all $q \in Q$, let $\mathcal{X}_q = \mathrm{Span}\{z \mid z \in W_q\}$ where the set W_q is defined as follows

$$\begin{aligned} W_q = & \{M_{x_{k+1}} M_{y_k} M_{x_k} \cdots M_{y_l} M_{z_l} M_{s_l} M_{y_{l-1}} \cdots M_{y_2} M_{y_1} \widetilde{B}_{(j_1,j_2)} \mid \\ & y_1, \ldots, y_k \in X_2, j_1 \in J_1, j_2 \in J_2, k \geq 0, \\ & q = \delta(\zeta(j_1), y_1 \cdots y_k), 1 \leq l \leq k+1, x_{k+1}, \ldots, x_l \in X_1^*, \\ & x_l = s_l z_l, z_l \in X_1^*, s_l \in X_1\} \cup \\ & \cup \{M_{x_{k+1}} M_{y_k} M_{x_k} \cdots M_{y_1} M_{x_1} \widetilde{B}_j \mid y_1, \ldots, y_k \in X_2, \\ & j \in J_1, x_{k+1}, \ldots, x_1 \in X_1^*, k \geq 0, q = \delta(\zeta(j), y_1 \cdots y_k)\} \end{aligned} \tag{3.15}$$

- For each $q \in Q$, $z \in X_1$, the maps $A_{q,z} : \mathcal{X}_q \to \mathcal{X}_q$, $z \in X_1$ are defined by $A_{q,z} = M_z|_{\mathcal{X}_q}$. That is, for all $x \in \mathcal{X}_q$, $z \in X_1$,

$$A_{q,z}x = M_z x$$

- For each $q \in Q$, the map $C_q : \mathcal{X}_q \to \mathbb{R}$ is defined by $C_q = \widetilde{C}|_{\mathcal{X}_q}$. That is, for all $x \in \mathcal{X}_q$,

$$C_q x = \widetilde{C} x$$

- For each $q \in Q$, $l \in J_2$, $z \in X_1$ let $B_{q,z,l} = M_z M_w \widetilde{B}_{j,l} \in \mathcal{X}_q$ for some $w \in X_2^*$ and $j \in J_2$ such that $\delta(\zeta(j), w) = q$.
- For all $q_1, q_2 \in Q$, $y \in X_2$ such that $q_1 = \delta(q_2, y)$ define the map $M_{q_1,y,q_2} : \mathcal{X}_{q_2} \to \mathcal{X}_{q_1}$ as follows. For each $x \in \mathcal{X}_{q_2}$,

$$M_{q_1,y,q_2}x = M_y x, x \in \mathcal{X}_{q_2}$$

- Define the map $\mu : J \to \bigcup_{q\in Q}\{q\} \times \mathcal{X}_q$ as follows.

$$\mu(j) = (\zeta(j), \widetilde{B}_j) \text{ for all } j \in J_1$$

Notice that $B_{q,z,j}$ is indeed well-defined for each $q \in Q$, $z \in X_1$, $j \in J_2$. If for some $g, j \in J_1$, $w, v \in X_2^*$, $q = \delta(\zeta(j), w) = \delta(\zeta(g), v)$, then $\kappa_g(v) = \kappa_j(w)$, since $\bar{\mathcal{A}}$ is a realization of $\mathcal{D}_\Omega$. But then $\kappa_g(v) = ((Z_g)_D(v), (v \circ (Z_{g,l})_C)_{l \in J_2}) = ((Z_j)_D(w), (w \circ (Z_{j,l})_C)_{l \in J_2}) = \kappa_j(w)$, i.e, $v \circ (Z_{g,l})_C = w \circ (Z_{j,l})_C$. Since R is a representation of Ψ_Ω we get that $v \circ (Z_{g,l})_C(zs) = (Z_{g,l})_C(vzs) = (Z_{j,l})_C(wzs) = \widetilde{C} M_s M_z M_w \widetilde{B}_{j,l} = \widetilde{C} M_s M_z M_v \widetilde{B}_{g,l}$ for each $s \in X^*, z \in X_1, l \in J_2$. Then observability of R implies that $M_z M_w \widetilde{B}_{j,l} = M_z M_v \widetilde{B}_{g,l}$, thus, $B_{q,z,l}$ is indeed well-defined. It should be clear now why we needed observability of R and reachability of $(\bar{\mathcal{A}}, \zeta)$. If R was not observable, we could have several choices for the vectors $B_{q,z,l}$. If $(\bar{\mathcal{A}}, \zeta)$ was not reachable, we would have trouble defining $\mathcal{X}_q$ for the unreachable discrete states $q \in Q$.

Notice that if $J_2 = \emptyset$, then the construction of $HR_{R,\bar{\mathcal{A}},\zeta}$ could be carried out for a non-observable representation R too. Assume that $J_2 = \emptyset$ and $(\mathcal{A}, \zeta)$ is a reachable realization of Ω_D. Assume that $\mathcal{A} = (Q, X_2, O, \delta, \lambda)$ and define $\bar{\mathcal{A}}$ by

$$\bar{\mathcal{A}} = (Q, X_2, O \times \bar{O}, \delta, \bar{\lambda}), \text{, where } \bar{\lambda}(q) = (\lambda(q), \emptyset)$$

It is easy to see that $(\bar{\mathcal{A}}, \zeta)$ is a realization of $\mathcal{D}_\Omega$ if $J_2 = \emptyset$. It is also easy to see that $\bar{\mathcal{A}}$ is uniquely determined by $\mathcal{A}$ and the construction of $HR_{R,\bar{\mathcal{A}},\zeta}$ can be carried out based purely on the information present in R and $(\mathcal{A}, \zeta)$. Then it is justified to denote $HR_{R,\bar{\mathcal{A}},\zeta}$ simply by $HR_{R,\mathcal{A},\zeta}$.

The construction of $HR_{R,\bar{\mathcal{A}},\zeta}$ in fact gives us a way to go from representations of Ψ_Ω and realizations of $\mathcal{D}_\Omega$ to hybrid representations of Ω.

Theorem 6. *Assume that R is an observable representation of Ψ_Ω and $(\bar{\mathcal{A}}, \zeta)$ is a reachable realization of $\mathcal{D}_\Omega$. Then $HR_{R,\bar{\mathcal{A}},\zeta}$ is a reachable hybrid representation of Ω.*

Proof. Let $HR = HR_{R,\bar{\mathcal{A}},\zeta}$. First we will show that HR is a representation of Ω. Notice that

$$\begin{aligned} M_{z_{k+1}} M_{\gamma_k} M_{z_k} \cdots M_{\gamma_1} M_{z_1} x = \\ = A_{q_k, z_{k+1}} M_{q_k, \gamma_k, q_{k-1}} A_{q_{k-1}, z_k} \cdots M_{q_1, \gamma_1, q_0} A_{q_0, z_1} x \in \mathcal{X}_{q_k} \\ \widetilde{C} M_{z_{k+1}} M_{\gamma_k} M_{z_k} \cdots M_{\gamma_1} M_{z_1} x = \\ = C_{q_k} A_{q_k, z_{k+1}} M_{q_k, \gamma_k, q_{k-1}} A_{q_{k-1} z_k} \cdots M_{q_1, \gamma_1, q_0} A_{q_0, z_1} x \end{aligned} \tag{3.16}$$

for all $x \in \mathcal{X}_{q_0}$, $q_0 \in Q$, $\gamma_1, \ldots, \gamma_k \in X_2$, $z_1, \ldots, z_{k+1} \in X_1^*$, $k \geq 0$, where $q_i = \delta(q_0, \gamma_1 \cdots \gamma_i)$, $i = 1, \ldots, k$. Moreover, for each $j \in J_1$, $\widetilde{B}_j \in \mathcal{X}_{\zeta(j)}$ and for each $w \in X_2^*$, $j_2 \in J_2$, $j \in J$, $s \in X_1$,

$$M_s M_w \widetilde{B}_{j,j_2} = B_{\delta(\zeta(j),w),s,j_2} \in \mathcal{X}_{\delta(\zeta(j),w)}$$

Notice that $\mu_D(j) = \zeta(j)$ for all $j \in J_1$. Since R is a representation of Ψ_Ω, we get that for all $j \in J_1, j_2 \in J_2$

$$\begin{aligned}
&(Z_{j,j_2})_C(w_1 w_2 \dots w_{l-1} s z_l w_l \cdots w_k z_{k+1}) = \\
&= \widetilde{C} M_{z_{k+1}} M_{w_k} \cdots M_{w_l} M_{z_l} M_s M_{w_{l-1}} \cdots M_{w_1} \widetilde{B}_{j,j_2} = \\
&= \widetilde{C} M_{z_{k+1}} M_{w_k} \cdots M_{w_l} M_{z_l} B_{q_{l-1},z,j_2} = \\
&= C_{q_k} A_{q_k,z_{k+1}} M_{q_k,w_k,q_{k-1}} A_{q_{k-1},z_k} \cdots M_{q_l,w_l,q_{l-1}} A_{q_{l-1},z_l} B_{q_{l-1},s,j_2} \qquad (3.17) \\
&(Z_j)_C(z_1 w_1 \cdots w_k z_{k+1}) = \\
&= \widetilde{C} M_{z_{k+1}} M_{w_k} \cdots M_{w_1} M_{z_1} \widetilde{B}_j = \\
&= C_{q_k} A_{q_k,z_{k+1}} M_{q_k,w_k,q_{k-1}} \cdots M_{q_1,w_1,q_0} A_{q_0,z_1} \mu(j)
\end{aligned}$$

for each $w = w_1 \cdots w_k$, $w_1, \dots, w_k \in X_2$, $z_1, \dots, z_{k+1} \in X_1^*, s \in X_1$, $k \ge 0$, $j_2 \in J_2$, $j \in J_1$, $1 \le l \le k+1$, where $q_i = \delta(q, w_1 \cdots w_i)$, $i = 0, \dots, k$, $q = \zeta(j)$.

If $(\bar{\mathcal{A}}, \zeta)$ is a realization of $\mathcal{D}_\Omega$, we get that for each $j \in J_1$, $w \in X_2^*$

$$(Z_j)_D(w) = \Pi_O \circ \kappa_j(w) = \Pi_O \circ \bar{\lambda}(\zeta(j), w) = \lambda(\zeta(j), w) = \lambda(\mu_D(j), w) \qquad (3.18)$$

From the definition and formulas (3.17) and (3.18) it follows that HR is a representation of Ω.

It is left to show that HR is reachable. Since $(\bar{\mathcal{A}}, \zeta)$ is reachable and $\mathcal{A}$ coincides with $\bar{\mathcal{A}}$ with the exception of the readout map, we get that $(\mathcal{A}, \zeta) = (\mathcal{A}_{HR}, \mu_D)$ is reachable. From the definition of HR it follows that for each $q \in Q$

$$\begin{aligned}
\mathcal{X}_q = \mathrm{Span}\{ & M_{z_{k+1}} M_{\gamma_k} M_{z_k} \cdots M_{\gamma_l} M_{z_l} M_s M_{\gamma_{l-1}} \cdots M_{\gamma_2} M_{\gamma_1} \widetilde{B}_{j,j_2} \mid \\
& \gamma_1, \dots, \gamma_k \in X_2, j \in J_1, k \ge 0, j_2 \in J_2, q = \delta(\zeta(f), \gamma_1 \cdots \gamma_k), \\
& 1 \le l \le k+1, z_{k+1}, \dots \dots z_l \in X_1^*, s \in X_1 \} \cup \\
& \cup \{ M_{z_{k+1}} M_{\gamma_k} M_{z_{k-1}} \cdots M_{\gamma_1} M_{z_1} \widetilde{B}_j \mid \\
& \gamma_k, \dots \gamma_k \in X_2, z_{k+1}, \dots, z_1 \in X_1^*, k \ge 0, q = \delta(\zeta(f), \gamma_1 \cdots \gamma_k), j \in J_1 \}
\end{aligned}$$

Using equality (3.16) we get that

$$\begin{aligned}
\mathcal{X}_q = \mathrm{Span}(\{ & A_{q_k,z_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_l,\gamma_{l+1},q_{l-1}} A_{q_{l-1},z_l} B_{q_{l-1},s,j_2} \mid \\
& q_k = q, q_0 = \mu_D(j), j \in J_1, \gamma_1, \dots, \gamma_k \in X_2, k \ge 0, z_l, \dots, z_{k+1} \in X_1^*, s \in X_1, \\
& 1 \le l \le k+1, j_2 \in J_2, q_i = \delta(q_0, \gamma_1 \cdots \gamma_i), i = 1, \dots, k \} \cup \\
& \cup \{ A_{q_k,z_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} A_{q_0,z_1} x_j, \mid \\
& q_k = q, z_1, \dots, z_{k+1} \in X_1^*, \gamma_1, \dots, \gamma_k \in X_2, j \in J_1, \\
& (q_0, x_j) = \mu(j), q_j = \delta(q_0, \gamma_1 \cdots \gamma_j), 1 \le j \le k, k \ge 0 \})
\end{aligned}$$

Thus, we get that that $W_{HR} = \bigoplus_{q \in Q} \mathcal{X}_q$, and thus by definition HR is reachable. $\square$

The remark before Theorem 6 on the construction of $HR_{R,\bar{\mathcal{A}},\zeta}$ in the case when $J_2 = \emptyset$ yields the following corollary.

Corollary 5. *If $J_2 = \emptyset$, R is a representation of Ψ_Ω and $(\mathcal{A},\zeta)$ is a reachable realization of Ω_D then the hybrid representation $HR_{R,\mathcal{A},\zeta}$ is a reachable hybrid representation of Ω.*

Existence of a finite Moore-automaton realization for $\mathcal{D}_\Omega$ is not easy to check. But we can give the following characterisation of existence of a finite Moore-automaton which is a realization of $\mathcal{D}_\Omega$. Define the sets $W_{O,\Omega} = \{v \circ (Z_{j_1,j_2})_C \mid v \in X_2^*, (j_1,j_2) \in J_1 \times J_2\}$ and $H_{O,\Omega} = \{(H_\Omega)_{.,(v,(j_1,j_2))} \mid v \in X_2^*, (j_1,j_2) \in J_1 \times J_2\}$. It is easy to see that $H_{O,\Omega}$ is simply the set of all columns of H_Ω indexed by $(v,(j_1,j_2))$ for each $v \in X_2^*$ and $(j_1,j_2) \in J_1 \times J_2$. It is also clear that there is a bijection $(H_\Omega)_{.,(v,(j_1,j_2))} \mapsto v \circ (Z_{j_1,j_2})_C$ from $H_{O,\Omega}$ to $W_{O,\Omega}$. With the notation above using Theorem 3 we get the following.

Lemma 10. *The indexed set $\mathcal{D}_\Omega$ has a finite Moore-automaton realization if and only if $card(W_{O,\Omega}) = card(H_{O,\Omega}) < +\infty$ and Ω_D has a finite Moore-automaton realization, that is, $card(W_{\Omega_D}) < +\infty$.*

That is, the lemma above states that existence of a Moore-automaton realization of $\mathcal{D}_\Omega$ is equivalent to existence of a Moore-automaton realization of Ω_D and to $card(H_{O,\Omega}) < +\infty$, i.e. that the number of different columns of the Hankel-matrix indexed by $(v,(j_1,j_2))$, $j_2 \in J_2, j_1 \in J_1, v \in X_2^*$ is finite. The latter in fact means that the indexed set $\{\Pi_{\bar{O}} \circ \kappa_j \in F(X_2^*, \bar{O}) \mid j \in J_1\}$ has a Moore-automaton realization.

Proof of Lemma 10. It is easy to see that $\mathcal{D}_\Omega$ has a Moore-automaton realization if and only if Φ_D and $\mathcal{K} = \{\Pi_{\bar{O}} \circ \kappa_j \mid j \in J_1\}$ have a realization by a finite Moore-automaton. Indeed, let $(\mathcal{A},\zeta)$, $\mathcal{A} = (Q, X_2, O \times \bar{O}, \delta, \lambda)$ be a realization of $\mathcal{D}_\Omega$. Then $(\mathcal{A}_1,\zeta)$ and $(\mathcal{A}_2,\zeta)$ are realizations of $\mathcal{K}$ and Ω_D respectively, where $\mathcal{A}_1 = (Q, X_2, \bar{O}, \delta, \Pi_{\bar{O}} \circ \lambda)$ and $\mathcal{A}_2 = (Q, X_2, O, \delta, \Pi_O \circ \lambda)$. Conversely, assume that $(\mathcal{A}_1,\zeta_1)$, $\mathcal{A}_1 = (Q_1, X_2, \bar{O}, \delta_1, \lambda_1)$ is a realization of $\mathcal{K}$ and $(\mathcal{A}_2,\zeta_2)$, $\mathcal{A}_2 = (Q_2, X_2, O, \delta_2, \lambda_2)$ is a realization of Ω_D. Then it is easy to see that $(\mathcal{A},\zeta)$, $\mathcal{A} = (Q_2 \times Q_1, X_2, O \times \bar{O}, \delta_2 \times \delta_1, \lambda_2 \times \lambda_1)$ is a realization of $\mathcal{D}_\Omega$, where $\delta_2 \times \delta_1(q_2, q_1, \gamma) = (\delta_2(q_2,\gamma), \delta_1(q_1,\gamma)), \gamma \in X_2, (q_2,q_1) \in Q_2 \times Q_1$ and $\lambda_2 \times \lambda_1((q_2,q_1)) = (\lambda_2(q_2), \lambda_1(q_1)) \in O \times \bar{O}, (q_2,q_1) \in Q_2 \times Q_1$.

By Theorem 3 Ω_D has a realization by a Moore-automaton if and only if $card(W_{\Omega_D}) < +\infty$ and $\mathcal{K}$ has a realization by a Moore-automaton if and only if $W_{\mathcal{K}} = \{w \circ \Pi_{\bar{O}} \circ \kappa_j \mid w \in X_2^*, j \in J_2\}$ is a finite set, i.e. $card(W_{\mathcal{K}}) < +\infty$. Notice that $w \circ (\Pi_{\bar{O}} \circ \kappa_j)(v) = (wv \circ (Z_{j,j_2})_C)_{j_2 \in J_2}$. It implies that $W_{\mathcal{K}}$ is finite if and only if $S = \{w \circ (Z_{j,j_2})_C \mid w \in X_2^*, j \in J_1, j_2 \in J_2\}$ is finite. Indeed, notice that $w \circ (\Pi_{\bar{O}} \circ \kappa_j) = v \circ (\Pi_{\bar{O}} \circ \kappa_g)$ if and only if $w \circ (\Pi_{\bar{O}} \circ \kappa_j)(\epsilon) = v \circ (\Pi_{\bar{O}} \circ \kappa_j)(\epsilon)$, or, in other words, $w \circ (Z_{j,j_2})_C = v \circ (Z_{g,j_2})_C, j_2 \in J_2$. The "only if" part is trivial. Assume that $w \circ (Z_{j,j_2})_C = v \circ (Z_{g,j_2})_C$, for all $j_2 \in J_2$. Then

$$w \circ (\Pi_{\bar{O}} \circ \kappa_j)(s) = (ws \circ (Z_{j,j_2})_C))_{j_2 \in J_2} = (s \circ (w \circ (Z_{j,j_2})_C))_{j_2 \in J_2} =$$
$$= (s \circ (v \circ (Z_{g,j_2}))_C)_{j_2 \in J_2} = (vs \circ (Z_{q,j_2})_C)_{j_2 \in J_2} = v \circ (\Pi_{\bar{O}} \circ \kappa_g)(s)$$

Thus, $W_{\mathcal{K}}$ is finite if and only if $\{w \circ (\Pi_{\bar{O}} \circ k_j)(\epsilon) = (w \circ (Z_{j,j_2})_C)_{j_2 \in J_2} \mid j \in J_2, w \in X_2^*\}$ is finite. Since J_2 is finite, it means that S is finite. Conversely, if S is finite, then the set $V = \{w \circ (\Pi_{\bar{O}} \circ \kappa_j)(\epsilon) = (w \circ (Z_{j,j_2})_C)_{j_2 \in J_2} \mid j \in J_1, w \in X_2^*\}$ is finite, and thus $W_{\mathcal{K}}$ is finite. But there is one to one correspondence between $w \circ (Z_{j,j_2})_C$, $w \in X_2^*, j \in J_1, j_2 \in J_2$ and elements of $H_{O,\Omega}$. That is, S is finite if and only if $H_{O,\Omega}$ is finite. □

Theorem 3, Theorem 1, Theorem 5, Theorem 6 and Lemma 10 imply the following theorem.

Theorem 7. *Let Ω be an indexed set of hybrid formal power series. Then the following are equivalent.*

(i) Ω is rational, that is, Ω has a hybrid representation

(ii) The indexed set of formal power series Ψ_Ω is rational and $\mathcal{D}_\Omega$ has a finite Moore-automaton realization.

(iii) rank $H_\Omega < +\infty$, *$card(H_{O,\Omega}) < +\infty$,and $card(W_{\Omega_D}) < +\infty$*

Proof. (i) $\implies$ (ii)
If HR is a representation of Φ, then from Theorem 5 it follows that R_{HR} is a representation of Ψ_Ω and $(\bar{\mathcal{A}}_{HR}, \mu_D)$ is a realization of $\mathcal{D}_\Omega$. Thus, Ψ_Ω is rational and $\mathcal{D}_\Omega$ has a realization by a Moore-automaton.

(ii) $\implies$ (i)
Assume that Ψ_Ω is rational and $\mathcal{D}_\Omega$ has a Moore-automaton realization. Then by Theorem 4 $\mathcal{D}_\Omega$ has a minimal Moore-automaton realization $(\mathcal{A}, \zeta)$ and this realization is reachable and observable. Similarly, by Theorem 2 if Ψ_Ω has a representation then there exists a minimal representation R of Ψ_Ω, and R is reachable and observable. Thus, $HR = HR_{(R,\mathcal{A},\zeta}$ is well defined and by Theorem 6 HR is a reachable realization of Φ.

(ii) $\iff$ (iii)
By Theorem 1, Ψ_Ω is rational if and only if rank $H_{\Psi_\Omega} =$ rank $H_\Omega < +\infty$. By

Lemma 10 $\mathcal{D}_\Omega$ has a Moore-automaton realization if and only if $card(W_{\mathcal{D}_\Omega}) < +\infty$ and $card(H_{\Omega,O}) < +\infty$. $\square$

Taking into account the discussion for the case when $J_2 = \emptyset$ we get the following corollary of the theorem above.

Corollary 6. *Assume that $J_2 = \emptyset$. Then Ω is rational if and only if Ψ_Ω is rational and Ω_D has a finite Moore-automaton realization. That is, Ω is rational if and only if* rank $H_\Phi < +\infty$ *and $card(W_{\Omega_D}) < +\infty$.*

3.3.3 Minimal Hybrid Representations

Our next step will be to characterise minimal hybrid representations. We will start with characterising reachability and observability of hybrid representations. Recall from Section 3.1 the notion of W-observability for formal power series representations R, where W is a subspace of the state-space of R. Consider the hybrid representation

$$HR = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j\in J_2, z\in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y\in X_2})_{q\in Q}, J, \mu)$$

Notice that for all $q \in Q$ the linear space $\mathcal{X}_q$ is a subspace of the state-space of R_{HR}. The following lemma characterises reachability and observability of HR.

Lemma 11. *The hybrid representation HR is reachable if and only if R_{HR} is reachable and $(\mathcal{A}, \mu_D)$ is reachable. The hybrid representation HR is observable if and only if $(\bar{\mathcal{A}}_{HR}, \mu_D)$ is observable and R_{HR} is $\mathcal{X}_q$ observable for all $q \in Q$.*

Proof. Let $HR = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j\in J_2, z\in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y\in X_2})_{q\in Q}, J, \mu)$ and let $R_{HR} = (\bigoplus_{q\in Q} \mathcal{X}_q \oplus \mathbb{R}^{|Q|\cdot|J_2|}, \{M_z\}_{z\in\widetilde{\Gamma}}, B, C)$. Recall that each $s \in X^*$ can be uniquely written as $s = z_1\gamma_1 z_2\gamma_2 \cdots z_k\gamma_k z_{k+1}$ for some $z_1, \dots, z_{k+1} \in X_1^*$, $k \geq 0$, $\gamma_1, \dots, \gamma_k \in X_2$. From (3.11) it follows that

$$\begin{aligned} W_{HR} = \mathrm{Span}(\{M_s M_y M_w B_{j,j_2} \mid s \in X^*, j \in J_1, j_2 \in J_2, y \in X_1 w \in X_2^*\}\cup \\ \cup\{M_s B_j \mid s \in X^*, j \in J_1\}) \end{aligned}$$

That is,

$$\begin{aligned} W_{R_{HR}} &= W_H + \mathrm{Span}\{M_w B_{j,j_2} \mid j_2 \in J_2, j \in J_1\} = \\ &= W_{HR} \oplus \mathrm{Span}\{e_{q,j_2} \mid j \in J_2, q = \delta(\mu_D(f), w), w \in X_2^*, z \in X_1\} \end{aligned}$$

In other words, $W_{R_{HR}} \cap \bigoplus_{q\in Q} \mathcal{X}_q = W_{HR}$. HR is reachable if and only if $W_{HR} = \bigoplus_{q\in Q} \mathcal{X}_q$ and $(\mathcal{A}, \mu_D)$ is reachable. Notice that $e_{q,j}$, $q \in Q$, $j \in J_2$, are linearly independent. Thus $\{e_{q,j} \mid q \in Q, j \in J_2\} = \{e_{q,j_2} \mid q = \delta(\mu_D(j), w), j \in J_1, w \in$

$X_2^*, j \in J_2\}$ is equivalent to $\mathbb{R}^{|Q|\cdot|J_2|} = \mathrm{Span}\{e_{q,j} \mid q \in Q, j \in J_2\} = \mathrm{Span}\{e_{q,j_2} \mid j_2 \in J_2, q = \delta(\mu_D(j), w), w \in X_2^*, j \in J_1\}$. But $(\mathcal{A}, \mu_D)$ is reachable implies that $\{e_{q,j} \mid q \in Q, j \in J_2\} = \{e_{q,j_2} \mid q = \delta(\mu_D(j), w), j \in J_1, w \in X_2^*, j \in J_2\}$. It is straightforward to see that $(\bar{\mathcal{A}}_{HR}, \mu_D)$ is reachable if and only if $(\mathcal{A}_{HR}, \mu_D)$ is reachable. Thus, if HR is reachable, then $W_{R_{HR}} = W_{HR} \oplus \mathbb{R}^{|Q|\cdot|J_2|} = \bigoplus_{q\in Q} \mathcal{X}_q \oplus \mathbb{R}^{|Q|\cdot|J_2|}$ Conversely, assume that R_{HR} is reachable and $(\bar{\mathcal{A}}_{HR}, \mu_D)$ is reachable. Then $(\mathcal{A}, \mu_D)$ is reachable and $W_{HR} = W_{R_{HR}} \cap \bigoplus_{q\in Q} \mathcal{X}_q = \bigoplus_{q\in Q} \mathcal{X}_q$. Thus, HR is reachable.

Next, we will show that HR is observable if and only if $\bar{\mathcal{A}}_{HR}$ is observable and R_{HR} is $\mathcal{X}_q$ observable for all $q \in Q$. It is easy to see that part(i) of Proposition 4 is equivalent to $(\lambda(q_1, w) = \lambda(q_2, w), w \in X_2^*$ and $T_{q_1,j} = T_{q_2,j}, \forall j \in J_2) \iff q_1 = q_2$. Notice that $T_{q_1,j} = T_{q_2,j}$ is equivalent to $w \circ T_{q_1,j} = w \circ T_{q_2,j}, w \in X_2^*$, and $w \circ T_{q,j} = T_{\delta(q,w),j}$ for all $j \in J_2$, $w \in X_2^*$. Thus, part (i) is equivalent to $(\lambda(q_1, w) = \lambda(q_2, w), (w \circ T_{q_1,j})_{j\in J_2} = (w \circ T_{q_2,j})_{j\in J_2}, w \in X_2^*) \iff q_1 = q_2$, or equivalently, $(\bar{\lambda}(q_1, w) = \bar{\lambda}(q_2, w), w \in X_2^*) \iff q_1 = q_2$. But the latter expression is equivalent to $(\bar{\mathcal{A}}_{HR}, \mu_D)$ being observable. That is, part(i) of Proposition 3 is equivalent to observability of $(\bar{\mathcal{A}}_{HR}, \mu_D)$. Consider part (ii) of Theorem Proposition 4. From formula (3.11) in the proof of Theorem 5 it follows that for each $q \in Q$, $\gamma_1, \ldots, \gamma_k \in X_2$, $k \geq 0$, $O_{q,\gamma_1\cdots\gamma_k} = \bigcap_{z_1,\ldots,z_{k+1}\in X_1^*, k\geq 0} (\ker CM_{z_{k+1}} M_{\gamma_k} M_{z_k} \cdots M_{\gamma_1} M_{z_1} \cap \mathcal{X}_q)$. Recall that each $s \in X^*$ can be uniquely written as $s = z_1\gamma_1 z_2\gamma_2 \cdots z_k\gamma_k z_{k+1}$ for some $z_1, \ldots, z_{k+1} \in X_1^*$, $k \geq 0$, $\gamma_1, \ldots, \gamma_k \in X_2$. That is ,

$$\bigcap_{w\in X_2^*} O_{q,w} = \mathcal{X}_q \cap \bigcap_{s\in X^*} \ker CM_s = \mathcal{X}_q \cap O_{R_{HR}}$$

That is, part (ii) of Proposition 4 is equivalent to $\mathcal{X}_q \cap O_{R_{HR}} = \{0\}$ for all $q \in Q$, that is, R_{HR} is $\mathcal{X}_q$ observable for each $q \in Q$.

But HR is observable if and only if part (i) and part (ii) of Proposition 4 holds. Thus HR is observable if and only if $(\bar{\mathcal{A}}_{HR}, \mu_D)$ is observable and R_{HR} is $\mathcal{X}_q$ observable for each $q \in Q$. □

Notice that if $J_2 = \emptyset$ then $(\bar{\mathcal{A}}_{HR}, \mu_D)$ is observable if and only if $(\mathcal{A}, \mu_D)$ is observable. That is, we get the following corollary.

Corollary 7. *If $J_2 = \emptyset$ then HR is observable if and only if $(\mathcal{A}, \mu_D)$ is observable and R_{HR} is $\mathcal{X}_q$ observable for all $q \in Q$.*

It is easy to see that the following result holds too.

Lemma 12. *If HR is a hybrid representation of some indexed set of hybrid formal power series Ω, then there exists a hybrid representation HR_r of Ω such that HR_r is reachable and $\dim HR_r \leq \dim HR$. Equality $\dim HR_r = \dim HR$ holds if and only if HR is reachable.*

Proof. Assume that

$$HR = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j \in J_2, z \in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y \in X_2})_{q \in Q}, J, \mu)$$

Define the hybrid representation HR_r by

$$HR_r = (\mathcal{A}^r, \mathcal{Y}, (\mathcal{X}_q^r, \{A_{q,z}^r, B_{q,z,j_2}^r\}_{j \in J_2, z \in X_1}, C_q^r, \{M_{\delta^r(q,y),y,q}^r\}_{y \in X_2})_{q \in Q^r}, J, \mu^r)$$

such that the following holds. The automaton $\mathcal{A}_r = (Q^r, X_2, O, \delta^r, \lambda^r)$ is the sub-automaton of $\mathcal{A}$ reachable from $\Pi_Q \circ \mathrm{Im}\mu$. That is $Q_r = \{q \in Q \mid \exists j \in J_1, w \in X_2^*, \delta(\mu_D(j), w) = q\}$ and $\delta^r(q,z) = \delta(q,z)$,$\lambda^r(q) = \lambda(q)$ for all $q \in Q_r$, $z \in X_2$. For each $q \in Q^r$ let $\mathcal{X}_q^r = \mathcal{X}_q \cap W_{HR}$ and let $A_{q,z}^r x = A_{q,z}x$, $M_{\delta^r(q,y),y,q}^r x = M_{\delta(q,y),y,q}x$, $C_q^r x = C_q x$ for all $q \in Q^r$, $z \in X_1$, $y \in X_2$. Since $A_{q,z}(W_{HR} \cap \mathcal{X}_q) \subseteq \mathcal{X}_q \cap W_{HR}$ an $M_{\delta(q,y),y,q}(W_{HR} \cap \mathcal{X}_q) \subseteq \mathcal{X}_{\delta(q,y)} \cap W_{HR}$ we get that $A_{q,z}^r : \mathcal{X}_q^r \to \mathcal{X}_q^r$ and $M_{\delta^r(q,y),y,q} : \mathcal{X}_q^r \to \mathcal{X}_{\delta^r(q,y)}$ are well-defined. It is easy to see that $B_{q,z,j} \in \mathcal{X}_q^r = \mathcal{X}_q \cap W_{HR}$. Let $\mu^r(j) = \mu(j)$ for all $j \in J_1$. It is also easy to see that $\mu_D(j) \in Q^r$ and $\mu_C(j) \in \mathcal{X}_{\mu_D(r)}^r$, and thus $\mu_D^r(j) \in \mathcal{X}_{\mu_D^r(j)}^r$. Thus, HR_r is a well-defined hybrid representation. Define the automaton morphism $\phi : (\mathcal{A}_r, (\mu_r)_D) \to (\mathcal{A}, \mu_D)$ by $\phi(q) = q$ for each $q \in Q^r$. It is easy to see that ϕ is indeed an automaton morphism. Define $T_C : \bigoplus_{q \in Q^r} \mathcal{X}_q^r \to \bigoplus_{q \in Q} \mathcal{X}_q$ by $T_C(x) = x$ for each $x \in \mathcal{X}_q^r$, $q \in Q^r$. It is easy to see that (ϕ, T_C) is a hybrid representation morphism. Thus, by Corollary 3 if HR is a representation of Ω then HR_r will also be a representation of Ω. (ϕ, T_C) is clearly injective, we get that $\dim H_r \leq \dim H$ by Proposition 6. It is easy to see that $W_{HR_r} = W_{HR} = \bigoplus_{q \in Q^r} \mathcal{X}_q^r$. Thus by Proposition 3 HR_r is reachable. □

Below we will investigate certain properties of hybrid representations of the form $HR_{R,\bar{\mathcal{A}},\zeta}$.

Lemma 13. *Let R be an observable representation of Ψ_Ω, let $(\bar{\mathcal{A}}, \zeta)$ be a reachable realization of $\mathcal{D}_\Omega$. Consider the hybrid representation $HR = HR_{R,\bar{\mathcal{A}},\zeta}$ and the associated representation R_{HR}. Then there exists a representation morphism $i_R : R_{HR} \to R$ such that $i_R(x) = x$ for all $x \in \mathcal{X}_q$, $q \in Q$.*

Proof. Assume that $R = (\mathcal{X}, \{F_z\}_{z \in X}, B, C)$. Assume that $\bar{\mathcal{A}} = (Q, X_2, O \times \bar{O}, \delta, \bar{\lambda})$,

$$HR = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j \in J_2, z \in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y \in X_2})_{q \in Q}, J, \mu)$$

where $\mathcal{A} = (Q, X_2, O, \delta, \lambda)$, $\lambda = \Pi_O \circ \bar{\lambda}$. Assume that $card(Q) = N$ for some $N \in \mathbb{N}$ and $Q = \{q_1, \ldots, q_N\}$. Assume that $R_{HR} = (\bigoplus_{q \in Q} \mathcal{X}_q \oplus \mathbb{R}^{N \cdot m}, \{M_z\}_{z \in X}, \widetilde{B}, \widetilde{C})$, where $m = card(J_2)$. Denote by $\mathcal{X}$ the vector space $\bigoplus_{q \in Q} \mathcal{X}_q$. Recall that $\mathcal{X}_q \subseteq \mathcal{X}$, thus the map $i_q : \mathcal{X}_q \ni x \mapsto x \in \mathcal{X}$ is well defined. Define $i_R : \bigoplus_{q \in Q} \mathcal{X}_q \oplus \mathbb{R}^{Nm} \to \mathcal{X}$ as follows. Let $i_R(x) = i_q(x) = x$ for all $x \in \mathcal{X}_q$, $q \in Q$. Let $i_R(e_{q,j}) = F_w B_{f,j}$ such that $\delta(\zeta(f), w) = q$ for some $w \in X_2^*$. Since $(\bar{\mathcal{A}}, \zeta)$ is reachable, such f and w exists. Assume that $\delta(\zeta(f), w) = \delta(\zeta(g), v)$. Then $(w \circ (Z_{f,j})_C)_{j \in J_2} = \Pi_{\bar{O}} \circ \lambda(\zeta(f), w) = \Pi_{\bar{O}} \circ \lambda(\zeta(g), v) = (v \circ (Z_{g,j})_C)_{j \in J_2}$ thus $w \circ (Z_{f,l})_C = v \circ (Z_{g,l})_C, l \in J_2$. But for each $s \in X^*$, $CF_sF_wB_{f,j} = (w \circ (Z_{f,j})_C)(s) = (v \circ (Z_{g,j})_C)(s) = CF_sF_vB_{g,j}$. Since R is observable, we get that $F_wB_{f,j} = F_vB_{g,j}$. That is, $i_R(e_{q,j})$ is well defined. We have to show that i_R is a representation morphism. Notice that $i_R(\widetilde{B}_{f,j}) = i_R(e_{\zeta(f),j}) = B_{f,j}$, for each $f \in J_1, j \in J_2$. It is easy to see that $i_R(\widetilde{B}_f) = \mu_C(f) = B_f$, for each $f \in J_1$, since $\widetilde{B}_f = \mu_C(f) \in \mathcal{X}_{\zeta(f)}$. We have to show that $Ci_R = \widetilde{C}$. For each $x \in \mathcal{X}_q, q \in Q$, $Ci_R(x) = Cx = C_qx = \widetilde{C}x$. On the other hand, for each $q \in Q$ there exists a $f \in J_1$ and $w \in X_2^*$ such that $\delta(\zeta(f), w) = q$. Thus, $i_R(e_{q,j}) = F_wB_{f,j}$ and $Ci_R(e_{q,j}) = CF_wB_{f,j} = (Z_{f,j})_C(w) = 0$, since $(Z_{f,j})_C(s) = 0$ for any $s \in X_2^*$. Hence $\widetilde{C}e_{q,j} = 0 = Ci_R(e_{q,j})$, $q \in Q, j \in J_2$. That is, $Ci_R = \widetilde{C}$. We have to show that $i_RM_z = F_zi_R$ holds for all $z \in X$. For each $\gamma \in X_2$, $i_RM_\gamma x = i_R(M_{\delta(q,\gamma),\gamma,q}x) = M_{\delta(q,\gamma),\gamma,q}x = F_\gamma x$ if $x \in \mathcal{X}_q$ for some $q \in Q$. If $x = e_{q,j}$ for some $q \in Q$,$j \in J_2$, then $i_R(M_\gamma e_{q,j}) = i_R(e_{\delta(q,\gamma),j}) = F_wB_{f,j}$. Assume that $\delta(\zeta(g), v) = q$. Then $i_R(e_{q,j}) = F_vB_{g,j}$. But $\delta(\zeta(q), v\gamma) = \delta(q, \gamma)$, thus $F_wB_{f,j} = F_{v\gamma}B_{g,j} = F_\gamma F_vB_{g,j}$. That is, $i_R(M_\gamma e_{q,j}) = F_\gamma i_R(e_{q,j})$. That is, $i_RM_\gamma = F_\gamma i_R$ holds for any $\gamma \in X_2$. As the last step we will prove that $i_RM_z = F_zi_R$ for all $z \in X_1$. Again, if $x \in \mathcal{X}_q$ for some $q \in Q$, then $i_RM_zx = A_{q,z}x = F_zx$. If $x = e_{q,j}$, then $i_RM_zx = i_R(B_{q,z,j}) = B_{q,z,j} = F_zF_wB_{f,j} = F_zi_R(x)$, where $\delta(\zeta(f), w) = q$. That is, $i_RM_z = F_zi_R$ holds for all $z \in X_1$. Thus, i_R is indeed a representation morphism. □

The lemma above has the following consequence.

Lemma 14. *Assume that R is minimal representation of Ψ_Ω and $(\bar{\mathcal{A}}, \zeta)$ is a minimal realization of $\mathcal{D}_\Omega$. Then the hybrid representation $HR = HR_{R,\bar{\mathcal{A}},\zeta}$ is reachable and observable.*

Proof. Since R is minimal, it is also observable. Since $(\bar{\mathcal{A}}, \zeta)$ is minimal it is reachable and observable. Thus, the hybrid representation HR is well-defined. Assume that

$$HR = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j \in J_2, z \in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y \in X_2})_{q \in Q}, J, \mu)$$

where $\mathcal{A} = (Q, X_2, O, \delta, \lambda)$. Assume that $R = (\mathcal{X}, \{M_z\}_{z \in X}, \widetilde{B}, \widetilde{C})$. The hybrid representation (H, μ) is reachable and $(\bar{\mathcal{A}}_{HR}, \mu_D) = (\bar{\mathcal{A}}, \zeta)$ is observable. Consider the representation

$$R_{HR} = (\bigoplus_{q \in Q} \mathcal{X}_q \oplus \mathbb{R}^{|Q|m}, \{M_z^{'}\}_{z \in \widetilde{\Gamma}}, \widetilde{B}^{'}, \widetilde{C}^{'})$$

where $card(J_2) = m$. Then by Lemma 13 there exists $i_R : R_{H,\mu} \to R$ such that for each $x \in \mathcal{X}_q, q \in Q$: $i_R(x) = x$ and thus

$$\widetilde{C}^{'} M_w^{'} x = \widetilde{C} M_w i_R(x) = \widetilde{C} M_w x$$

If $x \in O_{R_{HR}}$, then $x \in O_R = \{0\} \cap \mathcal{X}_q$. So we get that $\mathcal{X}_q \cap O_{R_{HR}} = \{0\}$, that is R_{HR} is $\mathcal{X}_q$ observable for each $q \in Q$. Thus, by Lemma 11 the hybrid representation HR is reachable and observable. □

As a next step we will investigate the relationship between hybrid representation morphisms and formal power series representation and Moore-automaton morphisms. The following technical lemmas characterise the relationship between the two concepts. In fact, any hybrid representation morphism induces a representation morphism and an automaton morphism.

Lemma 15. *Let HR_1, HR_2 be two hybrid representations and assume that*

$$HR_i = (\mathcal{A}^i, \mathcal{Y}, (\mathcal{X}_q^i, \{A_{q,z}^i, B_{q,z,j_2}^i\}_{j \in J_2, z \in X_1}, C_q^i, \{M_{\delta^i(q,y),y,q}^i\}_{y \in X_2})_{q \in Q^i}, J, \mu^i)$$

$i = 1, 2$. Let $T = (T_D, T_C) : HR_1 \to HR_2$ be a hybrid representation morphism. Then there exists a representation morphism $\widetilde{T} : R_{HR_1} \to R_{HR_2}$ such that $T_C(x) = \widetilde{T}(x)$ for all $x \in \mathcal{X}_q^1$, $q \in Q^1$ and $\widetilde{T}(e_{q,l}) = e_{T_D(q),l}$ for all $q \in Q_1$ and $l \in J_2$. The map $T_D : Q_1 \to Q_2$ is in fact an automaton morphism $T_D : (\bar{\mathcal{A}}_{HR_1}, (\mu_1)_D) \to (\bar{\mathcal{A}}_{HR_2}, (\mu_2)_D)$.

Proof. Assume that $\mathcal{A}^2 = (Q^2, X_2, O, \delta^2, \lambda^2)$. Define the linear morphism $\widetilde{T}x = T_C x$, if $x \in \mathcal{X}_q^1$ for some $q \in Q^1$, and $\widetilde{T} e_{q,j} = e_{T_D(q),j}$, for each $q \in Q, j \in J_2$. It is easy to see that $\widetilde{T} : \bigoplus_{q \in Q^1} \mathcal{X}_q^1 \oplus \mathbb{R}^{|Q^1|m} \to \bigoplus_{q \in Q^2} \mathcal{X}_q^2 \oplus \mathbb{R}^{|Q^2|m}$. where $m = card(J_2)$. Assume that R_{H_i,μ_i}, $i = 1, 2$ are of the form $R_{H_1,\mu_1} = (\bigoplus_{q \in Q^1} \mathcal{X}_q^1 \oplus \mathbb{R}^{|Q^1|m}, \{M_z\}_{z \in X}, B^1, C^1$ and $R_{H_2,\mu_2} = (\bigoplus_{q \in Q^2} \mathcal{X}_q^2 \oplus \mathbb{R}^{|Q^2|m}, \{F_z\}_{z \in X}, B^2, C^2)$. In order to show that $\widetilde{T}$ is a representation morphism we have to show that. $\widetilde{T} M_z x = F_z \widetilde{T} x, z \in X, C^1 = C^2 \widetilde{T} x$ and $B_j^2 = \widetilde{T}(B_j^1)$ for each $x \in \bigoplus_{q \in Q^1} \mathcal{X}_{q_1}^1 \bigoplus \mathbb{R}^{|Q^1|m}$.

First we assume that $x \in \mathcal{X}_q^1$ for some $q \in Q^1$. Then for all $z \in X_1$, $\widetilde{T} M_z x = \widetilde{T} A_{q,z}^1 x = T_C A_{q,z}^1 x = A_{T_D(q),z}^2 T_C(x)$. Since $T_C(x) \in \mathcal{X}_{T_D(q)}^2$ by definition of hybrid representation morphisms, we get that $\widetilde{T} M_z x = A_{T_D(q)}^2 T_C x = F_z \widetilde{T} x$ for all $z \in X_1$. For each $\gamma \in X_1$,

$$\begin{aligned} \widetilde{T} M_\gamma x = \widetilde{T} M_{\delta^1(q,\gamma),\gamma,q}^1 x = T_C M_{\delta^1(q,\gamma),\gamma,q}^1 x = \\ M_{\delta^2(T_D(q),\gamma),\gamma,T_D(q)}^2 T_C(x) = F_\gamma \widetilde{T} x \end{aligned}$$

It is easy to see that $C^1 x = C_q^1 x = C_{T_D(q)}^2 T_C(x) = C^2 \widetilde{T}(x)$.

Assume that $x = e_{q,j}$ for some $j \in J_2$, $q \in Q^1$. Then $\widetilde{T} M_z x = \widetilde{T}(B_{q,z,j}^1) = T_C(B_{q,z,j}^1) = B_{T_D(q),z,j}^2 = F_z e_{T_D(q),j} = F_z \widetilde{T}(e_{q,j})$. For each $\gamma \in X_2$, $\widetilde{T} M_\gamma e_{q,j} =$

$\widetilde{T}(e_{\delta^1(q,\gamma),j}) = e_{T_D(\delta^1(q,\gamma)),j} = e_{\delta^2(T_D(q),\gamma),j} = F_\gamma e_{T_D(q),j}$. It is easy to see that $C^1 x = 0 = C^2 e_{T_D(q),j} = C^2\widetilde{T}(x)$.

Finally, for each $f \in J_1$ $\widetilde{T}(B^1_f) = T_C(\mu_1)_C(f)) = (\mu_2)_C(f) = B^2_f$. For each $f \in J_1, j \in J_1$, $\widetilde{T}(B^1_{f,j}) = \widetilde{T}(e_{(\mu_1)_D(f),j}) = e_{T_D((\mu_1)_D(f)),j} = e_{(\mu_2)_D(f),j} = B^2_{f,j}$. Thus, $\widetilde{T}$ is indeed a representation morphism.

Finally, we will show that T_D is an automaton morphism from $(\bar{\mathcal{A}}_{H_1}, (\mu_1)_D)$ to $(\bar{\mathcal{A}}_{H_2}, (\mu_2)_D)$. From (3.10) it follows that $\bar{\mathcal{A}}_{H_i} = (Q^i, X_2, O \times \bar{O}, \delta^i, \bar{\lambda}^i)$, $\bar{\lambda}^i(q) = (\lambda^i(q), (T_{q,j})_{j\in J_2})$. In order to prove that T_D is an automaton morphism we have to show that $T_D(\delta^1(q,\gamma)) = \delta^2(T_D(q),\gamma)$, $\bar{\lambda}^1(q) = \bar{\lambda}^2(T_D(q))$ for all $q \in Q^1$ and $\gamma \in X_2$. But from formula (3.6) Proposition 7 we get that $(T_{q,j})_{j\in J_2} = (T_{T_D(q),j})_{j\in J_2}$ Notice that by definition of hybrid representation morphism $T_D : (\mathcal{A}_{HR_1}, (\mu_1)_D) \to (\mathcal{A}_{HR_2}, (\mu_2)_D)$ is an automaton morphism. That is, $T_D(\delta^1(q,\gamma) = \delta^2(T_D(q),\gamma)$ and $\lambda^1(q) = \lambda^2(T_D(q))$ for each $q \in Q^1$, $\gamma \in X_2$. Hence $\bar{\lambda}^1(q) = (\lambda^1(q), (T_{q,j})_{j\in J_2}) = (\lambda^2(T_D(q)), (T_{T_D(q),j})_{j\in J_2})$. □

The following lemma is in some sense the converse of the lemma above. Let $HR = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j\in J_2, z\in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y\in X_2})_{q\in Q}, J, \mu)$ be a hybrid representation over the index set J of Ω. Then the following lemma holds.

Lemma 16. *Assume that HR is a reachable representation of Ω. Assume that R is an observable representation of Ψ_Ω and $(\bar{\mathcal{A}}, \zeta)$ is a reachable realization of $\mathcal{D}_\Omega$. Assume that $T : R_{HR} \to R$ is a representation morphism and $\phi : (\bar{\mathcal{A}}_{HR}, \mu_D) \to (\bar{\mathcal{A}}, \zeta)$ is an automaton morphism. Then there exists a surjective hybrid representation morphism $H(T) = (\phi, T_C) : HR \to HR_{R,\bar{\mathcal{A}},\zeta}$ such that for all $x \in \mathcal{X}_q$, $q \in Q$, $T_C(x) = T(x)$.*

Proof. Assume that

$$HR_{R,\bar{\mathcal{A}},\zeta} = (\widetilde{\mathcal{A}}, (\widetilde{\mathcal{X}}_q, \{\widetilde{A}_{q,z}, \widetilde{B}_{q,z,j}\}_{j\in J_2, z\in X_1}, \widetilde{C}_q, \{\widetilde{M}_{\widetilde{\delta}(q,\gamma),\gamma,q}\}_{\gamma\in X_2})_{q\in Q}, J, \widetilde{\mu})$$

where $\bar{\mathcal{A}} = (\widetilde{Q}, X_2, O \times \bar{O}, \widetilde{\delta}, \widetilde{\lambda})$, and $\widetilde{\mathcal{A}} = (\widetilde{Q}, X_2, O, \widetilde{\delta}, \Pi_O \circ \widetilde{\lambda})$. Assume that

$$R = (\widetilde{\mathcal{X}}, \{F_z\}_{z\in X}, \bar{B}, \bar{C})$$

Assume that $\mathcal{A}_{HR} = \mathcal{A} = (Q, X_2, O, \delta, \lambda)$ and

$$R_{HR} = (\bigoplus_{q\in Q} \mathcal{X}_q \oplus \mathbb{R}^{|Q|m}, \{M_z\}_{z\in\widetilde{\Gamma}}, B, C)$$

where $card(J_2) = m$. It is easy to see that if $\phi : Q \to \widetilde{Q}$ is a automaton morphism $\phi : (\bar{\mathcal{A}}_{HR}, \mu_D) \to (\bar{\mathcal{A}}, \zeta)$, then ϕ can be viewed as an automaton morphism $\phi : (\mathcal{A}_{HR}, \mu_D) \to (\widetilde{\mathcal{A}}, \widetilde{\mu}_D)$ too. Indeed, $\Pi_{\widetilde{Q}} \circ \mu_{R,\bar{\mathcal{A}},\zeta} = \zeta$ and $\phi(\mu_D(f)) = \zeta(f)$. For each

$\gamma \in X_2, q \in Q$, $\phi(\delta(q,\gamma)) = \widetilde{\delta}(\phi(q),\gamma)$ and $\lambda(q) = \Pi_O \circ \bar{\lambda}(q) = \Pi_O \circ \widetilde{\lambda}(\phi(q)$. Thus, $\phi : (\bar{\mathcal{A}}_{HR}, \mu_D) \to (\widetilde{\mathcal{A}}, \zeta)$ is indeed an automaton morphism.

We have to show that T_C is well defined and $T_C(x) = Tx \in \widetilde{\mathcal{X}}_{\phi(q)}$ for each $q \in Q, x \in \mathcal{X}_q$. Define the linear map $T_C : \bigoplus_{q \in Q} \mathcal{X}_q \to \bigoplus_{q \in \widetilde{Q}} \widetilde{\mathcal{X}}_q$ by $T_C(x) = Tx \in \widetilde{\mathcal{X}}_{\phi(q)}$ for each $x \in \mathcal{X}_q$, $q \in Q$. First of all, we have to show that if $x \in \mathcal{X}_q$, then $Tx \in \widetilde{\mathcal{X}}_{\phi(q)}$. Since HR is reachable , by Lemma 11 we get that R_{HR} is reachable. That is, for all $x \in \mathcal{X}_q$ there exists $s_i, z_j \in X^*, v_j \in X_2^*, r_j \in X_1$, $f_i, g_j \in J_1$, $l_j \in J_2$ $\alpha_i, \beta_j \in \mathbb{R}$, $i = 1, \dots, r, j = 1, \dots d$ such that

$$x = \sum_{i=1}^{r} \alpha_i M_{s_i} B_{f_i} + \sum_{j=1}^{d} \beta_j M_{z_j} M_{r_j} M_{v_j} B_{g_j,l_j}$$

Assume that

$$s_i = \alpha_{i,1}\gamma_{i,1}\cdots\gamma_{i,k_i}\alpha_{i,k_i+1}$$

and

$$z_j = \beta_{j,1} w_{j,1} \cdots w_{j,h_j} \beta_{j,h_j+1}$$

where $\alpha_{i,1}, \dots, \alpha_{i,k_i+1}$, $\beta_{j,1}, \dots, \beta_{j,h_j+1} \in X_1^*$, $\gamma_{i,1}, \dots, \gamma_{i,k_i}, w_{j,1}, \dots, w_{j,k_j} \in X_2$, $k_i, h_i \geq 0$ for each $i = 1, \dots, r, j \in J_2$. Since $x \in \mathcal{X}_q$, from definition of R_{HR} we get that $\delta(\mu_D(f_i), \gamma_{i,1}\cdots\gamma_{i,k_i}) = q$ and $\delta(\mu_D(g_j), v_j w_{j,1}\cdots w_{j,h_j}) = q$. Thus,

$$\widetilde{\delta}(\phi(\mu_D(g_j)), \gamma_{i,1}\cdots\gamma_{i,k_i}) = \widetilde{\delta}(\zeta(g_j), \gamma_{i,1}\cdots\gamma_{i,k_i}) = \phi(q)$$

and $\phi(q) = \widetilde{\delta}(\phi(\mu_D(f_i)), \gamma_{i,1}\cdots\gamma_{i,k_i}) = \widetilde{\delta}(\zeta(f_i)), \gamma_{i,1}\cdots\gamma_{i,k_i})$. Notice that

$$\begin{aligned} T_C x = Tx &= \sum_{i=1}^{r} \alpha_i T M_{s_i} B_{f_i} + \sum_{j=1}^{d} \beta_j T M_{z_j} M_{r_j} M_{v_j} B_{g_j,l_j} \\ &= \sum_{i=1}^{r} \alpha_i F_{s_i} \widetilde{B}_{f_i} + \sum_{j=1}^{d} \beta_j F_{z_j} F_{r_j} F_{v_j} \widetilde{B}_{g_j,l_j} \end{aligned}$$

Thus, from the definition of $HR_{R,\bar{\mathcal{A}},\zeta}$ it follows that $T_C x \in \widetilde{X}_{\phi(q)}$.

It is easy to see that $T_C B_{q,z,j} = T M_z M_w B_{f,j} = F_e F_w \widetilde{B}_{f,j} = B_{\phi(q),z,j}$, for each $q \in Q$, $j \in J_2$, $z \in X_1$, $\delta(\mu_D(f), w) = q, f \in J_1$. Assume that $x \in \mathcal{X}_q$, $q \in Q$. Then $C_q x = Cx = \widetilde{C} Tx = C_{\phi(q)} T_C(x)$. It is easy to see that $T_C(A_{q,z}x) = T_C(M_z x) = T(M_z x) = F_z T(x) = \widetilde{A}_{q,z} T_C x$ for each $z \in X_1$. For each $\gamma \in X_2$, we get that $T_C(M_{\delta(q,\gamma),\gamma,q} x) = T_C(M_\gamma x) = T(M_\gamma x) = F_\gamma Tx = \widetilde{M}_{\widetilde{\delta}(\phi(q),\gamma),\gamma,\phi(q)} T_C x$. Finally, for each $f \in J_1$, $T_C \mu_C(f) = TB_f = \widetilde{B}_f = (\widetilde{\mu})_C(f)$. Thus, $H(T) = (\phi, T_C)$ is indeed an hybrid representation morphism. It is left to show that $H(T)$ is surjective. First, ϕ is surjective, since $(\bar{\mathcal{A}}, \zeta)$ is reachable. Indeed, for any $q \in \widetilde{Q}$ there exists $f \in J_1, w \in X_2^*$

such that $q = \widetilde{\delta}(\zeta(f), w)$. That is, $\phi(\delta(\mu_D(f), w)) = \widetilde{\delta}(\phi(\mu_D(f)), w) = \widetilde{\delta}(\zeta(f), w) = q$. Thus ϕ is surjective. We have to show that T_C is surjective. Consider $\widetilde{X}_s$ for some $s \in \widetilde{Q}$. From the definition of $\widetilde{X}_s$ it follows that it is a linear span of elements of the form $F_{z_{k+1}}F_{\gamma_{k+1}}F_{z_k}\cdots F_{\gamma_1}F_{z_1}\widetilde{B}_f$, $F_{z_{k+1}}F_{\gamma_{k+1}}F_{z_k}\cdots F_{\gamma_l}F_{z_l-1}F_vF_{\gamma_{l-1}}\cdots F_{\gamma_1}\widetilde{B}_{f,j}$, such that $z_1,\dots,z_{k+1} \in X_1^*, v \in X_1$, $j \in J_2$, $0 \le l \le k$ and $\widetilde{\delta}(\zeta(f), \gamma_1\cdots\gamma_k) = s$. It is easy to see that $\phi(\delta(\mu_D(f), \gamma_1\cdots\gamma_k) = \widetilde{\delta}(\zeta(f), \gamma_1\cdots\gamma_k) = s$. Let $q = \delta(\mu_D(f), \gamma_1\cdots\gamma_k)$. Define $x_1 = M_{z_{k+1}}M_{\gamma_{k+1}}M_{z_k}\cdots M_{\gamma_1}M_{z_1}B_f$,
$x_2 = M_{z_{k+1}}M_{\gamma_{k+1}}M_{z_k}\cdots M_{\gamma_l}M_{z_l-1}M_vM_{\gamma_{l-1}}\cdots M_{\gamma_1}B_{f,j}$. It follows that $x_1, x_2 \in \mathcal{X}_q$. It is also easy to see that $T_C(x_1) = Tx_1 = F_{z_{k+1}}F_{\gamma_{k+1}}F_{z_k}\cdots F_{\gamma_1}F_{z_1}\widetilde{B}_f$ and $T_C(x_2) = Tx_2 = F_{z_{k+1}}F_{\gamma_{k+1}}F_{z_k}\cdots F_{\gamma_l}F_{z_l-1}F_vF_{\gamma_{l-1}}\cdots F_{\gamma_1}\widetilde{B}_{f,j}$. Thus, $T_C(\bigoplus_{q\in Q}\mathcal{X}_q)$ contains a generator system of $\widetilde{\mathcal{X}}_s$ for each $s \in \widetilde{Q}$, that is, T_C is surjective. □

The discussion above for the case when $J_2 = \emptyset$ yields the following corollary of Lemma 14.

Corollary 8. *Assume that $J_2 = \emptyset$. Let R be any (not necessarily observable) representation of Ψ_Ω and let $(\widetilde{\mathcal{A}}, \zeta)$ any reachable realization Ω_D. Assume that $T : R_{HR} \to R$ is a representation morphism and $\phi : (\mathcal{A}, \mu_D) \to (\widetilde{\mathcal{A}}, \zeta)$ is an automaton morphism. Then there exists a hybrid representation morphism $H(T) : HR \to HR_{R,\widetilde{\mathcal{A}},\zeta}$ such that for all $x \in \mathcal{X}_q$,$q \in Q$, $T_C(x) = T(x)$.*

The results of Lemma 11–16 together with Theorem 4 and Theorem 2 characterising minimality of representations and automata yield the following Theorem.

Theorem 8. *If Ω has a hybrid representation, then it also has a minimal hybrid representation. Let HR be a hybrid representation of Ω. Then the following are equivalent.*

- *HR is minimal*
- *HR is reachable and observable*
- *For any reachable hybrid representation $HR^{'}$ of Ω there exists a surjective hybrid representation morphism $T : HR^{'} \to HR$. In particular, any two minimal hybrid representation of Ω are isomorphic.*

Proof. Notice that any minimal hybrid representation is reachable. Indeed, assume that HR is a minimal hybrid representation of Ω and HR is not reachable. Then by Lemma 12 there exists a representation HR_r of Ω such that $\dim HR_r < \dim HR$ and HR_r is reachable. Since HR is minimal, this is a contradiction.

First, we will show that if Ω has a hybrid representation, then Ω has a hybrid representation satisfying (iii). From Theorem 7 it follows that Ω has a hybrid representation if and only if Ψ_Ω has a representation and $\mathcal{D}_\Omega$ has a Moore-automaton realization. Let R be a minimal representation of Ψ_Ω and $(\bar{\mathcal{A}},\zeta)$ a minimal realization of $\mathcal{D}_\Omega$. By Theorem 2 and Theorem 4 such a minimal representation and a minimal realization always exist. Then by Lemma 14 $HR = HR_{R,\bar{\mathcal{A}},\zeta}$ is an observable and reachable representation of Ω.

We will show that (iii) holds for HR. Indeed, if $HR^{'}$ is a reachable hybrid representation of Ω, then $R_{HR'}$ is reachable and $(\bar{\mathcal{A}}_{HR'},\mu^{'}_D)$ is reachable. By Theorem 4 and Theorem 2 there exists surjective morphisms $T : R_{HR'} \rightarrow R$ and $\phi : (\bar{\mathcal{A}}_{HR'},\mu^{'}_D) \rightarrow (\bar{\mathcal{A}},\zeta)$. Then by Lemma 16 there exists a surjective hybrid representation morphism $(\phi, T_C) : HR^{'} \rightarrow HR$ such that $T_C x = Tx$ for all $x \in \mathcal{X}_q$, $q \in Q$.

Below we will show that (iii) implies (i). This will imply that HR is minimal, since HR satisfies (iii). Since HR exists whenever Ω has a hybrid representation, we get that if Ω has a hybrid representation, then it has a minimal minimal hybrid representation.

`(iii)` $\implies$ `(i)`

Assume that HR_m satisfies (iii). Assume now that $\widetilde{HR})$ is a hybrid representation of Ω. Then by Lemma 12 there exists a reachable hybrid representation HR_r of Ω, such that $\dim HR_r \leq \dim \widetilde{HR}$. Since HR_m satisfies (iii) we get that there exists a surjective hybrid representation morphism $T : HR_r \rightarrow HR_m$. It implies that $\dim HR_m \leq \dim HR_r \leq \dim \widetilde{HR}$. Thus, HR_m is a minimal hybrid representation of Ω.

Next we show that (ii) $\iff$ (iii), and (i) $\iff$ (ii).

`(ii)` $\implies$ `(iii)`

Consider the realization $HR = HR_{R,\bar{\mathcal{A}},\zeta}$ above. Let $HR^{'}$ be any reachable realization and consider the surjective hybrid morphism $S = (\phi, T_C)$ existence of which was proved above. If $HR^{'}$ is observable, then $(\bar{\mathcal{A}}_{HR'},\mu^{'}_D)$ is observable and $R_{HR'}$ is $\mathcal{X}^{'}_q, q \in Q^{'}$ observable, which implies that ϕ is bijective and $T|_{\mathcal{X}^{'}_q}$ is injective for all $q \in Q^{'}$. Since $T_C|_{\mathcal{X}^{'}_q} = T|_{\mathcal{X}^{'}_q}$ and $T_C x \in \mathcal{X}_q$ if and only if $x \in \mathcal{X}_{\phi^{-1}(q)}$ we get that T_C is an isomorphism. That is, S is an hybrid isomorphism, It is easy to see that $S^{-1} : HR \rightarrow HR^{'}$ is also a hybrid isomorphism, in particular, S^{-1} is surjective. For any reachable $\widetilde{HR}$ there exists a surjective hybrid morphism $T : \widetilde{HR} \rightarrow HR$. But then $S^{-1} \circ T : \widetilde{HR} \rightarrow HR^{'}$ is a surjective O-morphism. That is, $HR^{'}$ satisfies (iii). Thus (ii) implies (iii).

`(i)` $\implies$ `(ii)`

Indeed, let HR_m a minimal hybrid representation of Ω. From the discussion above it follows that HR_m has to be reachable. Then there exists a surjective $T : HR_m \rightarrow HR)$. But HR and HR_m are both minimal, thus $\dim HR = \dim HR_m$. It implies that T is a hybrid representation isomorphism. Notice that HR is observable. But then by HR_m has to be observable too. Thus, we get (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i). $\square$

Corollary 9. *Assume that R is a minimal representation of Ψ_Ω and $(\bar{\mathcal{A}}, \zeta)$ is a minimal realization of $\mathcal{D}_\Omega$ (Ω_D, if $J_2 = \emptyset$). Then $HR_{R,\bar{\mathcal{A}},\zeta}$ is a minimal hybrid representation of Ω.*

Chapter 4

Realization Theory of Switched Systems

Switched systems are one of the best studied subclasses of hybrid systems. A vast literature is available on various issues concerning switched systems, for a comprehensive survey see [44]. The current chapter develops realization theory for the following two subclasses of switched systems: linear switched systems and bilinear switched systems.

More specifically, the chapter tries to solve the following problems.

1. *Reduction to a minimal realization*
   Consider a linear (bilinear) switched system Σ, and a subset of its input-output maps Φ. Find a minimal linear (bilinear) switched system which realizes Φ.

2. *Existence of a realization with arbitrary switching*
   Find necessary and sufficient condition for the existence of a linear (bilinear) switched system realizing a given set of input-output maps.

3. *Existence of a realization with constrained switching*
   Assume that a set of admissible switching sequences is defined. Assume that the switching times of the admissible switching sequences are arbitrary. Consider a set of input-output maps Φ defined only for the admissible sequences. Find sufficient and necessary conditions for the existence of a linear (bilinear) switched system realizing Φ. Give a characterisation of the minimal realizations of Φ.

The motivation of the Problem 3 is the following. Assume that the switching is controlled by a finite automaton and the discrete modes are the states of this automaton.

Assume that the automaton is driven by external events, which can trigger a discrete-state transition at any time. We impose *no restriction* as to when an external event takes place. Then the traces of this automaton combined with the switching times (which are arbitrary) give us the admissible switching sequences.

If we can solve Problem 3 for such admissible switching sequences that the set of admissible sequences of discrete modes is a regular language, then we can solve the following problem. Construct a realization of a set of input-output maps by a linear (bilinear) switched system, such that switchings of that system are controlled by an automaton which is given in advance. Notice that the set of traces of an automaton is always a regular language.

The following results are proved in the chapter.

- A linear (bilinear) switched system is a minimal realization of a set of input-output maps if and only if it is observable and semi-reachable from the set of states which induce the input-output maps of the given set.
- Minimal linear (bilinear) switched systems which realize a given set of input-output maps are unique up to similarity.
- Each linear (bilinear) switched system Σ can be transformed to a minimal realization of any set of input-output maps which are realized by Σ.
- A set of input/output maps is realizable by a linear (bilinear) switched system if and only if it has a *generalised kernel representation* (*generalised Fliess-series expansion*) and the rank of its Hankel-matrix is finite. There is a procedure to construct the realization from the columns of the Hankel-matrix, and this procedure yields a minimal realization.
- Consider a set of input-output maps Φ defined on some subset of switching sequences. Assume that the switching sequences of this subset have arbitrary switching times and that their discrete mode parts form a regular language L. Then Φ has a realization by a linear (bilinear) switched system if and only if the Φ has a generalized kernel representation with constraint L (has a generalized Fliess-series expansion) and its Hankel-matrix is of finite rank. Again, there exists a procedure to construct a realization from the columns of the Hankel-matrix. The procedure yields an observable and semi-reachable realization of Φ. But this realization is not a realization with the smallest state-space dimension possible.

There are some earlier work on the realization theory of switched systems, see [50, 51, 53]. For realization theory for other classes of hybrid systems see [48, 54].

The brief overview of the results suggests that there is a remarkable analogy between the realization theories of linear and bilinear switched systems. In fact, this analogy is by no means a coincidence. Both the realization problem for linear and the realization problem for bilinear switched systems are equivalent to finding a (possibly minimal) representation for a set of formal power series. That is, realization theory of both linear and bilinear switched systems can be reformulated in terms of the theory of rational formal power series. This enables us to give a very concise and simple treatment of the realization problem for linear and bilinear switched systems. In fact, if one views switched systems as nonlinear systems and one is familiar with the realization theory of nonlinear systems, then the results of the chapter should not be too surprising. Exactly this similarity between realization theory of linear and bilinear switched systems in terms of results and mathematical tools is the motivation to present the realization theory of linear and bilinear switched systems in one chapter.

The approach to the realization theory taken in this chapter was inspired by works of M.Fliess, B. Jakubczyk and H. Sussmann, A.Isidori and E.Sontag [72, 36, 20, 22, 64, 84, 33]. The main tool used in the chapter is the theory of rational formal power series. Rational formal power series were used in systems theory earlier. Realization theory for bilinear systems is one of the major applications of rational formal power series, see [32].

4.1 Realization Theory of Linear Switched Systems

This section deals wit the realization theory of linear switched systems. First, definition and elementary properties of linear switched systems are presented. Linear switched systems have an extensive literature, for references see [50, 56, 70, 23, 69, 86, 44].

The current section uses the theory of formal power series presented in Section 3.1, Chapter 3 for developing realization theory of linear switched systems. The section is the most thorough account on realization theory of linear switched systems. In particular, all the results of Chapter 6 are implies by the results of this section. The outline of the section is the following. Subsection 4.1.1 presents the man concepts and some elementary results related to linear switched systems. Subsection 4.1.2 deals with the structure of input/output maps realizable by linear switched systems. Subsection 4.1.3 presents realization theory of linear switched systems for the case when arbitrary switching is allowed. Subsection 4.1.4 deals with the case when there is a set of admissible switching sequences, but there is no restriction on the switching times.

4.1.1 Linear Switched Systems

Recall from Section 2.4 the definition of linear switched systems. That is, a switched system Σ is called linear, if for each $q \in Q$ there exist linear mappings $A_q : \mathcal{X} \to \mathcal{X}$, $B_q : \mathcal{U} \to \mathcal{X}$ and $C_q : \mathcal{X} \to \mathcal{Y}$ such that

- $\forall u \in \mathcal{U}, \forall x \in \mathcal{X} : f_q(x, u) = A_q x + B_q u$
- $\forall x \in \mathcal{X} : h_q(x) = C_q x$

Recall that we adopted the following shorthand notation for linear switched systems

$$\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$$

Consider the linear switched systems

$$\Sigma_1 = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$$

and

$$\Sigma_2 = (\mathcal{X}_a, \mathcal{U}, \mathcal{Y}, Q, \{(A^a_q, B^a_q, C^a_q) \mid q \in Q\})$$

A linear map $S : \mathcal{X} \to \mathcal{X}_a$ is said to be a *linear switched system morphism* from Σ_1 to Σ_2 and it is denoted by $S : \Sigma_1 \to \Sigma_2$ if the the following holds

$$A^a_q S = S A_q, \quad B^a_q = S B_q, \quad C^a_q S = C_q \quad \forall q \in Q$$

The map S is called surjective (injective) if it is surjective (injective) as a linear map. The map S is said to be a linear switched system isomorphisms, if it is an isomorphisms as a linear map. By abuse of terminology, if (Σ_i, μ_i), $i = 1, 2$ are two linear switched system realizations and $S : \Sigma_1 \to \Sigma_2$ is a linear switched system morphism such that $S \circ \mu_1 = \mu_2$ then we will say that S is *linear switched system morphism* from realization (Σ_1, μ_1) to (Σ_2, μ_2) and we will denote it by $S : (\Sigma_1, \mu_1) \to (\Sigma_2, \mu_2)$. The linear switched systems realizations (Σ_1, μ_1) and (Σ_2, μ_2) are said to be *algebraically similar* or *isomorphic* if there exists an linear switched system isomorphism $S : (\Sigma_1, \mu_1) \to (\Sigma_2, \mu_2)$.

The results presented below can be found in the literature, for references see [69, 56].

Proposition 8. *For any LSS $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$ the following holds*

(1) $\forall u \in PC(T,\mathcal{U}), x_0 \in \mathcal{X}, w = (q_1,t_1)(q_2,t_2)\cdots(q_k,t_k) \in (Q\times T)^*$

$$x_\Sigma(x_0,u,w) = \exp(A_{q_k}t_k)\exp(A_{q_{k-1}}t_{k-1})\cdots\exp(A_{q_1}t_1)x_0 +$$
$$\int_0^{t_k}\exp(A_{q_k}(t_k-s))B_{q_k}u(\sum_1^{k-1}t_i+s)ds +$$
$$\exp(A_{q_k}t_k)\int_0^{t_{k-1}}\exp(A_{q_{k-1}}(t_{k-1}-s))B_{q_{k-1}}u(\sum_1^{k-2}t_i+s)ds +$$
$$\dots$$
$$\exp(A_{q_k}t_k)\exp(A_{q_{k-1}}t_{k-1})\cdots\exp(A_{q_2}t_2)\int_0^{t_1}\exp(A_{q_1}(t_1-s))B_{q_1}u(s)ds$$

and $y_\Sigma(x,u,w) = C_{q_k}x_\Sigma(x,u,w)$.

(2) *Reach*$(\Sigma,\{0\}) = \{A_{q_1}A_{q_2}\cdots A_{q_k}B_{q_{k+1}}u \mid u \in \mathcal{U}, q_1q_2\cdots q_{k+1} \in Q^+, k \geq 0\}$

(3) Two states $x_1, x_2 \in \mathcal{X}$ *are indistinguishable if and only if*

$$x_1 - x_2 \in \bigcap_{q_1,q_2,\dots,q_{k+1}\in Q, k\geq 0} \ker C_{q_{k+1}}A_{q_k}\cdots A_{q_1}$$

Σ *is observable if and only if*

$$\bigcap_{q_1,q_2,\dots,q_{k+1}\in Q, k\geq 0} \ker C_{q_{k+1}}A_{q_k}\cdots A_{q_1} = \{0\}$$

Remark Notice that if a linear switched system is reachable, the linear systems making up the switched systems need not be reachable . Moreover, the reachable set of the switched system may be bigger than the union of the reachable sets of the linear components. Indeed, consider the following switched system $\Sigma = (\mathbb{R}^3, \mathbb{R}, \mathbb{R}, \{q_1,q_2\}, \{(A_q,B_q,C_q) \mid q = q_1,q_2\})$

$$A_{q_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_{q_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C_{q_1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$A_{q_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_{q_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_{q_2} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Since $A_{q_1}B_{q_1} = [1,0,0]^T$, $A_{q_2}B_{q_1} = [0,0,1]^T$, we get that

$$\mathbb{R}^3 = \text{Span}\{B_{q_1}, A_{q_1}B_{q_1}, A_{q_2}B_{q_1}\} \subseteq \mathit{Reach}(\Sigma)$$

So *Reach*$(\Sigma) = \mathbb{R}^3$, i.e. the system is reachable. Yet, neither (A_{q_1}, B_{q_1}) nor (A_{q_2}, B_{q_2}) are reachable, moreover *Reach*$(A_{q_1},B_{q_1}) = \mathbb{R}^2$, *Reach*$(A_{q_2},B_{q_2}) = 0$, so *Reach*$(A_{q_1},B_{q_1}) \oplus$ *Reach*$(A_{q_2},B_{q_2}) \neq$ *Reach*(Σ).

4.1.2 Input-output Maps of Linear Switched Systems

This section deals with properties of input-output maps of linear switched systems. We define the notion of generalised kernel representation of a set of input-output maps, which turns out to be a notion of vital importance for the realization theory of linear switched systems. In fact, the realization problem is equivalent to finding a generalised kernel representation of a particular form for the specified set of input-output maps. The section also contains a number of quite technical statements, which are used in other parts of the paper.

Recall that for any $L \subseteq Q^+$ the set of admissible switching sequences is defined by $TL = \{(w, \tau) \in (Q \times T)^+ \mid w \in L\}$. Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$ be a set of maps of the form $PC(T,\mathcal{U}) \times TL \to \mathcal{Y}$. Define the languages $\text{suffix}L = \{u \in Q^* \mid \exists w \in Q^* : wu \in L\}$ and

$$\widetilde{L} = \{u_1^{i_1} \cdots u_k^{i_k} \in Q^* \mid u_1 \cdots u_k \in \text{suffix}L, u_j \in Q, i_j \geq 0, j = 1, \dots, k, i_1, i_k > 0\}$$

Definition 10 (Generalised kernel-representation with constraint L). *The set Φ is said to have* generalised kernel representation with constraint L *if for all $f \in \Phi$ and for all $w = w_1 w_2 \cdots w_k \in \widetilde{L}$, $w_1, \dots, w_k \in Q$, $k \geq 0$, there exist functions*

$$K_w^{f,\Phi} : \mathbb{R}^k \to \mathbb{R}^p \quad \textit{and } G_w^{\Phi} : \mathbb{R}^k \to \mathbb{R}^{p \times m}$$

such that the following holds.

1. *$\forall w \in \widetilde{L}, \forall f \in \Phi$: $K_w^{f,\Phi}$ is analytic and G_w^{Φ} is analytic*

2. *For each $f \in \Phi$ and $w, v \in Q^*$ such that $wqqv, wqv \in \widetilde{L}$, it holds that*

$$K_{wqqv}^{f,\Phi}(t_1, t_2, \dots, t_{|w|}, t, t', t_{|w|+2}, \dots t_{|w|+|v|+1}) = \\ K_{wqv}^{f,\Phi}(t_1, t_2, \dots t_{|w|}, t + t', t_{|w|+2} \dots t_{|w|+|v|+1})$$

$$G_{wqqv}^{\Phi}(t_1, t_2, \dots, t_{|w|}, t, t', t_{|w|+2}, \dots t_{|w|+|v|+1}) = \\ G_{wqv}^{\Phi}(t_1, t_2, \dots t_{|w|}, t + t', t_{|w|+2} \dots t_{|w|+|v|+1})$$

3. *$\forall vw \in \widetilde{L}, w \neq \epsilon, \forall f \in \Phi :$*

$$K_{vqw}^{f,\Phi}(t_1, \dots, t_{|v|}, 0, t_{|v|+1}, \dots, t_{|wv|}) = K_{vw}^{f,\Phi}(t_1, t_2, \dots, t_{|vw|})$$

   $\forall vw \in \widetilde{L}, v \neq \epsilon, w \neq \epsilon :$

$$G_{vqw}^{\Phi}(t_1, \dots, t_{|v|}, 0, t_{|v|+1}, \dots, t_{|wv|}) = G_{vw}^{\Phi}(t_1, \dots, t_{|vw|})$$

4. For each $f \in \Phi$, $(w_1,t_1)(w_2,t_2)\cdots(w_k,t_k) \in TL$, $u \in PC(T,\mathcal{U})$

$$f(u, w_1w_2\cdots w_k, t_1t_2\cdots t_k) = K^{f,\Phi}_{w_1w_2\cdots w_k}(t_1,t_2,\dots,t_k)+$$
$$+\sum_{i=1}^{k}\int_0^{t_i} G^{\Phi}_{w_i\cdots w_k}(t_i-s,t_{i+1},\dots,t_k)u(s+\sum_{j=1}^{i-1}t_j)ds$$

We say that Φ has a *generalised kernel representation* if it has a generalised kernel representation with the constraint $L = Q^+$. The reader may view the functions $K^{f,\Phi}_w$ as the part of the output which depends on the initial condition and the functions G^{Φ}_w as functions determining the dependence of the output on the continuous inputs.

Define the function $y_0^{\Phi} : PC(T,\mathcal{U}) \times TL \to \mathcal{Y}$ by

$$y_0^{\Phi}(u, w_1\cdots w_k, t_1\cdots t_k) := \sum_{i=1}^{k}\int_0^{t_i} G^{\Phi}_{w_i\cdots w_k}(t_i-s,t_{i+1},\dots,t_k)u(s+\sum_{j=1}^{i-1}t_j)ds$$

It follows from the fact that Φ has a generalised kernel representation that y_0^{Φ} can be expressed by $\forall f\in\Phi : y_0^{\Phi}(u,w,\tau) = f(u,w,\tau) - f(0,w,\tau)$

Another straightforward consequence of the definition is that the functions

$$\{K^{f,\Phi}_w, G^{\Phi}_w \mid f \in \Phi, w \in \mathrm{suffix}L\}$$

completely determine the functions $\{K^{f,\Phi}_w, G^{\Phi}_w \mid f\in\Phi, w\in\widetilde{L}\}$. Indeed, assume that $\widetilde{L} \ni w = z_1^{\alpha_1}\cdots z_k^{\alpha_k}$ such that $z_1,\dots,z_k \in Q$, $\alpha\in\mathbb{N}^k$, $\alpha_k > 0$ and $z_1\cdots z_k \in \widetilde{L}$. Then by using Part 2 and Part 3 of Definition 10 one gets

$$\begin{array}{rclcl} K^{f,\Phi}_w(t_1,\dots,t_{|w|}) & = & K^{f,\Phi}_{z_l\cdots z_k}(T_l,\dots,T_k) & = & K^{f,\Phi}_{z_1\cdots z_k}(T_1,\dots,T_k) \\ G^{\Phi}_w(t_1,\dots,t_{|w|}) & = & G^{\Phi}_{z_l\cdots z_k}(T_l,\dots,T_k) & & \end{array} \tag{4.1}$$

where $T_i = \sum_{j=1+\alpha_l+\cdots+\alpha_{i-1}}^{\alpha_l+\cdots+\alpha_i} t_j$, $i = l,\dots,k$, and $T_i = 0$, $i = 1,\dots,l-1$, $f\in\Phi$, $l = \min\{z \mid \alpha_z > 0\}$ and $\sum_{j=a}^{b} t_j$ is taken to be 0 if $a > b$. Now, for any $w\in\widetilde{L}$ there exist $d_1,\dots,d_l \in Q$ and $\xi\in\mathbb{N}^l$ such that $d_1\cdots d_l \in \mathrm{suffix}L$, $w = d_1^{\xi_1}\cdots d_l^{\xi_l}$ and $\xi_1,\xi_l > 0$. Applying (4.1) to w, $d_1\cdots d_l \in \mathrm{suffix}L \subseteq \widetilde{L}$ we get that $K^{\Phi,f}_w$ and G^{Φ}_w are uniquely determined by $K^{\Phi,f}_{d_1\cdots d_l}$ and $G^{\Phi}_{d_1\cdots d_l}$.

Using formula (4.1), the chain rule and induction it is straightforward to show that for each $w\in\widetilde{L}$, $w = z_1^{\alpha_1}\cdots z_k^{\alpha_k}$, $z_1\cdots z_k\in\widetilde{L}$, $\alpha_k > 0$, $l = \min\{z \mid \alpha_z > 0\}$ the following holds.

$$\begin{array}{rcl} \frac{d^{\beta_1}}{dt_1^{\beta_1}}\cdots\frac{d^{\beta_{|w|}}}{dt_{|w|}^{\beta_{|w|}}}K^{f,\Phi}_w(t_1,\dots,t_n) & = & \frac{d^{\gamma_1}}{d\tau_l^{\gamma_1}}\cdots\frac{d^{\gamma_{k-l+1}}}{d\tau_k^{\gamma_{k-l+1}}}K^{f,\Phi}_{z_l\cdots z_k}(\tau_l,\dots,\tau_k)|_{\underline{a}} \\ & = & \frac{d^{\gamma_1}}{d\tau_l^{\gamma_1}}\cdots\frac{d^{\gamma_{k-l+1}}}{d\tau_k^{\gamma_{k-l+1}}}K^{f,\Phi}_{z_1\cdots z_k}(\tau_1,\dots,\tau_k)|_{\underline{b}} \quad (4.2) \\ \frac{d^{\beta_1}}{dt_1^{\beta_1}}\cdots\frac{d^{\beta_{|w|}}}{dt_{|w|}^{\beta_{|w|}}}G^{\Phi}_w(t_1,\dots,t_n) & = & \frac{d^{\gamma_1}}{d\tau_l^{\gamma_1}}\cdots\frac{d^{\gamma_{k-l+1}}}{d\tau_k^{\gamma_{k-l+1}}}G^{\Phi}_{z_l\cdots z_k}(\tau_l,\dots,\tau_k)|_{\underline{a}} \end{array}$$

where $\beta \in \mathbb{N}^{|w|}, \gamma \in \mathbb{N}^{k-l+1}$, $\underline{a} \in T^{k-l+1}$, $\underline{b} \in T^k$ and $a_i = \sum_{j=1+\alpha_l+\cdots+\alpha_{l+i-2}}^{\alpha_l+\cdots\alpha_{i+l-1}} t_j$, $\gamma_i = \sum_{j=1+\alpha_l+\cdots+\alpha_{l+i-2}}^{\alpha_l+\cdots+\alpha_{l+i-1}} \beta_j$ for each $i = 1, \ldots, k-l+1$, $b_i = a_{i-l+1}$, for $i = l, \ldots, k$ and $b_i = 0$ for $i = 1, \ldots, l-1$. Substituting 0 for $t_1, \ldots, t_{|w|}$ we get

$$D^{\beta} K^{f,\Phi}_{w} = D^{\gamma} K^{f,\Phi}_{z_l \cdots z_k} = D^{(\mathbb{O}_{l-1},\gamma)} K^{f,\Phi}_{z_1 \cdots z_k} \text{ and } D^{\beta} G^{\Phi}_{w} = D^{\gamma} G^{\Phi}_{z_l \cdots z_k} \tag{4.3}$$

where $\mathbb{O}_{l-1} = (0, 0, \ldots, 0) \in \mathbb{N}^{l-1}$. The discussion above yields the following.

Proposition 9. *Let $z_1, z_2, \ldots, z_k, d_1, d_2, \ldots, d_l \in Q^*$. Let $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ and $\beta = (\beta_1, \ldots, \beta_l) \in \mathbb{N}^l$ Assume that $z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_k^{\alpha_k} = d_1^{\beta_1} d_2^{\beta_2} \cdots d_l^{\beta_l}$. If $q_2 z_1 z_2 \cdots z_k q_1 \in \widetilde{L}$ and $q_2 d_1 d_2 \cdots d_l q_1 \in \widetilde{L}$, then*

$$D^{(0,\alpha,0)} G^{\Phi}_{q_2 z_1 z_2 \cdots z_k q_1} = D^{(0,\beta,0)} G^{\Phi}_{q_2 d_1 d_2 \cdots d_l q_1}$$

If $z_1 z_2 \cdots z_k q_1$ and $d_1 d_2 \cdots d_l q_1 \in \widetilde{L}$ then

$$D^{(\alpha,0)} K^{f,\Phi}_{z_1 z_2 \cdots z_k q_1} = D^{(\beta,0)} K^{f,\Phi}_{d_1 d_2 \cdots d_l q_1}$$

Proof. Using (4.3) one gets that

$$D^{(0,\alpha,0)} G^{\Phi}_{q_2 z q_1} = D^{(0,\mathbb{I},0)} G^{\Phi}_{q_2 z_1^{\alpha_1} \cdots z_k^{\alpha_k} q_1} = D^{(0,\mathbb{I},0)} G^{\Phi}_{q_2 d_1^{\beta_1} \cdots d_l^{\beta_l} q_1} = D^{(0,\beta,0)} G^{\Phi}_{q_2 d q_1}$$

where $\mathbb{I} = (1, 1, \ldots, 1) \in \mathbb{N}^{\sum_1^k \alpha_i}$, $z = z_1 \cdots z_k, d = d_1 \cdots d_l$. Similarly $D^{(\alpha,0)} K^{f,\Phi}_{z_1 \cdots z_k q_1} = D^{(\alpha^+,0)} K^{f,\Phi}_{z_l \cdots z_k q_1} = D^{(\mathbb{I},0)} K^{f,\Phi}_{z_1^{\alpha_1} \cdots z_k^{\alpha_k} q_1} = D^{(\mathbb{I},0)} K^{f,\Phi}_{d_1^{\beta_1} \cdots d_l^{\beta_l} q_1} = D^{(\beta,0)} K^{f,\Phi}_{d_1 \cdots d_l q_1}$, where $l = \min\{z \mid \alpha_z > 0\}$ and $\alpha^+ = (\alpha_l, \ldots, \alpha_k)$. ☐

If Φ has a realization by a linear switched system, then Φ has a generalised kernel representation

Proposition 10. *For any LSS $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$, (Σ, μ) is a realization of Φ with constraint L if and only if Φ has a generalised kernel representation defined by*

$$G^{\Phi}_{w_1 w_2 \cdots w_k}(t_1, t_2, \ldots, t_k) = C_{w_k} \exp(A_{w_k} t_k) \exp(A_{w_{k-1}} t_{k-1}) \cdots \exp(A_{w_1} t_1) B_{w_1}$$

and

$$K^{f,\Phi}_{w_1 w_2 \cdots w_k}(t_1, t_2, \ldots, t_k) = C_{w_k} \exp(A_{w_k} t_k) \exp(A_{w_{k-1}} t_{k-1}) \cdots \exp(A_{w_1} t_1) \mu(f).$$

where $w_1 w_2 \cdots w_k \in \widetilde{L}$. Moreover, if (Σ, μ) is a realization of Φ, then

$$y^{\Phi}_0 = y_{\Sigma}(0, ., .)|_{PC(T,\mathcal{U}) \times TL}$$

Proof. (Σ, μ) is a realization of Φ if and only if for each $f \in \Phi$, $u \in PC(T, \mathcal{U})$, $w \in TL$ it holds that

$$f(u, w) = y_{\Sigma}(\mu(f), u, w) = C_{q_k} x_{\Sigma}(\mu(f), u, w)$$

where $w = w^{'}(q_k, t_k)$. The statement of proposition follows now directly from from part (1) of Proposition 8. $\square$

If the set Φ has a generalized kernel representation with constraint L, then the collection of analytic functions $\{K^{f,\Phi}_w, G^{\Phi}_w \mid w \in \text{suffix}L, f \in \Phi\}$ determines Φ. Since $K^{f,\Phi}_w$ is analytic, we get that it is determined locally by $\{D^{\alpha} K^{f,\Phi}_w \mid \alpha \in \mathbb{N}^{|w|}\}$. Similarly, G^{Φ}_w is determined locally by $\{D^{\alpha} G^{\Phi}_w \mid \alpha \in \mathbb{N}^{|w|}\}$.

By applying the formula $\frac{d}{dt}\int_0^t f(t,\tau)d\tau = f(t,t) + \int_0^t \frac{d}{dt} f(t,\tau)d\tau$ and Part 4 of Definition 10 one gets

$$D^{\alpha} K^{f,\Phi}_{q_1 q_2 \cdots q_k} = D^{\alpha} f(0, q_1 q_2 \cdots q_k, .) \tag{4.4}$$

$$D^{\alpha} G^{\Phi}_{q_l q_{l+1} \cdots q_k} e_z = D^{\beta} y^{\Phi}_0(e_z, q_1 q_2 \cdots q_k, .) \tag{4.5}$$

where $\mathbb{N}^k \ni \beta = (\underbrace{0, 0, \ldots, 0}_{l-1--times}, \alpha_1 + 1, \alpha_2, \ldots, \alpha_{k-l+1})$. Here e_z is the zth unit vector of $\mathbb{R}^m$, i.e $e_z^T e_j = \delta_{zj}$. Formulas (4.4) and (4.5) imply that all the high-order derivatives of the functions $K^{f,\Phi}_w, G^{\Phi}_w$ ($f \in \Phi$, $w \in \text{suffix}L$) at zero can be computed from high-order derivatives with respect to the switching times of the functions from Φ.

Define the set $S = \{(\alpha, w) \in \mathbb{N}^* \times Q^* \mid \alpha \in \mathbb{N}^{|w|}, w \in Q^*\}$. For each $w \in Q^*$, $q_1, q_2 \in Q$ define the sets

$$F_{q_1,q_2}(w) = \{(v, (\alpha, z)) \in Q^* \times S \mid vz \in L,$$
$$q_2 w q_1 = z_1 z_1^{\alpha_1} \cdots z_k^{\alpha_k} z_k, z_j \in Q, j = 1, \ldots, k, z = z_1 \cdots z_k\}$$

$$F_{q_1}(w) = \{(v, (\alpha, z)) \in Q^* \times S \mid vz \in L,$$
$$w q_1 = z_1^{\alpha_1} \cdots z_k^{\alpha_k} z_k, z_j \in Q, j = 1, \ldots, k, z = z_1 \cdots z_k\}$$

Define $\widetilde{L}_{q_1,q_2} = \{w \in Q^* \mid F_{q_1,q_2}(w) \neq \emptyset\}$ and $\widetilde{L}_q = \{w \in Q^* \mid F_q(w) \neq \emptyset\}$. Denote by $\mathbb{O}_l$ the tuple $(0, 0, \ldots, 0) \in \mathbb{N}^l$, $l \geq 0$. For any $\alpha \in \mathbb{N}^k$ let $\alpha^+ = (\alpha_1 + 1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$, $k \geq 0$.

The intuition behind the definition of the sets $F_{q_1,q_2}(w)$ and $F_{q_1}(w)$ is the following. Let (Σ, μ) be a realization of Φ. Then $(v, (\alpha, z)) \in F_{q_1,q_2}(w)$ if $D^{\alpha^+} y^{\Phi}_0(vz, e_j, .) = D^{(1,1,\ldots,1,0)} y_{\Sigma}(0, q_2 w q_1, e_j, .)$ for each $j = 1, \ldots, m$. Similarly, $(v, (\alpha, z)) \in F_{q_1}(w)$ if $D^{\alpha} f(vz, 0, .) = D^{(1,1,\ldots,1,0)} y_{\Sigma}(\mu(f), w q_1, 0)$ for each $f \in \Phi$. That is, $F_{q_1,q_2}(w)$ is non-empty if we can deduce from Φ some information on the output of Σ when the initial condition is 0 and the switching sequence is q_2wq_1. Similarly, $F_{q_1}(w)$ is non-empty, if we can derive from Φ some information on the output of Σ, if the initial condition is $\mu(f)$, the switching sequence is wq_1 and the continuous input is zero.

With the notation above, using the principle of analytic continuation and formulas (4.4) and (4.5), one gets the following

Proposition 11. *Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$. For any LSS*

$$\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q) \mid q \in Q\})$$

the pair (Σ,μ) is a realization of Φ with constraint L if and only if Φ has a generalized kernel representation with constraint L and the following holds

$$\begin{aligned}
&\forall w \in L, j=1,2,\ldots,m, f \in \Phi, \alpha \in \mathbb{N}^{|w|}: \\
&\quad D^{\alpha} y_0^{\Phi}(e_j,w,.) = D^{\beta} G^{\Phi}_{w_l\cdots w_k} e_j = C_{w_k} A_{w_k}^{\alpha_k} A_{w_{k-1}}^{\alpha_{k-1}} \cdots A_{w_l}^{\alpha_l -1} B_{w_l} e_j \\
&\quad D^{\alpha} f(0,w,.) = D^{\alpha} K^{f,\Phi}_{w} = C_{w_k} A_{w_k}^{\alpha_k} A_{w_{k-1}}^{\alpha_{k-1}} \cdots A_{w_l}^{\alpha_l} \mu(f)
\end{aligned} \tag{4.6}$$

where $l = \min\{h \mid \alpha_h > 0\}$, e_z is the zth unit vector of $\mathcal{U}$, $\beta = (\alpha_l - 1, \alpha_{l+1}, \ldots, \alpha_k)$ and $w = w_1 \cdots w_k$, $w_1,\ldots,w_k \in Q$. Formula (4.6) is equivalent to

$$\begin{aligned}
&\forall w \in \widetilde{L}, j=1,2,\ldots,m, q_1,q_2 \in Q, (v,(\alpha,z)) \in F_{q_1,q_2}(w): \\
&\quad D^{(\mathbb{O}_{|v|},\alpha^{+})} y_0^{\Phi}(e_j,vz,.) = D^{(0,\alpha,0)} G^{\Phi}_{q_2zq_1} e_j = C_{q_1} A_{z_k}^{\alpha_k} \cdots A_{z_1}^{\alpha_1} B_{q_2} e_j \\
&\forall w \in \widetilde{L}, q \in Q, (v,(\alpha,z)) \in F_q(w): \\
&\quad D^{(\mathbb{O}_{|v|},\alpha)} f(0,vz,.) = D^{(\alpha,0)} K^{f,\Phi}_{zq} = C_q A_{z_k}^{\alpha_k} \cdots A_{z_1}^{\alpha_1} \mu(f)
\end{aligned} \tag{4.7}$$

Proof. First we show that Φ is realized by (Σ,μ) if and only if Φ has a generalized kernel representation and (4.6) holds. By Proposition 10 (Σ,μ) is a realization of Φ if and only if Φ has a generalized kernel representation of the form

$$\begin{aligned}
G^{\Phi}_{w}(t_1,\ldots,t_k) &= C_{w_k} \exp(A_{w_k} t_k) \cdots \exp(A_{w_1} t_1) B_{w_1} \\
K^{f,\Phi}_{w}(t_1,\ldots,t_k) &= C_{w_k} \exp(A_{w_k} t_k) \cdots \exp(A_{w_1} t_1) \mu(f)
\end{aligned} \tag{4.8}$$

for each $w = w_1 \cdots w_k \in \widetilde{L}$, $w_1,\ldots,w_k \in Q$. From (4.1) it follows that it is enough to consider $\{K^{f,\Phi}_w, G^{\Phi}_w \mid w \in \mathrm{suffix}L, f \in \Phi\}$. Since $K^{f,\Phi}_w, G^{\Phi}_w$ are analytic functions, their high-order derivatives at zero determine them uniquely. Using (4.4), (4.5) we get that (4.8) is equivalent to (4.6).

Next we show that (4.6) is equivalent to (4.7). Notice that from (4.3) it follows that for any $z = z_1 \cdots z_k, z_1 = q_2, z_k = q_1$: $D^{\alpha} G^{\Phi}_{z_1\cdots z_k} = D^{(0,\alpha,0)} G^{\Phi}_{z_1z_1\cdots z_kz_k} = D^{(0,\alpha,0)} G^{\Phi}_{q_2zq_1}$ and $D^{\alpha} K^{f,\Phi}_{z} = D^{(\alpha,0)} K^{f,\Phi}_{zq_1}$. First, we will show that (4.7) implies (4.6). For any $w \in L$, $\alpha \in \mathbb{N}^{|w|}$, $w = w_1 \cdots, w_k$, $w_1,\ldots,w_k \in Q$ define $l = \min\{z \mid$

$\alpha_z > 0\}$, $v = w_1 \cdots w_{l-1}$, $z = w_l \cdots w_{|w|}$ and $x = w_l^{\alpha_l - 1} w_{l+1}^{\alpha_{l+1}} \cdots w_{|w|}^{\alpha_{|w|}}$. Then $(v, (\beta, z)) \in F_{w_l, w_{|w|}}(x)$ where $\beta = (\alpha_l - 1, \ldots, \alpha_{|w|})$. Notice that $(\mathbb{O}_{|v|}, \beta^+) = \alpha$. From (4.7) and the remark above we get that $D^{(\mathbb{O}_{|v|}, \beta^+)} y_0^{\Phi}(e_j, vz, .) =$
$= D^{(0,\beta,0)} G^{\Phi}_{w_l z w_{|w|}} e_j = D^{\beta} G^{\Phi}_z e_j = D^{\alpha} y_0^{\Phi}(e_j, w, .) = C_{w_{|w|}} A_{w_{|w|}}^{\alpha_{|w|}} \cdots A_{w_l}^{\alpha_l - 1} B_{w_l} e_j$.

Similarly, let $y = w_1^{\alpha_1} \cdots w_{|w|}^{\alpha_{|w|}}$. Then $(\epsilon, (\alpha, w)) \in F_{w_{|w|}}(y)$. Again, from the remark above and (4.7) we get that $D^{\alpha} f(0, w, .) = D^{(\alpha,0)} K^{f,\Phi}_{w w_{|w|}} = D^{\alpha} K^{f,\Phi}_w =$ $D^{\alpha} f(0, w, .) = C_{w_{|w|}} A_{w_{|w|}}^{\alpha_{|w|}} \cdots A_{w_1}^{\alpha_1} \mu(f)$. That is, (4.6) holds.

Conversely, (4.6) $\implies$ (4.7). Indeed, for any $w \in \widetilde{L}$, $q_1, q_2 \in Q$, $(v, (\alpha, z)) \in F_{q_1,q_2}(w)$ it holds that $vz \in L$, $z = z_1 \cdots z_k$, $z_1 = q_2, z_k = q_1$. Then (4.6) implies $D^{(\mathbb{O}_{|v|}, \alpha^+)} y_0^{\Phi}(e_j, vz, .) = D^{(0,\alpha,0)} G^{\Phi}_{q_2 z q_1} e_j = C_{z_k} A_{z_k}^{\alpha_k} \cdots A_{z_1}^{\alpha_1} B_{z_1}$ For any $(v, (\alpha, z)) \in F_q(w)$ it holds that $z = z_1 \cdots z_k$, $z_k = q$ and $vz \in L$. Then (4.6) implies
$D^{(\mathbb{O}_{|v|}, \alpha)} f(0, vz, .) = D^{(\alpha,0)} K^{f,\Phi}_{zq_1} = C_q A_{z_k}^{\alpha_k} \cdots A_{z_1}^{\alpha_1} \mu(f)$. That is, (4.6) implies (4.7). □

One may wonder whether a generalized kernel representation is unique, if it exists, and what is the relationship between a generalized kernel representation and such properties of input/output maps as linearity in continuous inputs, causality and etc. Below we will try to answer these questions.

Let $f \in F(PC(T, \mathcal{U}) \times TL, \mathcal{Y})$. We will say that f is *causal*, if for any $w = (q_1, t_1) \cdots (q_k, t_k) \in TL$ the following holds

$$\forall u, v \in PC(T, \mathcal{U}) : (\forall t \in [0, \sum_1^k t_i] : u(t) = v(t)) \implies f(w, u) = f(w, v)$$

That is, the value of $f(w, u)$ depends only on $u|_{[0, \sum_1^k t_i]}$.

Since $\mathcal{Y} = \mathbb{R}^p$, for each $f \in F(PC(T, \mathcal{U}) \times TL, \mathcal{Y})$ there exist functions $f_j : PC(T, \mathcal{U}) \times TL \to \mathbb{R}$ such that $f(u, w) = (f_1(u, w), \ldots, f_p(u, w))^T$. For each $t \in T$ define the map $P_t : PC(T, \mathcal{U}) \to PC(T, \mathcal{U})$ by

$$P_t(u)(s) = \begin{cases} u(s) & \text{if } s \le t \\ 0 & \text{otherwise} \end{cases}$$

For each $w \in TL$ define the map $f_j(w, .) : PC(T, \mathcal{U}) \to \mathbb{R}$ by $f_j(w, .)(u) = f_j(u, w)$. For each $1 \le p \le +\infty$ denote by $L^p([0, t_i], \mathbb{R}^{n \times m})$ the vector space of n by m matrices of functions from $L^p([0, t_i])$. I.e. $f : [0, t_i] \to \mathbb{R}^{n \times m}$ is an element of $L^p([0, t_i], \mathbb{R}^{n \times m})$, if $f = (f_{i,j})_{i=1,\ldots,n, j=1,\ldots,m}$ and $f_{i,j} \in L^p([0, t_i])$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. With the notation above we can formulate the following characterisation of input/output maps admitting a generalized kernel representation.

Theorem 9. *Let $\Phi \subseteq F(PC(T, \mathcal{U}) \times TL, \mathcal{Y})$. Then Φ admits a generalized kernel representation with constraint L if and only if the following conditions hold.*

1. *Each $f \in \Phi$ is causal and there exists a function $y^{\Phi} \in F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$ such that for each $f \in \Phi$*

$$\forall w \in TL, u \in PC(T,\mathcal{U}) : f(u,w) = f(0,w) + y^{\Phi}(u,w) \tag{4.9}$$

2. *For each $f \in \Phi, w = (q_1,t_1)\cdots(q_k,t_k) \in TL, j = 1,2,\ldots,p$ the map $y_j^{\Phi}(w,.) : PC([0,T_k],\mathcal{U}) \ni u \mapsto y_j^{\Phi}(w, u\#_{T_k}0) \in \mathbb{R}$ is a continuous linear functional, where $T_k = \sum_{j=1}^{k} t_j$. Here $PC([0,T_k],\mathcal{U})$ is viewed as a subspace of $L^1([0,T_k],\mathcal{U})$ and the topology considered on $PC([0,T_k],\mathcal{U})$ is the corresponding subspace topology.*

3. *For each $f \in \Phi$, $s \in (Q \times T)^+$, $w = (w_1,0)\cdots(w_k,0)$, $v = (v_1,0)\cdots(v_l,0) \in (Q \times T)^*$*

$$ws, vs \in TL \implies (\forall u \in PC(T,\mathcal{U}) : f(u,ws) = f(u,vs))$$

4. *For each $w = (q_1,t_1)\cdots(q_k,t_k) \in TL$, $1 \le l \le k$, $u \in PC(T,\mathcal{U})$*

$$y^{\Phi}(u,w) = y^{\Phi}(\mathrm{Shift}_{T_l}(u), v(q_l,t_l)\cdots(q_k,t_k)) + y^{\Phi}(P_{T_l}(u), w)$$

   where $T_l = \sum_{1}^{l-1} t_i$ and $v = (q_1,0)\ldots(q_{l-1},0)$.

5. *For each $f \in \Phi$, $w,v \in (Q \times T)^*$, $q \in Q$, if $w(q,t_1)(q,t_2)v, w(q,t_1+t_2)v \in TL$, then*

$$\forall u \in PC(T,\mathcal{U}) : f(u, w(q,t_1)(q,t_2)v) = f(u, w(q,t_1+t_2)v)$$

   For each $f \in \Phi$, $w,v \in (Q \times T)^*, |v| > 0$, $q \in Q$, if $w(q,0)v, wv \in TL$, then

$$\forall u \in PC(T,\mathcal{U}) : f(u, w(q,0)v) = f(u, wv)$$

6. *For each $q_1\cdots q_k \in L$, $u_1,\ldots u_k, \in \mathcal{U}$, $f \in \Phi$, the maps $f_{q_1\cdots q_k,u_1,\ldots,u_k} : T^k \to \mathcal{Y}$ defined below, are analytic.*

$$f_{q_1\cdots q_k,u_1,\ldots,u_k}(t_1,\ldots,t_k) = f(u,(q_1,t_1)\cdots(q_k,t_k)),$$

   where $u(t) = u_i$ if $t \in (\sum_{j=1}^{i-1} t_j, \sum_{j=1}^{i} t_j]$.

If Φ admits a generalized kernel representation, then the Φ admits an unique generalized kernel representation.

The proof of the theorem can be found in Subsection 4.1.5.

The theorem above gives an important characterisation of generalized kernel representation. It states that existence of a generalized kernel representation amounts to i) causality of the input-output maps, ii) switching sequences behaving as discrete inputs, iii) input-output maps being affine and continuous in the continuous inputs iv) input-output maps being analytic for constant inputs. In author's opinion, the theorem above demonstrates that existence of a generalized kernel representation is by no means an unnatural or a very restrictive condition. In particular, if the number of discrete modes is one, then existence of generalized kernel representation is equivalent to the conditions which are usually imposed on the input-output maps of linear (possibly infinite-infinite dimensional) systems. One may also compare the conditions of the above theorem with the so called realisability conditions from [50]. Notice that knowledge of analytic forms of $K^{f,\Phi}_w$ and G^{Φ}_w are not necessary for constructing a realization of Φ. All that is required is the knowledge that the functions $K^{f,\Phi}_w$, G^{Φ}_w exist. Therefore, it hardly makes sense to try to compute the functions $K^{f,\Phi}_w$ and G^{Φ}_w. Note that existence of an algorithm which computes these functions on the basis of Φ would imply the existence of a representation of Φ with finite data. Since elements of Φ are linear maps defined on the infinite-dimensional space $PC(T,\mathcal{U})$, existence of such a finite representation is quite unlikely.

4.1.3 Realization Theory of Linear Switched Systems: Arbitrary Switching

In this section the solution to the realization problem will be presented. That is, given a set of input-output maps we will formulate necessary and sufficient conditions for the existence of a linear switched system realizing that set. In addition, characterisation of minimal systems realizing the given set of input-output maps will be given. In this section we assume that there are no restrictions on the switching sequences. That is, in this section we study realization with the trivial constraint $L = Q^+$.

The main tool of this section is the theory of rational formal power series. The main idea of the solution is the following. We associate a set of formal power series Ψ_Φ with the set of input-output maps Φ . Any representation of Ψ_Φ yields a realization of Φ and any realization of Φ yields a representation of Ψ_Φ. Moreover, minimal representations give rise to minimal realizations and vice versa. Then we can apply the theory of rational formal power series to characterise minimal realizations.

Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$. Proposition 11 and formula (4.3) yield the following

Proposition 12. *The LSS* $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q) \mid q \in Q\})$ *is a realization of* Φ *if and only if* Φ *has a generalized kernel representation and there exists* $\mu : \Phi \to \mathcal{X}$

such that

$$\begin{array}{c} \forall w = w_1 \cdots w_k \in Q^+, q_1, q_2 \in Q, w_1, \ldots, w_k \in Q, z \in \{1,2,\ldots,m\}, f \in \Phi : \\ \begin{array}{rcccl} D^{(1,\mathbb{I}_k,0)} y_0^{\Phi}(e_z, q_2 w q_1, .) & = & D^{(0,\mathbb{I}_k,0)} G^{\Phi}_{q_2 w q_1} e_z & = & C_{q_1} A_{w_k} \cdots A_{w_1} B_{q_2} e_z \\ D^{(\mathbb{I}_k,0)} f(0, w q_1, .) & = & D^{(\mathbb{I}_k,0)} K^{f,\Phi}_{w q_1} & = & C_{q_1} A_{w_k} \cdots A_{w_1} \mu(f) \end{array} \end{array}$$

where $\mathbb{I}_k = (1,1,\ldots,1) \in \mathbb{N}^k$.

Proof. Applying (4.3) one gets the following equalities.

$$D^{\alpha} K^{f,\Phi}_{w} = D^{(\alpha,0)} K^{f,\Phi}_{w w_k} = D^{(\mathbb{I}_m,0)} K^{f,\Phi}_{w_1^{\alpha_1} w_2^{\alpha_2} \cdots w_k^{\alpha_k} w_k} \tag{4.10}$$

$$D^{\alpha} G^{\Phi}_{w} = D^{(0,\alpha,0)} G^{\Phi}_{w_1 w w_k} = D^{(0,\mathbb{I}_m,0)} G^{\Phi}_{w_1 w_1^{\alpha_1} w_2^{\alpha_2} \cdots w_k^{\alpha_k} w_k} \tag{4.11}$$

where $m = \sum_1^k \alpha_k$. The statement of the proposition follows now from Proposition 11. ∎

The proposition above allows us to reformulate the realization problem in terms of rationality of certain power series. Define formal power series $S_{q_1,q_2,z}, S_{f,q_1} \in \mathbb{R}^p \ll Q^* \gg$, ($q_1, q_2 \in Q, f \in \Phi$, $z \in \{1,2,\ldots,m\}$) by

$$S_{q_1,q_2,z}(w) = D^{(1,\mathbb{I}_{|w|},0)} y_0^{\Phi}(e_z, q_2 w q_1, .) \;,\; S_{f,q_1}(w) = D^{(\mathbb{I}_{|w|},0)} f(0, w q_1, .)$$

for each $w \in Q^*$. Notice that the functions $G^{\Phi}_w, K^{f,\Phi}_w$ are not involved in the definition of the series of $S_{q_1,q_2,z}$ and S_{f,q_1}. On the other hand, if Φ has a generalized kernel representation, then

$$S_{q_1,q_2,z}(w) = D^{(0,\mathbb{I}_{|w|},0)} G^{\Phi}_{q_2 w q_1} e_z \text{ and } S_{f,q_1}(w) = D^{(\mathbb{I}_{|w|},0)} K^{f,\Phi}_{w q_1}$$

For each $q \in Q, z = 1,2,\ldots,m$, $f \in \Phi$ define the formal power series $S_{q,z}, S_f \in \mathbb{R}^{p|Q|} \ll Q^* \gg$ by

$$S_{q,z} = \begin{bmatrix} S_{q_1,q,z} \\ S_{q_2,q,z} \\ \vdots \\ S_{q_N,q,z} \end{bmatrix} \;,\quad S_f = \begin{bmatrix} S_{f,q_1} \\ S_{f,q_2} \\ \vdots \\ S_{f,q_N} \end{bmatrix}$$

where $Q = \{q_1, q_2, \ldots, q_N\}$.

Define the set $J_{\Phi} = \Phi \cup \{(q,z) \mid q \in Q, z = 1,2,\ldots,m\}$. Define the *indexed set of formal power series associated with* Φ by

$$\Psi_{\Phi} = \{S_j \in \mathbb{R}^{p|Q|} \ll Q^* \gg \mid j \in J_{\Phi}\} \tag{4.12}$$

Define the *Hankel-matrix of* Φ H_{Φ} as the Hankel-matrix of the associated set of formal power series, i.e. $H_{\Phi} := H_{\Psi_{\Phi}}$.

Notice that the only information needed to construct the set of formal power series Ψ_Φ are the high-order derivatives at zero of the functions belonging to Φ. The fact that Φ has a generalized kernel representation is needed only to ensure the correctness of the construction. *No knowledge of the analytic forms of the functions $K^{f,\Phi}_w$, G^{Φ}_w is required in order to construct Ψ_Φ.*

Let $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q) \mid q \in Q\})$ be a LSS, and assume that (Σ,μ) is a realization of Φ. Define the *representation associated with* (Σ,μ) by

$$R_{\Sigma,\mu} = (\mathcal{X},\{A_q\}_{q\in Q},\widetilde{B},\widetilde{C})$$

where $\widetilde{C} : \mathcal{X} \to \mathbb{R}^{p|Q|}$, $\widetilde{C} = \begin{bmatrix} C_{q_1} \\ C_{q_2} \\ \vdots \\ C_{q_N} \end{bmatrix}$ and the indexed set $\widetilde{B} = \{B_j \in \mathcal{X} \mid j \in J_\Phi\}$ is defined by $\widetilde{B}_f = \mu(f), f \in \Phi$, and $\widetilde{B}_{q,l} = B_q e_l, l = 1,2,\ldots,m, q \in Q$, e_l is the lth unit vector in $\mathcal{U}$.

Conversely, consider a representation of Ψ_Φ

$$R = (\mathcal{X},\{A_q\}_{q\in Q},\widetilde{B},\widetilde{C})$$

Then define (Σ_R,μ_R) *the realization associated with* R by

$$\Sigma_R = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q) \mid q \in Q\})\ ,\ \mu_R(f) = \widetilde{B}_f$$

where $C_q : \mathcal{X} \to \mathcal{Y}$, $q \in Q$ are such that $\widetilde{C} = \begin{bmatrix} C_{q_1} \\ C_{q_2} \\ \vdots \\ C_{q_N} \end{bmatrix}$, and $B_q e_l = \widetilde{B}_{q,l}$ for each $l = 1,\ldots,m$. It is easy to see that C_q, $q \in Q$ are well defined, since

$$C_q = \begin{bmatrix} e^T_{q,1}\widetilde{C} \\ \vdots \\ e^T_{q,p}\widetilde{C} \end{bmatrix}$$

Here for $q = q_z \in Q$ for some $z = 1,\ldots,N$, $i = 1,\ldots,p$ it holds that $e_{q,i} \in \mathbb{R}^{p|Q|}$ and $(e_{q,i})_j = \begin{cases} 1 & \text{if } j = p*(z-1)+i \\ 0 & \text{otherwise} \end{cases}$. It is easy to see that $\Sigma_{R_{\Sigma,\mu}} = \Sigma$, $\mu_{R_{\Sigma,\mu}} = \mu$ and $R_{\Sigma_R,\mu_R} = R$. In fact, the following theorem holds.

Theorem 10. *Let $\Phi \subseteq F(PC(T,\mathcal{U})\times(Q\times T)^+,\mathcal{Y})$. Assume that Φ has a generalized kernel representation.*

(a) (Σ, μ) *is a realization of* $\Phi \iff R_{\Sigma,\mu}$ *is a representation of* Ψ_Φ

(b) $R = (\mathcal{X}, \{A_q\}_{q\in Q}, \widetilde{B}, \widetilde{C})$ *is a representation of* $\Psi_\Phi \iff (\Sigma_R, \mu_R)$ *is a realization of* Φ

Proof. First we prove part (a) of the theorem. By Proposition 12 (Σ, μ) is a realization of Φ if and only if for each $q_1, q_2, q \in Q$, $w = w_1 \cdots w_k \in Q^*$, $w_1, \dots, w_k \in Q, k \geq 0$

$$D^{(1,\mathbb{I}_k,0)} y_0(e_z, q_2 w q_1, .) = S_{q_1,q_2,z}(w) = C_{q_1} A_w B_{q_2} e_z$$

$$D^{(\mathbb{I}_k,0)} f(0, wq, .) = S_{f,q}(w) = C_q A_w \mu(f)$$

Here, the notation $A_w = A_{w_k} \cdots A_{w_1}$ introduced in Section 3.1 is used. That is,

$$\begin{aligned} S_{q_2,z}(w) &= \begin{bmatrix} C_{q_1}^T & C_{q_2}^T & \cdots & C_{q_N}^T \end{bmatrix}^T A_w B_{q_2} e_z = \widetilde{C} A_w \widetilde{B}_{q_2,z} \\ S_f(w) &= \begin{bmatrix} C_{q_1}^T & C_{q_2}^T & \cdots & C_{q_N}^T \end{bmatrix}^T A_w \mu(f) = \widetilde{C} A_w \widetilde{B}_f \end{aligned}$$

That is, $R_{\Sigma,\mu}$ is a representation of Ψ.

Since $R = R_{\Sigma_R,\mu_R}$, part (b) follows from part (a). □

The theorem has the following corollary.

Corollary 10. *Let the assumptions of Theorem 10 hold. If* (Σ, μ) *is a minimal realization of* Φ*, then* $R_{\Sigma,\mu}$ *is a minimal representation of* Ψ_Φ*. Conversely, if* R *is a minimal representation of* Ψ_Φ*, then* (Σ_R, μ_R) *is a minimal realization of* Φ*.*

Proof. Notice that $\dim \Sigma = \dim R_{\Sigma,\mu}$ and $\dim \Sigma_R = \dim R$. The statement of the corollary follows now from Theorem 10. □

Theorem 11 (Realization of input/output map). *For any set* $\Phi \subseteq F(PC(T, \mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$ *the following holds.*

(a) Φ *has a realization by a linear switched system if and only if* Φ *has a generalized kernel representation and* Ψ_Φ *is rational.*

(b) Φ *has a realization by a linear switched system if and only if* Φ *has a generalized kernel representation and* $\operatorname{rank} H_\Phi < +\infty$*.*

Proof. `Part (a)`
If Φ has a realization, then Φ has a generalized kernel representation, moreover, by Theorem 10, Ψ_Φ has a representation, i,e. Ψ_Φ is rational. If Φ has a generalized kernel representation and Ψ_Φ is rational, i.e. it has a representation, then by Theorem 10 Φ has a realization.

`Part (b)`
By Theorem 1 $\dim H_{\Phi} < +\infty$ is equivalent to Ψ_{Φ} being rational. The rest of the statement follows now from Part (a) $\square$

The theory of rational power series allows us to formulate necessary and sufficient conditions for a linear switched system to be minimal. Before formulating a characterisation of minimal realizations, additional work has to be done. Let $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q) \mid q \in Q\})$ be a linear switched system. Using Proposition 8 it is easy to see that for any $\mu : \Phi \to \mathcal{X}$

$$\begin{aligned} W_{R_{\Sigma,\mu}} &= \mathrm{Span}\{A_w x_0 \mid w \in Q^*, x_0 \in \mathrm{Im}\mu \text{ or } x_0 = B_q u, q \in Q, u \in \mathcal{U}\} \\ &= \mathrm{Span}\{A_{q_1}A_{q_2}\cdots A_{q_k}x_0 \mid q_1,q_2,\ldots,q_k \in Q, x_0 \in \mathrm{Im}\mu\} + \\ &\quad +\mathit{Reach}(\Sigma,\{0\}) \end{aligned}$$

and

$$O_{R_{\Sigma,\mathcal{X}_0}} = O_{\Sigma} = \bigcap_{q,w_1,w_2,\ldots,w_k \in Q, k\ge 0} \ker C_q A_{w_k}A_{w_{k-1}}\cdots A_{w_1}$$

Moreover, the following is true

Lemma 17. *$W_{R_{\Sigma,\mu}}$ is the smallest vector space containing $\mathit{Reach}(\Sigma,\mathrm{Im}\mu)$.*

Proof. Denote by WR the set $W_{R_{\Sigma,\mu}}$. Denote by $\mathcal{X}_0$ the image of μ.

First, we show that $\mathit{Reach}(\Sigma,\mathcal{X}_0)$ is contained in WR. From Proposition 8 it follows that

$$\begin{aligned} &\mathit{Reach}(\Sigma,\mathcal{X}_0) = \\ &\quad \{\exp(A_{q_k}t_k)\exp(A_{q_{k-1}}t_{k-1})\cdots\exp(A_{q_1}t_1)x_0 + x_{\Sigma}(0,u,(q_1,t_1)\cdots(q_k,t_k)) \\ &\quad \mid x_0 \in \mathcal{X}_0, (q_1,t_1)(q_2,t_2),\ldots,(q_k,t_k) \in (Q\times T)^*, k \ge 0, u \in PC(T,\mathcal{U})\} \end{aligned}$$

But $\exp(A_q t)x = \sum_0^{+\infty} \frac{t^k}{t!}A_q^k x \in \mathrm{Span}\{A_q^j x \mid j \in \mathbb{N}\}$, which implies that

$$\exp(A_{q_k}t_k)\cdots\exp(A_{q_1}t_1)x_0 \in \mathrm{Span}\{A_{w_1}A_{w_2}\cdots A_{w_k}x_0 \mid w_1,w_2,\ldots,w_k \in Q\}$$

Since $x(0,u,(q_1,t_1)\cdots(q_k,t_k)) \in \mathit{Reach}(\Sigma,\{0\})$, we get that $\mathit{Reach}(\Sigma,\mathcal{X}_0) \subseteq WR$.

We will show that WR is the smallest vector space containing $\mathit{Reach}(\Sigma,\mathcal{X}_0)$. Let W be a subspace of $\mathcal{X}$ containing $\mathit{Reach}(\Sigma,\mathcal{X}_0)$. For any $\alpha \in \mathbb{N}^{|w|}$, for any constant input function $u(t) = u \in \mathcal{U}$ $D^{\alpha}x(x_0,u,w,.) \in W$ must hold. But $x(x_0,u,w,\underline{t}) = x(x_0,0,w,\underline{t}) + x(0,u,w,\underline{t})$. It is straightforward to show that $\mathrm{Span}\{D^{\alpha}x(0,u,w,.) \mid w \in Q^+, \alpha \in \mathbb{N}^{|w|}, u \in \mathcal{U}\} = \mathit{Reach}(\Sigma,0)$. For $w \in Q^+, k := |w|$ define $\exp_w : T^k \to \mathcal{X}$ by

$$\exp_w(t_1,t_2,\ldots,t_k) = \exp(A_{w_k}t_k)\exp(A_{w_{k-1}}t_{k-1})\cdots\exp(A_{w_1}t_1)x_0$$

It is easy to see that $D^{\alpha}x(x_0,0,w,.) = D^{\alpha}\exp_w = A_{w_k}^{\alpha_k}A_{w_{k-1}}^{\alpha_{k-1}}\cdots A_{w_1}^{\alpha_1}x_0$, and therefore $\mathrm{Span}\{D^{\alpha}x(x_0,0,w,.) \mid w \in Q^+, \alpha \in \mathbb{N}^{|w|}, x_0 \in \mathcal{X}_0\} = \mathrm{Span}\{A_w x_0 \mid w \in Q^+\}$. Thus, we get that

$$\mathrm{Span}\{D^{\alpha}x(x_0,u,w,.) \mid w \in Q^+, \alpha \in \mathbb{N}^{|w|}, u \in \mathcal{U}, x_0 \in \mathcal{X}_0\} = WR$$

which implies that $WR \subseteq W$. □

The results above imply the following

Corollary 11. *Let $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$ and assume that (Σ, μ) is a realization of Φ. Then Σ is observable if and only if R is observable. Σ is semi-reachable from* $\mathrm{Im}\mu$ *if and only if R is reachable.*

It is a natural question to ask what the relationship is between linear switched system morphisms and representation morphisms. The following lemma answers this question.

Lemma 18. *$T : (\Sigma, \mu) \to (\Sigma^{'}, \mu^{'})$ is a linear switched system morphism if and only if $T : R_{\Sigma,\mu} \to R_{\Sigma^{'},\mu^{'}}$ is a representation morphism.*

Recall that $T : (\Sigma, \mu) \to (\Sigma^{'}, \mu^{'})$ is a linear switched system morphism if T is a linear map from the state-space of Σ to the state-space of $\Sigma^{'}$ satisfying certain properties. Recall that a representation morphism between two representations is a linear map between the state-spaces of the representations which satisfies certain properties. Since the state spaces of $R_{\Sigma,\mu}$ and $R_{\Sigma^{'},\mu^{'}}$ coincide with the state-space of Σ and $\Sigma^{'}$ respectively, it is justified to denote both the linear switched system morphism and the representation morphism by the same symbol, indicating that the underlying linear map is the same.

Proof of Lemma 18. Assume that the linear switched systems Σ and $\Sigma^{'}$ are of the form

$$\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\}) \text{ and } \Sigma^{'} = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A^{'}_q, B^{'}_q, C^{'}_q) \mid q \in Q\})$$

Then T is a switched linear system morphism if and only if $TA_q = A^{'}_q T$, $C_q = C^{'}_q T$, $TB_q = B^{'}_q$ and $T\mu(f) = \mu^{'}(f)$ for each $q \in Q, f \in \Phi$. But this is equivalent to $TA_q = A^{'}_q T, q \in Q, T\widetilde{B}_j = \widetilde{B}^{'}_j$ and $\widetilde{C} = \begin{bmatrix} C_{q_1}^T & \cdots & C_{q_N}^T \end{bmatrix}^T = \begin{bmatrix} (C^{'}_{q_1}T)^T & \cdots & (C^{'}_{q_N}T)^T \end{bmatrix}^T = \widetilde{C}^{'}T$, that is, to T being a representation morphism. □

Now we can state the main result of the section.

Theorem 12 (Minimal realizations). *If (Σ, μ) is a realization of Φ, then the following are equivalent.*

(i) (Σ, μ) *is minimal*

(ii) Σ *is semi-reachable from* $\mathrm{Im}\mu$ *and it is observable*

(iii) $\dim \Sigma = \dim H_{\Phi}$

(iv) If $(\Sigma^{'}, \mu^{'})$ *realizes* Φ *and* $\Sigma^{'}$ *is semi-reachable from* $\mathrm{Im}\mu^{'}$*, then there exists a surjective linear switched system morphism* $T : (\Sigma^{'}, \mu^{'}) \to (\Sigma, \mu)$*. In particular, all minimal realizations of* Φ *are algebraically similar.*

Proof. *(i)* $\iff$ *(ii)*
By Corollary 10 system (Σ, μ) is minimal if and only if $R := R_{\Sigma,\mu}$ is minimal. By Theorem 2 R is minimal if and only if R is reachable and observable. By Corollary 11 the latter is equivalent to Σ being semi-reachable from $\mathrm{Im}\mu$ and observable.

(i) $\iff$ *(iii)*
By Corollary 10 (Σ, μ) is minimal $\iff$ $R_{\Sigma,\mu}$ is minimal. By Theorem 2 $R_{\Sigma,\mu}$ is minimal $\iff$ $\dim R_{\Sigma,\mu} = \dim \Sigma = \mathrm{rank}\ H_{\Psi_{\Phi}} = \mathrm{rank}\ H_{\Phi}$.

(i) $\iff$ *(iv)*
Again we are using the fact that (Σ, μ) is minimal if and only if $R_{\Sigma,\mu}$ is minimal. By Theorem 2 R_{min} is minimal if and only if for any reachable representation R there exists a surjective representation morphism $T : R \to R_{min}$. It means that (Σ, μ) is minimal if and only if for any reachable representation R of Ψ_{Φ} there exists a surjective representation morphism $T : R \to R_{\Sigma,\mu}$. But any reachable representation R gives rise to a semi-reachable realization of Φ and vice versa. That is, we get that (Σ, μ) is minimal if and only if for any semi-reachable realization $(\Sigma^{'}, \mu^{'})$ of Φ there exists a surjective representation morphism $T : R_{\Sigma^{'},\mu^{'}} \to R_{\Sigma,\mu}$. By Lemma 18 we get that the latter is equivalent to $T : (\Sigma^{'}, \mu^{'}) \to (\Sigma, \mu)$ being a surjective linear switched system morphism. From Corollary 1 it follows that if $(\Sigma^{'}, \mu^{'})$ is a minimal realization of Φ, then there exists a representation isomorphism $T : R_{\Sigma^{'},\mu^{'}} \to R_{\Sigma,\mu}$ which means that (Σ, μ) is gives rise to the linear switched system isomorphism $T : (\Sigma^{'}, \mu^{'}) \to (\Sigma, \mu)$, that is, $\Sigma^{'}$ and Σ are algebraically similar. ☐

4.1.4 Realization Theory of Linear Switched Systems: Constrained Switching

In this section the solution of the realization problem with constraints will be presented. That is, given a set of constraints $L \subseteq Q^{+}$ and a set of input-output maps with domain $PC(T, \mathcal{U}) \times TL$ we will study linear switched systems realizing this set with constraint L. As in the previous section, the theory of formal power series will be our main tool in solving the realization problem.

Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$. Recall that (Σ, μ) *realizes* Φ with constraint L if for all $f \in \Phi$ it holds that $f = y_{\Sigma}(\mu(f), ., .)|_{PC(T,\mathcal{U})\times TL}$. In the sequel, unless stated otherwise, we assume that Φ has a generalised kernel representation with constraint L.

The solution of the realization problem for Φ goes as follows. As in the previous section, we associate a set of formal power series Ψ_Φ with the set of maps Φ. We will show that any representation of Ψ_Φ gives rise to a realization of Φ with constraint L. If L is regular, then any realization of Φ with constraint L gives rise to a representation of Ψ_Φ. Unfortunately minimal representations of Ψ_Φ do not yield minimal realizations of Φ. However, any minimal representation of Ψ_Φ yields an observable and semi-reachable realization of Φ.

Recall from Section 7.1.2 the definition of the languages $\widetilde{L}$, $\widetilde{L}_{q_1,q_2}$, $\widetilde{L}_q$ and the sets $F_{q_1,q_2}(w)$, $F_q(w)$. Let $E = (1, 1, \dots, 1) \in \mathbb{R}^{1\times p}$. Define the power series $Z_{q_1,q_2} \in \mathbb{R}^p \ll Q^* \gg$ by

$$Z_{q_1,q_2}(w) = \begin{cases} E^T & \text{if } w \in \widetilde{L}_{q_1,q_2} \\ 0 & \text{otherwise} \end{cases}$$

Define the power series $\Gamma_q \in \mathbb{R}^{p|Q|} \ll Q^* \gg$ by

$$\Gamma_q = \begin{bmatrix} Z_{q_1,q} \\ Z_{q_2,q} \\ \vdots \\ Z_{q_N,q} \end{bmatrix}$$

and $\Gamma \in \mathbb{R}^{p|Q|} \ll Q^* \gg$ by

$$\Gamma = \begin{bmatrix} Z_{q_1} \\ Z_{q_2} \\ \vdots \\ Z_{q_N} \end{bmatrix}$$

where $Z_q(w) = \begin{cases} E^T & \text{if } w \in \widetilde{L}_q \\ 0 & \text{otherwise} \end{cases}$ and $Q = \{q_1, \dots, q_N\}$. It is a straightforward exercise in automata theory to show that if L is regular, then the languages $\widetilde{L}_q$ and $\widetilde{L}_{q_1,q_2}$ are regular.

Lemma 19. *With the notation above, if $L \subseteq Q^+$ is a regular language, then $\widetilde{L}$, $\widetilde{L}_{q_1,q_2}$ and $\widetilde{L}_q$ are regular languages for each $q, q_1, q_2 \in Q$.*

Proof. Notice that $\widetilde{L}_{q_1,q_2} = \{w \in Q^* \mid q_1 w q_2 \in \widetilde{L}\}$ and $\widetilde{L}_q = \{w \in Q^* \mid wq \in \widetilde{L}\}$. It is easy to see that if $\widetilde{L}$ is regular, then so are $\widetilde{L}_{q_1,q_2}$ and $\widetilde{L}_q$. It is also easy to see that if L is regular then suffix$L$ is regular. Let $A = (S, Q, \delta, F, s_0)$ be a deterministic automaton accepting suffix$L$. Here S is the state-space, F is the set of accepting states, δ is the state-transition function, s_0 is the set of initial states. Recall, that the extended state-transition function is defined as follows. For each $s_0 \in S, w \in Q^*$, $\delta(s_0, w) = s$ if there exists $s_1, \dots, s_k = s \in Q$ such that $w = w_1 \cdots w_k \in Q^k$ and $s_i = \delta(s_{i-1}, w_i)$ for each $i = 1, \dots, k$.

Define the non-deterministic automaton $B = ((S \times Q) \cup \{s_0^{'}\}, Q, \delta_B, F \times Q, s_0^{'})$ in the following way. Let $\delta_B(s_0^{'}, x) \ni (s, x)$ if $\delta(s_0, wx) = s$ for some $w \in Q^*$. Let $(s^{'}, u) \in \delta_B((s, x), u)$ if either

(i) $u = x$ and $s^{'} = s$, or

(ii) there exists $wu \in Q^*$, such that $\delta(s, wu) = s^{'}$.

We will prove that B accepts $\widetilde{L}$. Denote $s \in \delta_B(z, x)$, $s, z \in (S \times Q) \cup \{s_0^{'}\}$ by $z \xrightarrow{x} s$. Then B accepts $z = z_1 \cdots z_k$ if and only if

$$s_0^{'} \xrightarrow{z_1} (s_1, z_1) \xrightarrow{z_2} \cdots \xrightarrow{z_k} (s_k, z_k)$$

where $s_k \in F$. This is equivalent to the existence of $0 < \alpha_1, \dots, \alpha_l \in \mathbb{N}$ and $w_0, \dots, w_l \in Q^*$ such that $\sum_{j=1}^{l} \alpha_j = k$, $\delta(s_0, w_0 z_1) = s_1$ and $(s_i, z_i) = (s_{i+1}, z_{i+1})$ for each $1 + \sum_1^d \alpha_j \leq i < \sum_1^{d+1} \alpha_j$ and $\delta(s_{\sum_1^d \alpha_j}, w_d z_{\sum_1^d \alpha_j}) = s_{1+\sum_1^d \alpha_j}$ for all $0 \leq d \leq l-1$. Define $u_d = z_{1+\sum_1^d \alpha_j}$. Then it is clear that in the original automaton A it holds that $\delta(s_0, w_0 u_0 w_1 u_1 \cdots w_l u_l) = s_k \in F$. That is, $w_0 u_0 \cdots w_l u_l \in$ suffix$L$ and

$$z = w_{0,1}^0 \cdots w_{0,m_0}^0 u_1^{\alpha_1} w_{1,1}^0 \cdots w_{m_1,1}^0 u_2^{\alpha_2} \cdots w_{l,1}^0 \cdots w_{l,m_l}^0 u_l^{\alpha_l}$$

where $w_i = w_{i,1} \cdots w_{i,m_i}$, $w_{i,1}, \dots, w_{i,m(i)} \in Q$. We get that B accepts exactly the elements of $\widetilde{L}$. $\square$

Corollary 12. *Define the indexed set of formal power series $\Omega = \{\Lambda_j \in \mathbb{R}^{pN} \ll Q^* \gg \mid j \in Q \times \{\emptyset\}\}$, where $\Lambda_q = \Gamma_q$ and $\Lambda_\emptyset = \Gamma$. If L regular then the indexed set of formal power series Ω is rational.*

Proof. Indeed, if L is regular, then $\widetilde{L}_{q_1,q_2}$ and $\widetilde{L}_q$ are regular languages. Then it is easy to see that for each $l = 1, \dots, pN$, such that $l = p * (z-1) + i$ for some $z = 1, \dots, N$, $i = 1, \dots p$, $(\Gamma)_l(w) = \begin{cases} 1 & \text{if } w \in L_{q_z} \\ 0 & \text{otherwise} \end{cases}$ and $(\Gamma_q)_l(w) = \begin{cases} 1 & \text{if } w \in L_{q_z,q} \\ 0 & \text{otherwise} \end{cases}$. That is, $(\Gamma_q)_l, \Gamma_l \in \mathbb{R} \ll Q^* \gg$ are rational formal power series for each $l = 1, \ldots, pN$. Consider the indexed set $\Theta = \{(\Lambda_{(l,j)} \mid (l,j) \in \{1, \ldots, pN\} \times (Q \cup \{\emptyset\})\}$, where $\Lambda_{(l,q)} = (\Lambda_q)_l = (\Gamma_q)_l$, $\Lambda_{(l,\emptyset)} = (\Lambda_\emptyset)_l = \Gamma_l$. Then by Corollary 2 from Section 3.1, Θ is rational. By Lemma 5 from Section 3.1, it implies that Ω is rational. $\square$

Consider a set of input-output maps $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$ with a $L \subseteq Q^*$. Assume that Φ has a generalised kernel representation.

Recall that for any $\alpha \in \mathbb{N}^k$, α^+ denotes $\alpha^+ = (\alpha_1 + 1, \alpha_2, \ldots, \alpha_k)$. We define the following formal power series. For $j = 1, 2, \ldots, m$ and $f \in \Phi$, $q_1, q_2 \in Q$,

$$S_{q_1,q_2,j}(w) = \begin{cases} D^{(\mathbb{O}_{|v|},\alpha^+)} y_0^{\Phi}(e_j, vz, .) & \text{if } w \in \widetilde{L}_{q_1,q_2} \text{ and} \\ & (v,(\alpha,z)) \in F_{q_1,q_2}(w) \\ 0 & \text{otherwise} \end{cases}$$

$$S_{q,f}(w) = \begin{cases} D^{(\mathbb{O}_{|v|},\alpha)} f(0, vz, .) & \text{if } w \in \widetilde{L}_q \text{ and } (v,(\alpha,z)) \in F_q(w) \\ 0 & \text{otherwise} \end{cases}$$

We will show that the series $S_{q_1,q_2,z}$ and $S_{q,f}$ are well-defined. Using formulas (4.4), (4.5) and (4.3) from Subsection 7.1.2 and the fact that $(v,(\alpha,z)) \in F_{q_1,q_2}(w) \implies z_1 = q_2, z_{|z|} = q_1$ and $(v,(\alpha,z)) \in F_q(w) \implies z_{|z|} = q$ we get the following

$$S_{q_1,q_2,j}(w) = \begin{cases} D^{\alpha} G^{\Phi}_z = D^{(0,\alpha,0)} G^{\Phi}_{q_2 z q_1} e_j & \text{if } w \in \widetilde{L}_{q_1,q_2} \text{ and} \\ & (v,(\alpha,z)) \in F_{q_1,q_2}(w) \\ 0 & \text{otherwise} \end{cases}$$

$$S_{q,f}(w) = \begin{cases} D^{(\mathbb{O}_{|v|},\alpha)} K^{f,\Phi}_{vz} = D^{\alpha} K^{f,\Phi}_z = D^{(\alpha,0)} K^{f,\Phi}_{zq} & \text{if } w \in \widetilde{L}_q \text{ and} \\ & (v,(\alpha,z)) \in F_q(w) \\ 0 & \text{otherwise} \end{cases}$$

That is, $S_{q_1,q_2,j}(w)$ and $S_{q,f}(w)$ do not depend on the choice of v in $(v,(\alpha,z)) \in F_{q_1,q_2}(w)$ or $(v,(\alpha,z)) \in F_q(w)$ respectively. We will argue that the value of $S_{q_1,q_2,z}(w)$ and $S_{q,f}(w)$ do not depend on the choice of (α,z). If $(v,(\alpha,z)),(u,(\beta,x)) \in F_{q_1,q_2}(w)$ then $x_1^{\beta_1} \cdots x_{|x|}^{\beta_{|x|}} = z_1^{\alpha_1} \cdots z_{|z|}^{\alpha_{|z|}} = w$, $z_1 = x_1 = q_2$, $z_{|z|} = x_{|x|} = q_1$ and $q_2 z q_1, q_2 x q_1 \in \widetilde{L}$, so by Proposition 9, $D^{(0,\alpha,0)} G^{\Phi}_{q_2 z q_1} = D^{(0,\beta,0)} G^{\Phi}_{q_2 x q_1}$. Similarly, if $(v,(\alpha,z)),(u,(\beta,x)) \in F_q(w)$, then $x_1^{\beta_1} \cdots x_{|x|}^{\beta_{|x|}} = z_1^{\alpha_1} \cdots z_{|z|}^{\alpha_{|z|}} = w$ and $zq, xq \in \widetilde{L}$, so by Proposition 9, $D^{(\alpha,0)} K^{f,\Phi}_{zq} = D^{(\beta,0)} K^{f,\Phi}_{q_2 x q_1}$.

Define the formal power series $S_{q,j}, S_f \in \mathbb{R}^{p|Q|} \ll Q^* \gg$, $j \in \{1,2,\ldots,m\}$, $q \in Q$ and $f \in \Phi$ by

$$S_{q,j} = \begin{bmatrix} S_{q_1,q,j} \\ S_{q_2,q,j} \\ \vdots \\ S_{q_N,q,j} \end{bmatrix}, \quad S_f = \begin{bmatrix} S_{q_1,f} \\ S_{q_2,f} \\ \vdots \\ S_{q_N,f} \end{bmatrix}$$

Define the *indexed set of formal power series associated with* Φ as $\Psi_\Phi = \{S_z \in \mathbb{R}^{p|Q|} \ll Q^* \gg \mid z \in J_\Phi\}$ where $J_\Phi = \Phi \cup (Q \times \{1, 2, \ldots, m\})\}$. Define the *Hankel-matrix* H_Φ as the Hankel-matrix of Ψ_Φ.

Consider the map $g : \Phi \cup (Q \times \{1, 2, \ldots, m\}) \to Q \times \{\emptyset\}$, where $g(f) = \emptyset, \forall f \in \Phi$ and $g((q, z)) = q$ for all $q \in Q, z = 1, \ldots, m$. Recall the indexed set of formal power series Ω from Corollary 12. Define the indexed set of formal power series $\Omega_\Phi = \{\Xi_j \in \mathbb{R}^{pN} \ll Q^* \gg \mid j \in J_\Phi\}$ by $\Xi_j = \Lambda_{g(j)}$, where $\Omega = \{\Lambda_j \mid j \in Q \cup \{\emptyset\}\}$. From Lemma 8 of Section 3.1 and Corollary 12 it follows that if L is regular, then Ω_Φ is rational. Let (Σ, μ) be a realization of Φ. Define $\Theta_{\Sigma,\mu} = \{y_\Sigma(\mu(f), ., .) \mid f \in \Phi\} \subseteq F(PC(T, \mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$. Define $U(\mu) : \Theta_{\Sigma,\mu} \to \Phi$ by $U(\mu)(y_\Sigma(\mu(f), .,)) = f$. The map $U(\mu)$ is well defined. Indeed, if $y_\Sigma(\mu(f_1), ., .) = y_\Sigma(\mu(f_2), ., .)$, then $f_1 = y_\Sigma(\mu(f_1), ., .)|_{PC(T,\mathcal{U})\times TL} = y_\Sigma(\mu(f_2), ., .)|_{PC(T,\mathcal{U})\times TL} = f_2$. It is easy to see that $(\Sigma, \mu \circ U(\mu))$ is a realization of $\Theta_{\Sigma,\mu}$. Assume that the set of formal power series associated to $\Theta_{\Sigma,\mu}$ as defined in Section 4.1.3, (4.12), is of the form

$$\Psi_{\Theta_{\Sigma,\mu}} = \{T_z \in \mathbb{R}^{p|Q|} \ll Q^* \gg \mid z \in \Theta_{\Sigma,\mu} \cup (Q \times \{1, 2, \ldots, m\})\}$$

From Theorem 11 it follows that $\Psi_{\Theta_{\Sigma,\mu}}$ is rational. Define the map $\psi : J_\Phi \to \Theta_{\Sigma,\mu} \cup (Q \times \{1, 2, \ldots, m\})$ by $\psi(f) = y_\Sigma(\mu(f), ., .), f \in \Phi$ and $\psi((q, z)) = (q, z), q \in Q, z = 1, \ldots, m$. Define $K_{\Sigma,\mu} = \{V_j \in \mathbb{R}^{p|Q|} \ll Q^* \gg \mid j \in J_\Phi\}$, $V_j = T_{\psi(j)}, j \in J_\Phi$. From Lemma 8 of Section 3.1 it follows that $K_{\Sigma,\mu}$ is rational.

Let $R = (\mathcal{X}, \{A_z\}_{z\in Q}, B, C)$ be a representation of Ψ_Φ. Define (Σ_R, μ_R) the *linear switched system realization associated with* R as in Section 4.1.3. That is,

$$\Sigma_R = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\}) \text{ and } \mu_R(f) = B_f$$

where $C_q : \mathcal{X} \to \mathcal{Y}$, $q \in Q$ are such that $C = \begin{bmatrix} C_{q_1} \\ \vdots \\ C_{q_N} \end{bmatrix}$ and $B_q e_j = B_{(q,j)}$ for all $q \in Q, j = 1, \ldots, m$. Assume that the resulting (Σ_R, μ_R) is a realization of Φ (in fact, this will be shown later). Let $(\Sigma, \mu) = (\Sigma_R, \mu_R \circ U(\mu_R))$. Then (Σ, μ) is a realization of Θ_{Σ_R,μ_R}. Let $\widetilde{R} = R_{\Sigma,\mu}$ – the representation associated to (Σ, μ) as defined in Section 4.1.3. Then it is easy to see that $\widetilde{R} = (\mathcal{X}, \{A_q\}_{q\in Q}, \widetilde{B}, C)$, where $\widetilde{B}_{y_{\Sigma_R}(\mu_R(f),.,.)} = \mu(y_{\Sigma_R}(\mu_R(f), ., .)) = \mu_R(f) = B_f$, $f \in \Phi$ and $\widetilde{B}_{(q,j)} = B_q e_j = B_{(q,j)}$, $q \in Q, j = 1, \ldots, m$. That is, R is observable if and only if $\widetilde{R}$ is observable. R is reachable if and only if $\widetilde{R}$ is reachable. It is also straightforward to see that $\mathrm{Im}\mu_R = \mathrm{Im}\mu_R \circ U(\mu_R) = \mathrm{Im}\mu$. Thus, by Corollary 11, the following holds. Σ_R is observable if and only if R is observable. (Σ_R, μ_R) is semi-reachable if and only if R is reachable.

Using the notation above and combining Proposition 11 and the definition of rational sets of power series one gets the following theorems.

Theorem 13. *Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$. Then (Σ,μ) is realization of Φ with constraint L if and only if Φ has a general kernel representation with constraint L and*

$$\Psi_\Phi = \Omega_\Phi \odot K_{\Sigma,\mu}$$

or, in other words

$$\begin{gathered}\forall f \in \Phi, q \in Q, z = 1,2,\ldots,m \\ S_f = T_{y_\Sigma(\mu(f),.,.)} \odot \Gamma \text{ and } S_{q,z} = T_{q,z} \odot \Gamma_q\end{gathered}$$

Proof. By Proposition 11 (Σ,μ) is a realization of Φ with constraint L, if and only if Φ has a generalised kernel representation with constraint L and

$$\begin{aligned}
&\forall w \in \widetilde{L}_{q_1,q_2}, (v,(\alpha,z)) \in F_{q_1,q_2}(w): \\
&\qquad D^{(0,\alpha,0)}G^{\Phi}_{q_2 z q_1} = C_{q_1}A^{\alpha_k}_{z_k}\cdots A^{\alpha_1}_{z_1}B_{q_2} = C_{q_1}A_w B_{q_2} \\
&\forall w \in \widetilde{L}_q, (v,(\alpha,z)) \in F_q(w): \\
&\qquad D^{(\alpha,0)}K^{f,\Phi}_{zq_1} = C_{q_1}A^{\alpha_k}_{z_k}\cdots A^{\alpha_1}_{z_1}\mu(f) = C_{q_1}A_w\mu(f)
\end{aligned}$$

But $(\Sigma, \mu \circ U(\mu))$ is also a realization of $\Theta = \Theta_{\Sigma,\mu}$ with constraint Q^+, so by Proposition 12 we get that

$$\begin{aligned}C_{q_1}A_wB_{q_2} = D^{(0,\mathbb{I}_{|w|},0)}G^{\Theta}_{q_2 w q_1} \text{ and } C_qA_w\mu(f) &= C_qA_w\mu(U(\mu)(y_\Sigma(\mu(f),.,.))) = \\ &= D^{(\mathbb{I}_{|w|},0)}K^{y_\Sigma(\mu(f),.,.),\Theta}_{wq}\end{aligned}$$

That is, for each $w \in \widetilde{L}_{q_1,q_2}$, $(v,(\alpha,z)) \in F_{q_1,q_2}(w)$, $q_1,q_2 \in Q$, $j = 1,\ldots,m$

$$T_{q_1,q_2,j}(w) = D^{(0,\mathbb{I}_{|w|},0)}G^{\Theta}_{q_2 w q_1}e_j = D^{(0,\alpha,0)}G^{\Phi}_{q_2 z q_1}e_j = S_{q_1,q_2,j}(w)$$

and for each $w \in \widetilde{L}_q$, $(v,(\alpha,z)) \in F_q(w)$

$$T_{q,y_\Sigma(\mu(f),.,.)}(w) = D^{(\mathbb{I}_{|w|},0)}K^{y_\Sigma(\mu(f),.,.),\Theta}_{wq} = D^{(\alpha,0)}K^{f,\Phi}_{zq} = S_{q,f}(w)$$

We get that

$$\begin{aligned}T_{q_1,y_\Sigma(\mu(f),.,.)}(w) &= S_{q_1,f}(w) && \text{if } w \in \widetilde{L}_{q_1} \\ T_{q_1,z_2,z}(w) &= S_{q_1,q_2,z}(w) && \text{if } w \in \widetilde{L}_{q_1,q_2}\end{aligned}$$

Notice that if $w \notin \widetilde{L}_{q_1,q_2}$, then $S_{q_1,q_2,z}(w) = 0$ and $Z_{q_1,q_2}(w) = 0$. Similarly, If $w \notin \widetilde{L}_{q_1}$, then $S_{q_1,f}(w) = 0 = Z_{q_1}(w)$. That is,

$$T_{q,z} \odot \Gamma_q = S_{q,z} \text{ and } T_{y_\Sigma(\mu(f),.,.)} \odot \Gamma = S_f$$

□

Define the language

$$comp(L) = \{w_1 \cdots w_k \in Q^* \mid \widetilde{L}_{w_k} = \emptyset\}$$

Intuitively, the language $comp(L)$ contains those sequences which can never be observed if the switching system is run with constraint L.

Theorem 14. *Assume that Φ has a generalised kernel representation with constraint L. If*

$$R = (\{A_q\}_{q \in Q}, B, C)$$

is a representation of Ψ_Φ, then (Σ_R, μ_R) realizes Φ. Moreover,

$$\forall f \in \Phi, \forall u \in PC(T, \mathcal{U}), w \in T(comp(L)) : y_{\Sigma_R}(\mu_R(f), u, w) = 0$$

Proof. Let $(\Sigma, \mu) = (\Sigma_R, \mu_R)$. If R is a representation of Φ, then

$$\begin{array}{rcl}
\forall w \in \widetilde{L}_{q_1,q_2}, (v, (\alpha, z)) \in F_{q_1,q_2}(w) & & \\
D^{(0,\alpha,0)} G^{\Phi}_{q_2 z q_1} e_j & = & S_{q_1,q_2,j}(w) = C_{q_1} A_w B_{q_2,j} \\
& = & C_{q_1} A^{\alpha_k}_{z_k} \cdots A^{\alpha_1}_{z_1} B_{q_2} e_j \\
\forall w \in \widetilde{L}_q, (v, (\alpha, z)) \in F_q(w) & & \\
D^{(\alpha,0)} K^{f,\Phi}_{zq} & = & S_{q,f}(w) = C_q A_w B_f \\
& = & C_q A^{\alpha_1}_{z_1} \cdots A^{\alpha_k}_{z_k} \mu(f)
\end{array} \tag{4.13}$$

Since Φ has a generalised kernel representation, Proposition 11 and (4.13) yield that (Σ, μ) is a realization of Φ with constraint L.

Let $\Phi^{'} = \Theta_{\Sigma,\mu}$. Then $(\Sigma, \mu \circ U(\mu))$ is a realization of $\Phi^{'}$. It is easy to see that for all $f \in \Phi$, $q_1, q_2 \in Q$, $z = 1, \ldots, m$,

$$\begin{array}{ll}
S_{q,f}(w) = C_q A_w \mu(f) = 0 & \text{if } w \notin \widetilde{L}_q \\
S_{q_1,q_2,z}(w) = C_{q_1} A_w B_{q_2} e_z = 0 & \text{if } w \notin \widetilde{L}_{q_1} \supseteq \widetilde{L}_{q_1,q_2}
\end{array}$$

As the second step we are going to show that for each $w \in comp(L)$, $y_\Sigma(\mu(f), ., .) \in \Phi^{'}$,

$$G^{\Phi^{'}}_w = 0 \text{ and } K^{y_\Sigma(\mu(f),.,.),\Phi^{'}}_w = 0 \tag{4.14}$$

Because of analyticity of these function it is enough to prove that for each $\alpha \in \mathbb{N}^{|w|}$: $D^\alpha G^{\Phi^{'}}_w = 0$, $D^\alpha K^{y_\Sigma(\mu(f),.,.),\Phi^{'}}_w = 0$. But from formulas (4.4), (4.5) and Proposition 11 we get that

$$D^\alpha G^{\Phi^{'}}_w = C_{w_k} A_v B_{w_1} \text{ and } D^\alpha K^{y_\Sigma(\mu(f),.,.),\Phi^{'}}_w = C_{w_k} A_v (\mu \circ U(\mu))(y_\Sigma(\mu(f), ., .)) = \\ = C_{w_k} A_v \mu(f)$$

$w = w_1, \cdots w_k$, $w_1, \ldots, w_k \in Q$, $v = w_1^{\alpha_1} \cdots w_k^{\alpha_k}$. But $w \in comp(L)$ implies $\widetilde{L}_{w_k} = \emptyset$, that is $u \notin \widetilde{L}_{w_k,w_l}$ and $v \notin \widetilde{L}_{w_k}$. Then it follows that $C_{w_k} A_v B_{w_1} = 0$ and $C_{w_k} A_v \mu(f) = 0$. It implies that $D^\alpha G_w^{\Phi'} = 0$ and $D^\alpha K_w^{f,\Phi'} = 0$.

It is easy to see that if $w_1 \cdots w_k \in comp(L)$, then for any $l \leq k$, $w_l \cdots w_k \in comp(L)$. Then from Definition 10, part 4 it follows that (4.14) implies $y_\Sigma(\mu(f), u, w) = 0$ for all $u \in PC(T, \mathcal{U})$ and $w \in T(comp(L))$. □

If L regular then the power series Γ, Γ_q, $(q \in Q)$ are rational. Then using Theorem 13 and Lemma 6 from Section 3.1 one gets the following.

Theorem 15. *Consider a language $L \subseteq Q^+$ and a set $\Phi \subseteq F(PC(T, \mathcal{U}) \times TL, \mathcal{Y})$ of input-output maps. Assume that L is regular. Then the following holds.*

(i) Φ has a realization by a linear switched system with constraint L if and only if Φ has a generalised kernel representation with constraint L and Ψ_Φ is rational, or equivalently $\dim H_\Phi < +\infty$.

(ii) Φ has a realization by a linear switched system with constraint L if and only if there exists a linear switched system realization (Σ, μ) of Φ with constraint L, such that (Σ, μ) is semi-reachable, it is observable, and

$$\forall f \in \Phi : y_\Sigma(\mu(f), ., .)|_{PC(T,\mathcal{U}) \times T(comp(L))} = 0 \tag{4.15}$$

Proof. `Part (i)`
If Φ has a generalised kernel representation with constraint L and Ψ_Φ is rational, then there exists a representation R of Ψ_Φ and by Theorem 14 (Σ_R, μ_R) is a realization of Φ. Conversely, assume that Φ is realized by (Σ, μ). Then by Theorem 13 Φ has a generalised kernel representation and with the notation of Theorem 13 it holds that $\Psi_\Phi = \Omega_\Phi \odot K_{\Sigma,\mu}$. Since $(\Sigma, \mu \circ U(\mu))$ is a realization of $\Theta_{\Sigma,\mu}$ without constraint, by Theorem 11 $\Psi_{\Theta_{\Sigma,\mu}}$ is rational. Then by Lemma 8 $K_{\Sigma,\mu}$ is rational too. If L is regular, then by Corollary 12 Ω is rational. Then by Lemma 8 Ω_Φ is rational. By Lemma 6 we get that $\Psi_\Phi = \Omega_\Phi \odot K_{\Sigma,\mu}$ is rational. From Theorem 1 it follows that Ψ_Φ is rational if and only if rank $H_{\Psi_\Phi} < +\infty$. By definition $H_\Phi = H_{\Psi_\Phi}$, so we get that Ψ_Φ is rational if and only if rank $H_\Phi < +\infty$.

`Part(ii)`
Φ has a realization with constraint L if and only if Φ has a generalised kernel representation with constraint L and Ψ_Φ is rational. Let $R = (\{A_q\}_{q \in Q}, B, C)$ be a minimal representation of Ψ_Φ. Consider $(\Sigma, \mu) = (\Sigma_R, \mu_R)$ – the linear switched system realization associated with R. Then by Theorem 14 (Σ, μ) is a realization of Φ with constraint L such that $\forall f \in \Phi, \forall u \in PC(T, \mathcal{U}), w \in T(comp(L)) : y_\Sigma(\mu(f), u, w) = 0$.

Since R is reachable and observable, we get that (Σ,μ) is semi-reachableand observable. ☐

Lemma 6 also yields the following result.

Theorem 16. *Consider a language $L \subseteq Q^+$ and a set $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$ of input-output maps. Assume L that is regular and that Φ has a realization by a linear switched system. Let (Σ,μ) be the realization of Φ from part (ii) of Theorem 15. If $(\widetilde{\Sigma},\widetilde{\mu})$ is an arbitrary linear switched system realizing Φ with constraint L, then*

$$\dim \Sigma \leq M \cdot \dim \widetilde{\Sigma} \tag{4.16}$$

where M depends only on L.

Proof. By Theorem 13 it holds that $\Psi_\Phi = K_{\Sigma,\mu} \odot \Omega_\Phi$. Since $R_{\Sigma,\mu}$ is a minimal representation of Ψ_Φ it holds that $\dim \Sigma = \dim R_{\Sigma,\mu} = \text{rank } H_{\Psi_\Phi}$. But from Lemma 6 one gets that

$$\text{rank } H_{\Psi_\Phi} = \text{rank } H_{K_{\Sigma,\mu}\odot\Omega_\Phi} \leq \text{rank } H_{K_{\Sigma,\mu}} \cdot \text{rank } H_{\Omega_\Phi}$$

Since $\text{rank } H_{K_{\Sigma,\mu}} = \text{rank } H_{\Psi_\Theta} \leq \dim \widetilde{\Sigma}$ and $M := \text{rank } H_\Omega$ depends only on L, we get the statement of the theorem. ☐

Notice that if L is finite then L is regular. It means that the results of this section in principle allow us to construct a realization of a set of input-output map by examining a finite number of sequences of discrete modes.

Remark

In fact, the result of the Theorem 16 is sharp in the following sense. One can construct an input-output y map and language L and realizations Σ_1 and Σ_2 such that the following holds. Both Σ_1 and Σ_2 realize y from the initial state zero with constraint L and they are both reachable from zero and observable, but $\dim \Sigma_1 = 1$ and $\dim \Sigma_2 = 2$. The construction goes as follows. Let $Q = \{1,2\}$, $L = \{q_1^k q_2 \mid k > 0\}$, $\mathcal{Y} = \mathcal{U} = \mathbb{R}$. Define $y : PC(T,\mathcal{U}) \times TL \to \mathcal{Y}$ by

$$y(u(.), \underbrace{q_1 \cdots q_1}_{m-times} q_2, t_1 \cdots t_m t_{m+1}) =$$
$$\int_0^{t_{m+1}} e^{2(t_{m+1}-s)} u(s + \sum_1^m t_i) ds + \int_0^{\sum_1^m t_i} e^{2t_{m+1}} e^{\sum_1^m t_i - s} u(s) ds$$

Define $\Sigma_1 = (\mathbb{R},\mathbb{R},\mathbb{R},Q,\{(A_{1,q},B_{1,q}C_{1,q}) \mid q \in \{q_1,q_2\}\})$ by

$$\begin{array}{lll} A_{1,q_1} = 1 & B_{1,q_1} = 1 & C_{1,q_1} = 1 \\ A_{1,q_2} = 2 & B_{1,q_2} = 1 & C_{1,q_2} = 1 \end{array}$$

Define $\Sigma_2 = (\mathbb{R}^2, \mathbb{R}, \mathbb{R}, Q\{(A_{2,q}, B_{2,q}, C_{2,q}) \mid q \in Q\})$ by

$$A_{2,q_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B_{2,q_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C_{2,q_1} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$A_{2,q_2} = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \quad B_{2,q_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C_{2,q_2} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Both Σ_1 and Σ_2 are reachable and observable as linear switched systems, therefore they are the minimal realizations of $y_{\Sigma_1}(0,.,.)$ and $y_{\Sigma_2}(0,.,.)$. Moreover, it is easy to see that

$$y_{\Sigma_1}(0,.,.)|_{PC(T,\mathcal{U})\times TL} = y = y_{\Sigma_2}(0,.,.)|_{PC(T,\mathcal{U})\times TL}$$

In fact, Σ_2 can be obtained by constructing the minimal representation of $\Psi_{\{y\}}$, i.e., Σ_2 is a minimal realization of y satisfying part (iii) of Theorem 15.

4.1.5 Proof of Theorem 9

Proof of Theorem 9. `only if part`

Assume that Φ has a generalized kernel representation. Then it is clear that for each $f \in \Phi$, f is causal, since for each $w = (q_1, t_1) \cdots (q_k, t_k) \in TL$ we get that $f_i(w, u) = e_i^T K^{f,\Phi}_{q_1 \cdots q_k}(t_1, \ldots, t_k) + \sum_{i=1}^k \int_0^{t_i} e_i^T G^{f,\Phi}_{q_i,\ldots,q_k}(t_i - s, \ldots, t_k) u(s + \sum_{j=1}^{i-1} t_j) ds$ $i = 1, \ldots, p$, that is, $f_i(w, u)$ depends only on $u|_{[0, \sum_1^k t_i]}$. It is also clear that the function $y^{\Phi} = y_0^{\Phi}$ defined by $y_0^{\Phi}(u, w) = \sum_{i=1}^k \int_0^{t_i} G^{f,\Phi}_{q_i,\ldots,q_k}(t_i - s, \ldots, t_k) u(s + \sum_{j=1}^{i-1} t_j) ds$ satisfies (4.9). Moreover, it is easy to see that $y_j^{\Phi}(w, .)$,$j = 1, \ldots, p$ is a continuous linear map from $PC([0, \sum_{j=1}^k t_j], \mathcal{U})$ to $\mathbb{R}^p$, since it is the sum of maps of the form $\phi_j : u \mapsto \int_0^{t_i} e_j^T G^{\Phi}_{q_i \cdots q_k}(t_i - s, \ldots, t_k) \mathrm{Shift}_{\sum_{j=1}^{i-1} t_j}(u)(s) ds$ $j = 1, \ldots, p$ and Shift_T is a continuous linear map on $PC(T, \mathcal{U})$, and $g_j(s) = e_j^T G^{\Phi}_{q_i \cdots q_k}(s, t_{i+1}, \ldots, t_k)$ is analytic, and thus the function $\widetilde{g}_j(s) = g_j(t_i - s)\chi(\{s \in [0, t_i]\})$ is in $L^{\infty}(T)$. But then $\phi_j(u) = \int_0^{t_i} \widetilde{g}_j(s) \mathrm{Shift}_{\sum_1^{i-1} t_i}(u)(s) ds$ and by [58] if follows that ϕ_j, $j = 1, \ldots, p$ is a a continuous linear map from $PC([0, \sum_1^k t_i], \mathcal{U})$ to $\mathbb{R}^p$ for Thus conditions 2 is satisfied. Let $z = (q_1, t_1) \cdots (q_h, t_h) \in (Q \times T)^+$, $w = (w_1, 0) \cdots (w_k, 0), v = (v_1, 0) \cdots (v_l, 0) \in (Q \times T)^*$. Let $x_1 = q_1 \cdots q_h$, $x_2 = w_1 \cdots w_k$ and $x_3 = v_1 \cdots v_l$. Assume that $wz, vz \in TL$. Then it is easy to see that $x_1 \in \mathrm{suffix} L$. Then $f(0, wz) = K^{f,\Phi}_{x_2 x_1}(0, \ldots, 0, t_1, \ldots, t_h) = K^{f,\Phi}_{x_1}(t_1, \ldots, t_h) = K^{f,\Phi}_{x_3 x_1}(0, \ldots, 0, t_1, \ldots, t_h)$. Notice that

$$\begin{aligned}
y_0^{\Phi}(u,wz) &= \sum_{i=1}^{k}\int_0^0 G^{\Phi}_{w_i\cdots w_k x_1}(\mathbb{O}_{l-i+1},\tau)u(s)ds+\\
&+\sum_{i=1}^{h}\int_0^{t_i} G^{\Phi}_{q_i\cdots q_h}(t_i-s,\dots,t_h)u_i(s)ds=\\
&=\sum_{i=1}^{h}\int_0^{t_i} G^{\Phi}_{q_i\cdots q_h}(t_i-s,\dots,t_h)u_i(s)ds=\\
&=\sum_{i=1}^{l}\int_0^{0} G^{\Phi}_{v_i\cdots v_l x_1}(\mathbb{O}_{l-i+1},\tau)u(s)ds+\\
&+\sum_{i=1}^{h}\int_0^{t_i} G^{\Phi}_{q_i\cdots q_h}(t_i-s,\dots,t_h)u_i(s)ds=\\
&=y_0^{\Phi}(u,vz)
\end{aligned}$$

where $\tau=(t_1,\dots,t_h)$, $\mathbb{O}_j=(0,0,\dots,0)\in\mathbb{N}^j, j=1,\dots,l$, $u_i=\mathrm{Shift}_{\sum_{j=1}^{i-1}t_i}(u)$. We get that $f(u,wz)=f(0,wz)+y_0^{\Phi}(u,wz)=f(0,vz)+y^{\Phi}(u,vz)=f(u,vz)$. That is, condition 3 is satisfied.

Let $w=(q_1,t_1)\cdots(q_k,t_k)\in TL$. It is also clear that if $z=(q_l,t_l)\cdots(q_k,t_k)$ and $1\le l\le k$, then

$$\begin{aligned}
y_0^{\Phi}(u,w) &= \sum_{i=l}^{k}\int_0^{t_i} G^{f,\Phi}_{q_i\cdots q_k}(t_i-s,\dots,t_k)\mathrm{Shift}_{T_{i-1,l}}(u_l)(s)ds+\\
&+\sum_{i=1}^{l-1}\int_0^{t_i} G^{f,\Phi}_{q_i,\dots,q_k}(t_i-s,\dots,t_k)u_{i-1}(s)ds = y_0^{\Phi}(u_l,(q_1,0)\cdots(q_{l-1},0)z)+\\
&+\sum_{i=1}^{k}\int_0^{t_i} G^{f,\Phi}_{q_i,\dots,q_k}(t_i-s,\dots,t_k)\mathrm{Shift}_{T_i}(v)(s)ds = y_0^{\Phi}(u_l,z)+y^{\Phi}(v,w)
\end{aligned}$$

where $T_i=\sum_{j=1}^{i-1}t_j$, $u_i=\mathrm{Shift}_{T_i}(u)$, $i=1,\dots,k$, $v=P_{T_l}u$, $T_{i,l}=\sum_{j=l}^{i}t_j$. That is, y^{Φ} satisfies condition 4. Let $w,v\in(Q\times T)^*$, and assume that $w(q,\tau_1)(q,\tau_2)v$, $w(q,\tau_1+\tau_2)v\in TL$. Assume that $w=(w_1,t_1)\cdots(w_l,t_l)$ and $v=(v_{l+1},t_{l+1})\cdots(v_k,t_k)$ where $v_i,w_j\in Q, i=l+1,\dots,k, j=1,\dots,l$. Let $T_i=\sum_{j=1}^{i}t_i$. Then using the properties of the functions $K^{f,\Phi}_z$, $G^{f,\Phi}_z, z \in \mathrm{suffix}L$ one gets.

$$\begin{aligned}
f(u,w(q,\tau_1)(q,\tau_2)v) &= K^{f,\Phi}_{wqqv}(t_1,\dots,t_l,\tau_1,\tau_2,\dots,t_k)+ \\
&\sum_{i=1}^{l}\int_0^{t_i} G^{\Phi}_{w_i\cdots w_l qqv}(t_i-s,\dots,\tau_1,\tau_2,\dots,t_k)u_i(s)ds + \\
&+\int_0^{\tau_1} G^{\Phi}_{qqv}(\tau_1-s,\tau_2,\dots,t_k)u_{l+1}(s)ds + y^{\Phi}_0(\mathrm{Shift}_{T_l+\tau_1+\tau_2}(u),v) + \\
&+\int_0^{\tau_2} G^{\Phi}_{qv}(\tau_2-s,\dots,t_k)u_{l+1}(s+\tau_1)ds = K^{f,\Phi}_{wqv}(t_1,\dots,t_l,\tau_1+\tau_2,\dots,t_k)+ \\
&\sum_{i=1}^{l}\int_0^{t_i} G^{\Phi}_{w_i\cdots w_l qv}(t_i-s,\dots,\tau_1+\tau_2,\dots,t_k)u_i(s)ds + \\
&+\int_0^{\tau_1+\tau_2} G^{\Phi}_{qv}(\tau_1+\tau_2-s,\dots,t_k)u_{l+1}(s)ds + y^{\Phi}_0(\mathrm{Shift}_{T_l+\tau_1+\tau_2}(u),v) = \\
&= f(u,w(q,\tau_1+\tau_2)v)
\end{aligned}$$

That is, Φ satisfies condition 5. If $|v|>0$, $w(q,0)v, wv \in TL$ and $w=(q_1,t_1)\cdots(q_l,t_l)$, $v=(q_{l+1},t_{l+1})\cdots(q_k,t_k)$, then we get that

$$\begin{aligned}
f(u,w(q,0)v) &= K^{f,\Phi}_{wv}(t_1,\dots,t_l,\dots,t_k)+ \\
&\sum_{i=1}^{l}\int_0^{t_i} G^{\Phi}_{w_i\cdots w_l qv}(t_i-s,\dots,t_l,0,\dots,t_k)\mathrm{Shift}_i(u)(s)ds \\
&+\int_0^{0} G^{\Phi}_{qv}(0-s,\dots,t_k)\mathrm{Shift}_l(u)(s)ds + y^{\Phi}_0(\mathrm{Shift}_{T_l+0}(u),v) = \\
&= K^{f,\Phi}_{wv}(t_1,\dots,t_l,\dots,t_k)+ \\
&\sum_{i=1}^{l}\int_0^{t_i} G^{\Phi}_{w_i\cdots w_l v}(t_i-s,\dots,t_k)\mathrm{Shift}_i(u)(s)ds + y^{\Phi}_0(\mathrm{Shift}_{T_l}(u),v) = \\
&f(u,wv)
\end{aligned}$$

where $T_i=\sum_{j=1}^{i-1}t_j$ and $\mathrm{Shift}_i=\mathrm{Shift}_{T_i}$, $i=1,\dots,k$. That is, Φ satisfies condition 5. Finally, it is easy to see that Φ satisfies condition 6. Indeed, $f_{q_1\cdots q_k,u_1\cdots u_k}(t_1,\dots,t_k) = K^{f,\Phi}_{q_1\cdots q_k}(t_1,\dots,t_k)+\sum_{i=1}^{k}(\int_0^{t_i} G^{\Phi}_{q_i\cdots q_k}(t_i-s,\dots,t_k)ds)u_i$. But by definition $K^{f,\Phi}_{q_1\cdots g_k}$ and $G^{\Phi}_{q_i\cdots q_k}$ are analytic, and thus $\int_0^{t_i} G^{\Phi}_{q_i\cdots q_k}(t_i-s,\dots,t_k)ds$ are analytic. That is, $f_{q_1\cdots q_k,u_1\cdots u_k}$ has to be analytic too.

`if part`

Assume that the set of maps Φ satisfies the conditions 1 – 6. First notice that condition 3 implies that each $f\in\Phi$ can be uniquely extended to a function in $F(PC(T,\mathcal{U})\times T(\mathrm{suffix}L),\mathcal{Y})$. From now on we will assume that $\Phi\subseteq F(PC(T,\mathcal{U})\times T(\mathrm{suffix}L),\mathcal{Y})$. Also notice that all the conditions 1-6 still hold for the extensions of elements of Φ to $F(PC(T,\mathcal{U}) \times T(\text{suffix}L), \mathcal{Y})$. Let $w = (q_1,t_1)\cdots(q_k,t_k) \in T(\text{suffix}L)$. We will construct function $K^{f,\Phi}_{q_l\cdots q_k}$ and $G^{f,\Phi}_{q_l\cdots q_k}$ for each $1 \le l \le k$. From condition 6 we get that for each $f \in \Phi$ it holds that $f_{q_1\cdots q_k,0\cdots 0} : T^k \to \mathcal{Y}$ is an analytic function. Let $K^{f,\Phi}_{q_l\cdots q_k}(t_l \cdots, t_k) = f_{q_1\cdots q_k,0\cdots 0}(0,0,\dots,0,t_l,t_{l+1},\dots,t_k)$. Then it is clear that $K^{f,\Phi}_{q_l\cdots q_k}$, $l = 1,\dots,k$ are analytic. Since f satisfies the condition 4 and 5 and $K^{f,\Phi}_{q_l\cdots q_k}(t_l,\dots,t_k = f((q_1,0)\cdots(q_{l-1},0)(q_l,t_l)\cdots(q_k,t_k),0)$ we get that $K^{f,\Phi}_{q_l\cdots q_k}$, $l = 1,\dots,k$ satisfies conditions 3 and 4 of Definition 10.

The definition of $G^{f,\Phi}_{q_l\cdots q_k}$ is a bit more involved. For each $l = 1,\dots,k$ $j = 1,\dots,p$ define the maps

$$y_{(q_l,t_l)\cdots(q_k,t_k),j} : PC([0,t_l],\mathcal{U}) \ni u \mapsto y^{\Phi}_j((q_1,t_1)\cdots(q_k,t_k),\widetilde{u})$$

where $\widetilde{u}(s) = \begin{cases} u(s-T_{l-1}) & \text{if } s \in [T_{l-1},T_l] \\ 0 & \text{otherwise} \end{cases}$ where $T_i = \sum_{j=1}^{i} t_j$. From condition 2 it follows that $y_{(q_l,t_l)\cdots(q_k,t_k),j}$ is a continuous linear functional on $PC([0,t_l],\mathcal{U})$. Since $PC([0,t_l],\mathcal{U})$ is dense in $L^1([0,t_l],\mathcal{U})$, we can extend it a unique way to a continuous linear functional on $L^1([0,t_l],\mathcal{U})$. By abuse of notation we will denote this functional by $y_{(q_l,t_l)\cdots(q_k,t_k),j}$ too. By Theorem 6.16 from [58] we get that there exists an a.s unique $g_{(q_l,t_l)\cdots(q_k,t_k),j} \in L^{\infty}([0,t_l],\mathbb{R}^{1\times m})$ such that

$$y_{(q_l,t_l)\cdots(q_k,t_k),j}(u) = \int_0^{t_l} g_{(q_l,t_l)\cdots(q_k,t_k),j}(s)u(s)ds$$

Let $y_w : u \mapsto \begin{bmatrix} y_{w,1}(u) & \cdots & y_{w,p}(u) \end{bmatrix}^T \in \mathbb{R}^p$ and define the map $g_w : s \mapsto \begin{bmatrix} (g_{w,1}(s))^T & \cdots & (q_{w,p}(s))^T \end{bmatrix}^T \in \mathbb{R}^{p\times m}$. Then

$$y_{(q_l,t_l)\cdots(q_k,t_k)}(u) = \int_0^{t_l} g_{(q_l,t_l)\cdots(q_k,t_k)}(s)u(s)ds$$

Note that if Φ satisfies conditions 1 – 6, then y^{Φ} satisfies conditions 3 - 6. We will use this fact to prove certain properties of $g_{(q_1,t_1)\cdots(q_k,t_k)}$.

For any $w,v \in (Q\times T)^*$,$|v| > 0$ one gets that if $v(q,\tau_1)(q,\tau_2)w$, $v(q,\tau_1+\tau_2)w \in T(\text{suffix}L)$, then it holds that $y_{v(q,\tau_1)(q,\tau_2)w}(u) = y^{\Phi}(\widetilde{u}, v(q,\tau_1)(q,\tau_2)w)$ $= y^{\Phi}(\widetilde{u}, v(q,\tau_1+\tau_2)w) = y_{v(q,\tau_1+\tau_2)w}(u)$. This implies that

$$g_{v(q,\tau_1)(q,\tau_2)w} = g_{v(q,\tau_1+\tau_2)w} \text{ a.s.} \tag{4.17}$$

Similarly, if $v(q,0)w, vw \in T(\text{suffix}L)$, $|w| > 0$, $|v| > 0$, then

$$y_{v(q,0)w}(u) = y^{\Phi}(\widetilde{u}, v(q,0)w) = y^{\Phi}(\widetilde{u}, vw) = y_{vw}(u)$$

which implies

$$g_{v(q,0)w} = g_{vw} \text{ a.s} \tag{4.18}$$

Moreover, if $(q,t_1)(q,t_2)w \in T(\text{suffix}L)$ and $(q,t_1+t_2)w \in T(\text{suffix}L)$, then for each $u \in PC([0,t_2],\mathcal{U})$ it holds that

$$y_{(q,t_1)(q,t_2)w}(u) = y^{\Phi}(\widetilde{u},(q,t_1)(q,t_2)w) = y^{\Phi}(\widetilde{u},(q,t_1+t_2)w) = \\ y_{(q,t_1+t_2)w}(u\#_{t_1}0) = \int_0^{t_1} g_{(q,t_1+t_2)w}(s)u(s)ds$$

By uniqueness of $g_{(q,t_1)(q,t_2)w}$ we get that

$$g_{(q,t_1)(q,t_2)w}(s) = g_{(q,t_1+t_2)w}(s) \text{ a.s. on } [0,t_1] \tag{4.19}$$

In addition, from condition 4 one gets for each $(q,t+s)w \in T(\text{suffix}L)$ that for each $u \in PC([0,s],\mathcal{U})$, $v \in PC([0,t+s],\mathcal{U})$, $v = 0\#_t u$,

$$y_{(q,t+s)w}(v) = y^{\Phi}(\widetilde{v},(q,t+s)w) = y^{\Phi}(\widetilde{v},(q,t)(q,s)w) = \\ y^{\Phi}(\text{Shift}_t\widetilde{v},(q,s)w) + y^{\Phi}(P_t\widetilde{v},(q,t)(q,s)w)$$

But $P_t\widetilde{v} = 0$ so $y^{\Phi}(P_t\widetilde{v},(q,t)(q,s)w) = 0$, and in addition $\text{Shift}_t\widetilde{v} = \widetilde{u}$, therefore we get $y_{(q,t+s)w}(v) = y^{\Phi}(\text{Shift}_t(\widetilde{v}),(q,s)w) = y_{(q,s)w}(u)$. That is,

$$y_{(q,s)w}(u) = \int_0^{t+s} g_{(q,t+s)w}(z)v(z)dz = \int_0^{s} g_{(q,t+s)w}(z+t)u(z)dz$$

From uniqueness of $g_{(q,s)w}$ we get

$$g_{(q,s)w}(\tau) = g_{(q,s+t)}(\tau+t) \text{ a.s} \tag{4.20}$$

From the equalities above we also get that we are free to change each of the maps $g_s, s \in T(\text{suffix}L)$ on some set of measure zero, so in fact we can choose the maps $g_s, s \in T(\text{suffix}L)$ is such a way that the formulas (4.17),(4.18), (4.19) and (4.20) holds not only almost surely, but exactly on the whole domain. If these equalities hold exactly, then $g_{(q,t)w}(s) = g_{(q,t-s)}(0)$. Let $q_l \cdots q_k \in \text{suffix}L$. Define $G_{q_l\cdots q_k} : T^k \to \mathbb{R}^{p\times m}$ by

$$G_{q_l\cdots q_k}(t_l,\ldots,t_k) = g_{(q_l,t_l)\cdots(q_k,t_k)}(0)$$

Formula (4.20) implies that $G_{q_l\cdots q_k}(t_l-s,\cdots,t_k) = g_{(q_l,t_l-s)\cdots(q_k,t_k)}(0) = g_{(q_l,t_l-s+s)\cdots(q_k,t_k)}(s) = g_{(q_l,t_l)\cdots(q_k,t_k)}(s)$. We immediately get that

$$y_{(q_l,t_l)\cdots(q_k,t_k)}(u) = \int_0^{t_l} G_{q_l\cdots q_k}(t_l-s,t_{l+1},\ldots,t_k)u(s)ds$$

Now, notice that for each $(q_1,t_1)\cdots(q_k,t_k) \in T(\text{suffix}L)$, by using condition 4 repeatedly, one can derive

$$y^{\Phi}(u,(q_1,t_1)\cdots(q_k,t_k)) = \sum_{i=1}^{k} y^{\Phi}(u_i,(q_i,t_i)\cdots(q_k,t_k))$$

where $u_i = P_{t_i}(\text{Shift}_{\sum_{j=1}^{i-1} t_j} u)$. That is, $u_i(s) = \begin{cases} u(s+\sum_{j=1}^{i-1} t_j) & \text{if } s \in [0,t_i] \\ 0 & \text{otherwise} \end{cases}$
That is, $u_i = \widetilde{v}_i$, $v_i = u_i|_{[0,t_i]} = (\text{Shift}_{\sum_{j=1}^{i-1} t_j} u)|_{[0,t_i]}$. Thus we get that for each $w = (q_1,t_1)\cdots(q_k,t_k) \in T(\text{suffix}L)$ and $u \in PC(T,\mathcal{U})$

$$y^{\Phi}(u,w) = \sum_{i=1}^{k} y_{(q_i,t_i)\cdots(q_k,t_k)}(v_i) = \sum_{i=1}^{k} \int_0^{t_i} G^{\Phi}_{q_i\cdots q_k}(t_i - s,\cdots t_k)u_i(s)ds$$

and

$$f(u,,w) = K^{f,\Phi}_{q_1\cdots q_k}(t_1,\ldots,t_k) + \sum_{i=1}^{k} \int_0^{t_i} G^{\Phi}_{q_i\cdots q_k}(t_i - s,\cdots t_k)u_i(s)ds \qquad (4.21)$$

where $u_i = \text{Shift}_{\sum_{j=1}^{i-1} t_j}(u)$. We already showed that $K^{f,\Phi}_w$ $w \in \text{suffix}L$ satisfies the conditions 1, 2 and 3 of Definition 10. Equalities (4.17),(4.18), (4.19) and (4.20) imply that G^{Φ}_w satisfies the conditions 2 and 3 too. Equation (4.21) implies that part 4 of Definition 10 is satisfied too. It is left to show that G^{Φ}_w can be chosen to be analytic for each $f \in \Phi$ and $w \in \text{suffix}L$. Assume that $w = q_1\cdots q_k$. Then condition 6 implies that the function $h_{u_1\cdots u_k} = f_{q_1\cdots q_k,u_1\cdots u_k} - f_{q_1\cdots q_k,0\cdots 0}$ is analytic for each $u_1,\cdots u_k \in PC(T,\mathcal{U})$ constant functions. But

$$h_{u_1\cdots u_k}(t_1,\ldots,t_k) = f(u,w) - f(0,w) = y^{\Phi}(u,w)$$

where $u(t) = u_i$ if $t \in (T_{i-1},T_i]$, $i = 1,\ldots,k$, $T_i = \sum_{j=1}^{i} t_j$. But then we get that

$$h_{u_1\cdots u_k}(t_1,\ldots,t_k) = \sum_{i=1}^{k} (\int_0^{t_i} G^{\Phi}_{q_i\cdots q_k}(t_i - s,t_{i+1},\ldots,t_k)ds)u_i$$

For each $i = 1\ldots,k$ taking $u_l = 0, j \neq l$ and $u_j = e_z = (0,0,\ldots,1,0,\ldots,0)^T$ we get that $h_{z,q_j\cdots q_k}(t_j,\ldots,t_k) := \int_0^{t_j} G^{\Phi}_{q_j\cdots q_k}(t_j - s,t_{j+1},\ldots,t_k)e_z ds$ is an analytic map. But $h_{z,q_j\cdots q_k}(0,t_{j+1},\ldots,t_k) = 0$, thus

$$h_{z,q_j\cdots q_k}(t_j,\ldots,t_k) = \int_0^{t_j} \frac{d}{ds} h_{z,q_j\cdots q_k}(t_j - s,\ldots,t_k)ds$$

Let $w(s) = G_{q_j\cdots q_k}(s,t_{j+1},\ldots,t_k)e_z - \frac{d}{ds}h_{z,q_j\cdots q_k}(s,t_{j+1},\ldots,t_k)$. That is, for each $t \in T$ we get that $\int_0^t w(t-s)ds = 0$, or equivalently $\int_0^t w(s)ds = 0$. It implies that

$\int_E w(s)ds = 0$ for each Borel-set $E \subseteq [0, N]$, $N \in \mathbb{N}$. Then we get that w=0 a.s., that is, $G_{q_j \cdots q_k}(t, t_{j+1}, \dots t_k)e_z = \frac{d}{dt_j} h_{z,q_j \cdots q_k}(s, t_{j+1}, \dots, t_k)$ for almost all s. For each $w \in \text{suffix}L$ let $h_w = (h_{1,w}, \dots, h_{m,w})$. It is easy to see that h_w are analytic and $G^{\Phi}_w(t_1, \dots, t_{|w|}) = h_w(t_1, \dots, t_{|w|})$ a.s. in t_1. That is, the set

$$A_w(t_2, \dots, t_{|w|}) = \{t \in T \mid G^{\Phi}_w(t, t_2, \dots, t_{|w|}) \neq h_w(t, t_2, \dots, t_{|w|})\}$$

is of measure zero. Thus, for any $a \in A_w(t_2, \dots, t_{|w|})$ there exists $x_n \notin A_w(t_2, \dots, t_{|w|})$, $\lim x_n = a$. Since h_w is continuous, it implies that h_w satisfies the conditions 2, 3, 4 of Definition 10, if G^{Φ}_w does. That is, we can take $G^{\Phi}_w := h_w$ and the resulting functions will satisfy the requirements for generalized kernel representation. We define the functions G^{Φ}_w and $K^{f,\Phi}_v$ only for $w \in \text{suffix}L, v \in L$. But it is easy to see that $\{G^{\Phi}_w, K^{f,\Phi}_w \mid f \in \Phi, w \in \widetilde{L}\}$ is uniquely determined by $\{G^{\Phi}_w, K^{f,\Phi}_v \mid f \in \Phi, w \in \text{suffix}L, v \in L\}$.

It is left to show that generalized kernel representations are unique. Assume that $\{K^{f,\Phi}_w, G^{\Phi}_w\}$ and $\{\widetilde{K}^{f,\Phi}_w, \widetilde{G}^{\Phi}_w\}$ are two different generalised kernel representations of Φ. By the remark above it is enough to show that $K^{f,\Phi}_w = \widetilde{K}^{f,\Phi}_w$ for each $w \in L, f \in \Phi$ and $G^{\Phi}_w = \widetilde{G}^{\Phi}_w$ $w \in \text{suffix}L$. There are two ways to proceed. One can use formula 4.4 to conclude that $\forall w \in L, \alpha \in \mathbb{N}^{|w|} : D^{\alpha}K^{f,\Phi}_w = D^{\alpha}\widetilde{K}^{f,\Phi}_w = D^{\alpha}f(0, w, .)$, and $\forall w \in \text{suffix}L, \alpha \in \mathbb{N}^{|w|}, j = 1, \dots, m, v \in Q^*, vw \in L :$ $D^{\alpha}G^{\Phi}_w e_j = D^{\alpha}\widetilde{G}^{\Phi}_w e_j = D^{(\mathbb{O}_{|v|}, \alpha^+)} y^{f,\Phi}_0(e_j, vw, .)$, where $\mathbb{O}_l = (0, 0, \dots, 0) \in \mathbb{N}^l, l \geq 0$, $\alpha^+ = (\alpha_1 + 1, \alpha_2, \dots, \alpha_k)$ for each $\alpha \in \mathbb{N}^k, k \geq 0$. That is, we get that the high-order derivatives at zero of $K^{f,\Phi}_w$ and $G^{f,\Phi}_w$ equal the respective high-order derivatives at zero of $\widetilde{K}^{f,\Phi}_w$ and $\widetilde{G}^{\Phi}_w$ respectively. Since $K^{f,\Phi}_w, G^{\Phi}_w, \widetilde{K}^{f,\Phi}_w, \widetilde{G}^{\Phi}_w$ are analytic, we get the required equalities.

Alternatively, we could use the proof of existence of a generalized kernel representation. Notice that $f(0, (q_1, t_1) \cdots (q_k, t_k)) = K^{f,\Phi}_{q_1 \cdots q_k}(t_1, \dots, t_k) = \widetilde{K}^{f,\Phi}_{q_1 \cdots q_k}(t_1, \dots, t_k)$ for all
$(q_1, t_1) \dots (q_k, t_k) \in T(\text{suffix}L)$ and $f \in \Phi$. On the other hand, from the proof above we can easily deduce that for each $w \in \text{suffix}L$. $G^{\Phi}_w = \widetilde{G}^{\Phi}_w$ almost everywhere, that is, $r_w = G^{\Phi}_w - \widetilde{G}^{\Phi}_w = 0$ a.s. But r_w is analytic, and if $r_w \neq 0$, then there exists an open set V such that $\forall v \in V : r_w(v) \neq 0$. But no non-empty open set is of measure zero, so we get that r_w is the constant zero function. But then $G^{\Phi}_w = \widetilde{G}^{\Phi}_w$. $\square$

4.2 Realization Theory of Bilinear Switched Systems

This section deals with the realization theory of bilinear switched systems. First, in Subsection 4.2.1 definition and certain elementary properties of bilinear switched systems will be presented. Then, in Subsection 4.2.2 the structure of the input/output maps of bilinear switched systems will be discussed. Subsection 4.2.3 presents the realization theory for bilinear switched systems for the case of arbitrary switching. Subsection 4.2.4 deals with realization theory for the case of switching with constraints.

4.2.1 Bilinear Switched Systems

Recall from Section 2.4 the definition of bilinear switched systems. That is, a switched system $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{f_q \mid q \in Q, u \in \mathcal{U}\},\{h_q \mid q \in Q\})$ is called bilinear if for each $q \in Q$ there exist linear mappings $A_q : \mathcal{X} \to \mathcal{X}$, $B_{q,j} : \mathcal{X} \to \mathcal{X}$, $j = 1, 2, \dots, m$, $C_q : \mathcal{X} \to \mathcal{Y}$ such that

- $\forall x \in \mathcal{X}, u = (u_1, \dots, u_m)^T \in \mathcal{U} = \mathbb{R}^m : f_q(x, u) = A_q x + \sum_{j=1}^{m} u_j B_{q,j} x$
- $\forall x \in \mathcal{X} : h_q = C_q x$.

Recall that we agreed on using the following shorthand notation

$$\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,\{B_{q,j}\}_{j=1,2,\dots,m},C_q) \mid q \in Q\})$$

to denote bilinear switched systems. Recall from [32, 33] that the state- and output-trajectory of a bilinear system can be expressed as infinite series of iterated integrals. A similar representation exists for switched bilinear systems. In order to formulate such a representation some notation has to be set up. Recall from Subsection 2.6 the notion of iterated integral $V_{w_1,\dots,w_k}[u](t_1, \dots, t_k)$ of $u \in PC(T,\mathcal{U})$ with respect to $w_1, \dots, w_k$.

For each $q \in Q$ and $w = j_1 \cdots j_k, k \geq 0, j_1, \cdots j_k \in \mathrm{Z}_m$ let us introduce the following notation

$$B_{q,0} := A_q,\ B_{q,\epsilon} := Id_{\mathcal{X}},\ ,B_{q,w} := B_{q,j_k} B_{q,j_{k-1}} \cdots B_{q,j_1}$$

where $Id_{\mathcal{X}}$ denotes the identity map on $\mathcal{X}$. With the notation above we can formulate the following result.

Proposition 13. *Using the notation above, for each $x_0 \in \mathcal{X}$, $u \in PC(T,\mathcal{U})$ and $s = (q_1,t_1)\cdots(q_k,t_k) \in (Q \times T)^*$ the state $x_\Sigma(x_0,u,s)$ and the output $y_\Sigma(x_0,u,s)$ can be expressed by the following absolutely convergent series.*

$$x_\Sigma(x_0,u,s) = \sum_{w_1,\ldots,w_k \in \mathrm{Z}_m^*} (B_{q_k,w_k}\cdots B_{q_1,w_1}x_0)V_{w_1,\ldots,w_k}[u](t_1,\ldots,t_k) \quad (4.22)$$

$$y_\Sigma(x_0,u,s) = \sum_{w_1,\ldots,w_k \in \mathrm{Z}_m^*} (C_{q_k}B_{q_k,w_k}\cdots B_{q_1,w_1}x_0)V_{w_1,\ldots,w_k}[u](t_1,\ldots,t_k)$$

Proof. To show absolute convergence of the series we will use the notion of a convergent generating series defined in Section 4.2.2. Using the notation of Section 4.2.2 define the series $c_{x_0} : \widetilde{\Gamma}^* \to \mathcal{X}$ by $c_{x_0}((q_1,w_1)\cdots(q_k,w_k)) = B_{q_k,w_k}\cdots B_{q_1,w_1}x_0$. Then $||c_{x_0}|| \leq ||x_0||M^{\sum_{i=1}^k |w_i|}$, where $M = \max\{||B_{q,j}|| \mid q \in Q, j \in \mathrm{Z}_m\}$. That is, c_{x_0} is a convergent generating series and by Lemma 20 the series

$$F_{c_{x_0}}(u,s) = \sum_{w_1,\ldots,w_k} \in (B_{q_k,w_k}\cdots B_{q_1,w_1}x_0)V_{w_1,\ldots,w_k}[u](t_1,\ldots,t_k)$$

is absolutely convergent, which also implies the absolute convergence of

$$\sum_{w_1,\ldots,w_k \in \mathrm{Z}_m^*} (C_{q_k}B_{q_k,w_k}\cdots\cdots B_{q_1,w_1}x_0)V_{w_1,\ldots,w_k}[u](t_1,\ldots,t_k)$$

It is left to show that the right-hand sides of (4.22) equal the respective left-hand sides. We will proceed by induction on k. If $k = 1$, then $x_\Sigma(x_0,u,(q_1,t))$ is the state under input u at time t with initial state x_0 of the bilinear system $\frac{d}{dt}x(t) = A_{q_1}x(t) + \sum_{j=1}^m (B_{q_1,j}x)u_j$. By classical results [32] on bilinear systems

$$x_\Sigma(x_0,u,(q_1,t)) = \sum_{w \in \mathrm{Z}_m^*} B_{q,w}x_0V_w[u](t)$$

and the series $\sum_{w \in \mathrm{Z}_m^*} B_{q,w}x_0V_w[u](t)$ is absolutely convergent. Assume that the statement of the proposition is true for all $k \leq N$. Notice that for each $s = (q_1,t_1)\cdots(q_N,t_N) \in (Q \times T)^*$ it holds that

$$x_\Sigma(x_0,u,s(q_{N+1},t_{N+1})) = x_\Sigma(x_\Sigma(x_0,\mathrm{Shift}_{\sum_1^N t_i}(u),s),(q_{N+1},t_{N+1}))$$

Using the induction hypothesis one gets

$$\begin{aligned} x_\Sigma(x_0,u,s(q_{N+1},t_{N+1}) &= \sum_{w_{N+1} \in \mathrm{Z}_m^*} B_{q_{N+1},w_{N+1}}x_\Sigma(x_0,u,s)V_{w_{N+1}}[u_N](t_{N+1}) \\ &= \sum_{w_{N+1} \in \mathrm{Z}_m^*} B_{q_{N+1},w_{N+1}}V_{w_{N+1}}[u_N](t_{N+1}) \times \\ &\times [\sum_{w_1,\ldots,w_N \in \mathrm{Z}_m^*} B_{q_N,w_N}\cdots B_{q_1,w_1}x_0V_{w_1,\ldots,w_N}[u](t_1,\ldots,t_N)\ ] = \\ &= \sum_{w_1,\ldots,w_{N+1} \in \mathrm{Z}_m^*} B_{q_{N+1},w_{N+1}}\cdots B_{q_1,w_1}x_0V_{w_1,\ldots,w_{N+1}}[u](t_1,\ldots,t_{N+1}) \end{aligned}$$

where $u_N = \mathrm{Shift}_{\sum_{i=1}^{N} t_i}(u)$. The rest of the statement of the proposition follows easily from the fact that

$$y_\Sigma(x_0, u, (q_1, t_1) \cdots (q_k, t_k)) = C_{q_k} x_\Sigma(x_0, u, (q_1, t_1) \cdots (q_k, t_k))$$

□

Reachability and observability properties of bilinear switched systems can be easily derived from the formulas above.

Proposition 14. *Let* $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\ldots,m}, C_q) \mid q \in Q\})$ *be a bilinear switched system. Then the following holds.*

(i) The linear span $W(\mathcal{X}_0) = \mathrm{Span}\{z \in \mathcal{X} \mid x \in Reach(\mathcal{X}_0, \Sigma)\}$ *of the states reachable from* $\mathcal{X}_0 \subseteq \mathcal{X}$ *is of the following form*

$$W(\mathcal{X}_0) =$$
$$\mathrm{Span}\{B_{q_k,w_k} \cdots B_{q_1,w_1} x_0 \mid q_k, \ldots q_1 \in Q, k \geq 0, w_k, \ldots, w_1 \in \mathrm{Z}_m^*, x_0 \in \mathcal{X}_0\}$$

(ii) Define the observability kernel O_Σ of Σ *by*

$$O_\Sigma = \bigcap_{q_1,\ldots,q_k \in Q, k \geq 0, w_1, \ldots, w_k \in \mathrm{Z}_m^*} C_{q_k} B_{q_k,w_k} \cdots B_{q_1,w_1}$$

$x_1, x_2 \in \mathcal{X}$ *are indistinguishable if and only if*

$$x_1 - x_2 \in O_\Sigma$$

Σ *is observable if and only if*

$$O_\Sigma = \{0\}$$

Proof. `Part (i)`
For each $\mathcal{X}_0 \subseteq \mathcal{X}$, $q_1, \ldots, q_k \in Q$ define the set $W_{q_1 \cdots q_k}(\mathcal{X}_0) \subseteq \mathcal{X}$ as

$$\mathrm{Span}\{x_\Sigma(x_0, u, (q_1, t_1) \cdots (q_k, t_k)) \mid u \in PC(T, \mathcal{U}), t_1, \ldots, t_k \in T, x_0 \in \mathcal{X}_0\}$$

Notice that $x_\Sigma(x_0, u, (q_1, t_1) \cdots (q_k, t_k)) = x_\Sigma(x_\Sigma(x_0, u, s), \mathrm{Shift}_{T_s}(u), (q_k, t_k))$ where $s = (q_1, t_1) \cdots (q_{k-1}, t_{k-1})$, $T_s = \sum_{i=1}^{k-1} t_i$. Using the fact that in the discrete mode q_k the system Σ behaves like a bilinear system and using the results from [32, 33] one gets that for each fixed $s = (q_1, t_1) \cdots (q_{k-1}, t_{k-1}) \in (Q \times T)^*$ and $u \in PC([0, \sum_1^{k-1} t_j], \mathcal{U})$ it holds that

$$W_{q_k}(\{x_\Sigma(x_0, u, s)\}) = \mathrm{Span}\{B_{q_k,w} x_\Sigma(x_0, u, s) \mid w \in \mathrm{Z}_m^*\}$$

That is,

$$W_{q_1,\ldots,q_k}(\mathcal{X}_0) = \mathrm{Span}\{B_{q_k,w}x \mid x \in W_{q_1,\ldots,q_{k-1}}(\mathcal{X}_0), w \in \mathrm{Z}_m^*\}$$

Taking into account that by [33] $W_q(\mathcal{X}_0) = \mathrm{Span}\{B_{q,w}x_0 \mid x_0 \in \mathcal{X}_0\}$ and $\mathrm{Span}\{x \mid x \in Reach(\Sigma, \mathcal{X}_0) = \mathrm{Span}\{x \mid x \in W_{q_1,\ldots,q_k}(\mathcal{X}_0), q_1, \ldots, q_k \in Q, k \geq 0\}$, the statement of the proposition follows.

`Part (ii)`

It is easy to deduce from (4.22) of Proposition 13 that $y_\Sigma(x, ., .)$ is linear in x, that is, $y_\Sigma(\alpha x_1 + \beta x_2, ., .) = \alpha_1 y_\Sigma(x_1, .,) + \beta y_\Sigma(x_2, ., .)$ That is, $y_\Sigma(x_1, ., .) = y_\Sigma(x_2, ., .)$ is equivalent to $y_\Sigma(x_1 - x_2, ., .) = 0$. Thus, it is enough to show that

$$x \in O_\Sigma \iff y_\Sigma(x, ., .) = 0$$

It is clear from Proposition 13 that $x_1 - x_2 \in O_\Sigma \implies y_\Sigma(x_1 - x_2, ., .) = 0$. It is left to show that $y_\Sigma(x, ., .) = 0 \implies x \in O_\Sigma$. Assume that $y_\Sigma(x, ., .) = 0$. Then for each fixed $w = (q_1, t_1) \cdots (q_k, t_k) \in (Q \times T)^*$, $u \in PC(T,\mathcal{U})$, $q \in Q$ it holds that $y_\Sigma(x_\Sigma(x, u, w), v, (q, t)) = y_\Sigma(x, u\#_{T_w} v, w(q, t)) = 0$ for any $v \in PC(T,\mathcal{U})$, where $T_w = \sum_1^k t_i$. Notice that for any $x_0 \in \mathcal{X}$ the map $PC(T,\mathcal{U}) \times T \ni (v, t) \mapsto y_\Sigma(x_0, v, (q, t))$ is the input-output map of the classical bilinear system $\frac{d}{dt}x(t) = A_q x + \sum_{j=1}^m u_j(t)(B_{q,j}x(t)), y(t) = C_q x(t)$ induced by the initial condition x_0. Thus by the classical result for bilinear systems, see [32], $y_\Sigma(x_\Sigma(x, u, w), v, (q, t)) = 0, \forall v \in PC(T,\mathcal{U})$ implies

$$x_\Sigma(x, u, w) \in \bigcap_{v \in \mathrm{Z}_m^*} \ker C_q B_{q,v}$$

Recall from the proof of part (i) the definition of $W_{q_1,\ldots,q_k}(\{x\})$. Since the choice of u and $t_1, \ldots, t_k$ are arbitrary, we get that $W_{q_1,\ldots,q_k}(\{x\}) \subseteq \bigcap_{v \in \mathrm{Z}_m^*} \ker C_q B_{q,v}$. Using the proof of part (i) we get that $W_{q_1,\ldots,q_k}(\{x\}) = \mathrm{Span}\{B_{q_k,w_k} \cdots B_{q_1,w_1}x \mid w_1, \ldots, w_k \in \mathrm{Z}_m^*\}$ which implies that

$$x \in \bigcap_{w, w_1, \ldots, w_k \in \mathrm{Z}_m^*} \ker C_q B_{q,w} B_{q_k,w_k} \cdots B_{q_1,w_1}$$

Since the choice of q and $q_1, \ldots, q_k \in Q$ is arbitrary, we get that $x \in O_\Sigma$. This completes the proof of the proposition. □

Let

$$\Sigma_1 = (\mathcal{X}_1, \mathcal{U}, \mathcal{Y}, Q, \{(A_q^1, \{B_{q,j}^1\}_{j=1,2,\ldots,m}, C_q^1) \mid q \in Q\})$$

and

$$\Sigma_2 = (\mathcal{X}_2, \mathcal{U}, \mathcal{Y}, Q, \{(A_q^2, \{B_{q,j}^2\}_{j=1,2,\ldots,m}, C_q^2) \mid q \in Q\})$$

be two bilinear switched systems. A linear map $T : \mathcal{X}_1 \to \mathcal{X}_2$ is called a *bilinear switched system morphism* from Σ_1 to Σ_2, denoted by $T : \Sigma_1 \to \Sigma_2$, if the following holds

$$TA^1_q = A^2_q T \quad C^1_q = C^2_q T \quad TB^1_{q,j} = B^2_{q,j}$$

By abuse of terminology T is said to be a bilinear switched system morphism from (Σ, μ) to $(\Sigma^{'}, \mu^{'})$, denoted by $T : (\Sigma, \mu) \to (\Sigma^{'}, \mu^{'})$, if $T : \Sigma \to \Sigma^{'}$ is a bilinear switched system morphism in the above sense and $T \circ \mu = \mu^{'}$. If T is a linear isomorphisms then (Σ_1, μ_1) and (Σ_2, μ_2) are said to be *isomorphic* or *algebraically similar*.

Note that switched systems defined above can be viewed as general non-linear systems with discrete inputs. In particular, bilinear switched systems can be viewed as ordinary bilinear systems with particular inputs. Indeed, let $Q = \{q_1, \dots, q_N\}$ and let $\widetilde{\mathcal{U}} = \mathbb{R}^N \oplus (\mathcal{U} \otimes \mathbb{R}^N)$. Denote the standard basis of $\mathbb{R}^N$ by $e_j, j = 1, \dots N$. We will denote e_j by e_{q_j}. Let $b_j, j = 1, \dots, m$ the standard basis of $\mathcal{U}$. Any $\widetilde{u} \in \widetilde{\mathcal{U}}$ has a unique representation $\widetilde{u} = \sum_{q \in Q} \widetilde{u}_q e_q + \sum_{j=1,\dots,m, q \in Q} \widetilde{u}_{j,q} b_j \otimes e_q$,

Consider the bilinear switched system $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\dots,m}, C_q) \mid q \in Q\})$. Define the following bilinear system with input space $\widetilde{\mathcal{U}}$ and output space $\mathcal{Y}$

$$\begin{aligned} \frac{d}{dt}x(t) &= \sum_{q \in Q} \widetilde{u}_q(t)(A_q x) + \sum_{q \in Q, j=1,\dots,m} \widetilde{u}_{q,j}(t)(B_{q,j} x) \\ y(t) &= \sum_{q \in Q} \widetilde{u}_q(t)(C_q x) \end{aligned}$$

Here $\widetilde{u}(t) \in \widetilde{\mathcal{U}}$ denoted the continuous input. The bilinear system above simulates Σ in the following sense. Let $w = (q_1, t_1) \cdots (q_k, t_k) \in (Q \times T)^+$, $u \in PC(T, \mathcal{U})$. Define $U_{u,w} := \widetilde{u} \in PC(T, \widetilde{\mathcal{U}})$ such that for each $i = 0, \dots, k-1$ $\forall \tau \in [\sum_{j=1}^{i} t_j, \sum_{j=1}^{i+1} t_j]$: $\widetilde{u}_{q_{i+1}}(\tau) = 1, \widetilde{u}_{q_{i+1},j}(\tau) = u_j(\tau)$ and $\widetilde{u}_q(\tau) = 0, \widetilde{u}_{j,q}(\tau) = 0, q \neq q_{i+1}$. Then $y_{\Sigma}(x, u, w)$ equals the output of the bilinear system above induced by $\widetilde{u}$ and initial state x. Using the correspondence above, one could try to reduce the realization problem for bilinear switched systems to the realization problem for classical bilinear systems and use the existing results on the realization theory of bilinear systems. In this paper we will not pursue this approach. The reason for that is the following. First, dealing with restricted switching would require dealing with the realization problem of bilinear systems with input constraints. The author is not aware of any work on this topic. Second, the author thinks that using bilinear realization theory would not substantially simplify the solution to realization problem for bilinear switched systems. Notice however, that the equivalence of realization problems mentioned above does explain the role of rational formal power series in realization theory of bilinear switched systems.

4.2.2 Input-output Maps of Bilinear Switched Systems

Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$ be a set of input-output maps defined for sequences of discrete modes belonging to $L \subseteq Q^{+}$. Let $\widetilde{\Gamma} = Q \times \mathbb{Z}_m^*$. Define the set

$$JL = \{(q_1,w_1)\cdots(q_k,w_k) \in \widetilde{\Gamma}^* \mid (q_1,w_1),\dots,(q_k,w_k) \in \widetilde{\Gamma}, k \geq 0, q_1\cdots q_k \in L\}$$

Define the relation $R \subseteq \widetilde{\Gamma}^* \times \widetilde{\Gamma}^*$ by requiring that $(q,w_1)(q,w_2)R(q,w_1w_2)$, and $(q,\epsilon)(q^{'},w)R(q^{'},w)$ hold for any $q \in Q, (q^{'},w) \in \widetilde{\Gamma}$ and $(q,w_1),(q,w_2) \in \widetilde{\Gamma}$. Let R^* be smallest congruence relation containing R. That is, R^* is the smallest relation such that $R \subseteq R^*$, R^* is symmetric, reflexive, transitive and $(v,v^{'}) \in R^*$ implies $(wvu, wv^{'}u) \in R^*$, for each $w,u \in \widetilde{\Gamma}^*$.

Definition 11 (Generating convergent series on JL). *A $c : JL \to \mathcal{Y}$ is called a generating convergent series on JL if the following conditions hold.*

(1) $(w,v) \in R^*, w,v \in JL \implies c(w) = c(v)$

(2) There exists $K, M > 0$ such that for each $(q_1,w_1)\cdots(q_k,w_k) \in JL$ and $(q_1,w_1)\dots(q_k,w_k) \in \widetilde{\Gamma}$

$$||c((q_1,w_1)\cdots(q_k,w_k))|| < KM^{|w_1|}\cdots M^{|w_k|}$$

The notion of generating convergent series is an extension of the notion of convergent power series from [67, 32]. If $|Q| = 1$ then a generating convergent series in the sense of Definition 11 can be viewed as a convergent formal power series in the sense of [67, 32].

Let $c : JL \to \mathcal{Y}$ be a generating convergent series. For each $u \in PC(T,\mathcal{U})$ and $s = (q_1,t_1)\cdots(q_k,t_k) \in TL$ define the series $F_c(u,s)$ by

$$F_c(u,s) = \sum_{w_1,\dots,w_k \in \mathbb{Z}_m^*} c((q_1,w_1)\cdots(q_k,w_k))V_{w_1,\dots,w_k}[u](t_1,\dots,t_k)$$

Notice that each generalised converge generating series $c : JL \to \mathcal{Y}$ determines a abstract globally convergent generating series $c_{abs} : I \to \mathcal{Y}$, where $I_k = Q^k \cap L$, $k \geq 0$, $I = \bigcup_{k=1}^{\infty} I_k \times (\mathbb{Z}_m^*)^k$ and $c_{abs}(((q_1,\dots,q_k),(w_1,\dots,w_k))) = c((q_1,w_1)\cdots(q_k,w_k))$

It is easy to see that c_{abs} is indeed an abstract globally convergent generating series. Indeed, for any $i = q_1\cdots q_k \in I_k$, $k \geq 1$ let $K_i = K$ and let M be the same as in Definition 11. Then for any $w_1, \ldots, w_k \in Z_m^*$

$$||c_{abs}(i, (w_1, \ldots, w_k))|| = ||c((q_1, w_1) \cdots (q_k, w_k))|| < KM^{|w_1|} \cdots M^{|w_k|} = \\ = K_i M^{|w_1|} \cdots M^{|w_k|}$$

Thus, c_{abs} is indeed an abstract globally convergent generating series.

Moreover, it follows that $F_c(u, s) = F_{c_{abs}}(u, ((q_1, \ldots, q_k), (t_1, \ldots, t_k)$. Thus, Lemma 1 implies the following.

Lemma 20. *If $c : JL \to \mathcal{Y}$ is a convergent generating series, then for each $u \in PC(T,\mathcal{U})$, $s = (q_1, t_1) \cdots (q_k, t_k) \in TL$ the series $F_c(u, s)$ is absolutely convergent.*

In fact we can define a function $F_c \in F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$ by

$$F_c : PC(T,\mathcal{U}) \times TL \ni (u, w) \mapsto F_c(u, w) \in \mathcal{Y}$$

The map F_c has some remarkable properties, listed below.

Lemma 21. *Let $c : JL \to \mathcal{Y}$ be a generating convergent series. Then the following holds.*

(i) For each $s = (q_1, t_1) \cdots (q_k, t_k) \in TL$, $u, v \in PC(T,\mathcal{U})$

$$(\forall t \in [0, \sum_1^k t_i] : u(t) = v(t)) \implies F_c(u, s) = F_c(v, s)$$

(ii) $\forall u \in PC(T,\mathcal{U}), w, s \in (Q \times T)^*, |s| > 0 :$

$$w(q, 0)s, ws \in TL \implies F_c(u, w(q, 0)s) = F_c(u, ws)$$

(iii) $\forall u \in PC(T,\mathcal{U}), w, v \in (Q \times T)^* :$

$$r = w(q, t_1)(q, t_2)v, \quad p = w(q, t_1 + t_2)v \in TL \implies F_c(u, r) = F_c(u, p)$$

(iv) Let $w = (w_1, 0) \cdots (w_k, 0), v = (v_1, 0) \cdots (v_l, 0) \in (Q \times T)^*$ and $s = (q_1, t_1) \cdots (q_h, t_h) \in (Q \times T)^+$

$$ws, vs \in TL \implies (\forall u \in PC(T,\mathcal{U}) : F_c(u, ws) = F_c(u, vs))$$

Proof. Part (i) and (ii) follow from the obvious facts that $V_w[u](t)$ depends only on $u|_{[0,t]}$ and $V_w[u](0) = 0$ for $|w| > 0$. Part (iv) follows from the fact that $V_w[u](0) = 0$ for $|w| > 0$ and thus $V_{w_1,\ldots,w_{k+h}}[u](0, \ldots, 0, t_1, \ldots, t_h) = 0$ if $\exists j \in \{1, \ldots, k\} : |w_j| \geq 0$, and

$$V_{w_1,\ldots,w_{k+h}}[u](0, \ldots, 0, t_1, \ldots, t_h) = V_{w_{k+1},\ldots,w_{k+h}}[u](t_1, \ldots, t_h)$$

if $w_{k+1} = \cdots = w_{k+h} = \epsilon$. The proof of Part (iii) is more involved. Recall Lemma 4. Using the lemma above and assuming that $w = (q_1, \tau_1)\cdots(q_i, \tau_i)$, $s = (q_{i+1}, \tau_{i+1})\cdots(q_k, \tau_k)$, $k \geq 0$, $\mathrm{T}_z = \sum_{j=1}^{z-1} t_j$ if $z \leq i$, $\hat{\mathrm{T}}_i = \sum_{j=1}^{i} t_i$ and $\mathrm{T}_{l+i} = \hat{\mathrm{T}}_i + t_1 + t_2 + \sum_{j=i+1}^{l+i-1} \tau_j$ we get

$$\begin{aligned}
F_c(u,r) = & \sum_{w_1,\ldots,w_k,s,z \in \mathrm{Z}_m^*} c((q_1,w_1)\cdots(q_i,w_i)(q,s)(q,z)(q_{i+1},w_{i+1})\cdots(q_k,w_k))\times \\
& \times V_s[\mathrm{Shift}_{\hat{\mathrm{T}}_i}(u)](t_1)V_z[\mathrm{Shift}_{t+\hat{\mathrm{T}}_i}(u)](t_2)\Pi_{j=1}^{k} V_{w_j}[\mathrm{Shift}_{\mathrm{T}_j}(u)](\tau_j) = \\
& = \sum_{w_1\ldots,w_k \in \mathrm{Z}_m^*} \sum_{w \in \mathrm{Z}_m^*} [c((q_1,w_1)\cdots(q_i,w_i)(q,w)(q_{i+1},w_{i+1})\cdots(q_k,w_k)) \times \\
& \times \Pi_{j=1}^{k} V_{w_j}[\mathrm{Shift}_{\mathrm{T}_j}(u)](\tau_j)] \sum_{sz=w} V_s[\mathrm{Shift}_{\hat{\mathrm{T}}_i}(u)](t_1)V_z[\mathrm{Shift}_{\hat{\mathrm{T}}_i+t_1}(u)](t_2) \\
& = \sum_{w_1,\ldots,w_k,w \in \mathrm{Z}_m^*} \{c((q_1,w_1)\cdots(q_i,w_i)(q,w)(q_{i+1},w_{i+1})\cdots(q_k,w_k)) \times \\
& \Pi_{j=1}^{k} V_{w_i}[\mathrm{Shift}_{\mathrm{T}_j}(u)](\tau_j)\} V_w[\mathrm{Shift}_{\hat{\mathrm{T}}_i}(u)](t_1+t_2) = F_c(u,p)
\end{aligned}$$

□

It is a natural to ask whether c determines F_c uniquely. It is easy to see that the function F_c correspond to the function $F_{c_{abs}}$ by

$$F_c(u,(q_1,t_1)\cdots(q_k,t_k)) = F_{c_{abs}}(u,(q_1\cdots q_k,(t_1,\ldots,t_k)))$$

It implies that if c, d are two generalised generating convergent series, then $F_c = F_d$ if and only if $F_{c_{abs}} = F_{d_{abs}}$. Thus, Lemma 3 implies the following

Lemma 22. *Let $L \subseteq Q^*$ and let $d, c : JL \to \mathcal{Y}$ be two convergent generating series. If $F_c = F_d$, then $c = d$.*

Now we are ready to define the concept of generalised Fliess-series representation of a set of input/output maps.

Definition 12 (Generalised Fliess-series expansion). *The set of input-output maps $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$ is said to admit* a generalised Fliess-series expansion *if for each $f \in \Phi$ there exist a generating convergent series $c_f : JL \to \mathcal{Y}$ such that $F_{c_f} = f$.*

Notice that if Φ has a generalised kernel representation with constraint L, then

Φ has a generalised Fliess-series expansion given as follows. For each $f \in \Phi$, let

$$c_f((q_1, w_1) \cdots (q_k, w_k)) = \begin{cases} D^{|w_k|,\dots,|w_1|} K^{f,\Phi}_{q_1 \cdots q_k} & \text{if } w_1, \dots, w_k \in \{0\}^* \\ D^{|w_k|,\dots,|w_l|-1} G^{f,\Phi}_{q_k \cdots q_l} e_j & \text{if } l = \min\{z \mid |w_z| > 0\}, w_k, \dots, w_{l+1} \in \{0\}^*, \\ & w_l = vj, v \in \{0\}^*, j \in \mathbb{Z}_m \setminus \{0\} \\ 0 & \text{otherwise} \end{cases}$$

From Lemma 22 we immediately get the following corollary.

Corollary 13. *Any $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$ admits at most one generalised kernel representation with constraint L.*

The following proposition gives a description of the Fliess-series expansion of Φ in the case when Φ is realized by a bilinear switched system.

Proposition 15. *(Σ, μ) is a bilinear switched system realization of Φ with constraint L if and only if Φ has a generalised Fliess-series expansion such that for each $f \in \Phi, (q_1, w_1) \cdots (q_k, w_k) \in JL$*

$$c_f((q_1, w_1) \cdots (q_k, w_k)) = C_{q_k} B_{q_k, w_k} \cdots B_{q_1, w_1} \mu(f) \tag{4.23}$$

Proof. If (Σ, μ) is a realization of Φ, then by Proposition 13 for each $f \in \Phi$, $w = (q_1, t_1) \cdots (q_k, t_k) \in TL$, $u \in PC(T, \mathcal{U})$

$$\begin{aligned} f(u, w) &= y_\Sigma(\mu(f), u, w) = \\ &= \sum_{w_1, \dots, w_k \in \mathbb{Z}_m^*} C_{q_k} B_{q_k, w_k} \cdots B_{q_1, w_1} V_{w_1, \dots, w_k}[u](t_1, \dots, t_k) \end{aligned}$$

That is, Φ admits a generalised Fliess-series expansion of the form given in (4.23). Conversely, if Φ admits a generalised Fliess-series expansion of the form (4.23), then using Proposition 13 one gets

$$\begin{aligned} & f(u, (q_1, t_1) \cdots (q_k, t_k)) = \\ &= \sum_{w_1, \dots, w_k \in \mathbb{Z}_m^*} c_f((q_1, w_1) \cdots (q_k, w_k)) V_{w_1, \dots, w_k}[u](t_1, \dots, t_k) = \\ &= \sum_{w_1, \dots, w_k \in \mathbb{Z}_m^*} C_{q_k} B_{q_k, w_k} \cdots B_{q_1, w_1} \mu(f) V_{w_1, \dots, w_k}[u](t_1, \dots, t_k) = \\ &= y_\Sigma(\mu(f), u, (q_1, t_1) \cdots (q_k, t_k)) \end{aligned}$$

That is, (Σ, μ) is a realization of Φ with constraint L. $\square$

4.2.3 Realization Theory of Bilinear Switched Systems: Arbitrary Switching

In this section realization theory for bilinear switched systems will be developed. We start with the case when the input/output maps are defined for all switching sequences. Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (Q \times T)^{+}, \mathcal{Y})$ be a set of input/output maps and assume that Φ has a generalised Fliess-series expansion. As in the case of linear switched systems, we will associate with Φ an indexed set of formal power series Ψ_{Φ}. It turns out that every representation of Ψ_{Φ} determines a realization of Φ and vice versa. We will be able to use the theory of formal power series to derive the results on realization theory.

Recall that $\widetilde{\Gamma} = Q \times \mathrm{Z}_m^{*}$. Let $\Gamma = \{(q,j) \mid q \in Q, j \in \mathrm{Z}_m\}$. Define $\phi : \widetilde{\Gamma} \rightarrow \Gamma$ by

$$\phi((q,w)) = (q,j_1)\cdots(q,j_k), \quad \phi((q,\epsilon)) = \epsilon$$

where $w = j_1 \cdots j_k \in Z_m^{*}$, $j_1,\ldots,j_k \in \mathrm{Z}_m, k \geq 0$. The map ϕ determines a monoid morphisms $\phi : \widetilde{\Gamma}^{*} \rightarrow \Gamma^{*}$ given by

$$\phi((q_1,w_1)\cdots(q_k,w_k)) = \phi((q_1,w_1))\cdots\phi((q_k,w_k))$$

for each $(q_1,w_1),\ldots,(q_k,w_k) \in \widetilde{\Gamma}, k \geq 0$. It is also clear that any element of Γ can be thought of as an element of $\widetilde{\Gamma}$, i.e. we can define the monoid morphism $i : \Gamma^{*} \rightarrow \widetilde{\Gamma}^{*}$ by $i(\epsilon) = \epsilon$ and $i((q_1,j_1)\cdots(q_k,j_k)) = (q_1,j_1)\cdots(q_k,j_k)$, $(q_1,j_1),\ldots,(q_k,j_k) \in \Gamma \subseteq \widetilde{\Gamma}$. It is also easy to see that $\phi(i(w)) = w, \forall w \in \Gamma^{*}$ and $w(q,\epsilon)R^{*}i(\phi(w))(q,\epsilon)$, $q \in Q$.

For each $f \in \Phi, q \in Q$ define the formal power series $S_{f,q} \in \mathbb{R}^{p} \ll \Gamma^{*} \gg$ as follows

$$S_{f,q}(s) = c_f(i(s)(q,\epsilon)) \;, \forall s \in \Gamma^{*}$$

It is easy to see that in fact $c_f(v(q,\epsilon)) = S_{f,q}(\phi(v)) = c_f(i(\phi(v))(q,\epsilon))$, since $(v(q,\epsilon), i(\phi(v))(q,\epsilon)) \in R^{*}$. Assume that $Q = \{q_1,\ldots,q_N\}$. Define the formal power series $S_f \in \mathbb{R}^{Np} \ll \Gamma^{*} \gg$ by

$$S_f = \begin{bmatrix} S_{f,q_1} \\ S_{f,q_2} \\ \vdots \\ S_{f,q_N} \end{bmatrix}$$

Define the *set of formal power series* Ψ_{Φ} *associated with* Φ as follows

$$\Psi_{\Phi} = \{S_f \in \mathbb{R}^{Np} \ll \Gamma^{*} \gg \mid f \in \Phi\}$$

Define the *Hankel-matrix* H_{Φ} of Φ as the Hankel-matrix of Ψ_{Φ}. i.e. $H_{\Phi} = H_{\Psi_{\Phi}}$.

Let $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\ldots,m}, C_q) \mid q \in Q\})$ be a bilinear switched system. Define the representation $R_{\Sigma,\mu}$ *associated with the realization* (Σ, μ) of Φ by

$$R_{\Sigma,\mu} = (\mathcal{X}, \{B_{(q,j)}\}_{(q,j)\in\Gamma}, I, \widetilde{C})$$

where $B_{(q,j)} = B_{q,j} : \mathcal{X} \to \mathcal{X}$, $q \in Q, j = 1, \ldots, m$, $B_{q,0} = A_q : \mathcal{X} \to \mathcal{X}$, $q \in Q$, $\widetilde{C} = \begin{bmatrix} C_{q_1} \\ C_{q_2} \\ \vdots \\ C_{q_N} \end{bmatrix} : \mathcal{X} \to \mathbb{R}^{pN}$ and $I_f = \mu(f) \in \mathcal{X}$, $f \in \Phi$.

Let $R = (\mathcal{X}, \{M_{(q,j)}\}_{(q,j)\in\Gamma}, I, \widetilde{C})$ be a representation of Ψ_Φ. Define the realization (Σ_R, μ_R) *associated with* R by

$$\Sigma_R = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\ldots,m}, C_q) \mid q \in Q\})$$

where $\mu_R(f) = I_f \in \mathcal{X}, f \in \Phi$, $B_{q,j} = M_{(q,j)} : \mathcal{X} \to \mathcal{X}$, $q \in Q, j = 1, \ldots, m$, $A_q = M_{(q,0)} : \mathcal{X} \to \mathcal{X}, q \in Q$ and the maps $C_q : \mathcal{X} \to \mathcal{Y}$, $q \in Q$ are such that $\widetilde{C} = \begin{bmatrix} C_{q_1} \\ \vdots \\ C_{q_N} \end{bmatrix}$. It is easy to see that $R_{\Sigma_R,\mu_R} = R$. It turns out that there is a close connection between realizations of Φ and representations of Ψ_Φ.

Proposition 16. *Assume that Φ admits a generalised Fliess-series expansion. Then, (a) (Σ, μ) realization of Φ if and only if $R_{\Sigma,\mu}$ is a representation of Ψ_Φ, (b) Conversely, R is a representation of Ψ_Φ if and only if (Σ_R, μ_R) is a realization of Φ.*

Proof. It is enough prove Part (a). Part (b) follows from Part (a) by using the equality $R_{\Sigma_R,\mu_R} = R$. Assume that $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\ldots,m}, C_q) \mid q \in Q\})$. Notice that the map $\phi : \widetilde{\Gamma}^* \to \Gamma^*$ is surjective and for each $w_1, \ldots, w_k \in \mathrm{Z}_m$ it holds that

$$B_{q,w_1\cdots w_k} = B_{q,w_k} B_{q,w_{k-1}} \cdots B_{q,w_1} = B_{(q,w_k)} \cdots B_{(q,w_1)} = B_{\phi(q,w_1\cdots w_k)}$$

Then it is easy to see that $R_{\Sigma,\mu}$ is a representation of Ψ_Φ if and only if for all $(q_1, w_1), \ldots, (q_k, w_k) \in \widetilde{\Gamma}$

$$\begin{aligned} c_f((q_1, w_1) \cdots (q_k, w_k)) &= c_f((q_1, w_1) \cdots (q_k, w_k)(q_k, \epsilon)) = \\ &= S_{f,q_k}(\phi((q_1, w_1)) \cdots \phi((q_k, w_k))) = C_{q_k} B_{\phi((q_1,w_1))} \cdots B_{\phi((q_1,w_1))} I_f = \\ &= C_{q_k} B_{q_k,w_k} \cdots B_{q_1,w_1} \mu(f) \end{aligned}$$

But by Proposition 15 this is exactly equivalent to (Σ, μ) being a realization of Φ. $\square$

From the discussion above using Theorem 1 one gets the following characterisation of realizability.

Theorem 17. *Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (Q \times T)^{+}, \mathcal{Y})$. The following are equivalent*

(i) Φ has a realization by a bilinear switched system

(ii) Φ has a generalised Fliess-series expansion and Ψ_{Φ} is rational

(iii) Φ has a generalised Fliess-series expansion and $\mathrm{rank}\ H_{\Phi} < +\infty$

Proof. First we show that (i) $\iff$ (ii). By Proposition 15 if (Σ, μ) a bilinear switched system realization of Φ, then Φ has a generalised Fliess-series expansion. From Proposition 16 we also get that $R_{\Sigma,\mu}$ is a representation of Ψ_{Φ}, i.e. Ψ_{Φ} is rational. Conversely, if Φ has a generalised Fliess-series expansion and R is a representation of Ψ_{Φ}, then from Proposition 16 it follows that (Σ_R, μ_R) is a realization of Φ. Since by Theorem 1 Ψ_{Φ} is rational if and only if $\mathrm{rank}\ H_{\Psi_{\Phi}} = \mathrm{rank}\ H_{\Phi} < +\infty$, we get that (ii) and (iii) are equivalent. □

The next step will be to characterise bilinear switched systems which are minimal realizations of Φ. In order to accomplish this task, we need to the following characterisation of observability and semi-reachability of bilinear switched systems.

Lemma 23. *Let Σ be a bilinear switched system. Assume that (Σ, μ) is a realization of Φ. Let $R = R_{\Sigma,\mu}$. (Σ, μ) is observable if and only if R is observable. (Σ, μ) is semi-reachable from* $\mathrm{Im}\ \mu$ *if and only if R is reachable.*

Proof. Notice that $B_{q,w} = B_{\phi((q,w))}$ and for each $(q_1, w_1), \ldots, (q_k, w_k) \in \widetilde{\Gamma}$

$$\ker \widetilde{C} B_{\phi((q_1,w_1))} \cdots B_{\phi((q_k,w_k))} = \bigcap_{q \in Q} \ker C_q B_{q_1,w_1} \cdots B_{q_k,w_k}$$

Notice that $\mathrm{Im}\mu = \{\mu(f) \mid f \in \Phi\} = \{I_f \mid f \in \Phi\}$. Then it follows from Proposition 14 that $O_{\Sigma} = O_R$ and $W_R = \mathrm{Span}\{x \mid x \in \mathit{Reach}(\Sigma, \mathrm{Im}\mu)\}$. Then the lemma follows from Proposition 14 and the definition of observability and reachability for representations. □

It is also easy to see that $\dim \Sigma = \dim R_{\Sigma,\mu}$ and $\dim R = \dim \Sigma_R$. In fact, Proposition 16 implies the following.

Lemma 24. *If R is a minimal representation of Ψ_{Φ} then (Σ_R, μ_R) is a minimal realization of Φ. Conversely, if (Σ, μ) is a minimal realization of Φ, then $R_{\Sigma,\mu}$ is a minimal representation of Ψ_{Φ}.*

The following lemma clarifies the relationship between representation morphisms and bilinear switched system morphisms.

Lemma 25. *$T : (\Sigma, \mu) \to (\Sigma^{'}, \mu^{'})$ is a bilinear switched system morphism if and only if $T : R_{\Sigma,\mu} \to (\Sigma^{'}, \mu^{'})$ is a representation morphism. Moreover, T is injective, surjective, an isomorphism as a bilinear switched system morphism if and only if T is injective, surjective, an isomorphism as a representation morphism.*

Proof. T is a bilinear switched system morphism if and only if

$$TA_q = A^{'}_q T \quad C_q = C^{'}_q T \quad TB_{q,j} = B^{'}_{q,j} T \quad T\mu(f) = \mu^{'}(f)$$

for each $q \in Q, j = 1, 2 \dots, m$ and $f \in \Phi$. This is equivalent to $TB_{(q,j)} = B^{'}_{(q,j)} T$ for each $j \in \mathrm{Z}_m$, $TI_f = T\mu(f) = \mu^{'}(f) = I^{'}_f$ and

$$\widetilde{C} = \begin{bmatrix} C_{q_1} \\ \vdots \\ C_{q_N} \end{bmatrix} = \begin{bmatrix} (C^{'}_{q_1} T) \\ \vdots \\ (C^{'}_{q_N} T) \end{bmatrix} = \widetilde{C}^{'} T$$

That is, T is a representation morphism. □

Using the theory of rational formal power series presented in Section 3.1 we get the following.

Theorem 18. *Let $\Phi \subseteq F(PC(T, \mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$. The following are equivalent*

(i) $(\Sigma_{min}, \mu_{min})$ *is a minimal realization of* Φ *by a bilinear switched system*

(ii) $(\Sigma_{min}, \mu_{min})$ *is semi-reachable from* $\mathrm{Im}\mu$ *and it is observable*

(iii) $\dim \Sigma_{\mathrm{min}} = \mathrm{rank}\ H_\Phi$

(iv) For any bilinear switched system realization (Σ, μ) of Φ, such that (Σ, μ) is semi-reachable from $\mathrm{Im}\mu$, there exist a surjective homomorphism $T : (\Sigma, \mu) \to (\Sigma_{min}, \mu_{min})$. In particular, all minimal realizations of Φ by bilinear switched systems are algebraically similar.

Proof. $(\Sigma_{min}, \mu_{min})$ is a minimal realization if and only if that $R_{min} = R_{\Sigma_{min},\mu_{min}}$ is minimal representation, that is, by Theorem 2 R_{min} is reachable and observable. By Lemma 23 the latter is equivalent to $(\Sigma_{min}, \mu_{min})$ being semi-reachable from Im μ and observable. That is, we get that (i) $\iff$ (ii). By Theorem 2 a representation R_{min} is minimal if and only if $\dim \Sigma_{min} = \dim R_{min} = \mathrm{rank}\ H_{\Phi_\Psi} = \mathrm{rank}\ H_\Phi$. That is, we showed that (i) $\iff$ (iii). To show that (i) $\iff$ (iv), notice that

$(\Sigma_{min}, \mu_{min})$ is a minimal realization if and only if $R_{\Sigma_{min},\mu_{min}}$ is a minimal representation. By Theorem 2 R_{min} is minimal if and only if for any reachable representation R there exists a surjective representation morphism $T : R \to R_{min}$. It means that $(\Sigma_{min}, \mu_{min})$ is minimal if and only if for any reachable representation R of Ψ_Φ there exists a surjective representation morphism $T : R \to R_{\Sigma_{min},\mu_{min}}$. But any reachable representation R gives rise to a semi-reachable realization of Φ and vice versa. That is, we get that $(\Sigma_{min}, \mu_{min})$ is minimal if and only if for any realization $(\Sigma^{'}, \mu^{'})$ of Φ such that $(\Sigma^{'}, \mu^{'})$ is semi-reachable from $\mathrm{Im}\mu$ there exists a surjective representation morphism $T : R_{\Sigma^{'},\mu^{'}} \to R_{\Sigma_{min},\mu_{min}}$. By Lemma 25 we get that the latter is equivalent to $T : (\Sigma^{'}, \mu^{'}) \to (\Sigma_{min}, \mu_{min})$ being a surjective bilinear switched system morphism. From Corollary 1 it follows that if $(\Sigma^{'}, \mu^{'})$ is a minimal realization of Φ, then there exists a representation isomorphism $T : R_{\Sigma^{'},\mu^{'}} \to R_{\Sigma_{min},\mu_{min}}$ which means that $(\Sigma_{min}, \mu_{min})$ is gives rise to the bilinear switched system isomorphism $T : (\Sigma^{'}, \mu^{'}) \to (\Sigma_{min}, \mu_{min})$, that is, $(\Sigma^{'}, \mu^{'})$ and $(\Sigma_{min}, \mu_{min})$ are algebraically similar. □

4.2.4 Realization Theory of Bilinear Switched Systems: Constrained switching

The case of restricted switching is slightly more involved. As in the case of arbitrary switching, we will associate a set Ψ_Φ of formal power series over Γ with the set of input-output maps $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$. Every representation of Ψ_Φ gives rise to a realization of Φ. If L is a regular language, then existence of a realization of Φ implies existence of a representation of Ψ_Φ. However, the dimension of the minimal representation of Ψ_Φ might be bigger than the dimension of a realization of Φ. Any minimal representation of Ψ_Φ gives rise to an observable and semi-reachable realization of Φ. But this observable and semi-reachable realization need not be a minimal one. Extraction of the right information from Φ and the construction of Ψ_Φ is much more involved in the case of restricted switching than in the case of arbitrary switching.

Recall the definition of the relation $R^* \subseteq \widetilde{\Gamma}^* \times \widetilde{\Gamma}^*$ from Subsection 4.2.2. Define the set $\widetilde{JL} \subseteq \widetilde{\Gamma}^*$ by

$$\widetilde{JL} = \{s \in \widetilde{\Gamma}^* \mid \exists w \in JL : (w, s) \in R^*\}$$

In fact, $\widetilde{JL}$ contains all those sequences in $\widetilde{\Gamma}^*$ for which we can derive some information based on the values of a convergent generating series for sequences from JL. More precisely, if $c : JL \to \mathcal{Y}$ is a generating convergent sequence, then c can be extended to a generating convergent series $\widetilde{c} : \widetilde{JL} \to \mathcal{Y}$ by defining $\widetilde{c}(s) = c(w)$

for each $s \in \widetilde{JL}, w \in JL, (s,w) \in R^*$. It is clear that for any $s \in \widetilde{JL}$ there exists a $w \in JL$ such that $(s,w) \in R^*$ and if $(s,w),(s,v) \in R^*$, $w,v \in JL$, then $c(w) = c(v) = \widetilde{c}(s)$, since c was assumed to be a generating convergent series. If $(s,x) \in R^*$, then $\widetilde{c}(s) = \widetilde{c}(x)$. Moreover, if $(s,w) \in R^*$ and $s = (z_1,x_1)\cdots(z_l,x_l)$ and $w = (q_1,v_1)\cdots(q_k,v_k)$, then from the definition of R it follows that $\sum_1^k |v_i| = \sum_1^l |x_i|$, that is, $||\widetilde{c}(s)|| = ||c(w)|| \le KM^{|v_1|}\cdots M^{|v_k|} = KM^{\sum_1^k |v_i|} = KM^{\sum_1^l |x_l|}$. That is, $\widetilde{c} : \widetilde{JL} \to \mathcal{Y}$ is indeed a generating convergent series. Moreover, on JL the sequence $\widetilde{c}$ coincides with c, that is, if $w \in JL$, then $\widetilde{c}(w) = c(w)$. By abuse of notation, we will denote $\widetilde{c}$ simply by c in the sequel.

For each $q \in Q$ define $JL_q = \{v(q,w) \in \widetilde{JL} \mid v \in \widetilde{\Gamma}^*, (q,w) \in \widetilde{\Gamma}\}$. Let $L_q = \{w \in \Gamma^* \mid \exists v \in JL_q : \phi(v) = w\}$. Notice that

$$w \in L_q \iff i(w)(q,\epsilon) \in JL_q$$

Indeed, if $i(w)(q,\epsilon) \in JL_q$, then $\phi(i(w)(q,\epsilon)) = \phi(i(w)) = w \in L_q$. Conversely, if $w \in L_q$, then $w = \phi(v)$ for some $v \in JL_q$. But then $v = u(q,z)$ and $(u(q,z)(q,\epsilon), u(q,z\epsilon) = v) \in R^*$ and $(v(q,\epsilon), i(w)(q,\epsilon)) \in R^*$ which implies $(v, i(w)(q,\epsilon)) \in R^*$. Since $v \in \widetilde{JL}$, we know that $i(w)(q,\epsilon) \in \widetilde{JL}$, that is, $i(w)(q,\epsilon) \in JL_q$.

Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$ be a set of input/output maps defined on sequences of discrete modes belonging to L. Assume Φ admits a generalised Fliess-series expansion. For each $q \in Q$, $f \in \Phi$ define the formal power series $T_{f,q} \in \mathbb{R}^p \ll \Gamma^* \gg$ by

$$T_{f,q}(s) = \begin{cases} c_f(i(s)(q,\epsilon)) & \text{if } s \in L_q \\ 0 & \text{otherwise} \end{cases}$$

Notice that for each $s \in L_q$ there exists a $w = u(q,v) \in JL$ such hat $T_{f,q}(s) = c_f(w)$. Indeed, $s \in L_q$ implies that there exists a $w = (q_1,x_1)\cdots(q_l,x_l)(q,x_{l+1}) \in JL$ such that $(w, i(s)(q,\epsilon)) \in R^*$. Thus $T_{f,q}(s) = c_f(i(s)(q,\epsilon)) = c_f(w)$. The intuition behind the definition of $T_{f,q}$ is the following. We store in $T_{f,q}$ the values of all those $c_f(s)$ which show up in the generalised Fliess-series expansion of $f(u,w)$, for some switching sequence $w \in TL$ such that w ends with discrete mode q. For all the other sequences from Γ^* we set the value of $T_{f,q}$ to zero.

Assume that $Q = \{q_1,\dots,q_N\}$. Define the formal power series $T_f \in \mathbb{R}^{Np} \ll \Gamma^* \gg$ by

$$T_f = \begin{bmatrix} T_{f,q_1} \\ T_{f,q_2} \\ \vdots \\ T_{f,q_N} \end{bmatrix}$$

Define the *set of formal power series* Ψ_Φ *associated with* Φ as follows

$$\Psi_\Phi = \{T_f \in \mathbb{R}^{Np} \ll \Gamma^* \gg \mid f \in \Phi\}$$

Define the *Hankel-matrix* H_Φ *of* Φ as the Hankel-matrix of Ψ_Φ, that is, $H_\Phi = H_{\Psi_\Phi}$.

For each $q \in Q$ define the formal power series $Z_q \in \mathbb{R}^p \ll \Gamma^* \gg$ by $Z_q(w) = \begin{cases} (1,1,\dots,1)^T & \text{if } w \in L_q \\ 0 & \text{otherwise} \end{cases}$. Let $Z \in \mathbb{R}^{Np} \ll \Gamma \gg$ be

$$Z = \begin{bmatrix} Z_{q_1} \\ \vdots \\ Z_{q_N} \end{bmatrix}$$

and let Ω be the indexed set $\{Z \mid f \in \Phi\}$, i.e $\Omega : \Phi \to \mathbb{R}^{Np} \ll \Gamma^* \gg$ and $\Omega(f) = Z, f \in \Phi$. With the notation above, the following holds.

Lemma 26. *Let* $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\dots,m}, C_q) \mid q \in Q\})$ *be a bilinear switched system. Assume that* (Σ, μ) *is a realization of* Φ *and* Φ *admits a generalised Fliess-series expansion. Let* $\Phi^{'} = \{y_\Sigma(\mu(f), ., .) \in F(PC(T,\mathcal{U}) \times (Q \times T)^+, \mathcal{Y}) \mid f \in \Phi\}$ *and let* $\Psi^{'}_\Phi$ *be the set of formal power series associated with* $\Phi^{'}$ *as defined in Subsection 4.2.3. That is,* $\Psi_{\Phi^{'}} = \{S_g \in \mathbb{R}^{Np} \ll \Gamma \gg \mid g \in \Phi^{'}\}$. *Let* $S_f = S_{y_\Sigma(\mu(f),.,.)}$ *and let* $\Theta = \{S_f \mid f \in \Phi\}$. *Then the following holds*

$$\Psi_\Phi = \Theta \odot \Omega$$

Proof. Define $\mu^{'} : \Phi^{'} \to \mathcal{X}$ by $\mu^{'}(y_\Sigma(\mu(f), ., .)) = \mu(f)$. Since (Σ, μ) is a realization of Φ, if for some $f_1, f_2 \in \Phi$ it holds that $y_\Sigma(\mu(f_1), ., .) = y_\Sigma(\mu(f_2), ., .)$, then $f_1 = y_\Sigma(\mu(f_1), ., .)|_{PC(T,\mathcal{U}) \times TL} = y_\Sigma(\mu(f_2), ., .)|_{PC(T,\mathcal{U}) \times TL} = f_2$. That is, $f_1 = f_2$ and thus $\mu^{'}$ is well-defined. It is also easy to see that $(\Sigma, \mu^{'})$ realizes $\Phi^{'}$, therefore $\Phi^{'}$ has a generalised Fliess-series expansion. For each $f \in \Phi$, denote by $c_f : \widetilde{JL} \to \mathcal{Y}$ the generating convergent series corresponding to f, i.e. $F_{c_f} = f$. Denote by $d_f : \widetilde{\Gamma}^* \to \mathcal{Y}$ the series corresponding to $y_\Sigma(\mu(f), ., .)$, i.e. $F_{d_f} = y_\Sigma(\mu(f), ., .)$. By Proposition 15 (Σ, μ) is a realization of Φ with constraint L, if and only if $\forall w(q, v) \in JL$: $c_f(w(q, v)) = C_q B_{q,v} B_{\phi(w)} \mu(f)$. Here we used the fact that if $w = (q_1, z_1) \cdots (q_k, z_k)$, then $B_{q_k,z_k} \cdots B_{q_1,z_1} = B_{\phi(w)}$. But $(\Sigma, \mu^{'})$ realizes $\Phi^{'}$, so by Proposition 15 it holds that $\forall s(q, x) \in \widetilde{JL} : d_f(s(q, x)) = C_q B_{q,x} B_{\phi(s)} \mu^{'}(y_\Sigma(\mu(f), ., .))$. Notice that if $(s(q, x), w(q, v)) \in R^*$, then $\phi(s(q, x)) = \phi(w(q, v))$, and therefore $B_{q,v} B_{\phi(w)} = B_{\phi(w(q,v))} = B_{\phi(s(q,x))} = B_{q,x} B_{\phi(s)}$. Notice that $\mu(f) = \mu^{'}(y_\Sigma(\mu(f), ., .))$. Thus for each $s(q, x) \in \widetilde{JL}$, $w(q, v) \in JL$ we get that $c_f(s(q, x)) = c_f(w(q, v)) = d_f(s(q, x))$. Thus, for each $q \in Q$, $f \in \Phi$, $s \in L_q$ we get that $T_{f,q}(s) = c_f(i(s)(q, \epsilon)) = d_f(i(s)(q, \epsilon)) = S_{f,q}(s)$. Notice that for each $s \notin L_q$, $T_{f,q}(s) = 0$ and $Z_q(s) = 0$. That is, $T_{f,q} = S_{f,q} \odot Z_q$ and therefore $T_f = S_f \odot Z$. $\square$

If L is regular, then Ω turns out to be a rational indexed set.

Lemma 27. *If L is regular, then $L_q, q \in Q$ are regular languages and Ω is a rational indexed set of formal power series.*

Proof. It is enough to show that if L is a regular language, then $L_q, q \in Q$ are regular languages. Indeed, if $L_q, q \in Q$ are regular, then $\{e_j^T Z_q\}, q \in Q, j = 1, \dots, p$ are rational sets of formal powers series, since $e_j^T Z_q(w) = 1 \iff w \in L_q$. Therefore, $\{Z = \begin{bmatrix} Z_{q_1}^T & \cdots & Z_{qN}^T \end{bmatrix}^T\}$ is a rational set, therefore Ω is a rational indexed set of formal power series by Lemma 5. Define $pr_Q : \Gamma^* \to Q^*$ by $pr_Q((q_1, j_1) \cdots (q_k, j_k)) = q_1 \cdots q_k$. Recall from Subsection 7.1.2 the definition of the sets $F_q(w)$ and $\widetilde{L}_q$. Lemma 19 says that if L is regular, then $\widetilde{L}_q$ is regular. We shall prove that $L_q = pr_Q^{-1}(\widetilde{L}_q)$. From this equality it follows that if $\widetilde{L}_q$ is regular, then L_q is regular. Indeed, pr_Q is a monoid morphism, and therefore can be realized by a regular transducer see [17]. Then the regularity of L_q follows from the classical result on regular transducers. Alternatively, if $\mathcal{A} = (S, Q, \delta, F)$ is a finite automaton accepting $\widetilde{L}_q$, then the deterministic finite automaton $\mathcal{A}' = (S, \Gamma, \delta', F)$ defined by $\delta'(s, (q, j)) = \delta(s, q), (q, j) \in \Gamma, s \in S$ accepts L_q.

We now proceed with the proof of the equality $L_q = pr_Q^{-1}(\widetilde{L}_q)$. First we show that $L_q \subseteq pr_Q^{-1}(\widetilde{L}_q)$. If $v = (q_1, j_1) \cdots (q_t, j_t) \in L_q$, then there exists $w(q, z) \in JL_q$, such that $\phi(w(q, p)) = v$. Let $w = (z_1, m_1) \cdots (z_k, m_k)$. Then $z_1 \cdots z_k q \in L$. Let $l = \min\{j \mid |m_j| > 0\}$. Let $s = z_1 \cdots z_{l-1}$, $x = z_l \cdots z_k$. From $\phi(w(q, z)) = v$ it follows that $z_l = q_1 = \cdots = q_{|m_l|}$, $z_{i+1} = q_{|m_i|+1} = \cdots = q_{|m_{i+1}|}$, for $i = l, l+1, \dots, k-1$, $q_{|m_k|+1} = \cdots q_t = q$, and $|p| + \sum_{i=1}^{k} |m_i| = t$. That is, we get that $q_1 \cdots q_t q = z_l^{|m_l|} \cdots z_k^{|m_k|} q^{|p|} q$ and $sxq = z_1 \cdots z_k q \in L$, that is, $(s, ((|m_1|, \dots, |m_k|, |p|), x) \in F_q(q_1 \cdots q_t)$, i.e. $q_1 \cdots q_t = pr_Q((q_1, j_1) \cdots (q_t, j_t)) \in \widetilde{L}_q$. That is, $L_q \subseteq pr_Q^{-1}(\widetilde{L}_q)$. Let $w \in \widetilde{L}_q$ and let $(u, (\alpha, h)) \in F_q(w)$. Assume that $u = q_1 \dots q_{|u|}$ and $h = z_1 \cdots z_k$, $q_1, \dots, q_{|u|}, z_1, \dots z_k \in Q$. Since $w = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$, we get that $v \in pr_Q^{-1}(w)$ if and only if $v = v_1 \cdots v_k$, $v_i = (z_i, j_{1,i}) \cdots (z_i, j_{\alpha_i, i}) \in \Gamma^*$, $|v_i| = \alpha_i$, $j_{i,j} \in \mathbb{Z}_m, i = 1, \dots, \alpha_j, j = 1, \dots, k$. Let $\underline{j_i} = j_{1,i} j_{2,i} \dots j_{\alpha_i, i}$, $s = (q_1, \epsilon) \cdots (q_{|u|}, \epsilon)(z_1, \underline{j_1}) \cdot \cdots \cdot (z_k, \underline{j_k})$. Since $uv \in L$, we have that $s \in JL$ and $z_k = q$ implies that $s \in JL_q$. But $\phi(s) = \phi((z_1, \underline{j_1}) \cdots (\phi(z_k, \underline{j_k})) = v_1 \cdots v_k \in L_q$. That is, $pr_Q^{-1}(\widetilde{L}_q) \subseteq L_q$, and consequently $L_q = pr_Q^{-1}(\widetilde{L}_q)$. ∎

Let $R = (\mathcal{X}, \{M_z\}_{z \in \Gamma}, I, C)$ be a representation of Ψ_Φ. Define the *bilinear switched system realization* (Σ_R, μ_R) *associated with* R as in Section 4.2.3. That is,

$$\Sigma_R = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\dots,m}, C_q) \mid q \in Q\}) \text{ and } \mu_R(f) = I_f$$

where $C_q : \mathcal{X} \to \mathcal{Y}$, $q \in Q$ are such that $C = \begin{bmatrix} C_{q_1} \\ \vdots \\ C_{q_N} \end{bmatrix}$, $B_{q,j} = M_{(q,j)}, A_q = M_{(q,0)}$, $q \in Q, j = 1, \ldots, m$. It is easy to see that (Σ_R, μ_R) is semi-reachable (observable) if and only if R is reachable (observable).

Recall from Subsection 4.1.4 the definition of $comp(L)$:

$$comp(L) = \{w_1 \cdots w_k \in Q^* \mid \widetilde{L}_{w_k} = \emptyset, w_1, \ldots, w_k \in Q\}$$

The following statement is an easy consequence of Proposition 15.

Theorem 19. *If Φ has a generalised Fliess-series expansion with constraint L and $R = (\mathcal{X}, \{B_z\}_{z\in\Gamma}, I, \widetilde{C})$ is a representation of Ψ_Φ, then (Σ_R, μ_R) is a realization of Φ. That is, if Ψ_Φ is rational, then Φ has a realization by a bilinear switched system. Moreover, for each $f \in \Phi$, $w \in T(comp(L))$*

$$\forall u \in PC(T, \mathcal{U}) : y_\Sigma(\mu(f), u, w) = 0$$

Proof. Let (Σ_R, μ_R) the realization associated with R. Assume that

$$\Sigma_R = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\ldots,m}, C_q) \mid q \in Q\})$$

Since R is a representation of Ψ_Φ, we get that for each $(q_1, w_1) \cdots (q_k, w_k) \in JL$, $f \in \Phi$

$$\begin{aligned} c_f((q_1, w_1) \cdots (q_k, w_k)) &= T_{f,q_k}(\phi((q_1, w_1) \cdots (q_k, w_k))) = \\ &= C_{q_k} B_{\phi((q_k,w_k))} \cdots B_{\phi((q_1,w_1))} I_f = C_{q_k} B_{q_k,w_k} \cdots B_{q_1,w_1} \mu(f) \end{aligned} \tag{4.24}$$

We used the definition of (Σ_R, μ_R) and the fact that $B_{(q,j_1)\cdots(q,j_l)} = B_{\phi((q,j_1\cdots j_l))}$ for each $q \in Q, j_1, \ldots, j_l \in \mathbb{Z}_m$. From Proposition 15 we get that (4.24) implies that (Σ_R, μ_R) is a realization of Φ.

Let $w = (q_1, t_1) \cdots (q_k, t_k) \in T(comp(L))$, that is, $\widetilde{L}_{q_k} = \emptyset$. Then for each $s = (q_1, w_1) \cdots (q_k, w_k) \in \widetilde{\Gamma}^*$ we get that $T_{f,q_k}(\phi(s)) = 0$, since $\phi(s) \notin L_{q_k}$. Indeed, $\widetilde{L}_{q_k} = \emptyset$ and from the proof of Lemma 27 we know that $L_q = pr_Q^{-1}(\widetilde{L}_q)$. If $\phi(s) \in L_{q_k}$, then we get that $pr_Q(\phi(s)) \in \widetilde{L}_{q_k} = \emptyset$, a contradiction. But $g = y_\Sigma(\mu(f), ., .)$ has a generalised Fliess-series expansion, and from Proposition 15 it follows that $c_g((q_1, w_1) \cdots (q_k, w_k)) = C_{q_k} B_{q_k,w_k} \cdots B_{q_1,w_1} \mu(f)$. Since R is a representation of Ψ_Φ, we also get that $C_{q_k} B_{q_k,w_k} \cdots B_{q_1,w_1} \mu(f) = C_{q_k} B_{\phi((q_k,w_k))} \cdots B_{\phi((q_1,w_1))} I_f = T_{f,q_k}(\phi((q_1, w_1) \cdots \phi(q_k, w_k)) = 0$. That is, if $q_1 \cdots q_k \in comp(L)$, then for each $w_1, \ldots, w_k \in \mathbb{Z}_m^*$ it holds that

$$c_g((q_1, w_1) \cdots (q_k, w_k)) = 0$$

Then the definition of F_{c_g} implies that $F_{c_g} = g = 0$ for each $q_1 \cdots q_k \in T(comp(L))$. □

We see that rationality of Ψ_Φ, i.e. the condition that rank $H_\Phi < +\infty$, is a sufficient condition for realisability of Φ. It turns out that if L is regular, this is also a necessary condition. From the discussion above, Lemma 26 and Lemma 6 one gets the following.

Theorem 20. *Assume that L is regular. Then the following are equivalent.*

(i) *Φ has a realization by a bilinear switched system*

(ii) *Φ has a generalised Fliess-series expansion and* rank $H_\Phi < +\infty$

(iii) *There exists a realization of Φ by a bilinear switched system (Σ, μ) such that Σ is observable and semi-reachable from* Imμ *and*

$$\forall f \in \Phi : y_\Sigma(\mu(f), ., .)|_{PC(T,\mathcal{U})\times T(compl(L))} = 0 \tag{4.25}$$

and for any $(\Sigma^{'}, \mu^{'})$ bilinear switched system realization of Φ

$$\dim \Sigma \leq \text{rank } H_\Omega \cdot \dim \Sigma^{'} \tag{4.26}$$

Proof. `(i)` $\iff$ `(ii)`
By Lemma 26, if (Σ, μ) is a realization of Φ, then Φ has a generalised Fliess-series expansion and $\Psi_\Phi = \Theta \odot \Omega$. Since (Σ, μ) is a realization of $\Phi^{'} = \{y_\Sigma(\mu(f), ., .) \mid f \in \Phi\}$ we get that $\Psi_{\Phi^{'}}$ is rational. Define the map $\Phi \ni f \mapsto i(f) = y_\Sigma(\mu(f), .,) \in \Phi^{'}$. Since $\Theta = \{S_{i(f)} \mid f \in \Phi\}$, Lemma 8 implies that Θ is rational. Since L is regular, by Lemma 27 Ω is rational, therefore by Lemma 6 $\Psi_\Phi = \Theta \odot \Omega$ is rational, that is, rank $H_\Phi < +\infty$. Conversely, if Φ admits a generalised Fliess-series expansion and rank $H_\Phi < +\infty$, i.e. Ψ_Φ is rational, then there exists a representation R of Ψ_Φ and by Theorem 19 (Σ_R, μ_R) is a realization of Φ

`(ii)` $\iff$ `(iii)`
It is clear that (iii) implies (i), which implies (ii). We will show that (ii) implies (iii). Assume that Φ admits a generalised Fliess-series expansion and Ψ_Φ is rational. Let R be the minimal representation of Ψ_Φ. Then (Σ_R, μ_R) is a realization of Φ, moreover Σ_R is observable and semi-reachable from Imμ. From Theorem 19 it follows that

$$y_\Sigma(\mu_R(f), ., .)|_{PC(T,\mathcal{U})\times T(comp(L))} = 0$$

Let $(\Sigma^{'}, \mu^{'})$ be a realization of Φ. Then $R^{'} = R_{\Sigma^{'},\mu^{'}}$ is a representation of $\Psi_{\Phi^{'}}$, where $\Phi^{'} = \{y_{\Sigma^{'}}(\mu^{'}(f), ., .) \mid f \in \Phi\}$. From Lemma 26 we know that $\Psi_\Phi = \Theta \odot \Omega$, where $\Theta = \{S_{y_{\Sigma'}(\mu'(f),.,.)} \mid f \in \Phi\}$. Assume that $R' = (\mathcal{X}', \{B'_z\}_{z\in\Gamma}, I', C')$. Then $\widetilde{R} = (\mathcal{X}', \{B'_z\}_{z\in\Gamma}, \widetilde{I}, C')$, where $\widetilde{I}_f = I_{y_{\Sigma'}(\mu'(f),.,.)}$, $f \in \Phi$, is a representation of Θ. But R is a minimal representation of Ψ_Φ, therefore $\dim R = \dim \Sigma_R = \text{rank } H_{\Psi_\Phi}$. From Lemma 6 it follows that $\text{rank } H_{\Psi_\Phi} = \text{rank } H_{\Theta\odot\Omega} \le (\text{rank } H_\Omega)(\text{rank } H_\Theta)$. Since $\dim \Sigma = \dim R' = \dim \widetilde{R} \ge \text{rank } H_\Theta$, we get that

$$\dim \Sigma_R \le \text{rank } H_\Omega \cdot \dim \Sigma'$$

Taking (Σ_R, μ_R) for (Σ, μ) completes the proof. $\square$

The following example demonstrates existence of a semi-reachable and observable realization of Φ, which is non-minimal.

Example

Let $Q = \{1, 2\}$, $L = \{q_1^k q_2 \mid k > 0\}$, $\mathcal{Y} = \mathcal{U} = \mathbb{R}$. Define the generating series $c : \widetilde{JL} \to \mathbb{R}$ by $c((q_1, w_1)(q_2, w_2)) = 2^k$, where $w_2 = 0^{j_0} z_1 \cdots z_l 0^{j_l}$, $k = \sum_{i=0}^{l} j_l$, $z_i \in \{1\}^*, i = 1, \dots, l$. Let $\Phi = \{F_c\}$. Define the system $\Sigma_1 = (\mathbb{R}, \mathbb{R}, \mathbb{R}, Q, \{(A_q, B_{q,1} C_q) \mid q \in \{q_1, q_2\}\})$ by $A_{q_1} = 1, B_{q_1,1} = 1, C_{q_1} = 1$ and $A_{q_2} = 2, B_{q_2,1} = 1, C_{q_2} = 1$. Define the system $\Sigma_2 = (\mathbb{R}^2, \mathbb{R}, \mathbb{R}, Q, \{(\widetilde{A}_q, \widetilde{B}_{q,1},$
$\widetilde{C}_q) \mid q \in Q\})$ by

$$\widetilde{A}_{q_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \widetilde{B}_{q_1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \widetilde{C}_{q_1} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\widetilde{A}_{q_2} = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \quad \widetilde{B}_{q_2,1} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad \widetilde{C}_{q_2} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Let $\mu_1 : F_c \mapsto 1$ and $\mu_2 : F_c \mapsto (1,0)^T \in \mathbb{R}^2$. Both (Σ_1, μ_1) and (Σ_2, μ_2) are semi-reachable from $\text{Im}\mu_1$ and $\text{Im}\mu_2$ respectively and they are observable, therefore they are the minimal realizations of $y_{\Sigma_1}(1, ., .)$ and $y_{\Sigma_2}((1,0)^T, ., .)$. Moreover, it is easy to see that $(\Sigma_i, \mu_i), i = 1, 2$ are both realizations of Φ with constraint L. Yet, $\dim \Sigma_1 = 1$ and $\dim \Sigma_2 = 2$. In fact, Σ_2 can be obtained by constructing the minimal representation of Ψ_Φ, i.e., Σ_2 is a realization of F_c satisfying part (iii) of Theorem 20.

Chapter 5

Reachability of Linear Switched Systems

This chapter deals with the reachability and the structure of the reachable set of linear switched systems. The issue of reachability for linear switched systems has been addressed in a number of papers, see [69, 86]. An exhaustive study of the reachability of linear switched systems is presented in [69]. On the level of results the current chapter doesn't offer anything more than [69]. The novelty lies in the methods which are used to prove these results. Namely, the current chapter uses techniques from differential geometric theory of nonlinear systems theory to derive the structure of the reachable set. The main tool is the theory of orbits, developed by H. Sussmann in [71], and realization theory for nonlinear systems by B. Jakubczyk [34]. The theory of orbits allows one to compute the structure of the set of states which are weakly reachable, i.e. reachable in positive or negative time from zero. This, in turn, allows the application of the classical nonlinear conditions for accessibility to the system restricted to the set of the weakly reachable states. Accessibility of the restricted system and the linear structure of the weakly reachable set makes it easy to determine the structure of the reachable set.

In the author's opinion, the proof presented in this chapter is more conceptual and it makes the connection between the classical systems theory and the theory of hybrid systems more transparent. The author also hopes that the methods employed in the chapter can be extended to more general classes of hybrid systems.

The outline of the chapter is the following. Section 5.1 gives the precise mathematical formulation of concepts and problems which are dealt with in this chapter. Some elementary properties of switched systems are also presented in Section 5.1.

This section also contains the statement of the main result. Section 5.2 contains the results from classical nonlinear systems theory, which are needed for the proof of the main result. Section 5.3 contains the proof main result of the chapter, the structure of the reachable set of linear switched systems. The chapter contains most of the results on nonlinear systems theory and differential geometry needed to derive the main results. Nevertheless some basic knowledge of these subjects is necessary to follow all the details. Good references on these topics are [78, 5].

Note that using the results of [60] could also be a potentially useful approach to determining the structure of the reachable set of linear switched systems. In fact, using those results might even lead to a less involved proof. In this chapter we will not pursue this approach. Note, however, that our discussion on the role of second countability in application of Jakubczyk's realization theorem was inspired by similar results in [60]

5.1 Preliminaries

This sections some elementary properties of switched systems. At the end of the section the main theorem of the chapter is formulated. Recall from Chapter 4, Section 4.1 the definition and basic properties of linear switched systems. Throughout this chapter we will study linear switched systems with the the fixed initial state 0. In particular, we will be interested in the set of reachable states from the initial state 0. In order to simplify notation, we will denote by $Reach(\Sigma)$ the set of states reachable from 0, i.e. $Reach(\Sigma) = Reach(\Sigma, \{0\})$. As a further simplification, we will denote the state-trajectory map x_Σ simply by x whenever it doesn't create confusion. That is, the expression $x(x_0, u, w)$ simply denotes $x_\Sigma(x_0, u, w)$.

Denote by $PC_{const}(T, \mathcal{U})$ the set of piecewise-constant input functions. A function $u(.) : T \to \mathcal{U}$ is called *piecewise-constant* if for each $[t_0, t_k] \subseteq T$ there exist $t_0 < t_1 < \cdots < t_{k-1} < t_k$ such that $u|_{[t_i, t_{i+1}]}$ is constant for all $i = 0 \dots k-1$. It is well-known that for each $u(.) \in PC(T, \mathcal{U})$ there exists a sequence $u_n(.) \in PC_{const}(T, \mathcal{U}), n \in \mathbb{N}$ such that $\lim_{n\to+\infty} u_n(.) = u(.)$ in $||.||_1$ norm. More precisely, for each $S \in T$, $S > 0$, $\lim_{n\to+\infty} u_n|_{[0,S]} = u|_{[0,S]}$ if both $u_n|_{[0,S]}$ and $u_{[0,S]}$ are viewed as elements of the space $L^1([0, S], \mathcal{U})$ of integrable measurable functions and the limit is taken in the usual topology (the topology induced by the norm $||.||_1$) of this space. Given a switched system Σ, by continuity of solutions of differential equations on inputs, see [26], we get that

$$\begin{gathered}\forall x \in \mathcal{X} : \forall w \in (Q \times T)^*, \forall u(.) \in PC(T, \mathcal{U}), \forall u_n(.) \in PC_{const}(T, \mathcal{U}) : \\ \lim_{n\to\infty} u_n(.) = u(.) \text{ in } ||.||_1 \implies \lim_{n\to\infty} x(x, u_n(.), w) = x(x, u(.), w) \text{ point-wise}\end{gathered} \tag{5.1}$$

The *set of states reachable by piecewise-constant input* is defined as

$$Reach_{const}(\Sigma) \quad = \quad \{x(0,u,w) \in \mathcal{X} \mid w \in (Q \times T)^*, u(.) \in PC_{const}(T,\mathcal{U})\}$$

From (5.1) one gets immediately following proposition

Proposition 17. *Given a switched system Σ, the set of states reachable by piecewise-constant input is dense in the set $Reach(\Sigma)$, i.e.*

$$Cl(Reach_{const}(\Sigma)) = Reach(\Sigma)$$

For any $u \in PC(T,\mathcal{U})$, $w,v \in (Q \times T)^*$ it holds that $x(0,u,w(q,t)(q,t^{'})v) = x(0,u,w(q,t+t^{'})v)$. Define $R \subseteq (Q \times T)^* \times (Q \times T)^*$ by $w(q,t)(q,t^{'})vRw(q,t+t^{'})v$ and let R^* be the smallest equivalence relation containing R.

Proposition 18. *For any $u \in PC_{const}(T,\mathcal{U})$ and $w \in (Q \times T)^*$ there exists $w^{'} = (q_1,t_1)\cdots(q_k,t_k), w^{'}R^*w$ such that $\forall i = 1,2,\ldots,k$ the function $u|_{[\sum_1^{i-1} t_j, \sum_1^i t_j]}$ is a constant.*

It is clear that for any $w = (q_1,t_1)(q_2,t_2)\cdots(q_k,t_k) \in (Q \times T)^*$ the value $x(x_0,u(.),w)$ depends on $u(.)|_{[0,\sum_1^k t_i)}$. Proposition 17 and Proposition 18 imply that without loss of generality it is enough to consider pairs (w,u) where $w = (q_1,t_1)\cdots(q_k,t_k) \in (Q \times T)^*$ and $u \in PC([0,\sum_1^k t_i],\mathcal{U})$, $u|_{[\sum_1^{i-1} t_j, \sum_1^i t_j)} = u_i \in \mathcal{U}$ for $i = 1,2,\ldots k$.

In the sequel we will use the following abuse of notation. For each $x \in \mathcal{X}$, $u \in \mathcal{U}^*$, $w \in Q^*$ and $\tau \in T^*$ such that $|t| = |w| = |u|$ we define

$$x(x,u,w,\tau) := x(x,\tilde{u},(w_1,t_1)(w_2,t_2)\cdots(w_k,t_k))$$

where $\tilde{u}|_{[\sum_1^{j-1} t_i, \sum_1^j t_i)} = u_j$ for $j = 1,2,\ldots,k$, and $\tilde{u}|_{[\sum_1^k t_i, +\infty)}$ is arbitrary. $x(x_0,.,.,.)$ will be considered as function with its domain in $(\mathcal{U} \times Q \times T)^*$ or equivalently in $\{(u,w,\tau) \in \mathcal{U}^* \times Q^* \times T^* \mid |u| = |w| = |\tau|\}$. It is easy to see that

$$Reach_{const}(\Sigma) = \{x(x_0,u,w,\tau) \mid (u,w,\tau) \in (\mathcal{U} \times Q \times T)^*\}$$

The main result of the chapter is the following.

Theorem 21. *Consider a switched linear system*
$\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q) \mid q \in Q\})$.

(a)

$$Reach(\Sigma) = \{A_{q_1}^{j_1} A_{q_2}^{j_2} \cdots A_{q_k}^{j_k} B_z u \mid q_1,q_2,\ldots q_k, z \in Q, j_1,j_2,\ldots j_k \geq 0, u \in \mathcal{U}\}$$

(b) *There exists a switching sequence $w \in (Q \times T)^+$ such that*

$$Reach(\Sigma) = \{x(0,u,w) \mid u \in PC(T,U)\}$$

5.2 Preliminaries on Nonlinear Systems Theory

Below the results of [71, 34, 78] will be reviewed. Basic knowledge of differential geometry is assumed. For references see [5]. In the sequel, unless stated otherwise, by manifold we mean a smooth finite-dimensional manifold, i.e. a topological space,which is a Hausdorff space, second countable and locally homeomorphic to open subsets of $\mathbb{R}^n$, and is endowed with a smooth (analytic) differentiable structure. Let M be a manifold. Then for each $x \in M$ the tangent space of M at x will be denoted by T_xM, the tangent bundle of M will be denoted by $TM = \bigcup T_xM$. Let X be a vector field of M. Then $X^t(x)$ denotes the flow of X passing through the point x at time t. The mapping $D : M \to 2^{TM}$ is called a *distribution* if for each $x \in M$, $D(x)$ is a subspace of T_xM. A sub-manifold N of M is an *integral sub-manifold* of the distribution D if for each $x \in N$ it holds that $D(x) = T_xN$. A sub-manifold N of M is called the *maximal integral sub-manifold* of D if N is connected, it is an integral sub-manifold of D and for each $N^{'}$ connected integral sub-manifold of D it holds that ($N^{'} \cap N \neq \emptyset \implies N^{'} \subseteq N$ and $N^{'}$is open in N). If N is a maximal integral sub-manifold of D and $x \in N$ then N is said to be the *maximal integral sub-manifold of D passing through x*. If for each $x \in M$ there exists a maximal integral sub-manifold of D passing through x then D is said to have the *maximal integral sub-manifold* property. There exists at most one maximal integral sub-manifold of D passing through $x \in M$.

Let $\mathcal{F} = \{X_\gamma | \gamma \in \Gamma\}$ be a family of vector fields. The *orbit of $\mathcal{F}$ through a point* $x \in M$ is the set

$$M_x^{\mathcal{F}} = \{X_1^{t_1} \circ X_2^{t_2} \circ \cdots X_k^{t_k}(x) | X_i \in \mathcal{F}, t_i \in \mathbb{R}, i = 1, \cdots, k\}$$

Let $\mathcal{F}$ be a family of vector fields over M. Define the distribution $D_{\mathcal{F}}$ as $D_{\mathcal{F}}(x) = span\{X(x) | X \in \mathcal{F}\}$. The distribution D is called $\mathcal{F}$-invariant if

(1) $\forall x \in M : D_{\mathcal{F}}(x) \subseteq D(x)$

(2) $\forall v \in D(x), \forall g : M \to M$

$$g(x) = X_1^{t_1} \circ X_2^{t_2} \circ \cdots \circ X_k^{t_k}(x) \implies \frac{dg}{dx}(x)v \in D(g(x))$$

where $X_i \in \mathcal{F}$, $t_i \in \mathbb{R}$, $i = 1, \cdots k$

Denote by $P_{\mathcal{F}}$ the smallest $\mathcal{F}$-invariant distribution containing $D_{\mathcal{F}}$. The main result of [71] is the following.

Theorem 22 (Existence of maximal integral manifold). *For each $x \in M$ the set $M_x^{\mathcal{F}}$ with a suitable topology and differentiable structure is a maximal integral* *sub-manifold of $P_{\mathcal{F}}$. $D_{\mathcal{F}}$ has maximal integral sub-manifold property if and only if $D_{\mathcal{F}} = P_{\mathcal{F}}$.*

Everything stated above also holds for analytic manifolds. For analytic manifolds the following, stronger result holds.

Proposition 19. *Let M be an analytic manifold, let $\mathcal{F}$ be a family of analytic vector fields. Denote the smallest involutive distribution containing $D_{\mathcal{F}}$ by $D^*_{\mathcal{F}}$. Then $D^*_{\mathcal{F}}$ has the maximal integral sub-manifold property. The maximal integral manifold of the distribution $D^*_{\mathcal{F}}$ passing through a point x is the orbit of $\mathcal{F}$ passing through x, i.e $M^{\mathcal{F}}_x$.*

Let M be a manifold, and let $\mathcal{F}$ be a family of vector fields over M. Let x be an element of M. The *reachable set* of $\mathcal{F}$ from x is defined as

$$Reach(\mathcal{F}, x) = \{X_1^{t_1} \circ X_2^{t_2} \circ \cdots \circ X_k^{t_k}(x) | X_i \in \mathcal{F}, t_i \geq 0, i = 1, \ldots, k\}$$

Below the main results of [34] will briefly be recalled. Let $(G, \cdot)$ be a group, $p : G \to \mathbb{R}^n$ be a function. Let $\cdot : G \times R \to G$ be a surjective mapping. The triple $\Gamma = (G, p, \mathbb{R}^n)$ is called an *abstract system*. Let $\underline{a} = (a_1, a_2, \ldots, a_p) \in G^p$, $\underline{b} = (b_1, b_2, \ldots b_k) \in G^k$ and define $\psi^{\underline{b}}_{\underline{a}} : \mathbb{R}^p \to \mathbb{R}^{kn}$ by

$$\psi^{\underline{b}}_{\underline{a}}(t) := \Big[p((t_1 \cdot a_1)(t_2 \cdot a_2) \cdots (t_p \cdot a_p) b_1), \quad \cdots, \quad p((t_1 \cdot a_1)(t_2 \cdot a_2) \cdots (t_p \cdot a_p) b_k) \Big]$$

The abstract system Γ is called *smooth* if $\psi^{\underline{b}}_{\underline{a}}$ is a smooth map for all $\underline{a} \in G^p, \underline{b} \in G^k$. Denote by $D\psi^{\underline{b}}_{\underline{a}}(t)$ the Jacobian of $\psi^{\underline{b}}_{\underline{a}}$ at $t \in \mathbb{R}^p$. Then the *rank* of p is defined to be $n = \sup_{\underline{a}, \underline{b}, t} D\psi^{\underline{b}}_{\underline{a}}(t)$ A *smooth representation* of Γ is a tuple $\Theta = (M, \{\phi_a \mid a \in G\}, h, x_0)$ where M is a smooth Hausdorff manifold, not necessarily second-countable, $\phi_a : M \to M$ are diffeomorphisms for which $\phi_{ab} = \phi_b \circ \phi_a$ and $\phi_1 = id_M$ holds, $h : M \to \mathbb{R}^n$ is a smooth map, $x_0 \in M$ is the initial state. Further, for all $\underline{a} = (a_1, a_2, \cdots, a_p) \in G^p$ define $\psi_{\underline{a}} : \mathbb{R}^p \to M$ by $\psi_{\underline{a}}(t) = \phi_{(t_1 a_1)(t_2 a_2) \cdots (t_p a_p)}(x_0)$. We require that $\psi_{\underline{a}}$ is smooth for all $\underline{a} \in G^p$ and that $p(a) = h(\psi_a(x_0))$. If $\Theta = (M, \{\phi_a \mid a \in G\}, h, x_0)$ is a representation of the abstract system Γ, then $\psi^{\underline{b}}_{\underline{a}} = \Big[h \circ \phi_{b_1} \circ \psi_{\underline{a}}, \quad \cdots, \quad h \circ \phi_{b_p} \circ \psi_{\underline{a}} \Big]$. A representation is called *reachable* if $M = \{\psi_a(x_0) \mid a \in G\}$ holds. A representation is called *transitive*, if $\forall x, y \in M (\exists g \in G : y = \phi_g(x))$ holds. If $x = \phi_{g_1}(x_0)$ and $y = \phi_{g_2}(x_0)$ then $y = \phi_{g_1^{-1} g_2}(x)$. It means that a representation is transitive if and only if it is reachable. A representation is called *distinguishable* if for all $x_1 \neq x_2 \in M$ it holds that $h(\phi_a(x_1)) \neq h(\phi_a(x_2))$ for all $a \in G$. A transitive and distinguishable representation is called *minimal*. Let $\Theta_1 = (M_1, \{\phi^1_a \mid a \in G\}, h^1, x^1_0)$ and $\Theta_2 = (M_2, \{\phi^2_a \mid a \in G\}, h^2, x^2_0)$ be two smooth representations. A smooth map $\chi : M_1 \to M_2$ is a homomorphism from the representation Θ_1 to the representation Θ_2 if the following conditions hold: $\chi(x_0^1) = x_0^2$, $h^2 \circ \chi = h^1$ and $\phi_a^2 \circ \chi = \chi \circ \phi_a^1$. In [34] the following theorem is proved.

Theorem 23. *Every smooth abstract system $(G, p, \mathbb{R}^n)$ with finite rank has a minimal smooth representation $\Theta = (M, \{\phi_a \mid a \in G\}, h, x_0)$ with $\dim M = \operatorname{rank}\ p$. If $\Theta^{'}$ is a minimal smooth representation of $(G, p, \mathbb{R}^n)$, then there exists a homomorphism χ [1] from Θ to $\Theta^{'}$ such that χ is a bijective map and $\operatorname{rank}\ \chi = \operatorname{rank}\ p$.*

5.3 Structure of the Reachable Set

Below we are going to apply the results from the previous section to determine the structure of the reachable set. The outline of the procedure is the following

- Given a linear switched system Σ, we associate a family of vector fields $\mathcal{F}$ over $\mathbb{R}^n$ with it.

- Determine the smallest distribution $D = P_{\mathcal{F}}$ invariant w.r.t the family of vector fields constructed above. Find another family of vector fields $\mathcal{F}^{'}$ which spans the distribution.

- Consider the orbit $M_0^{\mathcal{F}}$ of $\mathcal{F}$ passing through 0. By Theorem 22 it is the maximal integral sub-manifold of $P_{\mathcal{F}}$. But again by Theorem 22 and by uniqueness of maximal integral sub-manifold $M_0^{\mathcal{F}} = M_0^{\mathcal{F}^{'}}$.

- By direct computation we find the structure of $M_0^{\mathcal{F}^{'}}$ which turns out to be a subspace of $\mathbb{R}^n$ in the case of linear switched systems. Moreover, computation shows that $M_0^{\mathcal{F}^{'}} = D(0)$. Therefore, by taking $M_0^{\mathcal{F}^{'}}$ with subspace topology, and proper differentiable structure, it will be a regular sub-manifold of $\mathbb{R}^n$ and for each $x \in M_0^{\mathcal{F}^{'}}$ it holds that $D(x) = T_x M_0^{\mathcal{F}^{'}}$. Moreover, $\dim M_0^{\mathcal{F}^{'}} = \dim D(0)$.

- Consider the restriction $\Sigma^{'}$ of our switched system Σ to $M_0^{\mathcal{F}}$. Clearly, $Reach(\Sigma) = Reach(\Sigma^{'}) \subseteq M_0^{\mathcal{F}^{'}}$. Using the structure of $M_0^{\mathcal{F}} = M_0^{\mathcal{F}^{'}}$, Theorem 21 can be proved, either by using the results of [34] or by applying an elementary construction.

[1] In [34] χ is claimed to be a diffeomorphism. However, the author of the current paper failed to see how this stronger statement follows from the proof presented in [34], unless M is second-countable.

The rest of the subsection is devoted to carrying out the steps described above in a more formal way.

Consider a linear switched system Σ. Assume that for each $q \in Q$ and $u \in \mathcal{U}$ the dynamics is given by $\dot{x} = f_q(x,u) = A_q x + B_q u$. The family of vector fields $\mathcal{F}$ *associated with* Σ is defined as

$$\mathcal{F} = \{A_q x + B_q u | q \in Q, u \in \mathcal{U}\}$$

The proof of the lemma below is given in the Appendix 5.4

Lemma 28. *Consider a linear switched system Σ and the associated family of vector fields $\mathcal{F}$. The smallest involutive distribution containing $\mathcal{F}$ is of the following form*

$$\begin{aligned} D^*_{\mathcal{F}}(x) &= \mathrm{Span}\{A_{i_1}^{j_1} A_{i_2}^{j_2} \cdots A_{i_k}^{j_k} B_z u \mid i_1, i_2, \cdots i_k, z \in Q, j_1, j_2, \cdots j_k \geq 0, u \in \mathcal{U}\} \\ &\quad \cup \{[A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots] x | i_1, i_2, \cdots i_k \in Q\} \end{aligned} \tag{5.2}$$

Lemma 29. *Consider a linear switched system Σ and the family of associated vector fields $\mathcal{F}$.*

(a) The distribution $D^*_{\mathcal{F}}$ has the maximal integral manifold property. The maximal integral manifold of $D^*_{\mathcal{F}}$ passing through 0 is $M_0^{\mathcal{F}}$.

(b) $M_0^{\mathcal{F}}$ is of the form

$$W := \mathrm{Span}\{A_{i_1}^{j_1} A_{i_2}^{j_2} \cdots A_{i_k}^{j_k} B_z u \mid i_1, \ldots, i_k, z \in Q, j_1, \ldots, j_k \geq 0, u \in \mathcal{U}\} \tag{5.3}$$

Proof. `Part (a)`
Notice that $\mathbb{R}^n$ is an analytic vector field. Besides, each member of $\mathcal{F}$ is an analytic vector field. By Proposition 19 $D^*_{\mathcal{F}}$ has the integral manifold property and its maximal integral manifold passing through 0 is equal to $M_0^{\mathcal{F}}$. An alternative way to prove part (a) is to show that $D^*_{\mathcal{F}} = W$ is $\mathcal{F}$–invariant.

`Part (b)` Consider the following family of vector fields:

$$\begin{aligned} \mathcal{F}' &= \{A_{i_1}^{j_1} A_{i_2}^{j_2} \cdots A_{i_k}^{j_k} B_z u \mid i_1, \cdots i_k, z \in Q, u \in \mathcal{U}, j_1, \cdots j_k \geq 0\} \\ &\cup \{[A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots] | i_1, \cdots i_k \in Q\} \end{aligned}$$

Then for all $x \in \mathbb{R}^n$, $D^*_{\mathcal{F}}(x) = \mathrm{Span}\{X(x) | X \in \mathcal{F}'\} = D_{\mathcal{F}'}(x)$. Since $D^*_{\mathcal{F}}$ has the maximal integral manifold property, part (ii) of Theorem 22 implies that $P_{\mathcal{F}'} = D^*_{\mathcal{F}}$. By part (i) of Theorem 22 the maximal integral manifold of $D^*_{\mathcal{F}} = P_{\mathcal{F}'}$ passing through 0 is the orbit of $\mathcal{F}'$ passing through 0 i.e. $M_0^{\mathcal{F}'}$. But by the part (a) of this lemma we get that the maximal integral manifold of $D^*_{\mathcal{F}}$ passing through 0 is $M_0^{\mathcal{F}}$. So we get that $M_0^{\mathcal{F}} = M_0^{\mathcal{F}'}$.

On the other hand, we shall show that $M_0^{\mathcal{F}'}$ indeed has the structure given by (5.3).

Assume $X = A_{i_1}^{j_1} A_{i_2}^{j_2} \cdots A_{i_k}^{j_k} B_q u$. Then $X^t(z)) = z + tA_{i_1}^{j_1} A_{i_2}^{j_2} \cdots A_{i_k}^{j_k} B_q u$. So, if we identify each element of $X \in W$ with a constant vector field, then we get that $X^1(0) = X$, $\mathcal{F}' = W \cup \{[A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots] | i_1, \dots i_k \in Q\}$ and $W = \{X^1(0) \mid X \in W\} \subseteq M_0^{\mathcal{F}'}$. We need to prove that $M_0^{\mathcal{F}'} \subseteq W$. Since $0 \in M_0^{\mathcal{F}'} \cap W$ and

$$M_0^{\mathcal{F}'} = \{X_1^{t_1} \circ X_2^{t_2} \circ \dots X_k^{t_k}(0) \mid X_i \in \mathcal{F}', t_i \in \mathbb{R}, i = 1, \dots, k\}$$

it is sufficient to prove that W is invariant under $\mathcal{F}'$, i.e.

$$\forall X \in \mathcal{F}', \forall t \in \mathbb{R}, \forall z \in W : X^t(z) \in W$$

If $X = A_{i_1}^{j_1} A_{i_2}^{j_2} \cdots A_{i_k}^{j_k} B_q u$ then $X^t(z) = z + tX(0) \in W$. Assume that $X = [A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots] x$. Assume that $z \in W$. By definition of X^t and the Cayley-Hamilton theorem we get

$$\begin{aligned} X^t(z) &= \exp([A_{i_1}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots] t) z \\ &= \sum_{j=0}^{n-1} g_j(t) [A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots]^j z \end{aligned}$$

It is easy to see that $[A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots] \in \mathrm{Span}\{A_{z_1} A_{z_2} \cdots A_{z_k} \mid z_1, \dots, z_k \in Q\}$, which implies

$$z \in W \implies [A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots] z \in W$$

Then it follows easily that $z \in W \implies X^t(z) \in W$. □

Proof. Proof of Theorem 21 It is sufficient to prove that $Reach_{const}(\Sigma) = W$. Indeed, since W is a subspace of $\mathbb{R}^n$, it is closed in $\mathbb{R}^n$, so, in this case we get $W = Cl(W) = Cl(Reach_{const}(\Sigma)) = Reach(\Sigma)$. Let $\mathcal{F}$ be the family of vector fields associated to Σ. For $X_i = A_{q_i} x + B_{q_i} u_i \in \mathcal{F}, t_i \in \mathbb{R}$, $i = 1, 2, \dots, k$, $k \geq 0$ denote

$$X_1^{t_1} \circ X_2^{t_2} \circ \cdots \circ X_k^{t_k}(x_0) = x(x_0, u_1 u_2 \cdots u_k, q_1 q_2 \cdots q_k, t_1 t_2 \cdots t_k)$$

It follows that $Reach(\mathcal{F}, 0) = Reach_{const}(\Sigma)$. On the other hand $Reach(\mathcal{F}, 0) \subseteq M_0^{\mathcal{F}}$. From Lemma 29 we get that $M_0^{\mathcal{F}} = W$. Let $n = \dim W$ and let $b_1, \dots, b_n$ be a basis of W. Let $T : W \to \mathbb{R}^n$ be a linear isomorphism. It follows that for each $b_i, i = 1, \dots, n$ there exists vector fields $X_{i,1}, \dots X_{i,n_i} \in \mathcal{F}, n_i \geq 0$ such that $b_i = X_{i,n_i}^{t_{i,n_i}} \circ X_{i,n_i-1}^{t_{i,n_i-1}} \cdots X_{i,1}^{t_{i,1}}(0)$ for some $t_{i,1}, \dots, t_{i,n_i} \in \mathbb{R}$. Assume that $X_{i,j} = A_{q_{i,j}} x + B_{q_{i,j}} u_{i,j}$. Define $u_i = u_{i,1} \cdots u_{i,n_i}$, $w_i = q_{i,1} \cdots q_{i,n_i}$ $\tau_i = \tau_{i,1} \cdots \tau_{i,n_i}$. With the notation above we get that $x(0,u_i,w_i,\tau_i) = b_i$. For any sequence $s = s_1 \cdots s_k$ let $\overleftarrow{s} = s_k s_{k-1} \cdots s_1$, and $-s = (-s_1)(-s_2)\cdots(-s_k)$. Then define the sequences $w = \overleftarrow{w}_1\, w_1 \cdots \overleftarrow{w}_{n-1}\, w_{n-1}\, \overleftarrow{w}_n\, w_n$, $\tau = (-\overleftarrow{\tau}_1)\tau_1(-\overleftarrow{\tau}_2)\tau_2\cdots(-\overleftarrow{\tau}_{n-1})\tau_{n-1}(-\overleftarrow{\tau}_n)\tau_n$. Let $v_i = \mathbb{O}_1\mathbb{O}_1\cdots\mathbb{O}_{i-1}\mathbb{O}_{i-1}\mathbb{O}_i u_i \mathbb{O}_{i+1}\mathbb{O}_{i+1}\cdots\mathbb{O}_n\mathbb{O}_n$, where $\mathbb{O}_i = 00\cdots 0 \in \mathcal{U}^{|w_i|}, i = 1,\dots,n$. Then it is easy to see that

$$x(0,v_i,w,\tau) = b_i, \quad i = 1,\dots,n$$

Indeed, $x(0,v_i,w,\tau) = x(y_i, s_i^{'}, \beta_i^{'}, \gamma_i^{'})$, where $y_i = x(x(0,s_i,\beta_i,\gamma_i),u_i,w_i,\tau_i)$, $s_i = \mathbb{O}_1\mathbb{O}_1\cdots\mathbb{O}_{i-1}\mathbb{O}_{i-1}\mathbb{O}_i$, $\gamma_i = (-\overleftarrow{\tau}_1)\tau_1\cdots(-\overleftarrow{\tau}_{i-1})\tau_{i-1}(-\overleftarrow{\tau}_i)$, $\beta_i = \overleftarrow{w}_1\, w_1 \cdots \overleftarrow{w}_{i-1}\, w_{i-1}\, \overleftarrow{w}_i$, $s_i^{'} = \mathbb{O}_{i+1}\mathbb{O}_{i+1}\cdots\mathbb{O}_n\mathbb{O}_n$, $\gamma_i^{'} = (-\overleftarrow{\tau}_{i+1})\tau_{i+1}\cdots(-\overleftarrow{\tau}_n)\tau_n$, $\beta_i^{'} = \overleftarrow{w}_{i+1}\, w_{i+1}\cdots\overleftarrow{w}_n\, w_n$. It is easy to see that for any $(s,d) \in (Q\times\mathbb{R})^*$, $x(0,\mathbb{O}_{|s|},s,v) = 0$, $\mathbb{O}_{|s|} = 0\cdots 0 \in \mathcal{U}^{|s|}$. Thus, $x(0,s_i,v_i,\gamma_i) = 0$ and $y_i = x(0,u_i,w_i,\tau_i) = b_i$. It is easy to see that for all $(u,s,d) \in (\mathcal{U}\times Q\times\mathbb{R})^*$, $x(y,\overleftarrow{u}\, u,\overleftarrow{s}\, s,(-\overleftarrow{d})d) = y, y \in W$. That is, by noticing that $\overleftarrow{\mathbb{O}}_i = \mathbb{O}_i$, we get that $x(y,s_i^{'},\beta_i^{'},\gamma_i^{'}) = y, y\in W$, thus $x(0,v_i,w,\tau) = x(b_i,s_i^{'},\beta_i^{'},\gamma_i^{'}) = b_i$. Let $N = 2n$ and define the function $M : \mathbb{R}^N \to \mathbb{R}^{n\times n}$ by

$$M(\eta) = \Big[Tx(0,v_1,w,\eta), \quad \dots \quad ,Tx(0,v_n,w,\eta)\Big]$$

Then $\eta \mapsto \det M(\eta)$ is an analytic function and $\det M(\tau) \neq 0$. By the well-known property of analytic functions there exists a $\psi = (\psi_1,\dots,\psi_N) \in \mathbb{R}^N$, $\psi_1,\dots,\psi_N \geq 0$ such that $\det M(\psi) \neq 0$, that is, rank $M(\psi) = n$. It implies that $W = T^{-1}(\mathbb{R}^n) = \mathrm{Span}\{x(0,v_i,w,\psi) \mid i = 1,\dots,n\} = \{x(0,\sum_{i=1}^n \alpha_i v_i, w,\psi) \mid \alpha_1,\dots,\alpha_n \in \mathbb{R}\} \subseteq \mathit{Reach}(\Sigma)$, therefore

$$\{x(0,u,w,\psi) \mid u \in PC_{const}(T,\mathcal{U})\} = W = \mathit{Reach}(\Sigma)$$

That is, we get part (b) of the theorem, which implies part (a).

An alternative approach will be presented below. This approach uses the results from [34]. We proceed by proving part (b) of theorem, which already implies part (a). Define $G = (\mathcal{U}\times Q\times R)^*/\sim$, where $\sim$ is the smallest congruence relation such that $(u,q,0) \sim 1$ and $(u,q,t_1)(u,q,t_2) \sim (u,q,t_1+t_2)$. Denote by $[(u,w,\tau)] \in G$ the equivalence class represented by $(u,w,\tau) \in (\mathcal{U}\times Q\times R)^*$. The definition of G is essentially identical to the definition of the group of piecewise-constant inputs in [34]. Define the map $Z : \mathcal{X}\times(\mathcal{U}\times Q\times T)^* \to \mathcal{X}$ by $Z(z,u,w,\tau) := x(z,u,w,\tau)$. It is clear that the dependence of Z on the switching times is analytic, i.e. $\forall u \in U^*, w \in Q^*, x \in \mathcal{X} : Z(x,u,w,.) : T^{|w|} \to \mathcal{X}$ is analytic . From Proposition 8 it is clear that by the principle of analytic continuation $Z(x,u,w,.)$ can be extended to $\mathbb{R}^{|w|}$. From now on we will identify Z with this extension. Then it is easy to see that Z is in fact a function on G, since $(u,w,\tau) \sim (u^{'},w^{'},\tau^{'}) \implies Z(x,u,w,\tau) = Z(x,u^{'},w^{'},\tau^{'})$ for all $x \in \mathcal{X}$. Define

$$\Theta = (W, \{\phi \mid A \in G\}, 0, id)$$

where $W = M_0^{\mathcal{F}}$ as above and $\phi_{[(u,w,\tau)]}(x) = Z(x,u,w,\tau)$. Now, define $\cdot : G \times \mathbb{R} \to G$ by $[(u,w,\tau)] \cdot \alpha = [(\alpha u, w, \tau)]$. It is easy to see that Θ is a smooth representation of R with respect to $\cdot$, Θ is transitive and distinguishable, thus minimal. Recall the definition of the function $\psi_{\underline{a}}^{\underline{b}}$ from Section 5.2. Let $d = \text{rank}\ \ R = \sup_{\underline{a},\underline{b},\underline{\mu}} \text{rank}\ D\psi_{\underline{a}}^{\underline{b}}(\underline{\mu})$. We want to show that $d = \dim W = n$. Let $\Theta_m = (M_m, \{\phi_a^m \mid a \in G\}, h^m, x_0^m)$ be a minimal smooth representation of R w.r.t $\cdot$, such that $\dim M_m = d$ as described in Theorem 23. Let $\chi : M_m \to W$ the representation homomorphism described in Theorem 23. We shall prove that χ is a diffeomorphism. Since W is a second-countable Hausdorff-manifold, we get that W has a positive-definite Riemannian structure. Since χ is an immersion, Proposition 9.4.2 of [18] implies that M_m has a positive-definite Riemannian structure. We shall show that M_m is connected. If M_m is connected and has a positive-definite Riemanian structure, then M_m is a second countable Hausdorff manifold by Proposition 10.6.4 of [18]. But then bijectivity of χ implies that $\dim M_m = \dim W = d = n$. To see that M_m is connected, notice that for any $g = [(u,w,\tau)] \in G$ it holds that $R((0 \cdot g)[(s,v,t)]) = x(0, 0s, wv, \tau t) = R([(s,v,t)])$. That is, $h^m \circ \phi_{[(s,v,t)]}^m \circ \psi_g^m(0) = R([(s,v,t)]) = h^m \circ \phi_{[(s,v,t)]}^m(x_0)$. Since Θ_m is indistinguishable, it implies that $\psi_g^m(0) = x_0$. For any $x \in M_m$ there exists a $g^{'}$ such that $\phi_{g'}^m(x_0) = x$, by transitivity of Θ_p. But then there exists g, α such that $\alpha \cdot g = g^{'}$. Since Θ_m is a smooth representation, the map ψ_g^m is smooth, therefore continuous, which implies that $\psi_g^m(\mathbb{R})$ is connected. That is, $x_0 = \phi_g^m(0)$ and $x = \phi_g^m(\alpha)$ are in the same connected component of M_m. Since x is an arbitrary element of M_m, we get that M_m is connected.

Now, let $\underline{a} = (a_1, a_2, \ldots, a_k) \in G^k, \underline{b} = (b_1, b_2, \ldots, b_p) \in G^p, \underline{\mu} \in \mathbb{R}^k$ such that rank $D\psi_{\underline{a}}^{\underline{b}}(\underline{\mu}) = n$. Assume that $a_j = [(s_j, r_j, \gamma_j)] \in G$ and $b_i = [(v_i, w_i, \sigma_i)] \in G$. For all $z = z_1 z_2 \cdots z_k \in Q^*$ and $\tau = \tau_1 \tau_2 \cdots \tau_k$ denote by $\exp(A_z \tau)$ the expression $\exp(A_{z_k}\tau_k)\exp(A_{z_{k-1}}\tau_{k-1}) \cdots \exp(A_{z_1}\tau_1)$. For each $\underline{t} = (t_1, \ldots, t_k) \in \mathbb{R}^k$, let $M_j(\underline{t}) = x(0, s_j \underbrace{00\cdots 0}_{k-j-times}, r_j r_{j+1} \cdots r_k, t_j t_{j+1} \cdots t_k)$. We get that

$$\begin{aligned}
D\psi_{\underline{a}}^{b_i}(\mu) &= D_{\mu_1,\mu_2,\cdots,\mu_k}\phi_{b_i(a_1\cdot\mu_1)(a_2\cdot\mu_2)\cdots(a_k\cdot\mu_k)}(0) = D_{\mu_1,\mu_2,\ldots,\mu_k}[x(0,v_i,w_i,\sigma_i)+ \\
&+ \exp(A_{w_i}\sigma_i)x(0,(\mu_1 s_1)(\mu_2 s_2)\cdots(\mu_k s_k), r_1\cdots r_k, \gamma_1\cdots\gamma_k)] = \\
&= D_{\mu_1,\mu_2,\ldots,\mu_k}\exp(A_{w_i}\sigma_i)\sum_{j=1}^{k}\mu_j x(0, s_j \underbrace{00\cdots 0}_{k-jtimes}, r_j r_{j+1}\cdots r_k, \gamma_j\gamma_{j+1}\cdots\gamma_k) \\
&= \exp(A_{w_i}\sigma_i)M(\underline{\gamma})
\end{aligned}$$

where $\underline{\gamma} = (\gamma_1, \dots, \gamma_k)$ and $M(\underline{\gamma}) = \Big[M_1(\underline{\gamma}), \quad M_2(\underline{\gamma}), \quad \dots, \quad M_k(\underline{\gamma})\Big]$. Thus,

$$D\psi^{b}_{\underline{a}}(\mu) = \begin{bmatrix} \exp(A_{w_1}\sigma_1) & 0 & \cdots & 0 \\ 0 & \exp(A_{w_2}\sigma_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \exp(A_{w_k}\sigma_k) \end{bmatrix} \begin{bmatrix} M(\underline{\gamma}) \\ M(\underline{\gamma}) \\ \cdots \\ M(\underline{\gamma}) \end{bmatrix}$$

It follows that $n = \text{rank } D\psi^{b}_{\underline{a}}(\underline{\mu}) = \text{rank } M(\underline{\gamma})$. Notice that the dependence of $M(\underline{t})$ on $\underline{t}$ is analytic. Then it follows that we can choose $\underline{t} \in T^k$ such that rank $M(\underline{t}) = n$. Since $\sum_{j=1}^{k} \alpha_j M_j(\underline{t}) = x(0, (\alpha_1 s_1)\cdots(\alpha_k s_k), r_1 \cdots r_k, t_1 \cdots t_k)$ and $\dim \text{Im} M = \dim \mathit{Reach}(\Sigma)$, it follows that

$$\mathit{Reach}(\Sigma) = \text{Im } M = \{x(0, u(.), (r_1, t_1)(r_2, t_2)\cdots(r_k, t_k)) \mid u(.) \in PC(T, \mathcal{U})\}$$

□

5.4 Appendix

Proof. Proof of Lemma 28 The following two facts will be used in the proof.

- Let X, Y be vector fields over $\mathbb{R}^n$ of the form $X(x) = Ax$, $Y(x) = y$ for some $A \in \mathbb{R}^{n\times n}$ and $y \in \mathbb{R}^n$. Then in the usual coordinates $[X, Y](x) = -Ay$.
- For $i = 1, 2, \dots, k$ let X_i be vector fields of the form $X_i(x) = A_i x$. Then

$$[X_1, [X_2, \cdots [X_{k-1}, X_k] \cdots](x) \in \\ \text{Span}\{A_{\pi(1)}A_{\pi(2)} \cdots A_{\pi(k)} \mid \pi(1), \pi(2), \dots, \pi(k) \in \{1, 2, \dots, k\}$$

Clearly, $D^*_{\mathcal{F}} = \text{Span}\{[f_1, [f_2, [\cdots [f_{k-1}, f_k] \cdots] \mid f_i \in \mathcal{F} i = 1, 2, \cdots k\}$. Denote the right-hand side of (5.2) by D. First $D \subseteq D^*_{\mathcal{F}}$ will be proved. Since $A_q x + B_q \mathbf{0} = A_q x \in \mathcal{F}$ then we get that $[A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots]x \in D^*_{\mathcal{F}}$ for all $i_1, \cdots i_k \in Q$. Clearly $[A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}x + B_{i_k}u_k] \cdots](x)$ belongs to $D^*_{\mathcal{F}}$. But by linearity of the Lie-brackets we get

$$[A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}x + B_{i_k}u_k] \cdots](x) = \\ [A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots](x) - A_{i_1}A_{i_2} \cdots A_{i_{k-1}}B_{i_k}u_k$$

From this and the fact that

$$[A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots]x \in D^*_{\mathcal{F}}$$

we get that $A_{i_1}A_{i_2}\cdots A_{i_{k-1}}B_{i_k}u_k \in D^*_{\mathcal{F}}$ for all $i_1,\cdots i_k \in Q$ and $u_k \in \mathcal{U}$. So we get that $D \subseteq D^*_{\mathcal{F}}$. The reverse inclusion $D^*_{\mathcal{F}} \subseteq D$ will be shown by proving that for all $f_1,\cdots f_k \in \mathcal{F}$ the vector field $[f_1,[f_2,\cdots[f_{k-1},f_k]\cdots]$ belongs to D. This is done by induction on the length of expression. For $k=1$ it is true, since $\mathcal{F} \subseteq D$. Assume it is true for all expression of length $\le k$. Consider the expression $[f_1,[f_2,\cdots[f_k,f_{k+1}]\cdots]$. The vector field $[f_2,[f_3,\cdots[f_k,f_{k+1}]\cdots]$ belongs to D. By linearity of Lie-brackets it is enough to prove that for all $f = A_qx + B_qu$ and for all $Y = A_{i_1}A_{i_2}\cdots A_{i_l}B_zw$ or $Y = [A_{i_1},[A_{i_2},\cdots[A_{i_{l-1}},A_{i_{l-1}}]\cdots]$ it holds that $[f,Y] \in D$. For the first case we get

$$\begin{aligned}[A_qx + B_qu, Y] = [A_qx, Y] + [B_qu, Y] = [A_qx, A_{i_1}A_{i_2}\cdots A_{i_l}B_zw] +\\ +[B_qu, A_{i_1}A_{i_2}\cdots A_{i_l}B_zw] = -A_qA_{i_1}A_{i_2}\cdots A_{i_l}B_qw\end{aligned}$$

For the second case we get that

$$\begin{aligned}&[A_qx + B_qu, Y] = [A_qx, Y] + [B_qu, Y] = [A_qx, [A_{i_1},[A_{i_2},\cdots[A_{i_l}, Ai_{l-1}]\cdots]x]\\ &\quad +[B_qu, [A_{i_1},[A_{i_2},\cdots[A_{i_l}, A_{i_{l-1}}]\cdots]x] = [A_qx, [A_{i_1},[A_{i_2},\cdots[A_{i_l}, Ai_{l-1}]\cdots]x]\\ &\quad +[A_{i_1},[A_{i_2},\cdots[A_{i_l}, A_{i_{l-1}}]\cdots]B_qu \in D\end{aligned}$$

$\square$

Chapter 6

Realization Theory of Linear Switched Systems: an elementary construction

In this chapter an alternative approach to realization theory of linear switched systems will be presented. In contrast to the solution presented in Section 4.1, the solution formulated in this chapter does not require any use of formal power series. Instead, a direct construction of a linear switched system realization will be formulated. The issue of minimality will be approached using abstract systems theory.

Although the main results of the current chapter are special cases of the results proven in Section 4.1, the current chapter still contains interesting and useful ideas, which give an extra insight to realization theory of linear switched systems.

Unlike in Section 4.1, in this chapter we will consider the realization problem of a one single input-output map. We will look for linear switched systems which realize that input-output map from *zero initial state* . We could already see before that the zero initial state plays a special role for linear switched systems, similar to the zero initial state for linear systems. In particular, the set of states reachable from zero forms a vector space. Thus, semi-reachability and reachability coincide for linear switched systems with zero initial state.

More specifically, the chapter tries to answer the following two questions.

- Does there exist an algorithm, which, given a linear switched system Σ, constructs a minimal linear switched system $\Sigma^{'}$ such that Σ and $\Sigma^{'}$ are input-output equivalent.

- Given an input-output map y, what are the necessary and sufficient conditions for the existence of a linear switched system realizing the map y. Does there exist a procedure to construct a minimal linear switched system which realizes y.

The chapter presents a procedure for constructing a *minimal* (with the state-space of the smallest possible dimension, observable and controllable) linear switched system from a given linear switched system. The minimal linear switched system constructed by the procedure is equivalent as a realization to the original system. The procedure also gives a Kalman-like decomposition of the matrices of the original system. It is also proven that all minimal systems are *algebraically similar*, meaning that they are defined on vector spaces of the same dimension and their matrices can be transformed to each other by a basis transformation.

The chapter also deals with the inverse problem i.e., consider an input-output function and formulate necessary and sufficient conditions for the existence of a linear switched system which is a realization of the given input-output map. The chapter presents a set of conditions which are necessary and sufficient for the existence of such a realization. The proof of the sufficiency of these conditions also gives a procedure for constructing a minimal realization of the given input-output map. The necessary and sufficient conditions include a finite-rank condition which is reminiscent of the Hankel-matrix rank condition for linear systems. In fact, the classical conditions for the realisability of an input-output map by a linear system and the classical construction of the minimal linear system realizing the given input-output map are a special case of the results presented in the chapter.

In order to develop realization theory for linear switched systems, abstract realization theory for initialised systems (see [61]) has been used. In fact, even the definition of minimality for linear switched systems isn't that obvious. The approach taken in this chapter is to treat switched systems as a subclass of abstract initialised systems and use the concepts developed for abstract initialised systems.

Although the results on the realization theory of linear switched systems bear a certain resemblance to those of finite-dimensional linear systems, the former is by no means a straightforward extension of the latter. As the results of this and other papers demonstrate, the approach "apply the well-known linear system theory to each continuous system and combine the results in a smart way" doesn't always work. Reachability, observability and the realization theory of linear switched systems belong to the class of problems, for which classical linear system theory can't be applied. This also shows up on the results. For example, if a linear switched system is reachable, it doesn't mean that any of the linear systems constituting the switched system has to be reachable, nor does it imply that any point of the continuous state space can be reached by some continuous component. The same holds for the observability (in sense of indistinguishability) of linear switched systems. The reader who wishes to verify these statements is encouraged to consult [69]. In the light of these remarks it is not that surprising that a minimal linear switched system may have non-minimal continuous components. That is, if a linear switched system is minimal, it does not imply that any of its continuous components is minimal. On the other hand, the approach to the realization theory taken in the chapter bears a certain resemblance with the works on realization theory for nonlinear systems presented in [34, 35, 6]. In some sense linear switched systems have more in common with non-linear than with linear systems.

The outline of the chapter is the following. Section 6.1 describes some properties and concepts related to linear switched systems which are used in the rest of the chapter. Section 6.2 presents the minimisation procedure and the Kalman-decomposition for linear switched systems. The construction of the minimal linear switched system realizing a given input-output map can be found in Section 7.1.2

6.1 Linear Switched Systems: Basic Definition and Properties

The section is divided into several subsections. Subsection 6.1.1 contains the necessary definitions and results of switched systems. It also contains a reformulation of switched systems with fixed initial state as *initialised systems*. Due to this reformulation, some fundamental system theoretic concepts for switched systems, which were already defined in Section 2.4, need a slight reformulation too. This subsection also describes some basic properties of the input-output behaviour induced by switched systems. Subsection 6.1.1 deals with the definition and basic properties of minimal switched systems. Subsection 6.1.2 introduces linear switched systems and gives a brief overview of those properties of linear switched systems which are relevant for the realization theory.

6.1.1 Switched systems as initialised systems

Recall the notion of switched systems from Section 2.4 and the notion of linear switched systems from Section 4.1. As it was already indicated in the introduction, in this chapters we are mainly concerned with switched systems with fixed initial state and for linear switched systems this initial state is going to be the zero state.

Recall the notion of *initialised system* from [61]. In the sequel, we will identify switched systems with initialised systems. More precisely, with a given switched system $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{f_q \mid q \in Q, u \in \mathcal{U}\},\{h_q \mid q \in Q\})$ with a fixed initial state x_0. We will denote such a switched system with fixed initial state x_0 by $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{f_q \mid q \in Q, u \in \mathcal{U}\},\{h_q \mid q \in Q\},x_0)$. With each such switched system we associate the initialised system $\Sigma_{init} = (T,\mathcal{X},\mathcal{Y},\mathcal{U}\times Q,\phi,h,x_0)$ where ϕ and h are defined in the following way. The domain D_ϕ of the state-transition map is defined as the set of tuples $(\tau,\sigma,x,\omega) \in T\times T\times\mathcal{X}\times(\mathcal{U}\times Q)^{[\sigma,\tau)}$ such that $\pi_Q\circ\omega$ is piecewise constant. The mapping $\phi : D_\phi \to \mathcal{X}$ is defined as $\phi(\tau,\sigma,x_i,\omega) = x_\Sigma(x_i,\text{Shift}_{-\sigma}(\pi_{\mathcal{U}}\circ\omega),w)(\tau-\sigma)$ where $w = (q_1,t_1)(q_2,t_2)\cdots(q_k,t_k) \in (Q\times T)^+$ is any sequence such that $\widetilde{w} = \pi_Q\circ\omega$ holds. Since $x_\Sigma(x_0,u(.),w)$ depends on $\widetilde{w}$ rather than on w, the mapping ϕ above is well defined. The readout map $h : \mathcal{U}\times Q\times T\times X \to \mathcal{Y}$ is defined as $h(u,q,t,x) = h_q(x)$. It is easy to see that the initialised system corresponding to a switched system is time-invariant and complete. In the sequel whenever the term "initialised system" is used, we will mean time-invariant complete initialised system.

Note that in the definition of *initialised systems* in [61] the readout map depends on the time and state only. However it is easy to see that the whole theory also holds if one allows readout maps which depend on the input. For more on this see Chapter 2, Section 2.12 of [61].

The identification of switched systems with the initialised systems allows us to use the terminology and results of [61]. In particular, notions such as input-output behaviour, system morphism, response (input-output) map of a system from a state, the reachable set, reachability, observability (indistinguishability), canonical systems, system equivalence, minimal system, minimal representation, of an input-output map are well defined for initialised systems. Since switched systems form a subclass of initialised systems, these definitions can be directly applied to switched systems. However, for the sake of completeness these relevant notions will be repeated specifically for switched systems.

the reader is asked to consult [61]. With the abuse of terminology and notation, when referring to the input/output map and the trajectory of a switched system, we shall mean the mappings $y_\Sigma(x_0,.,.)$ and x_Σ. For a given switched system Σ, the *reachable set*, i.e the set of states reachable from the initial state x_0, will be denoted by $Reach(\Sigma)$, i.e. with the notation of Section 2.4, $Reach(\Sigma) = Reach(\Sigma,\{x_0\})$

Let $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{f_q \mid q \in Q, u \in \mathcal{U}\},\{h_q \mid q \in Q\},x_0)$ be a switched system. The map

$$y_\Sigma : PC(T,\mathcal{U}) \times (Q\times T)^+ \to \mathcal{Y}^T$$

defined by $y_\Sigma(u(.),w) = y_\Sigma(x_0,u(.),w)$ $(u(.) \in PC(T,\mathcal{U}),\ w \in (Q\times T)^+)$ is called the *input-output map* (or the *input-output behaviour*) induced by Σ. The switched system

Σ is said to be a *realization* of an input-output map $\psi : PC(T,\mathcal{U}) \times (Q \times T)^+ \to \mathcal{Y}^T$ if $y_\Sigma = \psi$, i.e. the input-output behaviour induced by Σ is identical to ψ. In the terminology of Section 2.4, Σ is a realization of ψ if $(\widetilde{\Sigma}, \mu)$ is a realization of the singleton set of input-output maps $\{\psi\}$, where $\widetilde{\Sigma} = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\})$ and $\mu(\psi) = x_0$, i.e. $\widetilde{\Sigma}$ is the same switched system as Σ, except that it has no fixed initial state.

A *system morphism* $\phi : \Sigma_1 \to \Sigma_2$ between switched systems

$$\Sigma_1 = (T, \mathcal{X}_1, \mathcal{U}, \mathcal{Y}, Q, \{f^1_q(., u) \mid u \in \mathcal{U}, q \in Q\}, \{h^1_q | q \in Q\}, x^1_0)$$

and

$$\Sigma_2 = (T, \mathcal{X}_2, \mathcal{U}, \mathcal{Y}, Q, \{f^2_q(., u) \mid u \in \mathcal{U}, q \in Q\}, \{h^2_q | q \in Q\}, x^2_0)$$

is a mapping $\phi : \mathcal{X}_1 \to \mathcal{X}_2$ such that

- $\phi(x^1_0) = x^2_0$
- for each $x \in \mathcal{X}_1$, $u(.) \in PC(T,\mathcal{U})$, $w \in (Q \times T)^+$ and $t \in \mathrm{dom}(\widetilde{w})$ it holds that $\phi(x_{\Sigma_1}(x, u(.), w)(t)) = x_{\Sigma_2}(\phi(x), u(.), w)(t)$
- for each $q \in Q$ and $x \in X_1$ it holds that $h^1_q(x) = h^2_q(\phi(x))$

An immediate consequence of the characterisation above is that whenever $\phi : \Sigma_1 \to \Sigma_2$ is a system morphism then it holds that $y_{\Sigma_1}(x, u(.), w) =$
$= y_{\Sigma_2}(\phi(x), u(.), w)$ for each $x \in \mathcal{X}_1$, $u(.) \in PC(T,\mathcal{U})$ and $w \in (Q \times T)^+$. Thus the switched systems Σ_1 and Σ_2 above induce the same input-output behaviour. Two switched systems

$$\Sigma_1 = (T, \mathcal{X}_1, \mathcal{U}, \mathcal{Y}, Q, \{f^1_q(., u) \mid u \in \mathcal{U}, q \in Q\}, \{h^1_q | q \in Q\}, x^1_0)$$

and

$$\Sigma_2 = (T, \mathcal{X}_2, \mathcal{U}, \mathcal{Y}, Q, \{f^2_q(., u) \mid u \in \mathcal{U}, q \in Q\}, \{h^2_q | q \in Q\}, x^2_0)$$

are called *(input-output) equivalent* if they induce the same input-output behaviour, i.e. $y_{\Sigma_1} = y_{\Sigma_2}$ holds.

Consequently, if two switched systems are related by a system morphism, then they are input-output equivalent. A system morphism is called *isomorphism* whenever it is bijective as a mapping between the state spaces. Two systems are called an *isomorphic* if there exists an isomorphism between them.

A switched system Σ is called *minimal*, if for each reachable switched system Σ' such that Σ' and Σ are input-output equivalent, there exists a unique surjective system morphism $\phi : \Sigma' \to \Sigma$.

A switched system $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{f_q \mid q \in Q, u \in \mathcal{U}\},\{h_q \mid q \in Q\},x_0)$ is *reachable* if

$$\begin{aligned} Reach(\Sigma) = \{x_\Sigma(x_0,u(.),w)(t) \mid u(.) \in PC(T,\mathcal{U}),\\ w \in (Q\times T)^+, t \in \mathrm{dom}(\widetilde{w})\} = \mathcal{X} \end{aligned}$$

A switched system $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{f_q \mid q \in Q, u \in \mathcal{U}\},\{h_q \mid q \in Q\},x_0)$ is called *observable* if for each $x_1,x_2 \in \mathcal{X}$ the equality $\forall w \in (Q\times T)^+, u(.) \in PC(T,\mathcal{U}) : y_\Sigma(x_1,u(.),w) = y_\Sigma(x_2,u(.),w)$ implies $x_1 = x_2$. That is, Σ is observable if and only if $\widetilde{\Sigma} = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{f_q \mid q \in Q, u \in \mathcal{U}\},\{h_q \mid q \in Q\})$ is observable according to the definition of Section 2.4. A reachable and observable switched system is called *canonical*.

Consider a switched system $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{f_q \mid q \in Q, u \in \mathcal{U}\},\{h_q \mid q \in Q\},x_0)$. The input-output behaviour induced by Σ is a map $y : PC(T,\mathcal{U}) \times (Q\times T)^+ \to \mathcal{Y}^T$. For each map $y : PC(T,\mathcal{U}) \times (Q\times T)^+ \to \mathcal{Y}^T$ we shall define a map $\widetilde{y} : (\mathcal{U}\times Q\times T)^+ \to \mathcal{Y}$ such that Σ is a realization of y if and only if Σ is a realization of $\widetilde{y}$ in the sense defined below.

Denote by $PC_{const}(T,\mathcal{U})$ the set of piecewise-constant input functions. It is well-known that for each $u(.) \in PC(T,\mathcal{U})$ there exists a sequence $u_n(.) \in PC_{const}(T,\mathcal{U}), n \in \mathbb{N}$ such that $\lim_{n\to+\infty} u_n(.) = u(.)$ in $||.||_1$ norm. Given a switched system Σ, by the continuity of the solutions of differential equations we get that

$$\lim_{n\to+\infty} x_\Sigma(x,u_n(.),w)(t) = x_\Sigma(x,u(.),w)(t)$$

and

$$\lim_{n\to+\infty} y_\Sigma(x,u_n(.),w)(t) = y_\Sigma(x,u(.),w)(t)$$

It is also easy to see that for any $u(.) \in PC_{const}(T,\mathcal{U})$ and for any $w \in (Q\times T)^+$ there exists a sequence $z = (q_1,t_1)(q_2,t_2)\cdots(q_k,t_k) \in (Q\times T)^+$ such that $\widetilde{w} = \widetilde{z}$ and $u|_{[\sum_1^i t_i, \sum_1^{i+1} t_i)}$ is constant for $i = 0,\dots,k-1$. This, of course, implies that $x_\Sigma(x,u(.),w) = x_\Sigma(x,u(.),z)$ and $y_\Sigma(x,u(.),w) = y_\Sigma(x,u(.),z)$. This simple fact lies in the heart of the proof of Proposition 20.

Let $\phi : PC(T,\mathcal{U}) \times (Q\times T)^+ \to \mathcal{Y}^T$. Define $\widetilde{\phi} : (\mathcal{U}\times Q\times T)^+ \to \mathcal{Y}$ as

$$\widetilde{\phi}((u_1,q_1,t_1)(u_2,q_2,t_2)\cdots(u_k,q_k,t_k)) = \phi(\widetilde{v},(q_1,t_1)(q_2,t_2)\cdots(q_k,t_k),\sum_1^k t_i)$$

where $v = (u_1,t_1)(u_2,t_2)\cdots(u_k,t_k) \in (\mathcal{U}\times T)^+$. Define the realization of a map $\psi : (\mathcal{U}\times Q\times T)^+ \to \mathcal{Y}$ in the following way

Definition 13. *Consider a function $\psi : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$ and a switched system*

$$\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\}, x_0)$$

The switched system Σ is a realization of ψ if $\widetilde{y}_\Sigma = \psi$.

The following proposition, proof of which is straightforward, gives the justification of the concept introduced in Definition 13

Proposition 20. *Consider a function $y : PC(T, \mathcal{U}) \times (Q \times T)^+ \to \mathcal{Y}^T$. If the input-output map y has a realization by a switched system then the following conditions hold*

1. *For each $w, z \in (Q \times T)^+$, $u \in PC(T, \mathcal{U})$ it holds that $\mathrm{dom}(y(u(.), w)) = \mathrm{dom}(\widetilde{w})$ and $\widetilde{z} = \widetilde{w} \implies y(u(.), w) = y(u(.), z)$.*
2. *For each $w \in (Q \times T)^+$ and $u_n, u(.) \in PC(T, \mathcal{U})$:*

$$\lim_{n\to\infty} u_n(.) = u(.) \implies \lim_{n\to\infty} y(u_n(.), w)(t) = y(u(.), w)(t), (\forall t \in \mathrm{dom}(\widetilde{w})).$$

If y is an arbitrary map which satisfies conditions 1 and 2, then a switched system Σ is a realization of y if and only if it is a realization of $\widetilde{y}$ in the sense of Definition 13

Definition of minimal switched systems

For linear systems the definition of minimality is clear, but for more general systems there is no standard definition of minimality. The definition of minimality used in this paper is analogous to that of abstract system theory, see [46, 16]. We first define minimality for initialised systems. In the sequel we will use the terminology of [61]. Let Θ be any subclass of initialised systems. An initialised system $\Sigma \in \Theta$ is called Θ–*minimal*, if for each reachable initialised system $\Sigma^{'} \in \Theta$ such that $\Sigma^{'}$ and Σ induce the same input-output behaviour, there exists a unique surjective system morphism $\phi : \Sigma^{'} \to \Sigma$. It is an easy consequence of the definition that all Θ–minimal systems realizing the same input-output behaviour are isomorphic. Denote by Ω the whole class of initial systems. It follows from Section 6.8, Theorem 30 of [61] that each canonical initialised system is Ω–minimal. It also follows from Section 6.8 of [61] that for each input-output map realizable by initialised systems there exists a canonical realization of that input-output map. Thus we get that for each input-output map realizable by initialised systems there exist a Ω–minimal initialised system realizing it. Since all minimal systems are isomorphic and reachability and observability are preserved by isomorphisms, we get that an initial system is Ω–minimal if and only if it is canonical, i.e. reachable and observable. Notice that existence of a minimal system realizing an input-output map is a property of the input-output map. Moreover, if an input-output map has a realization by an initialised system belonging to a certain class Θ (for example it has a realization by a switched system), then the input-output map need not have a Θ–minimal realization. It is easy to see that if $\Theta^{'} \subseteq \Theta$ then each Θ–minimal system belonging to $\Theta^{'}$ is $\Theta^{'}$–minimal. In particular, each canonical system $\Sigma \in \Theta$ is Θ–minimal.

Let Ω_{sw} be the class of switched systems, let $\Omega^{'} \subseteq \Omega_{sw}$ be a subclass of switched systems. The subclass $\Omega^{'}$ can be considered as a subclass of initialised systems. A switched system $\Sigma \in \Omega^{'}$ is called *minimal* if Σ is $\Omega^{'}$–minimal when considered as an initialised system. As a consequence any canonical switched system $\Sigma \in \Omega^{'}$ is $\Omega^{'}$–minimal. Later we will show that for linear switched systems (to be defined later) each minimal linear switched system has a state space of the smallest dimension among all linear switched systems realizing the same behaviour.

Notice that at the first glance the definition of minimality presented above differs from the definition of minimality formulated in Section 2.4. However, it will be shown in this chapter that for linear switched systems the *two definitions of minimality are equivalent*. More precisely, a linear switched system Σ with fixed initial state 0 is a minimal in the above sense if and only if the linear switched system realization (Σ, μ), $\mu : y_{\Sigma}(0, .) \mapsto 0$ is a minimal realization of $\{y_{\Sigma}(0, .)\}$ in the sense of Section 2.4.

6.1.2 Linear switched systems

A switched system $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\}, x_0)$ is called *linear switched system* if

- $x_0 = 0$
- For each $q \in Q$ there exist linear mappings

$$A_q : \mathcal{X} \to \mathcal{X} \quad B_q : \mathcal{U} \to \mathcal{X} \quad C_q : \mathcal{X} \to \mathcal{Y}$$

such that

$$f_q(x, u) = A_q x + B_q u \quad \text{and} \quad h_q(x) = C_q x.$$

That is, in this chapter by linear switched systems we will understand the same linear switched systems as defined Section 4.1, except that implicitly we will assume that the initial state of the system is fixed to be 0. Thus, by reachability we will mean reachability from 0, the set *Reach*(Σ) will stand for *Reach*$(\Sigma, \{0\})$, etc. In particular, we will use the same shorthand notation for denoting linear switched systems as defined in Section 4.1 and the notion of algebraic similarity will also be the same.

6.2 Minimisation of Linear Switched Systems

This section gives a procedure to construct a minimal linear switched system equivalent to a given linear switched system. Also a Kalman-like decomposition for linear switched systems will be presented. It will also be shown that two equivalent minimal linear switched systems are algebraically similar, and that a minimal linear switched system has a state space of smaller dimension than any other linear switched system realizing the same input-output map.

For a given linear switched system we will construct an equivalent canonical system. The steps of the construction are similar to the construction of the canonical initialised system equivalent to a given one. In its full generality the procedure is described in Section 6.8 of [61]. The challenge is to show that at each step of the general procedure we get a linear switched system. This will be done below.

Theorem 24. *Let $\widetilde{\Sigma}$ be an arbitrary linear switched system. Then there exists a canonical linear switched system $\widetilde{\Sigma}_{can}$ equivalent to $\widetilde{\Sigma}$.*

Proof. First, given a linear switched system $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) | q \in Q\})$, we take the restriction of Σ to its reachable set by defining the system

$$\Sigma_r = (Reach(\Sigma), \mathcal{U}, \mathcal{Y}, Q, \{(A^r_q, B^r_q, C^r_q) \mid q \in Q\})$$

where for each $q \in Q$ the map $A^r_q = A_q|_{Reach(\Sigma)} : Reach(\Sigma) \to Reach(\Sigma)$ is the restriction of A_q to $Reach(\Sigma)$, $B^r_q = B_q : \mathcal{U} \to Reach(\Sigma)$ and $C^r_q = C_q|_{Reach(\Sigma)} : Reach(\Sigma) \to \mathcal{Y}$ is the restriction of C_q to $Reach(\Sigma)$. It is easy to see that Σ^r is a well-defined linear switched system, it is reachable and it is equivalent to Σ. Indeed, by Proposition 8 for each $q \in Q$ it holds that $\mathrm{Im}(B_q) \subseteq Reach(\Sigma)$. So B^r_q is well defined for each $q \in Q$. Again from Proposition 8 it follows that to see that A^r_q is well defined it is enough to show that $A^r_q(A^{j_1}_{q_1} A^{j_2}_{q_2} \cdots A^{j_k}_{q_k} B_z u) \in Reach(\Sigma)$ for all $q_1, q_2, \ldots q_k, z \in Q, u \in \mathcal{U}$, $j_1, j_2, \ldots, j_k \geq 0$. But $A^r_q x = A_q x$ for all $x \in Reach(\Sigma)$, so we get

$$A^r_q(A^{j_1}_{q_1} A^{j_2}_{q_2} \cdots A^{j_k}_{q_k} B_z u) = A_q A^{j_1}_{q_1} A^{j_2}_{q_2} \cdots A^{j_k}_{q_k} B_z u \in Reach(\Sigma)$$

So, for each $q \in Q$ the map A^r_q is well defined. The map C^r_q is trivially well defined. Notice that the construction of Σ_r goes along the same lines as the construction of the reachable initialised system equivalent to a given one, as it is described in [61].

The next step is to construct an observable linear switched system from a reachable linear switched system in such a way that the new reachable and observable system is equivalent to the original one.

Let $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},,Q,\{(A_q,B_q,C_q) \mid q \in Q\})$ be a linear switched system. Define $O_{\Sigma} = \bigcap_{q_1,q_2,\ldots,q_k,z\in Q,j_1,j_2,\ldots,j_k\geq 0} \ker C_z A_{q_1}^{j_1}A_{q_2}^{j_2}\cdots A_{q_k}^{j_k}$. Let $W = O_{\Sigma}^{\perp}$ be the orthogonal complement of O_{Σ}. Assume that Σ is reachable. Consider the system $\Sigma^o = (W,\mathcal{U},\mathcal{Y},Q,\{(A^o_q,B^o_q,C^o_q) \mid q \in Q\})$ where $A^o_q = \tilde{A}_q|_W : W \to W$, and $\tilde{A}_q$ is defined by $z = \tilde{A}_q x \Longleftrightarrow A_q x = z + z^{'}, z \in W, z^{'} \in O_{\Sigma}$.
$C^o_q = C_q|_W : W \to \mathcal{Y}$, and $B^o_q : \mathcal{U} \to W$ is given by the rule $B^o_q u = z \Leftrightarrow B_q u = z + z^{'}$ such that $z \in W, z^{'} \in O_{\Sigma}$. Then the system Σ^o is well-defined, it is reachable and observable (i.e. canonical) and equivalent to Σ. The construction of Σ^o is a slight modification of the construction of the canonical initialised system presented in Section 6.8 of [61]. Note that W is isomorphic to $\mathcal{X}/O_{\Sigma}$. In fact, a linear switched system can be defined on $\mathcal{X}/O_{\Sigma}$ in such a way, that it will be isomorphic to Σ^o. This linear switched system defined on $\mathcal{X}/O_{\Sigma}$ corresponds to the canonical initialised system described in Section 6.8 of [61].

Using the notation above define $\widetilde{\Sigma}_{can}$ to be $(\widetilde{\Sigma}_r)^o$. Then $\widetilde{\Sigma}_{can}$ is indeed canonical and equivalent to $\widetilde{\Sigma}$. $\square$

Denote by Ω_{lin} the class of linear switched systems considered as a subclass of initialised systems. From Subsection 6.1.1 it follows that any canonical linear switched system is Ω_{lin}-minimal. We will show that any linear switched system Σ which is Ω_{lin}–minimal has state-space of the smallest dimension among all linear switched systems equivalent to it.

Lemma 30. *Consider two linear switched systems*

$$\Sigma_1 = (\mathcal{X}_1,\mathcal{U},\mathcal{Y},Q,\{(A^1_q,B^1_q,C^1_q) \mid q \in Q\})$$
$$\Sigma_2 = (\mathcal{X}_2,\mathcal{U},\mathcal{Y},Q,\{(A^2_q,B^2_q,C^2_q) \mid q \in Q\})$$

Assume that Σ_1 is reachable. Then for any system morphism $\phi : \Sigma_1 \to \Sigma_2$ the corresponding map $\phi : \mathcal{X}_1 \to \mathcal{X}_2$ is linear.

Proof. The fact that ϕ is a system morphism means that the following holds.

$$\forall u \in PC(T,\mathcal{U}), \forall w \in (Q\times T)^*, \forall t \in \mathrm{dom}(\widetilde{w}), \forall x \in X_1 :$$
$$\phi(x_{\Sigma_1}(x,u(.),w)(t)) = x_{\Sigma_2}(\phi(x),u(.),w)(t)$$

and, $\phi(0) = 0$, and $C^1_q x = C^2_q \phi(x)$. Now, we shall prove that ϕ is a linear map. Notice that by [69] there exists a $w = (q_1,t_1)(q_2,t_2)\cdots(q_k,t_k) \in (Q\times T)^+$ such that

$R_w = \{x_{\Sigma_1}(0, u(.), w)(t_k) \mid u(.) \in PC(T, \mathcal{U})\} = Reach(\Sigma_1)$
$= \mathcal{X}_1$. Then for each $x_1, x_2 \in \mathcal{X}_1$ we have that

$$\phi(\alpha x_1 + \beta x_2) = \phi(x_{\Sigma_1}(0, \alpha u_1(.) + \beta u_2(.), w)(t_k)) = x_{\Sigma_2}(0, \alpha u_1(.)+ \\ \beta u_2(.), w)(t_k) = \alpha x_{\Sigma_2}(0, u_1(.), w)(t_k) + \beta x_{\Sigma_2}(0, u_2(.), w)(t_k)$$

So, ϕ is indeed a linear map. □

An important consequence of this lemma is the following theorem

Theorem 25. *Let $\Sigma_{min} = (\mathcal{X}_{min}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q^{min}, B_q^{min}, C_q^{min}) \mid q \in Q\})$ be a linear switched system. Then Σ_{min} is a minimal linear switched system if and only if for any linear switched system $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$ such that Σ is equivalent to Σ_{min} the following holds*

$$\dim \mathcal{X}_{min} \leq \dim \mathcal{X} \tag{6.1}$$

Proof. `"only if" part`
Consider the linear switched system Σ_r, i.e. the restriction of Σ to $Reach(\Sigma)$. Clearly $\dim Reach(\Sigma) \leq \dim \mathcal{X}$. The system Σ_r is reachable and equivalent to Σ, hence it is equivalent to Σ_{min}. By definition of Ω_{lin}–minimality there exists a subjective system morphism $\phi : \Sigma_r \to \Sigma_{min}$. By Lemma 30 the map $\phi : Reach(\Sigma) \to \mathcal{X}_{min}$ is linear, and by the surjectivity of the system morphism it is surjective. That is,

$$\dim \mathcal{X}_{min} = \dim \operatorname{Im}(\phi) \leq \dim Reach(\Sigma) \leq \dim \mathcal{X}$$

`"if" part`
Assume Σ_{min} has the property (6.1). Then Σ_{min} must be reachable. Assume the opposite. The restriction of Σ_{min} to its reachable set would give a system equivalent to Σ_{min} with state space $Reach(\Sigma_{min})$. But $\dim Reach(\Sigma_{min}) < \dim \mathcal{X}_{min}$, which contradicts to (6.1). Let $\Sigma_{can} = (\mathcal{X}_{can}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q^{can}, B_q^{can}, C_q^{can}) \mid q \in Q\})$ be a canonical linear switched system equivalent to Σ_{min}. Such a system always exists by Theorem 24. The system Σ_{can} is minimal, so there exists a surjective system morphism $\phi : \Sigma_{min} \to \Sigma_{can}$. Then ϕ is a surjective linear map, so we get that $\dim \mathcal{X}_{can} \leq \dim \mathcal{X}_{min}$. But by (6.1) we have that $\dim \mathcal{X}_{can} \geq \dim \mathcal{X}_{min}$. It implies that $\dim \mathcal{X}_{can} = \dim \mathcal{X}_{min}$, that is, ϕ is an isomorphism. Since Σ_{can} is minimal and Σ_{min} is isomorphic to it, we get that Σ_{min} is minimal too. □

For reachable linear switched systems, isomorphism of systems is equivalent to algebraic similarity.

Theorem 26. *Two reachable linear switched systems*

$$\Sigma_1 = (\mathcal{X}_1, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$$
$$\Sigma_2 = (\mathcal{X}_2, \mathcal{U}, \mathcal{Y}, Q, \{(A^{'}_q, B^{'}_q, C^{'}_q) \mid q \in Q\})$$

are isomorphic if and only if they are algebraically similar

Proof. It is clear that if Σ_1 and Σ_2 are algebraically similar then Σ_1 and Σ_2 are isomorphic. Assume that $\phi : \Sigma_1 \to \Sigma_2$ is an isomorphism of systems. From Lemma 30 it follows that $\phi : \mathcal{X}_1 \to \mathcal{X}_2$ is a linear map. Since ϕ is isomorphism, we have that the linear map $\phi : \mathcal{X}_1 \to \mathcal{X}_2$ is bijective. We get that ϕ^{-1} is a linear bijective map too.

What we need to show is that for each $q \in Q$ the following holds.

$$A^{'}_q = \phi A_q \phi^{-1}, \quad B^{'}_q = \phi B_q \quad , C^{'}_q = C_q \phi^{-1}$$

It follows immediately from the fact that ϕ is a bijective system morphism that $C^{'}_q \phi = C_q$, which implies $C^{'}_q = C_q \phi^{-1}$.

We show that $A^{'}_q = \phi A_q \phi^{-1}$ for all $q \in Q$. For each $q \in Q$, $x_{\Sigma_1}(x, 0, (q, t))(t) = \exp(A_q t)x$ and $x_{\Sigma_2}(\phi(x), 0, (q, t))(t) = \exp(A^{'}_q t)\phi(x)$. So we get that $\phi(\exp(A_q t)x) = \exp(A^{'}_q t)\phi(x)$ for all $t > 0$. Taking the derivative of t at 0 we get that for all $x \in X_1$ it holds that $\phi(A_q x) = A^{'}_q \phi(x)$, which implies $A^{'}_q = \phi A_q \phi^{-1}$ for all $q \in Q$.

It is left to show that $B^{'}_q = \phi B_q$. Denote the constant function taking the value $u \in \mathcal{U}$ by $const_u$. Then $\phi(x_{\Sigma_1}(0, const_u, (q, t)))(t) = \phi(\int_0^t \exp(A_q(t-s))B_q u \, ds) = x_{\Sigma_2}(0, const_u, (q, t))(t) = \int_0^t \exp(A^{'}_q(t-s))B^{'}_q u \, ds$ for all $t > 0, u \in \mathcal{U}$. Again, after taking derivatives by t at $t = 0$ we get $\phi B_q u = B^{'}_q u$. That is, we get $B^{'}_q = \phi B_q$. So, Σ_1 and Σ_2 are indeed algebraically similar. □

Since all equivalent minimal linear switched systems are isomorphic, one gets the following result.

Corollary 14. *All minimal equivalent linear switched systems are algebraically similar.*

The following theorem sums up the results of the discussion above.

Theorem 27 (Existence and uniqueness of minimal realization). *For linear switched systems the following statements hold.*

1. *Given a linear switched system $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$ there exists a system $\Sigma_{min} = (\mathcal{Z}, \mathcal{U}, \mathcal{Y}, \{(A^{min}_q, B^{min}_q, C^{min}_q) \mid q \in Q\})$ such that Σ^{min} is minimal and equivalent to Σ. Such a minimal system is unique up to algebraic similarity.*

2. *A linear switched system is minimal if and only if it is canonical.*

3. *A linear switched system Σ_{min} is minimal if and only if for each equivalent linear switched system Σ the dimension of the state-space of Σ is not smaller than the dimension of the state-space of Σ_{min}*

Proof. The statement of part 1 follows from Theorem 24, the fact that each canonical linear switched system is minimal (see Subsection 6.1.1) and Corollary 14.

Let Σ be a minimal linear switched system. By Theorem 24 there exists a canonical system Σ_{can} equivalent to Σ. But by Section 6.1.1 Σ_{can} is minimal, therefore Σ_{can} and Σ are isomorphic. Since any isomorphism preserves reachability and observability we get that Σ_{min} is reachable and observable, hence canonical. So the statement of part 2 is proven.

The statement of part 3 follows directly from Theorem 25. □

The construction of the minimal representation described above yields the following Kalman-decomposition of a linear switched system.

Theorem 28. *Given a linear switched system $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q) \mid q \in Q\})$ there exists a basis transformation on $\mathcal{X}$ compatible with decomposition $X = W_{or} \oplus W_{rno} \oplus W_{onr} \oplus W_{nonr}$ where $W_{or} \oplus W_{rno} = Reach(\Sigma)$, $W_{onr} \oplus W_{nonr} = O_{\Sigma}$ such that in the new basis the matrix representation of maps A_q, B_q, C_q has the following form*

$$A_q = \begin{bmatrix} A_q^1 & 0 & A_q^2 & 0 \\ A_q^3 & A_q^4 & A_q^5 & A_q^6 \\ 0 & 0 & A_q^7 & 0 \\ 0 & 0 & A_q^8 & A_q^9 \end{bmatrix}, \quad B_q = \begin{bmatrix} B_q^1 \\ B_q^2 \\ 0 \\ 0 \end{bmatrix}, \quad C_q = \begin{bmatrix} C_q^1 & 0 & C_q^2 & 0 \end{bmatrix}$$

where

- $\Sigma_{or} = (W_{or},\mathcal{U},\mathcal{Y},Q,\{(A_q^1,B_q^1,C_q^1) \mid q \in Q\})$ *is minimal and equivalent to* Σ.
- $\Sigma_{rno} = (Reach(\Sigma),\mathcal{U},\mathcal{Y},Q,\{(\begin{bmatrix} A_q^1 & 0 \\ A_q^3 & A_q^4 \end{bmatrix}, \begin{bmatrix} B_q^1 \\ B_q^2 \end{bmatrix}, \begin{bmatrix} C_q^1 & 0 \end{bmatrix}) \mid q \in Q\})$ *is a reachable system equivalent to* Σ.
- $\Sigma_{rno} = (O_{\Sigma}^{\perp},\mathcal{U},\mathcal{Y},Q,\{(\begin{bmatrix} A_q^1 & A_q^2 \\ 0 & A_q^7 \end{bmatrix}, \begin{bmatrix} B_q^1 \\ 0 \end{bmatrix}, \begin{bmatrix} C_q^1 & C_q^2 \end{bmatrix}) \mid q \in Q\})$ *is an observable system equivalent to* Σ.

6.3 Constructing a Minimal Representation for Input-output Maps

Below necessary and sufficient conditions for the existence of realization by a linear switched system will be presented. Also a procedure will be described to construct a minimal representation for a realizable input-output map. The well-known condition for existence of realization by a linear system is a special case of the condition given here. The construction of a minimal linear representation of an input-output map is also a particular case of the procedure presented below. By Proposition 20 it is enough to determine conditions for realisability of input-output maps of the form $y : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$.

Below conditions on $y : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$ will be given, which will be proven necessary and sufficient for realisability of y in the sense of Definition 13. Before proceeding further some notation has to be introduced. Let $u_1 = u_{11}u_{12}\cdots u_{1k}, u_2 = u_{21}u_{22}\cdots u_{2k} \in \mathcal{U}^+$, then $\alpha u_1 + \beta u_2 = (\alpha u_{11} + \beta u_{21})(\alpha u_{12} + \beta u_{22})\cdots(\alpha u_{1k} + \beta u_{2k}) \in \mathcal{U}^+$ for $\alpha, \beta \in \mathbb{R}$. Let $u = u_1 u_2 \cdots u_k \in \mathcal{U}^+, w = w_1 w_2 \cdots w_k \in Q^+, \tau = \tau_1 \tau_2 \cdots t_k \in T^+$, then $y(u, w, \tau)$ is defined as

$$y(u, w, \tau) = y((u_1, w_1, \tau_1)(u_2, w_2, \tau_2)\cdots(u_k, w_k, \tau_k))$$

Let $\phi : \mathbb{R}^{k+r} \to \mathbb{R}^p$. Whenever we want to refer to the arguments of ϕ explicitly we will use the notation $\phi(t_1, t_2, \ldots, t_k, s_1, s_2, \ldots, s_r)$, or in vector notation $\phi(\underline{t}, \underline{s})$, where $\underline{t} = (t_1, t_2, \ldots, t_k)$ and $\underline{s} = (s_1, s_2, \ldots, s_r)$ are formal k and r-tuples respectively. If $\underline{a} \in \mathbb{R}^k$ then we use the notation $\phi(\underline{t}, \underline{s})|_{\underline{t}=\underline{a}}$ for the function $\mathbb{R}^r \ni \underline{b} \mapsto \phi(\underline{a}, \underline{b})$. For any $\alpha = (\alpha_k, \alpha_{k-1}, \cdots, \alpha_1) \in \mathbb{N}^k$ denote by $\frac{d^\alpha}{dt^\alpha}\phi$ the partial derivative

$$\frac{d^\alpha}{dt^\alpha}\phi = \frac{d}{dt_k^{\alpha_k} dt_{k-1}^{\alpha_{k-1}} \cdots dt_1^{\alpha_1}}\phi(t_k, t_{k-1}, \ldots, t_1, s_r, s_{r-1}, \ldots, s_1) : \mathbb{R}^{k+r} \to \mathbb{R}^p$$

If we want to refer to the components of $\alpha \in \mathbb{N}^k$ explicitly, we will use the notation $\frac{d^{(\alpha_k, \alpha_{k-1}, \cdots, \alpha_1)}}{dt^{(\alpha_k, \alpha_{k-1}, \cdots, \alpha_1)}}\phi = \frac{d^\alpha}{dt^\alpha}\phi$. If $\underline{t} = (t_1, t_2, \ldots, t_k)$ then denote by $\underline{t}^l$ the tuple $(t_l, t_{l+1}, \ldots, t_k)$ and by ${}^l\underline{t}$ the tuple $(t_1, t_2, \ldots, t_l)$ for $l < k$.

For any $u \in \mathcal{U}^+, w \in Q^+$ the function $y(u, w, \tau) : T^+ \to \mathcal{Y}$ will be identified with the function $T^{|w|} \ni (t_1, t_2, \ldots, t_k) \mapsto y(u, w, t_1 t_2 \cdots t_k)$

Consider the matrices $A_{q_1}, A_{q_2}, \cdots A_{q_k} \in \mathbb{R}^{n \times n}$ and define the function $\exp_{q_1 q_2 \cdots q_k} : T^k \to \mathbb{R}^{n \times n}$ by

$$\exp_{q_k q_{k-1} \cdots q_1}(t_1, t_2, \ldots, t_k) = \exp(A_{q_k} t_k)\exp(A_{q_{k-1}} t_{k-1}) \cdots \exp(A_{q_1} t_1)$$

Definition 14 (Realisability conditions). *Consider a map $y : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$. The map y is said to satisfy the* realisability conditions *if the following properties hold*

1. `Linearity of the input-output function`
   For all $u_1, u_2 \in \mathcal{U}^+$*,* $w \in Q^+$*,* $\tau \in T^+$ *such that* $|u_1| = |u_2| = |w| = |\tau|$ *and for all* $\alpha, \beta \in \mathbb{R}$ *it holds that*

$$y(\alpha u_1 + \beta u_2, w, \tau) = \alpha y(u_1, w, \tau) + \beta y(u_2, w, \tau)$$

2. `Zero-time behaviour`

$$y(u, w, \underbrace{00\cdots 0}_{|w|-times}) = 0$$

3. `Analyticity in switching times`
   For all $w \in Q^+, u \in \mathcal{U}^+$ *such that* $|w| = |u|$ *the function* $y(u, w, .) : T^{|w|} \to \mathcal{Y}$ *defined by* $(t_1, t_2, \ldots, t_{|w|}) \mapsto y(u, w, t_1 t_2 \cdots t_k)$ *is analytic.*

4. `Repetition of the same input`
   For all $w_1, w_2 \in Q^+, u_1, u_2 \in \mathcal{U}^+, \tau_1, \tau_2 \in T^*$ *such that* $|w_i| = |u_i| = |\tau_i|, (i = 1, 2)$ *and for all* $q \in Q, u \in \mathcal{U}, t_1, t_2 \in T$ *it holds that*

$$y(u_1 u u u_2, w_1 q q w_2, \tau_1 t_1 t_2 \tau_2) = y(u_1 u u_2, w_1 q w_2, \tau_1 (t_1 + t_2) \tau_2)$$

   The condition is equivalent to stating that for each $z, l \in (\mathcal{U} \times Q \times T)^+$

$$\widetilde{z} = \widetilde{l} \implies y(z) = y(l)$$

5. `Decomposition of concatenation of inputs`
   For each $w_1, w_2 \in Q^+, u_1, u_2 \in \mathcal{U}^+, \tau_1, \tau_2 \in T^+$ *such that* $|w_i| = |u_i| = |\tau_i|$*,* $(i = 1, 2)$ *it holds that*

$$y(u_1 u_2, w_1 w_2, \tau_1 \tau_2) = y(u_2, w_2, \tau_2) + y(u_1 \underbrace{00\cdots 0}_{|u_2|-times}, w_1 w_2, \tau_1 \tau_2)$$

6. `Elimination of zero duration`
   For all $w_1, w_2, v \in Q^+, \tau_1, \tau_2 \in T^+, u_1, u_2, u \in \mathcal{U}^+$ *such that*
   $|u_i| = |w_i| = |\tau_i|$ *and* $|v| = |u|$ *it holds that*

$$y(u_1 u u_2, w_1 v w_2, \tau_1 \underbrace{00\cdots 0}_{|u|-times} \tau_2) = y(u_1 u_2, w_1 w_2, \tau_1 \tau_2)$$

Proposition 21. *If a map* $y : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$ *is realizable by a linear switched system, then it satisfies the realisability conditions.*

Analyticity of the input-output maps allows to rephrase the property that a linear switched system realizes an input-output map in terms of the high-order derivatives of the input-output map.

Let A_q, B_q, C_q, $(q \in Q)$ be linear maps over suitable spaces and let $j_1, j_2, \ldots, j_k \geq 0$. If $l = \inf\{z \in \mathbb{N} | j_z > 0\} = -\infty$, i.e. $j_1 = j_2 = \cdots = j_k = 0$, then by $C_{q_k} A_{q_k}^{j_k} A_{q_{k-1}}^{j_{k-1}} \cdots A_{q_l}^{j_l - 1} B_{q_l}$ we mean simply the identically zero map.

Proposition 22. *Consider the linear switched system*

$$\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) | q \in Q\})$$

Then for each $w = q_1 q_2 \cdots q_k \in Q^+$, $u = u_1 u_2 \cdots u_k \in \mathcal{U}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$ *the following holds*

$$\frac{d^\alpha}{dt^\alpha} \widetilde{y}_\Sigma(u, w, \underline{t})|_{\underline{t}=0} = C_{q_k} A_{q_k}^{\alpha_k} A_{q_{k-1}}^{\alpha_{k-1}} \cdots A_{q_l}^{\alpha_l - 1} B_{q_l} u_l$$

where $l = \min\{z | \alpha_z > 0\}$.

Proof. Define the function $\widetilde{x}_\Sigma : (\mathcal{U} \times Q \times T)^+ \to \mathcal{X}$ in the following way. For $w = w_1 w_2 \cdots w_k \in Q^+$, $\tau = t_1 t_2 \cdots t_k \in T^+$ and $u = u_1 u_2 \cdots u_k \in \mathcal{U}^+$ define $\widetilde{x}_\Sigma(u, w, \tau)$ by $\widetilde{x}_\Sigma(u, w, \tau) = x_\Sigma(0, \widetilde{v}, z)(\sum_1^k t_i)$ where $v = (u_1, t_1)(u_2, t_2) \cdots (u_k, t_k)$ and $z = (w_1, t_1)(w_2, t_2) \cdots (w_k, t_k)$. It is easy to see that $\widetilde{x}_\Sigma$ satisfies the realisability conditions. We shall use this, the fact that $\widetilde{y}_\Sigma$ satisfies the realisability properties and the following basic property of linear switched systems (see [69])

$$\begin{aligned}
&\widetilde{y}_\Sigma(u_1 u_2 \cdots u_l \underbrace{0 \cdots 000}_{k-l-times}, q_1 q_2 \cdots q_k, t_1 t_2 \cdots t_k) = C_{q_k} \exp(A_{q_k} t_k) \times \\
&\quad \exp(A_{q_{k-1}} t_{k-1}) \cdots \exp(A_{q_{l+1}} t_{l+1}) \widetilde{x}_\Sigma(u_1 u_2 \cdots u_l, q_1 q_2 \cdots q_l, t_1 t_2 \cdots t_l) \\
&\quad = C_{q_k} \exp_{q_k q_{k-1} \cdots q_{l+1}}(t_k, t_{k-1}, \ldots, , t_{l+1}) \widetilde{x}_\Sigma(u_1 u_2 \cdots u_l, q_1 q_2 \cdots q_l, t_1 t_2 \cdots t_l)
\end{aligned}$$

From condition 5 of the realisability conditions one gets

$$\begin{aligned}
&\widetilde{y}_\Sigma(u_1 u_2 \cdots u_l u_{l+1} \cdots u_k, q_1 q_2 \cdots q_l q_{l+1} \cdots q_k, t_1 t_2 \cdots t_l t_{l+1} \cdots t_k) = \\
&\quad \widetilde{y}_\Sigma(u_{l+1} \cdots u_k, q_{l+1} \cdots q_k, t_{l+1} \cdots t_k) + \widetilde{y}_\Sigma(u_1 \cdots u_l 00 \cdots 0, w, t_1 t_2 \cdots t_k)
\end{aligned}$$

where $w = q_1 q_2 \cdots q_k$. Combining the two expressions above one gets

$$\begin{aligned}
&\frac{d^{\alpha}}{dt^{\alpha}}\widetilde{y}_{\Sigma}(u,w,\underline{t})|_{\underline{t}=0} = \frac{d^{\alpha}}{dt^{\alpha}}\widetilde{y}_{\Sigma}(u_1u_2\cdots u_l 00\cdots 0, w, \underline{t})|_{\underline{t}=0} \\
&= \frac{d^{\alpha}}{dt^{\alpha}}(C_{q_k}\exp_{q_kq_{k-1}\cdots q_{l+1}}(\underline{t}^{l+1})\widetilde{x}_{\Sigma}(u_1u_2\cdots u_l, q_lq_2\cdots q_1, {}^{l}\underline{t}))|_{\underline{t}=0} \\
&= \frac{d^{\alpha}}{dt^{\alpha}}C_{q_k}\exp_{q_kq_{k-1}\cdots q_{l+1}}(\underline{t}^{l+1}) \times \\
&\quad(\widetilde{x}_{\Sigma}(u_l,q_l,t_l) + \widetilde{x}_{\Sigma}(u_1u_2\cdots u_{l-1}0, q_1q_2\cdots q_{l-1}q_l, {}^{l}\underline{t}))|_{\underline{t}=0} \\
&= \frac{d^{(\alpha_k,\alpha_{k-1},\cdots,\alpha_l)}}{dt^{(\alpha_k,\alpha_{k-1},\cdots,\alpha_l)}}C_{q_k}\exp_{q_k,q_{k-1},\cdots q_{l+1}}(\underline{t}^{l+1}) \times \\
&\quad(\widetilde{x}_{\Sigma}(u_l,q_l,t_l) + \exp(A_{q_l}t_l)\widetilde{x}_{\Sigma}(u_1u_2\cdots u_{l-1}, q_1q_2\cdots q_{l-1}, {}^{l}\underline{t}))|_{\underline{t}=0}
\end{aligned}$$

where $l = \min\{z \mid \alpha_z > 0\}$. In the derivation above the condition 5 of the realisability conditions was applied to $\widetilde{x}_{\Sigma}$. Since
$\widetilde{x}_{\Sigma}(u_1u_2\cdots u_{l-1}, q_1q_2\cdots q_{l-1}, 00\cdots 0) = 0$ we get that

$$\begin{aligned}
&\frac{d^{\alpha}}{dt^{\alpha}}\widetilde{y}_{\Sigma}(u,w,\underline{t})_{\underline{t}=0} = \frac{d^{(\alpha_k,\alpha_{k-1},\ldots,\alpha_l)}}{dt^{(\alpha_k,\alpha_{k-1},\ldots,\alpha_l)}}(C_{q_k}\exp_{q_k,q_{k-1},\ldots,q_{l+1}}(\underline{t}^{l+1})\widetilde{x}_{\Sigma}(u_l,q_l,t_l)|_{\underline{t}=0} \\
&= \frac{d^{(\alpha_k,\alpha_{k-1},\ldots,\alpha_l)}}{dt^{(\alpha_k,\alpha_{k-1},\ldots,\alpha_l)}}(C_{q_k}\exp_{q_k,q_{k-1},\ldots,q_{l+1}}(\underline{t}^{l+1}) \\
&\quad\int_0^{t_l}\exp(A_{q_l}(t_l-s))B_{q_l}u_l\,ds)|_{\underline{t}=0} \\
&= (\frac{d^{(\alpha_k,\alpha_{k-1},\ldots,\alpha_{l+1})}}{dt^{(\alpha_k,\alpha_{k-1},\ldots,\alpha_{l+1})}}C_{q_k}\exp_{q_k,q_{k-1},\ldots,q_{l+1}}(\underline{t}^{l+1}) \times \\
&\quad(\frac{d}{dt_l^{\alpha_l-1}}(\exp(A_{q_l}t_l)B_{q_l}u_l) + \int_0^{t_l}\frac{d}{dt_l^{\alpha_l}}\exp(A_{q_l}(t_l-s))B_{q_l}u_l\,ds)|_{\underline{t}=0} \\
&= \frac{d^{(\alpha_k,\alpha_{k-1},\ldots,\alpha_{l+1})}}{dt^{(\alpha_k,\alpha_{k-1},\ldots,\alpha_{l+1})}}(C_{q_k}\exp(A_{q_k}t_k) \times \\
&\quad\times\exp(A_{q_{k-1}}t_{k-1})\cdots\exp(A_{q_{l+1}}t_{l+1})A_{q_l}^{\alpha_l-1}B_{q_l}u_l)|_{\underline{t}=0} \\
&= C_{q_k}A_{q_k}^{\alpha_k}A_{q_{k-1}}^{\alpha_{k-1}}\cdots A_{q_l}^{\alpha_l-1}B_{q_l}u_l.
\end{aligned}$$

In the last equation the fact was used that $\frac{d}{dt^j}Z\exp(At)L|_{t=0} = ZA^jL$ holds for any A, L, Z matrices of compatible dimensions. $\square$

Proposition 22 , and the fact that $\widetilde{y}_{\Sigma}(u,w,,t_1t_2\cdots t_l)$ is analytic in $(t_1,t_2,\cdots,t_l)$ implies the following corollary.

Corollary 15. *Let $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q) \mid q \in Q\})$ be a linear switched system. Consider a map $y : (\mathcal{U}\times Q\times T)^+ \to \mathcal{Y}$ and assume that for each $w \in Q^+$, $u \in \mathcal{U}^+$, $|u| = |w|$ the map $(t_1,t_2,\ldots,t_{|w|}) \mapsto y(u,w,t_1t_2\cdots t_{|w|})$ is analytic. Then*

Σ is a realization of y if and only if

$$\begin{aligned}&\forall u = u_1 u_2 \cdots u_k \in \mathcal{U}^+, \forall w = q_1 q_2 \cdots q_k \in Q^+, \forall \alpha \in \mathbb{N}^k \\ &\frac{d^\alpha}{dt^\alpha} y(u, w, \underline{t})|_{\underline{t}=0} = C_{q_k} A_{q_k}^{\alpha_k} A_{q_{k-1}}^{\alpha_{k-1}} \cdots A_{q_l}^{\alpha_l - 1} B_{q_l} u_l\end{aligned} \tag{6.2}$$

where $l = \min\{z | \alpha_z > 0\}$

The corollary above says that the matrices of the form $C_{q_k} A_{q_k}^{\alpha_k} A_{q_{k-1}}^{\alpha_{k-1}} \cdots A_{q_1}^{\alpha_1} B_z$ $(q_1, q_2, \dots, q_k, z \in Q,\ \alpha \in \mathbb{N}^k)$ determine the input-output behaviour of linear switched systems. In fact, for the case of one discrete mode these matrices are the Markov-parameters of the system. The matrices (6.2) can be viewed as a generalisation of the concept of Markov parameters.

Now we shall introduce a few concepts, which are needed to formulate the generalisation of the Hankel-matrix for linear switched systems. Let $\mathcal{Y} = \mathbb{R}^p$, $T = \mathbb{R}_+$ and Q be an arbitrary finite set. Define the following set

$$\begin{aligned}Z = \{\phi : Q^+ \to Y^{T^+} \mid \forall w \in Q^+ : \mathrm{dom}(\phi(w)) = T^{|w|} \\ \text{and } \phi(w) : T^{|w|} \to Y \text{ is analytic } \}\end{aligned}$$

Then Z is a vector space with respect to point-wise addition and multiplication by scalar, i.e. $\forall \phi_1, \phi_2 \in Z, \forall w \in Q^+, t \in T^{|w|}$:

$$(\alpha\phi_1 + \beta\phi_2)(w, t) := \alpha\phi_1(w, t) + \beta\phi_2(w, t)\ ,\ \alpha, \beta \in \mathbb{R}$$

Define the set D as follows

$$D = \{f : (Q \times \mathbb{N})^+ \to \mathcal{Y}\}$$

It is easy to see that D is a vector space with respect to point-wise addition and multiplication by real numbers, i.e.

$$\forall f_1, f_2 \in D, \forall w \in (Q \times \mathbb{N})^+ : (\alpha f_1 + \beta f_2)(w) := \alpha f_1(w) + \beta f_2(w)\ ,\ \alpha, \beta \in \mathbb{R}$$

Define the mapping $F : Z \to D$ in the following way

$$F(\phi)((q_1, \alpha_1)(q_2, \alpha_2) \cdots (q_k, \alpha_k)) = \frac{d^\alpha}{dt^\alpha} \phi(q_1 q_2 \cdots q_k)(\underline{t})|_{\underline{t}=0} \tag{6.3}$$

That is, the function F stores the germs of functions from Z in sequences of the form $(Q \times \mathbb{N})^+ \to \mathcal{Y}$.

For each $f \in Z$ and for each sequence $w \in Q^+$ the value of $F(f)$ at $(w, \alpha_1\alpha_2 \cdots \alpha_{|w|})$ equals the partial derivative $\frac{d^\alpha}{dt^\alpha}$ at $(0, 0, \dots, 0) \in T^{|w|}$ of the analytic function $f(w) : T^{|w|} \to \mathcal{Y}$. Thus, the proof of the following theorem is straightforward.

Proposition 23. *The mapping $F : Z \to D$ defined above is an injective vector space homomorphism.*

Now we are ready to define the generalised Hankel-matrix. Consider a mapping $y : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$ and assume that it satisfies the realisability conditions. For each $(w, u) = (w_1, u_1)(w_2, u_2) \cdots (w_k, u_k) \in (Q \times \mathcal{U})^+$ and $\alpha \in \mathbb{N}^k$ define the mapping $\frac{d^\alpha}{dt^\alpha} y_{(w,u)} : Q^+ \to Y^{T^+}$ in the following way. For all $v \in Q^+$ let $\mathrm{dom}(\frac{d^\alpha}{dt^\alpha} y_{(w,u)}(v)) = T^{|v|}$. For each fixed $\tau \in T^{|v|}$

$$\frac{d^\alpha}{dt^\alpha} y_{(w,u)}(v)(\tau) = \frac{d^\alpha}{dt^\alpha} y(u \underbrace{00 \cdots 0}_{|v|-times}, wv, \underline{t}\tau)|_{\underline{t}=0}$$

Then by analyticity of $y(u00 \cdots 0, wv, .)$ the mapping $\frac{d^\alpha}{dt^\alpha} y_{(w,u)}$ belongs to Z. Consider the following subspace of Z

$$\mathcal{X}_y = \mathrm{Span}\{\frac{d^\alpha}{dt^\alpha} y_{(w,u)} \mid (w, u) \in (Q \times \mathcal{U})^+, \alpha \in \mathbb{N}^{|w|}\} \tag{6.4}$$

The Hankel-matrix of y can be defined in the following way

Definition 15 (Hankel-matrix). *Consider a mapping $y : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$ such that y satisfies the realisability condition. Using the notation above define the map $H_y = F|_{\mathcal{X}_y} : \mathcal{X}_y \to D$. The map H_y will be called the Hankel-map (or Hankel-matrix) of the mapping y.*

It is easy to see that H_y is a linear mapping, therefore it makes sense to speak about its rank, $\mathrm{rank}\ H_y := \dim \mathrm{Im} H_y \in \mathbb{N} \cup \{\infty\}$.

Lemma 31. *Consider the mapping $y : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$ and assume that y has a realization by a linear switched system. Then y satisfies the realisability conditions and* $\mathrm{rank}\ H_y < +\infty$.

Proof. Assume that the linear switched system
$\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) | q \in Q\})$ is a realization of y. Then by Corollary 15

$$\begin{aligned} H_y(\frac{d^\alpha}{dt^\alpha} y_{(w,u)})((q_1, \beta_1)(q_2, \beta_2) \cdots (q_l, \beta_l)) = \\ = \frac{d^\beta}{d\tau^\beta} \frac{d^\alpha}{dt^\alpha} y(u00 \cdots 0, wq_1 q_2 \cdots q_l, \underline{t\tau})|_{\underline{t}=0, \underline{\tau}=0} \\ = C_{q_l} A_{q_l}^{\beta_l} A_{q_{l-1}}^{\beta_{l-1}} \cdots A_{q_1}^{\beta_1} A_{w_k}^{\alpha_k} \cdots A_{w_b}^{\alpha_b - 1} B_{w_b} u_b \end{aligned}$$

where $b = \min\{z | \alpha_z > 0\}$.

Let $r = \dim Reach(\Sigma) < +\infty$. Choose a basis $e_1, e_2, \ldots, e_r$ of $Reach(\Sigma)$. Assume that $e_i = A_{q(i)_{k(i)}}^{\alpha(i,k(i))} A_{q(i)_{k(i)-1}}^{\alpha(i,k(i)-1)} \cdots A_{q(i)_1}^{\alpha(i,1)-1} B_{q(i)_1} u(i)$. For each $i = 1, 2, \ldots, r$ define

$$f_i = \frac{d^{(\alpha(i,k(i)), \alpha(i,k(i)-1), \ldots \alpha(i,1))}}{dt^{(\alpha(i,k(i)), \alpha(i,k(i)-1), \ldots \alpha(i,1))}} y_{(q(i)_1 q(i)_2 \cdots q(i)_{k(i)}, u(i) \underbrace{00 \cdots 0}_{k(i)-1-times})}$$

Then we claim that $H_y(f_i)$ generates $\mathrm{Im}H_y$. Indeed, take an arbitrary $f = \frac{d^\alpha}{dt^\alpha} y_{(w,u)}$ Define $\widetilde{f} = A_{w_k}^{\alpha_k} A_{w_{k-1}}^{\alpha_{k-1}} \cdots A_{w_l}^{\alpha_l - 1} B_{w_l} u_l$ where $l = \min\{z | \alpha_z > 0\}$. Then there exist scalars $\gamma_i \in \mathbb{R}$ such that $\widetilde{f} = \sum_{z=1}^{r} \gamma_i e_i$. But for each $x = (q_1, d_1)(q_2, d_2) \cdots (q_e, d_e) \in (Q \times \mathbb{N})^+$ it holds that $H_y(f)(x) = C_{q_e} A_{q_e}^{d_e} \cdots A_{q_1}^{d_1} \widetilde{f}$. Then $H_y(f_i)(x) = C_{q_e} A_{q_e}^{d_e} \cdots A_{q_1}^{d_1} e_i$, so we get that

$$(\sum_{j=1}^{r} \gamma_j H_y(f_j))(x) = \sum_{j=1}^{r} \gamma_j C_{q_e} A_{q_e}^{d_e} \cdots A_{q_1}^{d_1} e_j = C_{q_e} A_{q_e}^{d_e} \cdots A_{q_1}^{d_1} \widetilde{f} = H_y(f)(x)$$

so that we get that

$$H_y(f) = \sum_{j=1}^{r} \gamma_j H_y(f_j)$$

That is, the set $\{H_y(f_i) \mid i = 1, 2, \ldots, r\}$ is a finite generator of $\mathrm{Im}H_y$. □

Now we are ready to state the main theorem of the section.

Theorem 29. *Consider a map $y : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$. The map y is realizable by a linear switched system if and only if it satisfies the realisability conditions and its Hankel-map is of finite rank, i.e. $n = \mathrm{rank}\ H_y < +\infty$. If y is realizable, and* $\mathrm{rank}\ H_y < +\infty$ *then there exists a minimal linear switched system*

$$\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$$

which realizes it and $\dim \mathcal{X} = n = \mathrm{rank}\ H_y$. *This minimal representation is unique up to algebraic similarity.*

Proof. Lemma 31 and Proposition 21 imply the necessity of the condition. The last statement of the theorem follows from Corollary 14 In order to prove sufficiency, a minimal linear switched system will be constructed that realizes y. The proof will be divided into several steps.

(1) Consider $H = \mathrm{Im}H_y$. For each $q \in Q$ define the following linear maps $\mathcal{A}_q : H \to H$, $\mathcal{C}_q : H \to \mathcal{Y}$ and $\mathcal{B}_q : \mathcal{U} \to H$ as follows

$$\begin{aligned} &\forall (q_1, j_1)(q_2, j_2) \cdots (q_k, j_k) : \\ &\quad (\mathcal{A}_q \phi)((q_1, j_1)(q_2, j_2) \cdots (q_k, j_k)) := \phi((q, 1)(q_1, j_1)(q_2, j_2) \cdots (q_k, j_k)) \end{aligned}$$

$$\mathcal{B}_q u := H_y(\frac{d}{dt} y_{(q,u)}), \quad \mathcal{C}_q \phi := \phi((q, 0))$$

It is clear that $\mathcal{B}_q$ and $\mathcal{C}_q$ are well defined linear mappings. It is left to show that $\mathcal{A}_q$ is well defined. It is clear that $\mathcal{A}_q : H \to D$ is linear. We need to show that $\mathcal{A}_q(H) \subseteq H$. In fact, the following is true: for all $f = \frac{d^\alpha}{dt^\alpha} y_{(w,u)} \in \mathcal{X}_y$ it holds that

$$\mathcal{A}_q(H_y(f)) = H_y(\frac{d^{(1,\alpha)}}{dt^{(1,\alpha)}} y_{(wq,u0)}) \tag{6.5}$$

Indeed, denote by ϕ the right-hand side of (6.5). Then

$$\begin{aligned}
&\phi((q_1,\beta_1)(q_2,\beta_2)\cdots(q_z,\beta_z)) = \\
&\quad = \frac{d^{\beta}}{d\tau^{\beta}}\frac{d^{(1,\alpha)}}{dt^{(1,\alpha)}} y(u0\underbrace{00\cdots 0}_{z-times}, wqq_1q_2\cdots q_z, t_1t_2\cdots t_kt_{k+1}\tau_1\tau_2\cdots\tau_z)|_{\underline{t}=0,\underline{\tau}=0} \\
&\quad = \frac{d^{(\beta,1)}}{d\tau^{(\beta,1)}}\frac{d^{\alpha}}{dt^{\alpha}} y(u000\cdots 0, wqq_1q_2\cdots q_z, t_1t_2\cdots t_k\tau_1\tau_2\tau_2\cdots\tau_{z+1})|_{\underline{t}=0,\underline{\tau}=0} \\
&\quad = H_y(f)((q,1)(q_1,\beta_1)(q_2,\beta_2)\cdots(q_z,\beta_z))
\end{aligned}$$

(2) For each $q_1q_2\cdots q_k, z \in Q^+$, $\alpha \in \mathbb{N}^k$ and $u \in \mathcal{U}$ the following holds

$$\mathcal{A}_{q_k}^{\alpha_k}\cdots\mathcal{A}_{q_1}^{\alpha_1}\mathcal{B}_z u = H_y(\frac{d^{(\alpha,1)}}{dt^{(\alpha,1)}} y_{(zq_1q_2\cdots q_k, u\underbrace{00\cdots 0}_{k-times})}) \tag{6.6}$$

It is easy to see that $\frac{d^{(1,\alpha)}}{dt^{(1,\alpha)}} y_{(wqq,vu0)} = \frac{d^{(\alpha_m+1,\alpha_{m-1},\ldots,\alpha_1)}}{dt^{(\alpha_m+1,\alpha_{m-1},\ldots,\alpha_1)}} y_{(wq,vu)}$, $m = |wq|$. The correctness of (6.6) follows now from the repeated application of (6.5). We also get the following equalities.

$$\mathcal{A}_{q_k}^{\alpha_k}\mathcal{A}_{q_{k-1}}^{\alpha_{k-1}}\cdots\mathcal{A}_{q_1}^{\alpha_1-1}\mathcal{B}_{q_1}u_1 = H_y(\frac{d^{\alpha}}{dt^{\alpha}} y_{(q_1q_2\cdots q_k, u_1\underbrace{00\cdots 0}_{k-1-times})}) \tag{6.7}$$

$$\mathcal{C}_q\mathcal{A}_{q_k}^{\alpha_k}\mathcal{A}_{q_{k-1}}^{\alpha_{k-1}}\cdots\mathcal{A}_{q_1}^{\alpha_1-1}\mathcal{B}_{q_1}u_1 = \frac{d^{\alpha}}{dt^{\alpha}} y(q_1q_2\cdots q_kq, u_1\underbrace{00\cdots 0}_{k-times}, \underline{t}s)|_{\underline{t}=0,s=0} \tag{6.8}$$

where $\alpha_1 > 0$.

(3) Using condition 5 of realizability conditions one gets for any $k \geq l \in \mathbb{N}$

$$\begin{aligned}
&\frac{d^{\alpha}}{dt^{\alpha}} y_{(q_1 q_2 \cdots q_k, u_1 u_2 \cdots u_k)}(v)(\tau) = \\
&= \frac{d^{\alpha}}{dt^{\alpha}} y(q_1 q_2 \cdots q_k v, u_1 u_2 \cdots u_k \underbrace{0 \cdots 00}_{|v|-times}, \underline{t}\tau)|_{\underline{t}=0} \\
&= \frac{d^{\alpha}}{dt^{\alpha}} (y(q_{l+1} q_{l+2} \cdots q_k v, u_{l+1} \cdots u_k \underbrace{0 \cdots 00}_{|v|-times}, \underline{t}^{l+1}\tau) + \\
&\quad + y(q_l q_{l+1} \cdots q_k v, u_l \underbrace{00 \cdots 0}_{|v|+k-l-times}, \underline{t}^l \tau)) \\
&\quad + y(q_1 q_2 \cdots q_k v, u_1 u_2 \cdots u_{l-1} 0 \underbrace{0 \cdots 00}_{|v|+k-l-times}, \underline{t}\tau))|_{\underline{t}=0} \\
&= \frac{d^{\alpha}}{dt^{\alpha}} y(q_{l+1} q_{l+2} \cdots q_k v, u_{l+1} \cdots u_k \underbrace{0 \cdots 00}_{|v|-times}, \underline{t}^{l+1}\tau) \\
&\quad + \frac{d^{(\alpha_k, \alpha_{k-1}, \ldots, \alpha_l)}}{dt^{(\alpha_k, \alpha_{k-1}, \ldots, \alpha_l)}} y_{(q_l q_{l+1} \cdots q_k, u_l \underbrace{00 \cdots 0}_{k-l-times})}(v)(\tau) \\
&\quad + \frac{d^{\alpha}}{dt^{\alpha}} y_{(q_1 q_2 \cdots q_k, u_1 u_2 \cdots u_{l-1} 0 \underbrace{00 \cdots 0}_{k-l-times})}(v)(\tau)
\end{aligned}$$

Assume that $l = \min\{z | \alpha_z > 0\}$. Now, since the function

$$y(q_{l+1} q_{l+2} \cdots q_k v, u_{l+1} u_{l+2} \cdots u_k \underbrace{0 \cdots 00}_{|v|-times}, t_{l+1} t_{l+2} \cdots t_k \tau)$$

doesn't depend on t_l, we get that

$$\frac{d^{\alpha}}{dt^{\alpha}} (y(q_{l+1} q_{l+2} \cdots q_k v, u_{l+1} \cdots u_k \underbrace{0 \cdots 00}_{|v|-times}, \underline{t}\tau)|_{\underline{t}=0} = 0$$

For the third term of the sum

$$\begin{aligned}
&\forall w = w_1 w_2 \cdots w_z \in Q^+, \tau = \tau_1 \tau_2 \cdots \tau_z \in T^z : \\
&\frac{d^{\alpha}}{dt^{\alpha}} y_{(q_1 q_2 \cdots q_k, u_1 u_2 \cdots u_{l-1} 0 \underbrace{00 \cdots 0}_{k-l-times})}(w)(\tau) \\
&= \frac{d^{\alpha}}{dt^{\alpha}} y(u_1 u_2 \cdots u_{l-1} 0 \underbrace{00 \cdots 0}_{k-l-times} \underbrace{00 \cdots 0}_{z-times}, q_1 q_2 \cdots q_k w_1 w_2 \cdots w_z, \underline{t}\tau)|_{\underline{t}=0} \\
&= \frac{d^{(\alpha_k, \alpha_{k-1}, \ldots, \alpha_l)}}{dt^{(\alpha_k, \alpha_{k-1}, \ldots, \alpha_l)}} y(0 \underbrace{00 \cdots 0}_{k-l-times} \underbrace{00 \cdots 0}_{z-times}, q_l \cdots q_k w_1 w_2 \cdots w_z, \underline{t}\tau)|_{\underline{t}=0} = 0
\end{aligned}$$

In the last two steps the condition 6 of the realizability conditions and the equality $y(00\cdots 0, w, \tau) = 0$ were applied. So, we get that the following holds:

$$\frac{d^{\alpha}}{dt^{\alpha}} y_{(q_1 q_2 \cdots q_k, u_1 u_2 \cdots u_k)} = \frac{d^{(\alpha_k, \alpha_{k-1}, \ldots, \alpha_l)}}{dt^{(\alpha_k, \alpha_{k-1}, \ldots, \alpha_l)}} y_{(q_l q_{l+1} \cdots q_k, u_l \underbrace{00\cdots 0}_{k-l-times})}$$

Taking into account equalities (6.7) and (6.8) one immediately gets

$$H_y(\frac{d^{\alpha}}{dt^{\alpha}} y_{(q_1 q_2 \cdots q_k, u_1 u_2 \cdots u_k)}) = \mathcal{A}_{q_k}^{\alpha_k} \mathcal{A}_{q_{k-1}}^{\alpha_{k-1}} \cdots \mathcal{A}_{q_l}^{\alpha_l - 1} \mathcal{B}_{q_l} u_l \tag{6.9}$$

and

$$\frac{d^{\alpha}}{dt^{\alpha}} y(q_1 q_2 \cdots q_k, u_1 u_2 \cdots u_k, \underline{t})|_{\underline{t}=0} = \mathcal{C}_{q_k} \mathcal{A}_{q_k}^{\alpha_k} \mathcal{A}_{q_{k-1}}^{\alpha_{k-1}} \cdots \mathcal{A}_{q_l}^{\alpha_l - 1} \mathcal{B}_{q_l} u_l \tag{6.10}$$

(4) Consider vector spaces

$$W = \mathrm{Span}\{\mathcal{A}_{q_k}^{\alpha_k} \mathcal{A}_{q_{k-1}}^{\alpha_{k-1}} \cdots \mathcal{A}_{q_1}^{\alpha_1} \mathcal{B}_z u \mid u \in \mathcal{U}, q_1, q_2, \ldots q_k, z \in Q, \alpha \in \mathbb{N}^k\}$$

and

$$O = \bigcap_{q_1, q_2, \ldots, q_k, z \in Q, \alpha \in \mathbb{N}^k} \ker \mathcal{C}_z \mathcal{A}_{q_k}^{\alpha_k} \mathcal{A}_{q_{k-1}}^{\alpha_{k-1}} \cdots \mathcal{A}_{q_1}^{\alpha_1}$$

From (6.6) and (6.9) it follows that $H = H_y(\mathcal{X}_y) = W$. We will show that $O = \{0\}$. Let $f = \frac{d^{\alpha}}{dt^{\alpha}} y_{(x,v)} \in \mathcal{X}_y$. Then

$$\begin{aligned} \mathcal{C}_{w_z} \mathcal{A}_{w_z}^{\beta_z} \mathcal{A}_{w_{z-1}}^{\beta_{z-1}} \cdots \mathcal{A}_{w_1}^{\beta_1} H_y f &= \mathcal{C}_{w_z} H_y(\frac{d^{\beta}}{d\tau^{\beta}} \frac{d^{\alpha}}{dt^{\alpha}} y_{(xw, v \underbrace{0\cdots 0}_{z-times})}) \\ &= H_y(f)((w_1, \beta_1)(w_2, \beta_2) \cdots (w_z, \beta_z)) \end{aligned}$$

For each $z \in O$ there exist $f_1, f_2, \ldots f_r$ and $\alpha_i \in \mathbb{R}, i = 1, 2, \ldots r$ such that $f_i = \frac{d^{(\alpha(i,k(i)), \alpha(i,k(i)-1), \ldots, \alpha(i,1))}}{dt^{(\alpha(i,k(i)), \alpha(i,k(i)-1), \ldots, \alpha(i,1))}} y_{(w_i, u_i)}$ and $z = \sum_{i=1}^{r} \gamma_i H_y(f_i)$. For each

$$(w, \beta) = (w_1, \beta_1)(w_2, \beta_2) \cdots (w_k, \beta_k) \in (Q \times \mathbb{N})^+$$

it holds that
$\mathcal{C}_{w_k} \mathcal{A}_{w_k}^{\beta_k} \mathcal{A}_{w_{k-1}}^{\beta_{k-1}} \cdots \mathcal{A}_{z_1}^{\beta_1} z = 0$. But

$$\begin{aligned} \mathcal{C}_{w_k} \mathcal{A}_{w_k}^{\beta_k} \mathcal{A}_{w_{k-1}}^{\beta_{k-1}} \cdots \mathcal{A}_{z_1}^{\beta_1} \sum_{i=1}^{r} \gamma_i H_y f_i &= \sum_{i=1}^{r} \gamma_i \mathcal{C}_{w_k} \mathcal{A}_{w_k}^{\beta_k} \mathcal{A}_{w_{k-1}}^{\beta_{k-1}} \cdots \mathcal{A}_{z_1}^{\beta_1} H_y(f_i) \\ &= \sum_{i=1}^{r} \gamma_i H_y(f_i)((w, \beta)) = z(w, h) \end{aligned}$$

So for each $(w, \beta) \in (Q \times \mathbb{N})^+$ we get that $z((w, \beta)) = 0$, that is, $z = 0$.

(5) Since $n = \dim H$ there is a $T : H \to \mathbb{R}^n$ vector space isomorphism. Define on $\mathbb{R}^n$ the following linear switched system $\Sigma = (\mathbb{R}^n, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) | q \in Q\})$ where

$$A_q = T\mathcal{A}_q T^{-1}, \quad B_q = T\mathcal{B}_q, \quad C_q = \mathcal{C}_q T^{-1}$$

Then for each $q_1, q_2, \dots q_k \in Q$, $u \in \mathcal{U}$, $\alpha \in \mathbb{N}^k$ we get that

$$C_{q_k} A_{q_k}^{\alpha_k} \cdots A_{q_1}^{\alpha_1 - 1} B_{q_1} u = \mathcal{C}_{q_k} \mathcal{A}_{q_k}^{\alpha_k} \cdots \mathcal{A}_{q_1}^{\alpha_1 - 1} \mathcal{B}_{q_1} u$$

This and (6.10) together with Corollary 15 imply that Σ is indeed a realization of y. Also, we get that $Reach(\Sigma) = TW = TH = \mathbb{R}^n$, so Σ is reachable. Again, $TO = O_\Sigma = \{0\}$, so Σ is observable. That is, Σ is a minimal linear switched system that realizes y and its state space is of dimension n. □

As a consequence of the theorem we get the following corollary

Corollary 16. *Let* $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) | q \in Q\})$ *be a linear switched system. Let* $y := \widetilde{y}_\Sigma$*. Then* $\operatorname{rank} H_y \leq \dim \mathcal{X}$*. The system* Σ *is minimal if and only it holds that* $\operatorname{rank} H_y = n = \dim \mathcal{X}$.

Chapter 7

Realization Theory of Linear and Bilinear Hybrid Systems

In this chapter we will present realization theory for linear and bilinear hybrid systems. The material of this chapter is partly based on [48, 54]. Let us first recall the realization problem for linear and bilinear hybrid systems.

1. *Reduction to a minimal realization* Consider a linear (bilinear) hybrid system H, and a subset of its input-output maps Φ. Find a minimal linear (bilinear) hybrid system which realizes Φ.

2. *Existence of a realization* Find necessary and sufficient condition for existence of a linear (bilinear) hybrid system realizing a specified set of input-output maps.

3. *Partial realization* Find a procedure for constructing a linear (bilinear) hybrid system realization of a set of input-output maps from finite data.

Except the partial realization problem, which we be treated in Section 10.5, all the problems listed above will be discussed in this chapter. More precisely, we will present the following results.

- A linear (bilinear) hybrid system is a minimal realization of a set of input-output maps if and only if it is observable and semi-reachable. Minimal linear (bilinear) hybrid systems which realize a given set of input-output maps are unique up to isomorphism. Each linear (bilinear) hybrid system H realizing a set of input-output maps Φ can be transformed to a minimal realization of Φ.

- A set of input/output maps is realizable by a linear hybrid system if and only if it has a *hybrid kernel representation*, the rank of its Hankel-matrix is finite, the discrete parts of the input/output maps are realizable by a finite Moore-automaton and certain other finiteness conditions hold. A set of input/output maps is realizable by a bilinear hybrid system if and only if it has a *hybrid Fliess-series expansion*, the rank of its Hankel-matrix is finite and the discrete parts of the input/output maps are realizable by a finite Moore-automaton. There is a procedure to construct the linear (bilinear) hybrid system realization from the columns of the Hankel-matrix, and this procedure yields a minimal realization.

Notice that the results above are very similar to those for hybrid formal power series. This is not a coincidence, in fact, the results announced above will be proven by using theory of hybrid formal power series. It turns out that there is one-to-one correspondence between linear and bilinear hybrid systems and hybrid representations of certain families of hybrid formal power series. This correspondence will enable us to reduce the realization problem for linear and bilinear hybrid systems to the problem of existence and minimality of hybrid representations for a certain family of hybrid formal power series. Moreover, such system theoretic properties of hybrid systems as observability, semi-reachability and minimality have their counterparts in hybrid representations. That is, there is one-to-one correspondence between reachable, observable, minimal hybrid representations and semi-reachable, observable, minimal linear and bilinear hybrid systems. Thus, theory of hybrid formal power series can be used to characterise minimality of linear and bilinear hybrid systems. It can be also used to derive partial realization theory of linear and bilinear hybrid systems, see Section 10.5.

It is also possible to develop realization theory for linear and bilinear hybrid system without using hybrid formal power series. It was done in [48, 54]. Compared to the direct approach the use of hybrid formal power series helps to avoid unnecessary repetition of proofs and concepts. It also results in a much more elegant and concise treatment of realization theory for linear and bilinear hybrid systems.

In fact, the main motivation for discussing realization theory for both linear and bilinear hybrid systems in one chapter is that in both cases the same framework of hybrid formal power series can be used.

The outline of the chapter is the following. Section 7.1 describes realization theory of linear hybrid systems. Section 7.2 presents realization theory of bilinear hybrid systems.

7.1 Realization Theory for Linear Hybrid Systems

In this section realization theory of linear hybrid systems will be discussed. As it was mentioned in the introduction to the chapter, the theory of hybrid formal series will be the main tool for developing realization theory of linear hybrid systems.

In fact, one can pursue a direct approach for realization theory of linear hybrid systems, without resorting to theory of hybrid formal power series. This was done in [54]. A quick comparison of the direct approach and the one with hybrid formal power series reveals that in the former one in fact repeats the proofs of Section 3.3 on hybrid formal power series. Thus, the direct approach does not seem to yield a construction simpler than the current one.

The outline of the section is the following. Subsection 7.1.1 presents certain concepts and elementary results related to linear hybrid systems. Subsection 7.1.2 describes the structure of input-output maps of linear hybrid systems. Finally, Subsection 7.1.3 develops realization theory for linear hybrid systems.

7.1.1 Linear Hybrid Systems

Recall from Chapter 2, Section 2.3 the definition of linear hybrid systems. In this section we will introduce some additional notation and terminology, which will be used specifically for linear hybrid systems. Let

$$H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, B_q, C_q)_{q \in Q}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2, \gamma)\})$$

be a linear hybrid systems. With abuse of notation denote by $\mathcal{X}$ the set $\mathcal{X} = \bigoplus_{q \in Q} \mathcal{X}_q$. Recall from Section 2.3 that $\mathcal{A}_H$ refers to the Moore automaton $\mathcal{A}$ of H.

Recall the definition of the continuous state-trajectory $x_H : \mathcal{H} \times PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T \to \bigcup_{q \in Q} \mathcal{X}_q$. Notice that $\bigcup_{q \in Q} \mathcal{X}_q$ can be viewed as a subset of $\mathcal{X} = \bigoplus_{q \in Q} \mathcal{X}_q$. Thus, x_H can be viewed as a map which takes its values in $\mathcal{X}$. In the sequel we will view x_H as a map taking its values in $\mathcal{X}$. We can derive an explicit expression for the continuous state trajectory x_H using the well-known expression for trajectories of linear systems

Proposition 24. *For all $h_0 \in \mathcal{H}$, $h_0 = (q_0, x_0)$, $u \in PC(T,\mathcal{U})$, $w \in (\Gamma \in T)^*$,*

$w = (\gamma_1, t_1) \cdots (\gamma_k, t_k)$, $\gamma_1, \ldots, \gamma_k \in \Gamma$, $k \geq 0$, $t_{k+1} \in T$,

$$\begin{aligned} & x_H(h_0, u, w, t_{k+1}) = e^{A_{q_k} t_{k+1}} M_{q_k,\gamma_k,q_{k-1}} e^{A_{q_{k-1}} t_k} \cdots M_{q_1,\gamma_1,q_0} e^{A_{q_0} t_1} x_0 + \\ & + \sum_{i=0}^{k} e^{A_{q_k} t_{k+1}} M_{q_k,\gamma_k,q_{k-1}} e^{A_{q_{k-1}} t_k} \cdots \\ & \cdots e^{A_{q_{i+1}} t_{i+2}} M_{q_{i+1},\gamma_i,q_i} \int_0^{t_{i+1}} e^{A_{q_i}(t_{i+1}-s)} B_{q_i} u_i(s) ds \end{aligned} \tag{7.1}$$

where $q_{i+1} = \delta(q_i, \gamma_{i+1})$, $u_i(s) = u(\sum_{j=1}^{i} t_j + s)$, $0 \leq i \leq k$.

Proof. We proceed by induction. If $k = 0$, then $x_H(h_0, u, \epsilon, t_1)$ is a state-trajectory of the linear system $\frac{d}{dt}x(t) = A_{q_0}x(t) + B_{q_0}u(t)$ and thus

$$x_H(h_0, u, \epsilon, t_1) = e^{A_{q_0} t_1} x_0 + \int_0^{t_1} e^{A_{q_0}(t_1-s)} B_{q_0} u(s) ds$$

Assume that the statement of the proposition is true for $k \leq N$. That is,

$$\begin{aligned} & x_H(h_0, u, (\gamma_1, t_1) \cdots (\gamma_N, t_N), t_{N+1}) = e^{A_{q_N} t_{N+1}} M_{q_N,\gamma_N,q_{N-1}} \cdots M_{q_1,\gamma_1,q_0} e^{A_{q_0} t_1} x_0 + \\ & \sum_{l=l}^{N+1} e^{A_{q_N} t_{N+1}} M_{q_N,\gamma_N,q_{N-1}} \cdots M_{q_l,\gamma_l,q_{l-1}} \int_0^{t_l} e^{A_{q_{l-1}} t_l - s} B_{q_{l-1}} u(s + \sum_{j=1}^{l-1} t_j) ds \end{aligned}$$

Consider any $\gamma_{N+1} \in \Gamma$, $t_{N+2} \in T$. Recall that

$$x_H(h_0, u, (\gamma_1, t_1) \cdots (\gamma_{N+1}, t_{N+1}), t_{N+2}) = x(t_{N+2})$$

where $x(t)$ is the state trajectory of the linear system $\frac{d}{dt}x(t) = A_{q_{N+1}}x(t) + B_{q_{N+1}}u(t + \sum_{j=1}^{N+1} t_j)$ and $x(0) = M_{q_{N+1},\gamma_{N+1},q_N} x_H(h_0, u, (\gamma_1, t_1) \cdots (\gamma_N, t_N), t_{N+1})$. Thus, $x(t) = e^{A_{q_{N+1}} t} x_H(h_0, u, (\gamma_1, t_1) \cdots (\gamma_N, t_N), t_{N+1}) + \int_0^t e^{A_{q_{N+1}}(t-s)} B_{q_{N+1}} u(s + \sum_{j=1}^{N+1} t_j) ds$. Combining the expression for $x(t)$ with the induction hypothesis we get that

$$\begin{aligned} & x(t) = e^{A_{q_{N+1}} t} M_{q_{N+1},\gamma_{N+1},q_N} (e^{A_{q_N} t_{N+1}} M_{q_N,\gamma_N,q_{N-1}} \cdots M_{q_1,\gamma_1,q_0} e^{A_{q_0} t_1} x_0 + \\ & \sum_{l=0}^{N} e^{A_{q_N} t_{N+1}} M_{q_N,\gamma_N,q_{N-1}} \cdots \\ & \cdots e^{A_{q_{l+1}} t_{l+2}} M_{q_{l+1},\gamma_{l+1},q_l} \int_0^{t_{l+1}} e^{A_{q_l}(t_l - s)} B_{q_l} u(s + \sum_{j=1}^{l} t_j) ds) + \\ & + \int_0^{t_{N+2}} e^{A_{q_{N+1}}(t_{N+2}-s)} B_{q_{N+1}} u(s + \sum_{j=1}^{N+1} t_j) ds \end{aligned}$$

It is easy to see that the expression above is equivalent to the formula in the statement of the proposition. ∎

Let $\mathcal{H}_0$ be a subset of $\mathcal{H}$. Recall the definition of the set $Reach(H, \mathcal{H}_0)$. The linear hybrid system H is said to be *semi-reachable* from $\mathcal{H}_0$ if $\mathcal{X}$ is the vector space of the smallest dimension containing $Reach(H, \mathcal{H}_0)$ and the automaton $\mathcal{A}_H$ is reachable from $\Pi_Q(\mathcal{H}_0)$. That is, H is semi-reachable from $\mathcal{H}_0$ if $\mathcal{A}_H$ is reachable from $\Pi_Q(\mathcal{H}_0)$ and $\mathcal{X} = \mathrm{Span}\{z \mid z \in Reach(H, \mathcal{H}_0)\}$. Recall the notion of a hybrid system realization. Hybrid system realizations of the form (H, μ) where H is a linear hybrid system will be called linear hybrid system realizations. We say that a linear hybrid system realization (H, μ) is semi-reachable if H is semi-reachable from $\mathrm{Im}\mu$.

Recall the definition of hybrid morphisms. For linear hybrid systems we will use a related but slightly different notion of system morphism, which we will call *linear hybrid morphisms*. The goal of this new definition is to capture the linear structure of linear hybrid systems. Let (H, μ) and $(H^{'}, \mu^{'})$ be two realizations such that $dom(\mu) = dom(\mu^{'})$, i.e. the domain of definition of μ and $\mu^{'}$ coincide and

$$\begin{aligned} H &= (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, B_q, C_q)_{q \in Q}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2, \gamma)\}) \\ H^{'} &= (\mathcal{A}^{'}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}^{'}_q, A^{'}_q, B^{'}_q, C^{'}_q)_{q \in Q^{'}}, \{M^{'}_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q^{'}, \gamma \in \Gamma, q_1 = \delta^{'}(q_2, \gamma)\}) \end{aligned}$$

where $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ and $\mathcal{A}^{'} = (Q^{'}, \Gamma, O, \delta^{'}, \lambda^{'})$. A pair $T = (T_D, T_C)$ is called a *linear hybrid morphism* from (H, μ) to $(H^{'}, \mu^{'})$, denoted by $T : (H, \mu) \to (H^{'}, \mu^{'})$, if the the following holds. The map $T_D : (\mathcal{A}, \mu_D) \to (\mathcal{A}^{'}, \mu^{'}_D)$, where $\mu_D(f) = \Pi_Q(\mu_D(f))$, $\mu^{'}_D(f) = \Pi_{Q^{'}}(\mu^{'}_D(f))$, is an automaton morphism and $T_C : \bigoplus_{q \in Q} \mathcal{X}_q \to \bigoplus_{q \in Q^{'}} \mathcal{X}^{'}_q$ is a linear morphism, such that

- $\forall q \in Q : T_C(\mathcal{X}_q) \subseteq \mathcal{X}^{'}_{T_D(q)}$,
- $T_C A_q = A^{'}_{T_D(q)} T_C \quad T_C B_q = B^{'}_{T_D(q)} \quad C_q = C^{'}_{T_D(q)} T_C$ for each $q \in Q$,
- $T_C M_{q_1,\gamma,q_2} = M^{'}_{T_D(q_1),\gamma,T_D(q_2)} T_C$, $\forall q_1, q_2 \in Q, \gamma \in \Gamma, \delta(q_2, \gamma) = q_1$,
- $T_C(\Pi_{\mathcal{X}_q}(\mu(f))) = \Pi_{\mathcal{X}^{'}_{T_D(q)}}(\mu^{'}(f))$ for each $q = \mu_D(f), f \in dom(\mu)$.

The linear hybrid morphism T is said to be injective, surjective or bijective if both T_D and T_C are respectively injective, surjective and bijective. Bijective linear hybrid morphisms are called linear hybrid isomorphisms. Two linear hybrid system realizations are *isomorphic* if there exists a linear hybrid isomorphism between them. Notice that linear hybrid morphisms can be defined between realizations (H, μ) and $(H^{'}, \mu^{'})$ only if μ and $\mu^{'}$ have the same domain of definition.

Notice that the linear map $T_C : \bigoplus_{q \in Q} \mathcal{X}_q \to \bigoplus_{q \in Q^{'}} \mathcal{X}^{'}_q$ is uniquely determined by its restriction to $\bigcup_{q \in Q} \mathcal{X}_q$, which we will denote by $M(T_C)$. It is easy to see that in fact $M(T_C)$ takes it values in $\bigcup_{q \in Q^{'}} \mathcal{X}^{'}_q$. The following proposition is an easy consequence of the remarks above.

Proposition 25. *With the notation above, if $T = (T_D, T_C)$ is a linear hybrid morphism, then $\psi(T) = (T_D, M(T))$ is a hybrid morphism. Moreover, T is a linear hybrid isomorphism if and only if $\psi(T)$ is a hybrid isomorphism.*

Recall that with any hybrid morphism $S : (H_1, \mu_1) \to (H_2, \mu_2)$ one can associate a map $\phi(S) : \mathcal{H}_1 \to \mathcal{H}_2$. If $T : (H_1, \mu_1) \to (H_2, \mu_2)$ is an linear hybrid morphism between linear hybrid system realizations, then by the proposition above we can associate with it a hybrid morphism $\psi(T)$, with which, in turn, we can associate the map $\phi(\psi(T))$. Whenever it doesn't create confusion we will denote $\phi(\psi(T))$ simply by $\phi(T)$ or T. Then the following holds.

Proposition 26. *The map T is a linear hybrid isomorphism if and only if $\phi(T)$ is bijective as a map from $\mathcal{H}_1$ to $\mathcal{H}_2$.*

Proof. Indeed, assume that $\phi(T) : \mathcal{H}_1 \to \mathcal{H}_2$ is bijective. Then for all $q \in Q^{'}$ there exists uniquely a $q \in Q$ such that $T((q, 0)) = (T_D(q), T_C(0)) = (q^{'}, 0)$, i.e., $T_D(q) = q^{'}$. Thus, T_D is bijective. For any $x \in \mathcal{X}^{'}_{q^{'}}$ there exists a unique $z \in \mathcal{X}_q$ such that $T((q, z)) = (T_D(q), T_C z) = (q, x)$, i.e., $T_C z = x$. Thus, T_C is surjective. We will show that T_C is injective. Indeed, assume that $T_C y = x$. Then $y = y_{q_1} + \cdots + y_{q_{|Q|}}$, where $y_{q_i} \in \mathcal{X}_{q_i}$, $i = 1, \ldots, |Q|$. But $T_C(y_{q_i}) \in \mathcal{X}^{'}_{T_D(q_i)}$, thus $T_C(y_{q_i}) = 0$ for all $i = 1, \ldots, |Q|, q_i \neq q$. Thus, $y \in \mathcal{X}_q$, and thus $y = z$.

If T is a linear hybrid isomorphism, then by Proposition 25 $\psi(T)$ is a hybrid isomorphism, thus by Proposition 1, $\phi(T) = \phi(\psi(T))$ is a bijection. □

Proposition 27. *Let (H_1, μ_1) and (H_2, μ_2) be two linear hybrid systems. Assume that $T : (H_1, \mu_1) \to (H_2, \mu_2)$ is a linear hybrid morphism. Then the following holds.*

- *If T is injective, then* $\dim H_1 \leq \dim H_2$.
- *If T is surjective, then* $\dim H_2 \leq \dim H_1$.
- *If T is either injective or surjective and* $\dim H_1 = \dim H_2$, *then T is a linear hybrid isomorphism*

Proof. Let

$$H_1 = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, B_q, C_q)_{q \in Q}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2, \gamma)\})$$

and

$$H_2 = (\mathcal{A}^{'}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}^{'}_q, A^{'}_q, B^{'}_q, C^{'}_q)_{q \in Q^{'}}, \{M^{'}_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q^{'}, \gamma \in \Gamma, q_1 = \delta^{'}(q_2, \gamma)\})$$

Then $T_C : \bigoplus_{q\in Q} \mathcal{X}_q \to \bigoplus_{q\in Q'} \mathcal{X}^{'}_q$ is a linear morphism. Assume that T is injective. Then T_C and T_D are injective. Then $card(Q) = card(T_D(Q)) \leq card(Q^{'})$ and

$$\text{rank } T_C = \sum_{q\in Q} \dim \mathcal{X}_q = \dim \bigoplus_{q\in Q} \mathcal{X}_q \leq \sum_{q\in Q'} \dim \mathcal{X}^{'}_q$$

Thus

$$\dim H_1 = (card(Q), \sum_{q\in Q} \dim \mathcal{X}_q) \leq (card(Q^{'}), \sum_{q\in Q'} \dim \mathcal{X}^{'}_q)$$

Similarly, if T is surjective, then T_C and T_D are surjective. Thus,

$$\sum_{q\in Q} \dim \mathcal{X}_q \geq \text{rank } T_C = \sum_{q\in Q'} \dim \mathcal{X}^{'}_q$$

and $card(Q) \geq card(T_D(Q)) = card(Q^{'})$. Thus, $\dim H_1 \geq \dim H_2$. Assume that T is injective and $\dim H_1 = \dim H_2$. Then

$$\text{rank } T_C = \sum_{q\in Q} \dim \mathcal{X}_q = \sum_{q\in Q'} \dim \mathcal{X}^{'}_q \text{ and } card(T_D(Q)) = card(Q) = card(Q^{'})$$

Similarly, if T is surjective and $\dim H_1 = \dim H_2$, then

$$\text{rank } T_C = \sum_{q\in Q'} \dim \mathcal{X}^{'}_q = \sum_{q\in Q} \dim \mathcal{X}_q \text{ and } card(T_D(Q)) = card(Q^{'}) = card(Q)$$

Thus, if T is injective or surjective and $\dim H_1 = \dim H_2$, then T_C and T_D are bijections, and thus T is a linear hybrid isomorphism. □

The following proposition gives an important system theoretic characterisation of linear hybrid morphisms.

Proposition 28. *Let (H_i, μ_i), $i = 1, 2$ be two linear hybrid systems and let $T : (H_1, \mu_1) \to (H_2, \mu_2)$ be a linear hybrid morphism. Then the following holds.*

$$\phi(T) \circ \xi_{H_1}(h, .) = \xi_{H_2}(\phi(T)(h), .) \text{ and } \upsilon_{H_1}(h, .) = \upsilon_{H_2}(\phi(T)(h), .), \forall h \in \mathcal{H}_1$$

If T is a linear hybrid isomorphism, then (H_1, μ_1) is semi-reachable if and only if (H_2, μ_2) is semi-reachable and (H_1, μ_1) is observable if and only if (H_2, μ_2) is observable.

Proof. All the statements of the proposition is a straightforward consequence of Proposition 2, except the one about semi-reachability. Assume that T is a linear hybrid isomorphism. Then T_C and T_D are bijective maps. Let $\psi(T) = (T_D, M(T))$.

Notice that $T_C(x) = M(T)(x)$ for each $x \in \mathcal{X}_q$, $q \in Q$. Thus, from the proof of Proposition 2, equation (2.2) it follows that $T_C x_{H_1}(h,u,s,t) = M(T)(x_{H_1}(h,u,s,t)) = x_{H_2}(\psi(T)(h),u,s,t) = x_{H_2}(\phi(T)(h),u,s,t)$ It is easy to see that $\phi(T)((\mu_1(f)) = \mu_2(f)$ and thus $\phi(T)(\mathrm{Im}\mu_1) = \mathrm{Im}\mu_2$. Then by linearity of T_C it follows that

$$T_C(\mathrm{Span}\{z \mid z \in \mathit{Reach}(H_1, \mathrm{Im}\mu_1)\}) = \mathrm{Span}\{T_C x_{H_1}(h,u,s,t) \mid h \in \mathrm{Im}\mu_1, u \in PC(T,\mathcal{U}), s \in (\Gamma \times T)^*, t \in T\} =$$
$$= \mathrm{Span}\{x_{H_2}(h,u,s,t) \mid h \in \phi(T)(\mathrm{Im}\mu_1), u \in PC(T,\mathcal{U}), s \in (\Gamma \times T)^*, t \in T\}) =$$
$$= \mathrm{Span}\{z \mid z \in \mathit{Reach}(H_2, \mathrm{Im}\mu_2)\}$$

On the other hand, (H_1, μ_1) is semi-reachable if and only if

$$\dim \mathrm{Span}\{z \mid z \in \mathit{Reach}(H_1, \mathrm{Im}\mu_1)\} = \dim \bigoplus_{q \in Q} \mathcal{X}_q$$

Since T_C is a linear isomorphism, we get that the latter equality is equivalent to

$$\dim \bigoplus_{q \in Q'} \mathcal{X}_q' = \dim T_C(\bigoplus_{q \in Q} \mathcal{X}_q) = \dim T_C(\mathrm{Span}\{z \mid z \in \mathit{Reach}(H_1, \mathrm{Im}\mu_1)\}) =$$
$$= \dim \mathrm{Span}\{z \mid z \in \mathit{Reach}(H_2, \mathrm{Im}\mu_2)\}$$

That is, it is equivalent to (H_2, μ_2) being semi-reachable. □

7.1.2 Input-output Maps of Linear Hybrid Systems

This section deals with properties of input-output maps of linear hybrid systems. Let $f \in F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ be an input-output map. Define $f_C = \Pi_{\mathcal{Y}} \circ f : PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T \to \mathcal{Y}$ and $f_D = \Pi_O \circ f : PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T \to O$. That is, $f(u,w,t) = (f_C(u,w,t), f_D(u,w,t))$ for all $u \in PC(T,\mathcal{U})$, $w \in (\Gamma \times T)^*$, $t \in T$. Below we will define the notion of hybrid kernel representations, existence of which is an important necessary condition for existence of a linear hybrid realization.

Definition 16 (hybrid kernel representation). *A set $\Phi \subseteq F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ is said to admit a* hybrid kernel representation *if there exist functions $K^f_w : \mathbb{R}^{k+1} \to \mathbb{R}^p$ and $G^f_{w,j} : \mathbb{R}^j \to \mathbb{R}^{p \times m}$ for each $f \in \Phi, w \in \Gamma^*, |w| = k, j = 1, 2, \dots, k+1$, such that*

1. *$\forall w \in \Gamma^*, \forall f \in \Phi, j = 1, 2, \dots, |w|+1$: K^f_w is analytic and $G^f_{w,j}$ is analytic*
2. *For each $f \in \Phi$, the function f_D depends only on Γ^*, i.e.*

$$\forall u_1, u_2 \in PC(T,\mathcal{U}), w \in \Gamma^*, \tau_1, \tau_2 \in T^{|w|}, t_1, t_2 \in T :$$
$$f_D(u_1, (w, \tau_1), t_1) = f_D(u_2, (w, \tau_2), t_2)$$

The function f_D will be regarded as a function $f_D : \Gamma^* \rightarrow O$.

3. *For each $f \in \Phi$, $w = \gamma_1\gamma_2\cdots\gamma_k \in \Gamma^*$, $t_{k+1} \in T$, $\gamma_1,\dots,\gamma_k \in \Gamma$, $\underline{t} = (t_1,\dots,t_k) \in T^k$:*

$$f_C(u,(w,\underline{t}),t_{k+1})) = K_w^f(t_1,\dots,t_k,t_{k+1}) + \\ + \sum_{i=0}^{k}\int_0^{t_{i+1}} G^f_{w,k+1-i}(t_{i+1}-s,t_{i+2},\dots,t_{k+1})\sigma_i u(s)ds$$

where $\sigma_j u(s) = u(s+\sum_{i=1}^{j} t_i)$.

Using the notation above, define for each $f \in \Phi$ the function $y_0^f : PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T \rightarrow \mathcal{Y}$ by

$$y_0^f(u,(w,\underline{t}),t_{k+1}) = \\ = \sum_{i=0}^{k}\int_0^{t_{i+1}} G^f_{w,k+1-i}(t_{i+1}-s,t_{i+2},\dots,t_{k+1})\sigma_i u(s)ds$$

where $\underline{t} = (t_1,\dots,t_k)$. It follows that $y_0^f(u,(w,\tau),t) = f_C(u,(w,\tau),t) - f_C(0,(w,\tau),t)$.

The intuition behind the definition fo y_0^f is the following. If (H,μ) is a realization of Φ, then for each $f \in \Phi$, $y_0^f = \Pi_{\mathcal{Y}} \circ \upsilon_H((\Pi_Q(\mu(f)),0),.)$. In fact, the following holds.

Lemma 32. *Consider a linear hybrid system realization (H,μ)*

$$H = (\mathcal{A},\mathcal{U},\mathcal{Y},(\mathcal{X}_q,A_q,B_q,C_q)_{q\in Q},\{M_{q_1,\gamma,q_2} \mid q_1,q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2,\gamma)\})$$

Then (H,μ) is a realization of Φ if and only if Φ has a hybrid kernel representation of the form

$$\begin{aligned} K_w^f(t_1,\dots,t_{k+1}) &= C_{q_k}e^{A_{q_k}t_{k+1}}M_{q_k,\gamma_k,q_{k+1}}\cdots e^{A_{q_0}t_0}\mu_C(f) \\ G^f_{w,k+2-j}(t_j,\dots,t_{k+1}) &= C_{q_k}e^{A_{q_k}t_{k+1}}M_{q_k,\gamma_k,q_{k-1}}\cdots \\ &\cdots e^{A_{q_j}t_{j+1}}M_{q_j,\gamma_j,q_{j-1}}e^{A_{q_{j-1}}t_j}B_{q_{j-1}} \\ f_D(u,(w,\tau),t) &= \lambda(\mu_D(f),w) \text{ for each } u \in PC(T,\mathcal{U}), \tau \in T^k, t \in T \end{aligned} \tag{7.2}$$

for each $w = \gamma_1\cdots\gamma_k, \gamma_1,\dots,\gamma_k \in \Gamma$, $k \geq 0$, $j = 1,\dots,k+1$, $f \in \Phi$. If (H,μ) is a realization of Φ, then $y_0^f = \Pi_{\mathcal{Y}} \circ \upsilon_H((\mu_D(f),0),.)$.

Proof. (H,μ) is a realization of Φ if and only if

$$f = \upsilon_H(\mu(f),.) \tag{7.3}$$

Let $y_H(h,.) = \Pi_{\mathcal{Y}} \circ \upsilon_H(h,.)$ for all $h \in \mathcal{H}$. Thus, (7.3) is equivalent to $f_C = y_H(\mu(f),.)$ and $f_D = \Pi_O \circ \upsilon_H(\mu(f),.)$. But for each $u \in PC(T,\mathcal{U}), w \in \Gamma^*, \tau \in T^{|w|}, t \in T$

$$\Pi_O \circ \upsilon_H((\mu_D(f),0),u,(w,\tau),t) = \lambda(\mu_D(f),w)$$

Thus, (7.3) implies $f_D(u,(w,\tau),t) = \lambda(\mu_D(f),w)$. It is easy to see that

$$y_H((q,x),u,(w,\underline{t}),t_{k+1}) = C_{q_k} e^{A_{q_k} t_{k+1}} M_{q_k,\gamma_k,q_{k+1}} \cdots e^{A_{q_0} t_0} x +$$
$$\sum_{j=1}^{k} \int_0^{t_j} C_{q_k} e^{A_{q_k} t_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots e^{A_{q_j} t_{j+1}} M_{q_j,\gamma_j,q_{j-1}} e^{A_{q_{j-1}} t_j - s} B_{q_{j-1}} u(s + \sum_{i=1}^{j-1} t_i) ds$$

where $w = \gamma_1 \cdots \gamma_k$, $\gamma_1, \dots, \gamma_k \in \Gamma$, $k \geq 0$, $\underline{t} = t_1 \cdots t_k$, $q_i = \delta(q, \gamma_1 \cdots \gamma_k)$, $i = 0, \dots, k$, $q_0 = q$, $(q,x) \in \mathcal{H}$.

Thus, we get that (7.3) is equivalent to

$$\begin{aligned}
&\forall w \in \Gamma^*, \underline{t} = t_1 \dots t_{k+1} \in T^{k+1}, t_{k+1} \in T, |w| = k, u \in PC(T,\mathcal{U}) : \\
&f_C(u,(w,\underline{t}),t_{k+1}) = y_H(\mu_f,u,(w,\underline{t}),t_{k+1}) = \\
&= K^f_w(t_1,t_2,\dots,t_{k+1}) + \\
&+ \sum_{j=1}^{k+1} \int_0^{t_j} G^f_{w,j}(t_j - s, t_{j+1}, \dots, t_{k+1}) u(s + \sum_{i=1}^{j-1} t_i) ds \\
&f_D(u,(w,\underline{t}),t_{k+1}) = \lambda(\mu_D(f),w)
\end{aligned} \tag{7.4}$$

Thus, Φ has a hybrid kernel representation of the form (7.2). The last statement of the lemma follows from the fact that

$$y_H((\mu_D(f),0),u,(w,\underline{t}),t_{k+1}) =$$
$$= \sum_{j=1}^{k+1} \int_0^{t_j} C_{q_k} e^{A_{q_k} t_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_j,\gamma_j,q_{j-1}} e^{A_{q_{j-1}} t_j - s} B_{q_{j-1}} u(s + \sum_{i=1}^{j-1} t_i) ds =$$
$$\sum_{j=1}^{k+1} \int_0^{t_j} G^f_{w,j}(t_j - s, t_{j+1}, \dots, t_{k+1}) u(s + \sum_{i=1}^{j-1} t_i) ds = y^f_0(u,(w,\underline{t}),t_{k+1})$$

for all $u \in PC(T,\mathcal{U})$, $w = \gamma_1 \cdots \gamma_k$, $\gamma_1, \dots, \gamma_k \in \Gamma$, $k \geq 0$, $\underline{t} = t_1 \cdots t_k \in T^k, t_{k+1} \in T$. □

If the set Φ has a hybrid kernel representation, then the collection of analytic functions $\{K^f_w, G^f_{w,j} \mid w \in \Gamma^*, j = 1,2,\dots,|w|+1, f \in \Phi\}$ determines $\{f_C \mid f \in \Phi\}$. Since K^f_w is analytic, we get that the collection $\{D^\alpha K^f_w, D^\beta G^f_{w,j} \mid \alpha \in \mathbb{N}^{|w|}, \beta \in \mathbb{N}^j\}$ determines K^f_w and $G^f_{w,j}$ locally.

For each $f \in \Phi$, $u \in PC(T,\mathcal{U})$, $w \in \Gamma^*$ define the maps

$$f_C(u,w,.) : T^{|w|+1} \ni (t_1,\dots,t_{|w|+1}) \mapsto f_C(u,(w,t_1\cdots t_{|w|}),t_{|w|+1})$$
$$y_0^f(u,w,.) : T^{|w|+1} \ni (t_1,\dots,t_{|w|+1}) \mapsto y_0^f(u,(w,t_1\cdots t_{|w|}),t_{|w|+1})$$

By applying the formula $\frac{d}{dt}\int_0^t f(t,\tau)d\tau = f(t,t) + \int_0^t \frac{d}{dt}f(t,\tau)d\tau$ and Definition 16 one gets

$$D^{\alpha}K_w^f = D^{\alpha}f_C(0,w,.) \quad , D^{\xi}G_{w,l}^f e_z = D^{\beta}y_0^f(e_z,w,.) \tag{7.5}$$

where $w = \gamma_1\cdots\gamma_k$, $l \le k+1$, $\mathbb{N}^{k+1} \ni \beta = (\underbrace{0,0,\dots,0}_{k-l+1-times},\xi_1+1,\xi_2,\dots,\xi_l)$, and e_z is the zth unit vector of $\mathbb{R}^m$, i.e $e_z^T e_j = \delta_{zj}$. The formula above implies that all the high-order derivatives of the functions $K_w^f, G_{w,j}^f$ ($f \in \Phi$, $w \in \Gamma^*$, $j = 1,2,\dots|w|+1$) at zero can be computed from high-order derivatives of the functions from Φ with respect to the relative arrival times of discrete events.

The discussion above yields the following result.

Lemma 33. *If Φ has a hybrid kernel representation, then the functions K_w^f, $G_{w,j}^f$, $f \in \Phi$, $w \in \Gamma^*$, $j = 1,\dots,|w|+1$ are uniquely defined. That is, if $\widetilde{K}_w^f$, $\widetilde{G}_{w,j}^f$ are analytic functions such that condition 3 holds, then*

$$\widetilde{K}_w^f = K_w^f \text{ and } \widetilde{G}_{w,j}^f = G_{w,j}^f$$

$f \in \Phi$, $w \in \Gamma^*$, $j = 1,\dots,|w|+1$.

Proof. Indeed, assume that both K_w^f, $G_{w,j}^f$ and $\widetilde{K}_w^f$, $\widetilde{G}_{w,j}^f$ are analytic functions which satisfy condition 3. Then by (7.5) for each $\alpha \in \mathbb{N}^{|w|+1}$, $\beta \in \mathbb{N}^{|w|+2-j}$, $j = 1,\dots,|w|+1$

$$D^{\alpha}K_w^f = D^{\alpha}f_C(0,w,.) = D^{\alpha}\widetilde{K}_w^f$$
$$D^{\beta}G_{w,|w|+2-j}^f e_z = D^{\eta}y_0^f(e_z,w,.) = D^{\beta}\widetilde{G}_{w,|w|+2-j}^f e_z$$

where $e_z \in \mathcal{U}$ is the zth unit vector, $z = 1,\dots,m$,
$\eta = (\underbrace{0,0,\dots,0}_{j},\beta_1+1,\beta_2,\dots,\beta_{|w|+2-j}) \in \mathbb{N}^{|w|+1}$. Thus we get that $D^{\alpha}K_w^f = D^{\alpha}\widetilde{K}_w^f$ and $D^{\beta}G_{w,j}^f = D^{\beta}\widetilde{G}_{w,j}^f$ holds for each $\alpha \in \mathbb{N}^{|w|+1}, \beta \in \mathbb{N}^{|w|+2-j}$. Since the functions K_w^f, $G_{w,j}^f$, $\widetilde{K}_w^f$ and $\widetilde{G}_{w,j}^f$ are analytic we get that $K_w^f = \widetilde{K}_w^f$ and $G_{w,j}^f = \widetilde{G}_{w,j}^f$. $\square$

From the discussion above one gets the following.

Proposition 29. *Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$. Let (H,μ) be a linear hybrid system realization*

$$H = (\mathcal{A},\mathcal{U},\mathcal{Y},(\mathcal{X}_q,A_q,B_q,C_q)_{q\in Q},\{M_{q_1,\gamma,q_2} \mid q_1,q_2 \in Q,\gamma \in \Gamma, q_1 = \delta(q_2,\gamma)\})$$

where $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$. *The pair* (H, μ) *is a realization of* Φ *if and only if* Φ *has a hybrid kernel representation and for each* $w \in \Gamma^*$, $f \in \Phi$, $j = 1, 2, \dots, m$ *and* $\alpha \in \mathbb{N}^{|w|+1}$ *the following holds*

$$D^{\alpha} y_0^f(e_j, w, .) = D^{\beta} G^f_{w,k+2-l} e_j = C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_l,\gamma_l,q_{l-1}} A_{q_{l-1}}^{\alpha_l - 1} B_{q_{l-1}} e_j$$
$$D^{\alpha} f_C(0, w, .) = D^{\alpha} K^f_w = C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} A_{q_0}^{\alpha_1} x_0$$
$$f_D(w) = \lambda(q_0, w)$$

where $l = \min\{h \mid \alpha_h > 0\}$, e_z *is the zth unit vector of* $\mathcal{U}$, $\beta = (\alpha_l - 1, \dots, \alpha_{|w|+1})$ *and* $w = \gamma_1 \cdots \gamma_k$, $\gamma_1, \dots, \gamma_k \in \Gamma$, $q_j = \delta(q_0, \gamma_1 \cdots \gamma_j)$ *and* $\mu(f) = (q_0, x_0)$.

Proof. By Lemma 32 (H, μ) is a realization of Φ if and only if Φ admits a hybrid kernel representation of the form (7.2). By (7.5) we get that $D^{\alpha} y_0^f(e_j, w, .) = D^{\beta} G^f_{w,l} e_j$ and $D^{\alpha} f(0, w, .) = D^{\alpha} K^f_w$. Using the notation of Lemma 32 define the functions

$$\phi_{f,w} : (t_1, \dots, t_{k+1}) \mapsto C_{q_k} e^{A_{q_k} t_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} e^{A_{q_0} t_1} \mu_C(f)$$
$$\psi_{f,w,l,j} : (t_l, \dots, t_{k+1}) \mapsto C_{q_k} e^{A_{q_k} t_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_l,\gamma_l,q_{l-1}} e^{A_{q_{l-1}} t_l} B_{q_{l-1}} e_j$$

for each $f \in \Phi$, $w \in \Gamma^*$, $|w| = k$, $j = 1, \dots, m$, $l = 1, \dots, k+1$, $w = \gamma_1 \cdots \gamma_k$. It is easy to see that $\psi_{f,w,l,j}, \phi_{f,w}$ are analytic.

$$D^{(\alpha_l - 1, \dots, \alpha_{k+1})} \psi_{f,w,l,j} = C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_l,\gamma_l,q_{l-1}} A_{q_{l-1}}^{\alpha_l - 1} B_{q_{l-1}} e_j$$
$$D^{(\alpha_1, \dots, \alpha_{k+1})} \phi_{f,w} = C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} A_{q_0}^{\alpha_1} \mu_C(f)$$

It is easy to see that $K^f_w = \phi_{f,w}$ and $G^f_{w,l} e_j = \psi_{f,w,l,j}$ are equivalent to $D^{\alpha} K^f_w = D^{\alpha} \phi_{f,w}$ and $D^{\beta} G^f_{w,l} e_j = D^{\beta} \psi_{f,w,l,j}$ for all $\alpha \in \mathbb{N}^{|w|+1}$ and $\beta \in \mathbb{N}^{|w|+2-l}$. Thus, using the notation of the statement of the proposition we get that (7.2) is equivalent to

$$\begin{aligned}
\forall f \in \Phi, w \in \Gamma^*, |w| = k, j = 1, \dots, m, \alpha \in \mathbb{N}^{k+1} :\\
D^{\alpha} y_0^f(e_j, w, .) = D^{\beta} G^f_{w,k+2-l} e_j = D^{\beta} \psi_{f,w,l,j} =\\
= C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_l,\gamma_l,q_{l-1}} A_{q_{l-1}}^{\alpha_l - 1} B_{q_{l-1}} e_j\\
D^{\alpha} f_C(0, w, .) = D^{\alpha} K^f_w =\\
= D^{\alpha} \phi_{f,w} = C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} A_{q_0}^{\alpha_1} \mu_C(f)\\
f_D(w) = \lambda(q_k) = \lambda(q_0, w)
\end{aligned}$$

□

Below we will present sufficient and necessary conditions for existence of hybrid kernel representation for a set of input-output maps $\Phi \subseteq F(PC(T, \mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$. Before formulating the conditions some notation has to be introduced.

Recall from [58] the definition of $L^p([a,b])$ spaces for intervals $[a,b] \subseteq \mathbb{R}$, and $1 \leq p \leq +\infty$. For each $1 \leq p \leq +\infty$, $t \in T$ denote by $L^p([0,t],\mathbb{R}^{n\times m})$ the vector space of n by m matrices of functions from $L^p([0,t_i])$. I.e. $f : [0,t] \to \mathbb{R}^{n\times m}$ is an element of $L^p([0,t],\mathbb{R}^{n\times m})$, if $f = (f_{i,j})_{i=1,\ldots,n,j=1,\ldots,m}$ and $f_{i,j} \in L^p([0,t_i])$, $i = 1,\ldots,n$, $j = 1,\ldots,m$. Notice that $PC([0,t],\mathcal{U}) \subseteq L^p([0,t],\mathcal{U})$ for all $t \in T$. Denote by $||.||_p$ the usual norm on $L^p([0,t],\mathbb{R})$. If $f \in L^p([0,t],\mathbb{R}^{n\times m})$, then denote by M_f the $n\times m$ matrix defined by $(M_f)_{i,j} = ||f_{i,j}||_p$ for all $i = 1,\ldots,n$, $j = 1,\ldots,m$. Let s be any norm on $\mathbb{R}^{n\times m}$. Then it is easy to see that $||.||_{p,s} : L^p([0,t],\mathbb{R}^{n\times m}) \to \mathbb{R}_+$, $||f||_{p,s} = s(M_f)$ is a norm on $L^p([0,t],\mathbb{R}^{n\times m})$. Recall that on $\mathbb{R}^{n\times m}$ all norms are equivalent, that is, if s_1, s_2 are two norms on $\mathbb{R}^{n\times m}$, then there exists $m, M > 0$ such that $ms_1(T) \leq s_2(T) \leq Ms_1(T)$ for all $T \in \mathbb{R}^{n\times m}$. But then it implies that $m||f||_{p,s_1} \leq ||f||_{p,s_2} \leq M||f||_{p,s_1}$ for all $f \in L^p([0,t],\mathbb{R}^{n\times m})$. Thus, all the norms $||.||_{p,s}$ for a fixed p induce the same topology. In the sequel we will assume that some norm s is fixed on $\mathbb{R}^{n\times m}$ and by abuse of notation we will denote $||.||_{p,s}$ simply by $||.||_p$. It is an easy consequence of the classical theory that $PC([0,t],\mathcal{U})$ is dense in $L^p([0,t],\mathcal{U})$, $1 \leq p < +\infty$ in the topology induced by the norm $||.||_p$.

For $f,g \in PC(T,\mathcal{U})$ define for any $\tau \in T$ the concatenation $f\#_\tau g \in PC(T,\mathcal{U})$ of f and g by

$$f\#_\tau g(t) = \begin{cases} f(t) & \text{if } t \leq \tau \\ g(t) & \text{if } t > \tau \end{cases}$$

Assume that $\Phi \subseteq F(PC(T,\mathcal{U}) \times (\Gamma\times T)^* \times T, \mathcal{Y}\times O)$. For each $f \in \Phi$ denote by y^f_0 the map $y^f_0 : (u,w,t) \mapsto f_C(u,w,t) - f_C(0,w,t)$.

Let $f \in \Phi$, $w \in \Gamma^*$, $k = |w|$, $\underline{t} = (t_1,\ldots,t_k)$, $t_{k+1} \in T$, $j = 1,\ldots,p$. Define the map

$$y^f_{(w,\underline{t}),t_{k+1}} : PC([0,S_k],\mathcal{U}) \ni u \mapsto y^f(u\#_{S_k}0,(w,\underline{t}),t) \in \mathbb{R}^p$$

where $S_k = \sum_{j=1}^{k+1} t_j$. For each $l = 1,\ldots,k+1$ define the following map

$$y^f_{l,(w,\underline{t}),t_{k+1}} : PC([0,t_l],\mathcal{U}) \ni u \mapsto y^f_0(\widetilde{u}_l,(w,\underline{t}),t) \in \mathbb{R}^p$$

where e_j is the jth unit vector of $\mathbb{R}^p$ and

$$\widetilde{u}_l(t) = \begin{cases} u(t - \sum_{j=1}^{l-1} t_j) & \text{if } t \in (\sum_{j=1}^{l-1} t_j, \sum_{j=1}^{l} t_j] \\ 0 & \text{otherwise} \end{cases}$$

Define the map

$$\psi^f_w : T^{k+1} \ni (t_1,\ldots,t_{k+1}) \mapsto f_C(0,(\gamma_1,t_1)(\gamma_2,t_2)\ldots(\gamma_k,t_k),t_{k+1}) \in \mathbb{R}^p$$

where w is assumed to be of the form $w = \gamma_1\cdots\gamma_k$, $\gamma_1,\ldots,\gamma_k \in \Gamma$. For each $u \in \mathcal{U}$ identify u with the constant map $[0,t_l] \ni s \mapsto u$ and define the map

$$\psi^f_{l,u,w} : T^{|w|+1} \ni (t_1,\ldots,t_k,t_{k+1}) \mapsto y^f_{l,(w,(t_1,\ldots,t_k)),t_{k+1}}(u) \in \mathbb{R}^p$$

Now we are ready to formulate sufficient and necessary conditions for existence of a hybrid kernel representation.

Theorem 30. *Φ has a hybrid kernel representation if and only if the following holds*

1. *For all $f \in \Phi$ the map f_D depends only on Γ^*, that is, for all $u, u^{'} \in PC(T,\mathcal{U})$, $w \in \Gamma^*$, $\tau, \tau^{'} \in T^{|w|}$, $t, t^{'} \in T$*

$$f_D(u,(w,\tau),t) = f_D(u^{'},(w,\tau^{'}),t)$$

   That is, f_D can be viewed as a map $f_D : \Gamma^* \to O$.

2. *For each $f \in \Phi$, for each $w \in \Gamma^*$, $|w| = k$, $k \geq 0$, $\underline{t} = (t_1, \dots, t_k) \in T^k$, $t_{k+1} \in T$*

$$\forall u, v \in PC(T,\mathcal{U}) : (u(\tau) = v(\tau) \text{ for all } \tau \in [0, \sum_{j=1}^{k+1} t_j]) \implies$$
$$f_C(u,(w,\underline{t}),t_{k+1}) = f_C(v,(w,\underline{t}),t_{k+1})$$

3. *For each $f \in \Phi$, $w \in \Gamma^*$, $k = |w|$, $k \geq 0$, $\underline{t} = t_1 \cdots t_k$, $t_{k+1} \in T$, the maps*

$$y^f_{(w,\underline{t}),t_{k+1}} : PC([0, \sum_{j=1}^{k+1} t_j], \mathcal{U}) \to \mathbb{R}^p$$

   are linear and contnious in $||.||_1$ norm.

4. *For each $f \in \Phi$, $w \in \Gamma^*$, $k = |w|$, $l = 1, \dots, k+1$, $u \in \mathcal{U}$ the maps $\psi^f_{l,w,u} : T^{k+1} \to \mathbb{R}^p$ and $\psi^f_w : T^{k+1} \to \mathbb{R}^p$ are analytic*

5. *For each $f \in \Phi$, $u \in PC(T,\mathcal{U})$, $v, w \in \Gamma^*$, $|w| = l$, $|v| = k$ $\underline{t} = t_1 \cdots t_k \in T^k$, $\underline{s} = s_1 \cdots s_l \in T^l$, $t_{k+1} \in T$, $S \in [\sum_{j=1}^l s_j, \sum_{j=1}^l s_j + t_1]$*

$$y^f_0(0\#_S u, (wv, \underline{st}), t_{k+1}) = y^f_0(u, (wv, \underline{0\tau}), \tau_{k+1})$$

   where $\underline{0} = 00\cdots0 \in T^l$ and $\tau_1 = t_1 - (S - \sum_{j=1}^l s_j)$, $\tau_{i+1} = t_{i+1}$ for all $i = 1, \dots, k$ and $\underline{\tau} = (\tau_1, \dots, \tau_k)$.

Proof. `only if`
Assume that Φ has a hybrid kernel representation.

`Condition 1`
It is easy to see that condition 1 is the same as the condition 2 of Definition 16.

`Condition 2`

Condition 2 follows from condition 3 of Definition 16. Indeed,

$$\begin{aligned} f_C(u,(w,\underline{t}),t_{k+1}) = K^f_w(t_1,\ldots,t_{k+1}) + \\ +\sum_{l=1}^{k+1}\int_0^{t_l} G^f_{w,k+2-l}(t_l-s,t_{l+1},\ldots,t_{k+1})u(s+T_l)ds = \\ = K^f_w(t_1,\ldots,t_{k+1}) + \\ +\sum_{l=1}^{k+1}\int_0^{t_l} G^f_{w,k+2-l}(t_l-s,t_{l+1},\ldots,t_{k+1})v(s+T_l)ds = \\ = f_C(v,(w,\underline{t}),t_{k+1}) \end{aligned}$$

where $T_l=\sum_{z=1}^{l-1}t_z$, $l=1,\ldots,k+1$.

`Condition 3`

Notice that for all $u\in PC([0,\sum_{j=1}^{k+1}t_j],\mathcal{U})$

$$\begin{aligned} y^f_0(u\#_{T_{k+2}}0,(w,\underline{t}),t_{k+1}) = y^f_0(u\#_{T_{k+2}}0,(w,\underline{t}),t_{k+1}) = y^f_{(w,\underline{t}),t_{k+1}}(u) = \\ \sum_{l=1}^{k+1}\int_0^{t_l} G^f_{w,k+2-l}(t_l-s,t_{l+1},\ldots,t_{k+1})u(s+T_l)ds \end{aligned} \tag{7.6}$$

where $T_{k+2}=\sum_{j=1}^{k+1}t_j$. Thus, $y^f_{(w,\underline{t}),t_{k+1}}$ is indeed linear map for each $w\in\Gamma^*$, $|w|=k$, $\underline{t}=(t_1,\ldots,t_k)\in T^k, t_{k+1}\in T$. Since $G^f_{w,k+2-l}$ is analytic, the map

$$\psi:[0,T_{k+2}]\ni s\mapsto G^f_{w,k+2-l}(s-\sum_{j=1}^{l-1}t_j,t_{l+1},\ldots,t_{k+1}) \text{ if } s\in[\sum_{j=1}^{l-1}t_j,\sum_{j=1}^{l}t_j]$$

is in $L^\infty([0,T_{k+2}],\mathbb{R}^{1\times m})$. Notice that by the formula above

$$y^f_{(w,\underline{t}),t_{k+1}}(u)=\int_0^{T_{k+2}}\psi(s)u(s)ds$$

Thus, using a slight reformulation of the well-known result from functional analysis ([58]) we get that $y^f_{(w,\underline{t}),t_{k+1}}$ is indeed linear and continuous in $||.||_1$ norm. That is, condition 3 holds.

`Condition 4`

Formula (7.6) implies that for each $u\in\mathcal{U}$,

$$\psi^f_{l,w,u}(t_1,\ldots,t_{k+1})=(\int_0^{t_l}G^f_{w,k+2-l}(t_l-s,t_{l+1},\ldots,t_{k+1})ds)u$$

By analyticity of $G^f_{w,k+2-l}$ it implies that $\psi^f_{l,w,u}$ is analytic. Similarly, notice that

$$\psi^f_w(t_1,\ldots,t_{k+1})=f_C(0,(w,\underline{t}),t_{k+1})=K^f_w(t_1,\ldots,t_{k+1})$$

Since K^f_w is analytic, we get that ψ^f_w is analytic. Thus, we have shown that condition 4 holds.

`Condition 5`

Notice that

$$\begin{aligned}
&y^f_0(0\#_S u,(wv,\underline{st}),t_{k+1}) = \\
&\sum_{j=1}^{l}\int_0^{s_i} G^f_{wv,k+l+2-j}(s_i-s,s_{i+1},\dots,s_l,t_1,\dots,t_{k+1})(0\#_S u)(s+S_j)ds+ \\
&+\sum_{j=1}^{k+1}\int_0^{t_i} G^f_{wv,k+2-j}(t_j-s,t_{j+1},\dots,t_{k+1})(0\#_S u)(s+T_j)ds = \\
&\sum_{j=1}^{k+1}\int_0^{t_i} G_{wv,k+2-j}(t_j-s,t_{j+1},\dots,t_{k+1})(0\#_S u(s+T_j)ds = \\
&\int_{S-\sum_{j=1}^{l}s_j}^{t_1} G^f_{wv,k+2-j}(t_1-s,t_2,\dots,t_{k+1})u(s)ds+ \\
&+\sum_{j=2}^{k+1} G^f_{wv,k+2-j}(t_j-s,t_{j+1},\dots,t_{k+1})u(s+T_j)ds = \\
&\sum_{j=1}^{l}\int_0^{0} G^f_{wv,k+l+2-j}(0-s,0,\dots,0,t_1,\dots,t_{k+1})u(s)ds+ \\
&\sum_{j=1}^{k+1}\int_0^{\tau_i} G_{wv,k+2-j}(\tau_j-s,\tau_{j+1},\dots,\tau_{k+1})u(s+Z_j)ds = \\
&y^f_0(u,(wv,\underline{0\tau}),\tau_{k+1})
\end{aligned}$$

where $S_j=\sum_{i=1}^{j-1}s_j$, $j=1,\dots,l$, $T_j=\sum_{j=1}^{l}s_j+\sum_{i=1}^{j-1}t_j$, $j=1,\dots,k+1$ and $Z_j=\sum_{i=1}^{j-1}\tau_j$ for $j=1,\dots,k+1$.

`if part`

Assume that conditions 1–5 hold. We will show that Φ admits a hybrid kernel representation. Notice that 1 is equivalent to condition 2 of Definition 16. Thus, it is enough to show that there exist analytic functions K^f_w and $G^f_{w,l}$ for each $f\in\Phi, w\in\Gamma^*, l=1,\dots,|w|+1$ such that condition 3 of Definition 16 holds. For each $f\in\Phi$, $w\in\Gamma^*$, $w=|k|$ let

$$K^f_w=\psi^f_w$$

and for each $l=1,\dots,k+1$ define the maps $G^f_{w,l}:T^l\to\mathbb{R}^{p\times m}$ as follows. For each fixed $t_2,\dots,t_l\in T$ define the maps

$$g_{l,w,i,t_2,\dots,t_l}: T\in s\mapsto \psi^f_{k+2-l,w,e_i}(\underbrace{0,0,\dots,0}_{k+1-l-\text{times}},s,t_2,\dots,t_l)$$

for each $i = 1, \ldots, m$. Denote by $g^{'}_{l,w,i,t_2,\ldots,t_l}$ the derivative of $g_{l,w,i,t_2,\ldots,t_l}$ and for each $t_1, \ldots, t_l \in T$ define

$$G^f_{w,l}(t_1, t_2, \ldots, t_l) = \begin{bmatrix} g^{'}_{l,w,1,t_2,\ldots,t_l}(t_1) & g^{'}_{l,w,2,t_2,\ldots,t_l}(t_1) & \cdots & g^{'}_{l,w,m,t_2,\ldots,t_l}(t_1) \end{bmatrix}^T$$

It is easy to see that both K^f_w and $G^f_{w,l}$ are analytic maps.

For each $\underline{\tau} = (\tau_1, \ldots, \tau_k) \in T^k$, $\tau_{k+1} \in T$ define the map

$$Z^f_{w,\underline{\tau},\tau_{k+1}} : PC([0, \sum_{j=1}^{k+1} \tau_j], \mathcal{U}) \to \mathbb{R}^p$$

by

$$Z_{w,\underline{\tau},\tau_{k+1}}(u) = \sum_{j=1}^{k+1} \int_0^{\tau_j} G^f_{w,k+2-j}(\tau_j - s, \tau_{j+1}, \ldots, \tau_{k+1}) u(s + \sum_{i=1}^{j-1} \tau_i) ds$$

If we can show that for each $w \in \Gamma^*$, $|w| = k$, $k \geq 0$, $\underline{\tau} \in T^k, \tau_{k+1} \in T$

$$Z^f_{w,\underline{\tau},\tau_{k+1}} = y^f_{(w,\underline{\tau}),\tau_{k+1}}, \tag{7.7}$$

then existence of a hybrid kernel representation follows easily. Indeed, notice that

$$f(u, (w, \underline{\tau}), \tau_{k+1}) = f(0, (w, \underline{\tau}), \tau_{k+1}) + y^f(u, (w, \underline{\tau}), \tau_{k+1})$$

and $f(0, (w, \underline{\tau}), \tau_{k+1}) = \psi^f_w(\tau_1, \ldots, \tau_{k+1})$. By condition 3

$$y^f(u, (w, \underline{\tau}), \tau_{k+1}) = y^f(u\#_{\sum_{j=1}^{k+1}\tau_j} 0, (w, \underline{\tau}), \tau_{k+1}) = y^f_{(w,\underline{\tau}),\tau_{k+1}}(\widetilde{u})$$

where $\widetilde{u}(s) = u(s), \forall s \in [0, \sum_{j=1}^{k+1} \tau_j]$. Thus, if (7.7) is true, then

$$f(u, (w, \underline{\tau}), \tau_{k+1}) = K^f_w(\tau_1, \ldots, \tau_{k+1}) + \\ + \sum_{j=1}^{k+1} \int_0^{\tau_j} G^f_{w,k+2-j}(\tau_j - s, \tau_{j+1}, \ldots, \tau_{k+1}) u(s + \sum_{i=1}^{j-1} \tau_j) ds$$

i.e., condition 3 of Definition 16 holds.

Notice that for each $z = 1, \ldots, m$

$$\psi^f_{k+2-l,w,e_z}(0, 0, \ldots, 0, 0, t_{l+1}, t_{l+2}, \ldots, t_{k+1}) = \\ y^f(0\#_0(e_z\#_0 0), (w, 00\cdots 00t_{l+1}\cdots t_k), t_{k+1})$$

But $0\#_0(e_z\#_0)0 = 0$ thus by linearity of $y^f_{(w,00\cdots 00t_{l+1}\cdots t_k),t_{k+1}}$ we get that

$$y^f(0\#_0(e_z\#_0 0), (w, 00\cdots 00t_{l+1}\cdots t_k), t_{k+1}) = y^f_{(w,00\cdots 00t_{l+1}\cdots t_k),t_{k+1}}(0) = 0$$

thus $\psi^f_{k+2-l,w,e_z}(0,0,\ldots,0,0,t_{l+1},\ldots,t_{k+1})=0$. Thus,

$$\begin{aligned}
\psi^f_{k+2-l,w,e_z}(0,0,\ldots,0,\tau_l,\ldots,\tau_{k+1}) = \\
= \int_0^{\tau_l} -\frac{d}{ds}\psi^f_{k+2-l,w,e_z}(\tau_l-s,\tau_{l+1},\ldots,\tau_{k+1})ds = \\
= \int_0^{\tau_l} g^{'}_{k+2-l,w,z,t_{l+1},\ldots,t_{k+1}}(\tau_l-s)ds = \\
= \int_0^{\tau_l} G^f_{k+2-l,w}(\tau_l-s,\tau_{l+1},\ldots,\tau_{k+1})e_z ds
\end{aligned} \tag{7.8}$$

It is also easy to see that that for any $u=(u_1,\ldots,u_m)^T\in\mathcal{U}$, $\underline{\tau}=(\tau_1,\ldots,\tau_k)\in T^k$, $\tau_{k+1}\in T$, $l=1,\ldots,k+1$, $s\in[0,\tau_l]$

$$\begin{aligned}
y^f_{(w,\underline{\tau}),\tau_{k+1}}(0\#_{T_l+s}(u\#_{\tau_l-s}\#0)) = \\
= \sum_{i=1}^m u_i\psi^f_{l,w,e_i}(0,\ldots,0,\tau_l-s,\tau_{l+1},\ldots,\tau_{k+1})
\end{aligned} \tag{7.9}$$

where $T_l=\sum_{j=1}^{l-1}\tau_j$. Indeed, by condition 5 we get that

$$y^f_{(w,\underline{\tau}),\tau_{k+1}}(0\#_{T_l+s}(u\#_{\tau_l-s}0)) = y^f_{(w,00\cdots 0(\tau_l-s)\cdots\tau_k),\tau_{k+1}}(u\#_{\tau_l-s}0)$$

and by linearity of $y^f_{(w,00\cdots 0(\tau_l-s)\tau_{l+1}\cdots\tau_k),\tau_{k+1}}$ we get that

$$\begin{aligned}
y^f_{(w,00\cdots 0(\tau_l-s)tau_{l+1}\cdots\tau_k),\tau_{k+1}}(u\#_{\tau_l-s}0) = \\
\sum_{i=1}^m u_i y^f_{(w,00\cdots 0(\tau_l-s)\tau_{l+1}\cdots\tau_k),\tau_{k+1}}(e_i\#_{\tau_l-s}0) = \\
= \sum_{i=1}^m u_i\psi^f_{k+2-l,w,e_i}(0,0,\ldots,0,\tau_l-s,\ldots,\tau_{k+1})
\end{aligned}$$

Thus, using (7.8)

$$\begin{aligned}
y^f_{(w,\underline{\tau}),\tau_{k+1}}(0\#_{T_l+s}(u\#_{\tau_l-s}\#0)) = \\
= \int_0^{\tau_l-s} G_{k+2-l,w}(\tau_l-s-\tau,\tau_{l+1},\ldots,\tau_{k+1})ud\tau = \\
= \int_0^{\tau_l} G_{k+2-l,w}(\tau_l-\tau,\tau_{l+1},\ldots,\tau_{k+1})(0\#_{T_l+s}(u\#_{\tau_l-s}0))(T_l+\tau)d\tau = \\
= Z^f_{(w,\underline{\tau}),\tau_{k+1}}(0\#_{T_l+s}(u\#_{\tau_l-s}0))
\end{aligned}$$

Notice that for each $s_1,s_2\in[0,\tau_l]$, $s_1<s_2$,

$$\begin{aligned}
0\#_{T_l+s_1}(u\#_{s_2-s_1}0) = 0\#_{T_l+s_1}((u\#_{s_2-s_1}0)\#_{\tau_l-s_2}0) = \\
0\#_{T_l+s_1}(u\#_{t_l-s_1}0) - 0\#_{T_l+s_1}(0\#_{s_2-s_1}(u\#_{\tau_l-s_2}0)) = \\
= 0\#_{T_l+s_1}(u\#_{t_l-s_1}0) - 0\#_{T_l+s_2}(u\#_{t_l-s_2}0)
\end{aligned}$$

Thus, by condition 5 and 3 we get that

$$
\begin{aligned}
& y^f(0\#_{T_l+s_1}(u\#_{s_2-s_1}0),(w,\underline{\tau}),\tau_{k+1}) = \\
=& y^f(0\#_{T_l+s_1}(u\#_{\tau_l-s_1}0),(w,\underline{\tau}),\tau_{k+1}) - y^f(0\#_{T_l+s_2}(u\#_{\tau_l-s_2}0),(w,\underline{\tau}),\tau_{k+1}) = \\
&= \int_0^{\tau_l-s_1} G^f_{w,k+2-l}(\tau_l - s_1 - \tau, \tau_{l+1},\dots,\tau_{k+1})ud\tau - \\
&- \int_0^{\tau_l-s_2} G^f_{w,k+2-l}(\tau_l - s_2 - \tau, \tau_{l+1},\dots,\tau_{k+1})ud\tau = \\
&= \int_0^{s_2-s_1} G^f_{w,k+2-l}(\tau_l - s_1 - \tau, \tau_{l+1},\dots,\tau_{k+1})ud\tau = \\
&= \int_0^{\tau_l} G^f_{w,k+2-l}(\tau_l - \tau, \tau_{l+1},\dots,\tau_{k+1})(0\#_{s_1}(u\#_{s_2-s_1}0)(\tau)d\tau
\end{aligned}
$$

Therefore, we get that for each $s_1, s_2 \in [0,\tau_l]$, $s_1 < s_2$

$$
y^f(0\#_{T_l+s_1}(u\#_{s_2-s_1}0),(w,\underline{\tau}),\tau_{k+1}) = Z^f_{(w,\underline{\tau}),\tau_{k+1}}(0\#_{s_1+T_l}(u\#_{s_2-s_1}0)) \tag{7.10}
$$

Let $T_{k+1} = \sum_{j=1}^{k+1}\tau_j$. For any piecewise-constant function $u : T \to \mathcal{U}$ there exist $n(1),\dots,n(k+1) \in \mathbb{N}$, $s_{i,j} \in T, i = 1,\dots,k+1, j = 1,\dots,k(i)$, such that $u(s) = u_{i,j} \in \mathcal{U}$ if $s \in [s_{i,j}, s_{i,j+1})$ or $s \in [s_{n(i)}, t_i]$ where $0 = s_{i,1} < s_{i,2} < \dots < s_{i,n(i)} < t_i$ and $i = 1,\dots,k+1$. Then it follows that

$$
u = \sum_{j=1}^{k+1}\sum_{j=1}^{n(i)} 0\#_{T_i}(0\#_{S_{i,j}}\#(u_{i,j}\#_{s_{i,j+1}}0)) =
$$

where $S_{i,j} = \sum_{z=1}^{j} s_{i,z}$, $i = 1,\dots,k+1$, $j = 1,\dots,n(i)$. Thus, by linearity of y^f and Z^f and by formula (7.10)

$$
\begin{aligned}
y^f(u,(w,\underline{\tau}),\tau_{k+1}) &= \sum_{j=1}^{k+1}\sum_{i=1}^{n(i)} y^f(0\#_{T_i+S_{i,j}}(u_{i,j}\#_{s_{i,j+1}}0),(w,\underline{\tau}),\tau_{k+1}) = \\
&= \sum_{j=1}^{k+1}\sum_{i=1}^{n(i)} Z^f_{(w,\underline{\tau}),\tau_{k+1}}(0\#_{T_i+S_{i,j}}(u_{i,j}\#_{s_{i,j+1}}0) = Z^f_{(w,\underline{\tau}),\tau_{k+1}}(u)
\end{aligned}
$$

That is,

$$
y^f_{(w,\underline{\tau}),\tau_{k+1}}(u) = Z^f_{(w,\underline{\tau}),\tau_{k+1}}(u) \text{ for all piecewise-constant } u
$$

Since both $y^f_{(w,\underline{\tau}),\tau_{k+1}}$ and $Z^f_{(w,\underline{\tau}),\tau_{k+1}}$ are continuous linear maps and any element of $PC(T,\mathcal{U})$ can be represented as a limit in $||.||_1$ of a sequence of piecewise-constant maps, we get that $y^f_{(w,\underline{\tau}),\tau_{k+1}} = Z^f_{(w,\underline{\tau}),\tau_{k+1}}$. By the remark above it implies the "if" part of the theorem. □

7.1.3 Realization of Input-output Maps by Linear Hybrid Systems

In this section the solution to the realization problem will be presented. That is, given a set of input-output maps we will formulate necessary and sufficient conditions for the existence of a linear hybrid system realizing that set. In addition, characterisation of minimal systems realizing the specified set of input-output maps will be given. We will use the theory of hybrid formal power series developed in Section 3.3.

The main idea behind the realization construction is the following. We associate a family of hybrid formal power series with the specified set of input-output maps. It turns out that if the set of input-output maps admits a hybrid kernel representation, then there is a one-to-one correspondence between the linear hybrid systems realization of the set of input-output maps and the hybrid representations of the hybrid formal power series. Moreover, minimal linear hybrid realizations correspond to minimal hybrid representations. Thus, we can use the theory of hybrid representations developed in Section 3.3 to develop realization theory for linear hybrid systems.

The outline of the subsection is the following. We start with presenting necessary and sufficient conditions for observability and semi-reachability of linear hybrid systems. Then we will proceed with defining the family of hybrid formal power series associated with the set of input-output maps and the correspondence between linear hybrid realizations and hybrid representations. As it was explained before, this correspondence will be used to formulate necessary and sufficient conditions for existence of a linear hybrid realization and to characterise minimality.

Observability and semi-reachability of linear hybrid systems

The following two theorems characterise observability and semi-reachability of linear hybrid systems. Observability of related classes of hybrid systems was investigated in [81, 8, 11]. Let

$$H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, B_q, C_q)_{q \in Q}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2, \gamma)\})$$

be a linear hybrid system. The following theorem characterises observability of linear hybrid systems.

Theorem 31. *H is observable if and only if*

(i) For each $s_1, s_2 \in Q$, $s_1 = s_2$ if and only if for all $\gamma_1, \dots \gamma_k \in \Gamma, j_1, \dots, j_{k+1} \geq$

$0, 0 \leq l \leq k, k \geq 0$:

$$\lambda(s_1, \gamma_1 \cdots \gamma_k) = \lambda(s_2, \gamma_1 \cdots \gamma_k) \ \textit{and}$$
$$C_{q_k} A_{q_k}^{j_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_{l+1},\gamma_{l+1},q_l} A_{q_l}^{j_{l+1}} B_{q_l} =$$
$$= C_{v_k} A_{v_k}^{j_{k+1}} M_{v_k,\gamma_k,v_{k-1}} \cdots M_{v_{l+1},\gamma_{l+1},v_l} A_{q_l}^{j_{l+1}} B_{v_l}$$

where $q_j = \delta(s_1, \gamma_1 \cdots \gamma_j)$ *and* $v_j = \delta(s_2, \gamma_1 \cdots \gamma_j), j = 0, 1, \ldots, k$.

(ii) For each $q \in Q$ *it holds that* $O_{H,q} := \bigcap_{w \in \Gamma^*} O_{q,w} = \{0\} \subseteq \mathcal{X}_q$ *where* $\forall w = \gamma_1 \cdots \gamma_k \in \Gamma^*, \gamma_1, \ldots, \gamma_k \in \Gamma, k \geq 0$*:*

$$O_{q,w} = \bigcap_{j_1,\ldots,j_k \geq 0} \ker C_{q_k} A_{q_k}^{j_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} A_{q_0}^{j_1}$$

where $q \in Q, q_l = \delta(q_0, \gamma_1 \cdots \gamma_l), 0 \leq l \leq k, k \geq 0$.

A quick look at Proposition 4 from Chapter 3 reveals that the conditions for observability of linear hybrid systems described in the theorem above are very similar to the conditions for observability of hybrid representations. It is by no means a coincidence and it is related to the correspondence between linear hybrid realizations and hybrid representations. More precisely, there is a direct correspondence between observability of linear hybrid systems and observability of certain hybrid representations. We will present this correspondence later on in this section.

Proof. For any $(q, x) \in \mathcal{H}$, define $y_H((q, x), .) = \Pi_{\mathcal{Y}} \circ \upsilon_H((q, x), .)$. For each $u \in PC(T, \mathcal{U})$, $w \in \Gamma^*$ define the function $y_H((q, x), u, w, .) : (t_1, \ldots, t_{k+1}) \ni T^{k+1} \mapsto y_H((q, x), u, (w, \underline{t}), t_{k+1})$, where $k = |w|$, $\underline{t} = (t_1, \ldots, t_{k+1})$. It is easy to see that $y_H((q, x), 0, w, .)$ is linear in x, that is, $y_H((q, ax_1+bx_2), 0, w, .) = ay_H((q, x_1), 0, w, .)+ by_H(((q, x_2), 0, w, .)$ for all $a, b \in \mathbb{R}$. On the other hand, $y_H((q, 0), u, w, .)$ is linear is u, that is, $y_H((q, 0), \alpha u_2 + \beta u_2, w, .) = \alpha y_H((q, 0), u_1, w, .) + \beta y_H((q, 0), u_2, w, .)$ for all $\alpha, \beta \in \mathbb{R}$. Moreover, $y_H((q, x), u, w, .) = y_H((q, x), 0, w, .) + y_H((q, 0), u, w, .)$.

First we show that $\upsilon_H((s_1, 0), .) = \upsilon_H((s_2, 0), .)$ if and only if for each $\gamma_1, \ldots, \gamma_k \in \Gamma$, $k \geq 0$, $l = 0, \ldots, k$, $j_1, \ldots, j_{k+1} \geq 0$,

$$\lambda(s_1, \gamma_1 \cdots \gamma_k) = \lambda(s_2, \gamma_1 \cdots \gamma_k) \text{ and}$$
$$C_{q_k} A_{q_k}^{j_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_{l+1},\gamma_{l+1},q_l} A_{q_l}^{j_{l+1}} B_{q_l} =$$
$$= C_{v_k} A_{v_k}^{j_{k+1}} M_{v_k,\gamma_k,v_{k-1}} \cdots M_{v_{l+1},\gamma_{l+1},v_l} A_{q_l}^{j_{l+1}} B_{v_l}$$

where $q_j = \delta(s_1, \gamma_1 \cdots \gamma_j)$ and $v_j = \delta(s_2, \gamma_1 \cdots \gamma_j), j = 0, 1, \ldots, k$. Indeed,

$$\upsilon_H((s_1, 0), .) = \upsilon_H((s_2, 0), .)$$

is equivalent to the fact that

$$\begin{aligned}(\lambda(s_1,w), y_H((s_1,0),u,(w,\tau),t_{k+1})) = \upsilon_H((s_1,0),u,(w,\underline{t}),t_{k+1}) = \\ = \upsilon_H((s_2,0),u,(w,\underline{t}),t_{k+1}) = \\ = (\lambda(s_2,w), y_H((s_2,0),u,(w,\underline{t}),t_{k+1})\end{aligned}$$

holds for all $u \in PC(T,\mathcal{U}), w \in \Gamma^*, \underline{t} \in T^k, k = |w|, t_{k+1} \in T$. That is, it is equivalent to $\lambda(s_1,w) = \lambda(s_2,w)$ for all $w \in \Gamma^*$ and $y_H((s_1,0),.) = y_H((s_2,0),)$. For each $s_i, i = 1,2$, let $f_i = \upsilon_H((s_i,0),.)$ and consider the following singleton set consisting of one single input-output map $\Phi_{s_i} = \{f_i\}$. Define the map $\mu_{s_i} : \Phi_{s_i} \ni f \mapsto (s_i,0) \in \mathcal{H}$. It is easy to see that (H, μ_{s_i}) is a realization of Φ_{s_i}. Thus, by Lemma 32 Φ_{s_1}, Φ_{s_2} admit a hybrid kernel representation and $y_H((s_i,0),.) = y_0^{f_i,\Phi_{s_i}}$ for $i = 1,2$. From the definition of $y_0^{f_i,\Phi_{s_i}}$ it is clear that $y_0^{f_i,\Phi_{s_1}} = y_0^{f_i,\Phi_{s_2}}$ is equivalent to requiring that $G^{f_1,\Phi_{s_1}}_{w,|w|-l+2} = G^{f_2,\Phi_{s_2}}_{w,|w|-l+2}$ holds for each $w \in \Gamma^*, l = 1,\dots,|w|+1$. Then by analyticity of $G^{f_i,\Phi_{s_i}}_{w,|w|-l+2}$, $i = 1,2$ that the latter is equivalent to $D^\alpha G^{f_1,\Phi_{s_1}}_{w,|w|-l+2} e_j = D^\alpha G^{f_2,\Phi_{s_2}}_{w,|w|-l+2} e_j$ for all $\alpha \in \mathbb{N}^{|w|-l+2}, w \in \Gamma^*, l = 1,\dots |w|+1$, $j = 1,\dots,m$. From Lemma 32 by uniqueness of hybrid kernel representation (Lemma 33) we get that the last equality is equivalent to

$$\begin{aligned}C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_l,\gamma_l,q_{l-1}} A_{q_{l-1}}^{\alpha_l} B_{q_{l-1}} e_j = \\ = D^\alpha y_H((s_1,0),e_j,w,.) = D^\alpha y_H((s_2,0),e_j,w,.) = \\ = C_{v_k} A_{v_k}^{\alpha_{k+1}} M_{v_k,\gamma_k,v_{k-1}} \cdots M_{v_l,\gamma_l,v_{l-1}} A_{v_{l-1}}^{\alpha_l} B_{v_{l-1}} e_j\end{aligned}$$

where $q_j = \delta(s_1, \gamma_1 \cdots \gamma_j)$ and $v_j = \delta(s_2, \gamma_1 \cdots \gamma_j), j = 0,1,\dots,k.$, e_j is the jth unit vector of $\mathbb{R}^m$, $\alpha \in \mathbb{N}^{k+1}$, $w \in \Gamma^*$, $k = |w|$, $l = 1,\dots,k+1$. That is, part (i) of the theorem is equivalent to

$$\forall s_1, s_2 \in Q \upsilon_H((s_1,0),.) = \upsilon_H((s_2,0),.) \iff s_1 = s_2$$

Next, we will show that $\upsilon_H((q,x_1),.) = \upsilon_H((q,x_2),.)$ is equivalent to

$$\forall w \in \Gamma^* : x_1 - x_2 \in O_{q,w} = \bigcap_{j_1,\dots,j_k \geq 0} \ker C_{q_k} A_{q_k}^{j_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} A_{q_0}^{j_1}$$

where $q_0 \in Q, q_l = \delta(q, \gamma_1 \cdots \gamma_l), 1 \leq l \leq k, k \geq 0$, $w = \gamma_1 \cdots \gamma_k, \gamma_1,\dots,\gamma_k \in \Gamma$.

Indeed, $\upsilon_H((q,x_1),.) = \upsilon((q,x_2),.)$ if and only if $y_H((q,x_1),.) = y_H((q,x_2),.)$. The equality $y_H((q,x),u,w,.) = y_H((q,0),0,w,.) + y_H((q,0),u,w,.)$ implies that $y_H((q,x_1),.) = y_H((q,x_2),.)$ if and only if $y_H((q,x_1),0,w,.) = y_H((q,x_2),0,w,.)$ holds for all $w \in \Gamma^*$, or, equivalently, $y_H((q,x_1-x_2),0,w,.) = 0$.

Consider the set $\Phi_{s,x_1-x_2} = \{f\}$, $f = \upsilon_H((s,x_1-x_2),.)$. Define $\mu_{s,x_1-x_2} : f \mapsto (s,x_1-x_2)$. It is easy to see that (H, μ_{s,x_1-x_2}) is a realization of Φ_{s,x_1-x_2}. By Lemma

32 Φ_{s,x_1-x_2} admits a hybrid kernel representation. By the definition of hybrid kernel representation, $f_C(0,(w,\tau),t) = y_H((s,x_1-x_2),0,(w,\tau),t) = K_w^{f,\Phi_{s,x_1-x_2}}(\tau,t)$. Thus, $y_H((s,x_1-x_2,0,w,.) = 0$ is equivalent to $K_w^{f,\Phi_{s,x_1-x_2}} = 0$ for each $w \in \Gamma^*$. The latter, by analyticity of $K_w^{f,\Phi_{s,x_1-x_2}}$, Lemma 32 and formula (7.5) is equivalent to

$$D^{\alpha} K_w^{f,\Phi_{s,x_1-x_2}} = D^{\alpha} y_H((q,x_1-x_2),0,w,.) = \\ = C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} A_{q_0}^{\alpha_1}(x_1-x_2) = 0$$

for all $\alpha \in \mathbb{N}^{|w|}, w \in \Gamma^*$, where $q_0 \in Q, q_l = \delta(q,\gamma_1\cdots\gamma_l), 1 \le l \le k, k \ge 0$, $w = \gamma_1\cdots\gamma_k, \gamma_1,\ldots,\gamma_k \in \Gamma$. That is, $x_1 - x_2 \in O_{q,w}$ for all $w \in \Gamma^*$.

That is, Part (ii) of the theorem is equivalent to

$$\forall q \in Q : \upsilon_H((q,x_1),.) = \upsilon_H((q,x_2),.) \iff x_1 = x_2,$$

for each $x_1, x_2 \in \mathcal{X}_q$

We will show that the conditions of the theorem imply observability. Assume that the conditions (i) and (ii) hold. We will show that then H is observable. Assume that (s_1,x_1) and (s_2,x_2) are indistinguishable, that is, $\upsilon_H((s_1,x_2),.) = \upsilon_H((s_2,x_2),.)$. The latter equality is implies that $\upsilon_H((s_1,0),.) = \upsilon_H((s_2,0),.)$. Indeed, $\upsilon_H((s_1,x_1),.) = \upsilon_H((s_2,x_2),.)$ implies that $y_H((s_1,x_1),.) = y_H((s_2,x_2),.)$. The latter equality implies that $y_H((s_1,x_1),0,w,.) = y_H((s_1,x_1),0,w,.)$ for all $w \in \Gamma^*$. But

$$y_H((s_1,x_1),u,w,.) = y_H((s_1,x_1,0,w,.) + y_H((s_1,0),u,w,.) = \\ = y_H(s_2,x_2),u,w,.) = y_H((s_2,x_2,0,w,.) + y_H((s_2,0),u,w,.)$$

which implies that $y_H((s_1,0),.) = y_H((s_2,0),.)$. Since $\Pi_O \circ \upsilon_H((s_1,0),.) = \Pi_O \circ \upsilon_H((s_1,x_1),.) = \Pi_O \circ \upsilon_H((s_2,x_2),.) = \Pi_O \circ \upsilon_H((s_2,0),.)$, we get that $\upsilon_H((s_1,0),.) = \upsilon_H((s_2,0),.)$. But then $s_1 = s_2 = s$ by part (i) of the theorem. From $\upsilon_H((s_1,x_1),.) = \upsilon_H((s_2,x_2),.)$ we get that $\upsilon_H((s,x_1),.) = \upsilon_H((s,x_2),.)$, but by part (ii) of the theorem it implies that $x_1 = x_2$. That is, $(s_1,x_1) = (s_2,x_2)$. That is, H is observable.

Assume H is observable. Then for any $s_1,s_2 \in Q$, $\upsilon_H((s_1,0),.) = \upsilon_H((s_2,0),.)$ is equivalent to $s_1 = s_2$. But this is equivalent to part (i) of the theorem. Similarly, $\upsilon_H((s,x_1),.) = \upsilon_H((s,x_2),.)$ is equivalent to $s_1 = s_2$. But this is equivalent to part (ii) of the theorem. That is, if H is observable, then part (i) and part(ii) of the theorem hold. ☐

The following theorem characterises semi-reachability of (H,μ).

Theorem 32. *(H,μ) is semi-reachable if and only if (A_H,μ_D), $\mu_D = \Pi_Q \circ \mu$, is reachable and $\dim W_H = \sum_{q\in Q} \dim \mathcal{X}_q$, where*

$$\begin{aligned} W_H = \mathrm{Span}\{ & A_{q_k}^{j_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_{l+1},\gamma_{l+1},q_l} A_{q_l}^{j_{l+1}} B_{q_l} u, \\ & A_{q_k}^{j_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} A_{q_0}^{j_1} x_f \mid \\ & j_1,\ldots,j_{k+1} \geq 0, u \in \mathcal{U}, \gamma_1,\ldots,\gamma_k \in \Gamma, (q_f,x_f) = \mu(f), f \in \Phi, \\ & q_j = \delta(q_0,\gamma_1\cdots\gamma_j), 0 \leq l,j \leq k, k \geq 0\} \\ & \subseteq \bigoplus_{q\in Q} \mathcal{X}_q \end{aligned}$$

Proof. Let $\mathcal{X} = \bigoplus_{q\in Q} \mathcal{X}_q$. We will show that

$$W_H = \mathrm{Span}\{x_H(h_0,u,s,t) \mid h_0 \in \mathrm{Im}\mu, u \in PC(T,\mathcal{U}), s \in (\Gamma\times T)^*, t \in T\} \subseteq \mathcal{X}$$

Denote the right-hand side of the equality above by V. Define the map $x_H(h_0,u,w,.) : T \ni (t_1,\ldots,t_{k+1}) \mapsto x_H(h_0,u,(w,t_1t_2\cdots t_k),t_{k+1}) \in \mathcal{X}$, for each $u \in PC(T,\mathcal{U}), w \in \Gamma^*, |w| = k$, $h_0 \in \mathrm{Im}\mu$. Thus we get that $x_H(h_0,e_j,w,.)(T^{k+1}) \subseteq V$, for each $w \in \Gamma^*, |w| = k$. Since V is a finite-dimensional vectors pace, we get that

$$D^\alpha x_H(h_0,e_j,w,.) \in V$$

and

$$D^\alpha x_H(h_0,0,w,.) \in V$$

for each $\alpha \in \mathbb{N}^{|w|+1}, w \in \Gamma^*$, $j = 1,\ldots,m$. Assume that $h_0 = (q,x_0)$ It is easy to see that $x_H(h_0,e_j,w,.) = x_H(h_0,0,w,.) + x_H((q,0),e_j,w,.)$. That is $x_H((q,0),e_j,w) = x_H(h_0,e_j,w,.) - x_H(h_0,0,w,.) \in V$. That is, we get that $D^\alpha x_H(h_0,0,w,.) \in V$ and $D^\alpha x_H((q,0),e_j,w,.) \in V$ holds for each $w \in \Gamma^*$, $j = 1,\ldots,m$, $\alpha \in \mathbb{N}^{|w|+1}$. It is easy to see from (7.1) in Section 7.1 that

$$\begin{aligned} D^\alpha x_H(h_0,0,w,.) &= A_{q_k}^{\alpha_{k+1}} M_{q_k,\gamma_k,q_{k-1}} A_{q_{k-1}}^{\alpha_k} \cdots M_{q_1,\gamma_1,q_0} A_{q_0}^{\alpha_1} x_0 \\ D^\alpha x_H(0,e_j,w,.) &= A_{q_k}^{\alpha_{k+1}} M_{q_k,\gamma_k,q_{k-1}} A_{q_{k-1}}^{\alpha_k} \cdots M_{q_l,\gamma_l,q_{l-1}} A_{q_{l-1}}^{\alpha_l - 1} B_{q_{l-1}} e_j \end{aligned}$$

where $q_i = \delta(q,\gamma_1\cdots\gamma_i)$, $w = \gamma_1\cdots\gamma_k$. $h_0 = (q,x_0)$, $\alpha_l > 0$ and $\alpha_{l-1} = \alpha_{l-2} = \cdots = \alpha_1 = 0$. That is, $W_H = \mathrm{Span}\{D^\alpha x_H(h_0,0,w,.), D^\alpha((q,0),e_j,w,.) \mid j = 1,\ldots,m, h_0 = (q,x_0), h_0 \in \mathrm{Im}\mu\}$. Thus, we get that $W_H \subseteq V$.

On the other hand, it is easy to see that

$$\exp(A_{q_k}t_{k+1}) M_{q_k,\gamma_k,q_{k-1}} \exp(A_{q_{k-1}}t_k) \cdots \exp(A_{q_0}t_1) x_0 \in W_H$$

and

$$\exp(A_{q_k}t_k)M_{q_k,\gamma_k,q_{k-1}}\cdots M_{q_l,\gamma_1,q_{l-1}}\exp(A_{q_{l-1}}t_l)B_je_j \in W_H$$

for each $\gamma_1\cdots\gamma_k \in \Gamma^*, k \le 0$, $(q,x_0) \in \mathrm{Im}\mu$, $1 \le l \le k$, $j = 1,\ldots,m$, where $q_i = \delta(q,\gamma_1\cdots\gamma_i)$. Thus, we get that $x_H(h_0,u,s,t) \in W_H$, for all $h_0 \in \mathrm{Im}\mu$, $u \in PC(T,\mathcal{U})$, $s \in (Q\times T)^*, t \in T$. That is, $W_H = V$. The rest of the theorem follows from the definition of semi-reachability. □

Later we will show that observability and semi-reachability of linear hybrid systems can be checked algorithmically. Using the results above, we can give a procedure, which transforms any realization (H,μ) of Φ to a semi-reachable realization (H_r,μ_r) such that $\dim H_r \le \dim H$. The procedure goes as follows.

Lemma 34. *Assume (H,μ) is a realization of Φ,*

$$H = (\mathcal{A},\mathcal{U},\mathcal{Y},(\mathcal{X}_q,A_q,B_q,C_q)_{q\in Q},\{M_{q_1,\gamma,q_2} \mid q_1,q_2 \in Q,\gamma\in\Gamma,q_1 = \delta(q_2,\gamma)\})$$

Let $\mathcal{A}_r = (Q^r,\Gamma,O,\delta^r,\lambda^r)$ be the sub automaton of $\mathcal{A}_H$ reachable from $\Pi_Q(\mathrm{Im}\mu)$ and for each $q \in Q_r$ let

$$\mathcal{X}^r_q = W_H \cap \mathcal{X}_q\ ,\ A^r_q = A_q|_{\mathcal{X}^r_q}\ ,\ C^r_q = C_q|_{\mathcal{X}^r_q}\ ,\ B^r_q = B_q\ ,\ M^r_{q_1,\gamma,q_e} = M_{q_1,\gamma,q_e}|_{\mathcal{X}^r_q}$$

Let $(H_r,\mu_r) = (\mathcal{A}^r,\mathcal{U},\mathcal{Y},(\mathcal{X}^r_q,A^r_q,B^r_q,C^r_q)_{q\in Q^r},\{M^r_{q_1,\gamma,q_2} \mid q_1,q_2 \in Q^r,\gamma\in\Gamma,q_1 = \delta^r(q_2,\gamma)\})$. Then (H_r,μ_r) is semi-reachableand it is a realization of Φ too. Moreover $\dim H_r \le \dim H$.

Just as it was the case with Theorem 31 the lemma above can be proven using the correspondence between linear hybrid systems and hybrid representations. We will not present that approach here and below we will discuss a direct proof instead. But we will come back to it later in the next section.

Proof. Define the automaton morphism $\phi : (\mathcal{A}_r,(\mu_r)_D) \to (\mathcal{A},\mu_D)$ by $\phi(q) = q$ for each $q \in Q^r$. It is easy to see that ϕ is indeed an automaton morphism. Define $T_C : \bigoplus_{q\in Q^r}\mathcal{X}^r_q \to \bigoplus_{q\in Q}\mathcal{X}_q$ by $T_C(x) = x$ for each $x \in \mathcal{X}^r_q$, $q \in Q^r$. It is easy to see that (ϕ,T_C) is a O-morphism. Thus, for all $f \in \Phi$, $\upsilon_{H^r}(\mu_r(f),.) = \upsilon_H(T(\mu_r(f),.) = \upsilon_H(\mu(f),.)$ by Proposition 2. Thus, if (H,μ) is a realization of Φ, then (H_r,μ_r) is a realization of Φ too. Since (ϕ,T_C) is clearly injective, we get that $\dim H_r \le \dim H$ by Proposition 27. It is easy to see that $W_{H_r} = W_H = \bigoplus_{q\in Q^r}\mathcal{X}^r_q$. Thus by Theorem 32 (H_r,μ_r) is semi-reachable. □

Realization theory of linear hybrid systems

Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ be a set of input-output maps. Assume that Φ has a hybrid kernel representation. Then Proposition 29 allows us to reformulate the realization problem in terms of rationality of certain hybrid formal power series. The construction of these hybrid formal power series goes as follows.

Let $\widetilde{\Gamma} = \Gamma \cup \{e\}, e \notin \Gamma$. Every $w \in \widetilde{\Gamma}$ can be written as $w = e^{\alpha_1}\gamma_1 e^{\alpha_2}\gamma_2 \cdots \gamma_k e^{\alpha_{k+1}}$ for some $\gamma_1, \ldots, \gamma_k \in \Gamma$, $\alpha_1, \ldots, \alpha_{k+1} \geq 0$. For each $f \in \Phi$ define the formal power series $(Z_f)_C, (Z_{f,j})_C \in \mathbb{R}^p \ll \widetilde{\Gamma}^* \gg$, $j = 1, \ldots, m$ as follows.

$$\begin{aligned}(Z_f)_C(e^{\alpha_1}\gamma_1 e^{\alpha_2} \cdots \gamma_k e^{\alpha_{k+1}}) &= D^\alpha f_C(0, w, .)\\ (Z_{f,j})_C(e^{\alpha_1}\gamma_1 e^{\alpha_2} \cdots \gamma_k e^{\alpha_{k+1}}) &= D^\alpha y_0^f(e_j, w, .)\end{aligned}$$

where $w = \gamma_1 \cdots \gamma_k$ and $\alpha = (\alpha_1, \ldots, \alpha_{k+1}) \in \mathbb{N}^k$. Notice that $(Z_{f,j})_C(v) = 0$ for all $v \in \Gamma^*$. Notice that the complete knowledge of the functions $K_w^{f,\Phi}$ and $G_{w,l}^{f,\Phi}$ is not needed in order to construct the formal power series $(Z_f)_C, (Z_{f,j})_C$. In fact, one can think of $(Z_f)_C$ as an object containing all the information on the behaviour of f with the zero continuous input. The series $(Z_{f,j})_C$, $j = 1, \ldots, m$, contains all the information on the behaviour of the pair $(q, 0)$, where q is the discrete part of the hybrid state inducing f in some realization of Φ (if there is any).

Let $J = I_\Phi = \Phi \cup (\Phi \times \{1, 2, \ldots, m\})$. That is, J can be interpreted as a hybrid power series index set, where $J_1 = \Phi$ and $J_2 = \{1, \ldots, m\}$. The alphabet $\widetilde{\Gamma}$ decomposes into two disjoint subsets Γ and $\{e\}$. With the notation of Section 3.3, let $X = \widetilde{\Gamma}$, $X_1 = \{e\}$, $X_2 = \Gamma$. Define the hybrid formal power series Z_f and $Z_{f,j}$, $j = 1, \ldots, m$ by

$$Z_f = (Z_C, f_D) \text{ and } Z_{f,j} = ((Z_{f,j})_C, f_D)$$

That is, the discrete-valued part of the hybrid formal power series Z_f and $Z_{f,j}, j \in \{1, \ldots, m\}$ is the map f_D, i.e. the discrete-valued part of $f \in \Phi$. Notice that Φ has to have a hybrid kernel representation for f_D to be a map from Γ^* to O. The continuous valued parts of Z_f and $Z_{f,j}$ are the formal power series $(Z_f)_C$ and $(Z_{f,j})_C$ respectively. Thus, the continuous valued parts store the high-order derivatives at zero of $f_C(0, .)$ and $y_0^f(e_j, .)$, $j = 1, \ldots, m$. By analyticity of $f_C(0, .)$ and $y_0^f(e_j, .)$ these high-order derivatives determine the functions uniquely. Thus, by the particular structure of f imposed by existence of a hybrid kernel representation we get that $(Z_f)_C$ and $(Z_{f,j})_C$, $j = 1, \ldots, m$ determine f_C completely, thus the hybrid formal power series Z_f together with $Z_{f,j}$ determine f completely.

Note that we used heavily the assumption that Φ has a hybrid kernel representation while construction the hybrid formal power series Z_f and $Z_{f,j}$, $j = 1, \ldots, m$. In particular, if Φ does not have a hybrid kernel representation, then the derivatives of

$f(0,.)$ or $y_0^f(e_j,.)$ need not exist or f_D might depend on switching times or continuous inputs instead of sequences of discrete inputs only.

We will use the hybrid formal power series above to associate with Φ a suitable family of hybrid formal power series. Define the *set of hybrid formal power series associated with* Φ by

$$\Psi_\Phi = \{Z_j \in \mathbb{R}^p \ll \widetilde{\Gamma}^* \gg \times F(\Gamma^*, O) \mid j \in I_\Phi\}$$

It is easy to see that Ψ_Φ is a well-posed indexed set of hybrid formal power series. Define the *Hankel-matrix* H_Φ of Φ as $H_\Phi = H_{\Psi_\Phi}$. Notice that if Φ is finite, then Ψ_Φ has finitely many elements.

Let (H,μ) be a hybrid system realization with $\mu : \Phi \to \bigcup_{q\in Q}\{q\} \times \mathcal{X}_q$. Define the *hybrid representation* $HR_{H,\mu}$ *associated with* (H,μ) by

$$HR_{H,\mu} = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j\in J_2, z\in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y\in X_2})_{q\in Q}, J, \mu)$$

where $J = I_\Phi$, $J_1 = \Phi$, $J_2 = \{1,\dots,m\}$, $X_1 = \{e\}$, $X_2 = \Gamma$ and for each $q \in Q$, $j = 1,\dots,m$

$$A_{q,e} = A_q \text{ and } B_{q,e,j} = B_q e_j$$

where e_j is the jth unit vector of $\mathcal{U}$.

Conversely, let

$$HR = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j\in J_2, z\in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y\in X_2})_{q\in Q}, J, \mu)$$

be a hybrid representation with index set I_Φ such that $X_1 = \{e\}$, $X_2 = \Gamma$, $J_1 = \Phi$, $J_2 = \{1,\dots,m\}$. Define the *linear hybrid realization* (H_{HR}, μ_{HR}) *associated with* HR as follows

$$H_{HR} = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, B_q, C_q)_{q\in Q}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2,\gamma)\})$$

and

$$\mu_{HR} = \mu$$

where for each $q \in Q$

$$A_q = A_{q,e} \text{ and } B_q = \begin{bmatrix} B_{q,e,1} & B_{q,e,2} & \cdots B_{q,e,m} \end{bmatrix}$$

It is easy to see that $(H_{HR_{H,\mu}}, \mu_{HR_{H,\mu}}) = (H,\mu)$ and $HR_{H_{HR},\mu_{HR}} = HR$ for any hybrid representation HR and linear hybrid realization (H,μ). It is also easy to see that $\dim H = \dim HR_{H,\mu}$.

The following theorem follows easily from Proposition 29 and plays a crucial role in realization theory of linear hybrid system.

Theorem 33. *A linear hybrid system (H,μ) is a realization of Φ if and only if Φ has a hybrid kernel representation and $HR_{H,\mu}$ is a hybrid representation of Ψ_{Φ}. Conversely, if Φ has a hybrid kernel representation and HR is a hybrid representation of Ψ_{Φ} then (H_{HR},μ_{HR}) is a linear hybrid system realization of Φ.*

Proof. Assume that (H,μ) is a hybrid realization and let

$$H = (\mathcal{A},\mathcal{U},\mathcal{Y},(\mathcal{X}_q,A_q,B_q,C_q)_{q\in Q},\{M_{q_1,\gamma,q_2} \mid q_1,q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2,\gamma)\})$$

By Proposition 29, (H,μ) is a realization of Φ if and only if Φ has a hybrid kernel representation and for all $\alpha = (\alpha_1,\dots,\alpha_{k+1}) \in \mathbb{N}^{k+1}$, $w = \gamma_1\cdots\gamma_k$, $\gamma_1,\dots,\gamma_k \in \Gamma^*$, $k \geq 0$, $j = 1,\dots,m$, $f \in \Phi$,

$$\begin{aligned}
&D^{\alpha}y_0^f(e_j,w,.) = D^{\beta}G^{f,\Phi}_{w,k+2-l}e_j = C_{q_k}A_{q_k}^{\alpha_{k+1}}M_{q_k,\gamma_k,q_{k-1}}\cdots M_{q_l,\gamma_l,q_{l-1}}A_{q_{l-1}}^{\alpha_l-1}B_{q_{l-1}}e_j\\
&D^{\alpha}f_C(0,w,.) = D^{\alpha}K^{f,\Phi}_w = C_{q_k}A_{q_k}^{\alpha_{k+1}}M_{q_k,\gamma_k,q_{k-1}}\cdots M_{q_1,\gamma_1,q_0}A_{q_0}^{\alpha_1}x_0\\
&f_D(w) = \lambda(q_0,w)
\end{aligned}$$

where $\beta = (\alpha_l - 1,\alpha_{l+1},\dots,\alpha_{k+1})$, $l = \min\{z \mid \alpha_z > 0\}$. Taking into account the definition of $Z_{f,j}$, Z_f for all $f \in \Phi, j = 1,\dots,m$ we get that the formula above is equivalent to $f \in \Phi$,

$$\begin{aligned}
&(Z_{f,j})_C(\gamma_1\cdots\gamma_{l-1}e^{\alpha_l}\gamma_l e^{\alpha_{l+1}}\cdots\gamma_k e^{\alpha_{k+1}}) = C_{q_k}A_{q_k}^{\alpha_{k+1}}M_{q_k,\gamma_k,q_{k-1}}\cdots\\
&\cdots M_{q_l,\gamma_l,q_{l-1}}A_{q_{l-1}}^{\alpha_l-1}B_{q_{l-1}}e_j\\
&(Z_f)_C)(e^{\alpha_1}\gamma_1 e^{\alpha_2}\cdots\gamma_k e^{\alpha_{k+1}}) = C_{q_k}A_{q_k}^{\alpha_{k+1}}M_{q_k,\gamma_k,q_{k-1}}\cdots M_{q_1,\gamma_1,q_0}A_{q_0}^{\alpha_1}x_0\\
&(Z_f)_D(w) = \lambda(q_0,w)
\end{aligned} \tag{7.11}$$

Consider the hybrid representation

$$HR_{H,\mu} = (\mathcal{A},\mathcal{Y},(\mathcal{X}_q,\{A_{q,z},B_{q,z,j_2}\}_{j\in J_2,z\in X_1},C_q,\{M_{\delta(q,y),y,q}\}_{y\in X_2})_{q\in Q},J,\mu)$$

Taking into account that $A_q = A_{q,e}$, $B_{q,e,j} = B_q e_j$ we get that (7.11) is equivalent to $HR_{H,\mu}$ being a representation of Ψ_{Φ}. That is, (H,μ) satisfies the conditions of Proposition 29 if and only if Φ has a hybrid kernel representation and $HR_{H,\mu}$ is a representation of Ψ_{Φ}.

Thus, if (H,μ) is a realization of Φ, then Φ has hybrid kernel representation and $HR_{H,\mu}$ is a representation of Ψ_{Φ}. Conversely, assume that HR is a representation of Ψ_{Φ} and Φ admits a hybrid kernel representation. Since $HR = HR_{H_{HR},\mu_{HR}}$ we get that (H,μ) satisfies the condition of Proposition 29 and thus (H,μ) is a realization of Φ. □

The theorem above allows us to reduce the realization problem for linear hybrid systems to existence of a hybrid representation of a indexed set of hybrid formal power series. Moreover, Theorem 31 and Theorem 32 allow us to relate observability and semi-reachability of linear hybrid systems to observability and reachability of hybrid representations.

Theorem 34. *A linear hybrid system realization (H,μ) is observable if and only if $HR_{H,\mu}$ is observable. A linear hybrid system realization (H,μ) is semi-reachable if and only if $HR_{H,\mu}$ is reachable.*

Proof. Let (H,μ) be a linear hybrid system realization, assume that

$$H = (\mathcal{A},\mathcal{U},\mathcal{Y},(\mathcal{X}_q,A_q,B_q,C_q)_{q\in Q},\{M_{q_1,\gamma,q_2} \mid q_1,q_2\in Q,\gamma\in\Gamma,q_1=\delta(q_2,\gamma)\})$$

Let $HR = HR_{H,\mu}$, which by definition will be of the form

$$HR = (\mathcal{A},\mathcal{Y},(\mathcal{X}_q,\{A_{q,z},B_{q,z,j_2}\}_{j\in J_2,z\in X_1},C_q,\{M_{\delta(q,y),y,q}\}_{y\in X_2})_{q\in Q},J,\mu)$$

with $X_2=\Gamma$, $X_1=\{e\}$, $J_1=\Phi$, $J_2=\{1,\dots,m\}$ and $A_{q,e}=A_q$, $B_{q,e,j}=B_qe_j$. Recall from Theorem 32 the vector space W_H and recall from Proposition 3 the vector space W_{HR}. It is easy to see that $W_H=W_{HR}$. By Theorem 32 (H,μ) is semi-reachable if and only if $(\mathcal{A},\mu_D)$ is reachable and $W_{HR}=W_H=\bigoplus_{q\in Q}\mathcal{X}_q$, but by Proposition 3 the latter is equivalent to HR being reachable.

Recall from Theorem 31 the conditions for observability of H. Notice that for each $q\in Q$, for all $\gamma_1,\dots\gamma_k\in\Gamma, j_1,\dots,j_{k+1}\ge 0, 0\le l\le k, k\ge 0$, $z=1,\dots,m$,

$$\begin{aligned}
&C_{q_k}A_{q_k}^{j_{k+1}}M_{q_k,\gamma_k,q_{k-1}}\cdots M_{q_{l+1},\gamma_{l+1},q_l}A_{q_l}^{j_{l+1}}B_{q_l}e_z = \\
&= C_{q_k}A_{q_k,e^{j_{k+1}}}M_{q_k,\gamma_k,q_{k-1}}\cdots M_{q_{l+1},\gamma_{l+1},q_l}A_{q_l,e^{j_{l+1}}}B_{q_l,e,z} = \\
&= T_{s_1,z}(\gamma_1\cdots\gamma_l e^{j_{l+1}+1}\gamma_{l+1}\cdots\gamma_k e^{j_{k+1}})
\end{aligned}$$

where $q_i=\delta(q,\gamma_1\cdots\gamma_i)$, $i=0,\dots,k$. From this it follows that condition (i) of Theorem 31 is equivalent to

$$\begin{aligned}
([\forall w\in\Gamma^*:\lambda(s_1,w)=\lambda(s_2,w)] \text{ and } T_{s_1,j}=T_{s_2,j}, j\in\{1,\dots,m\})& \\
\iff s_1=s_2&
\end{aligned}$$

That is, condition (i) of Theorem 31 is equivalent to condition (i) of Proposition 4. Similarly, it is easy to see that $O_{H,q}=O_{HR,q}$ for all $q\in Q$ and thus condition (ii) of Theorem 31 is equivalent to condition (ii) of Proposition 4. That is, H is observable if and only if HR is observable. □

Notice that Theorem 34 above implies Lemma 34. Indeed, let (H,μ) be a hybrid realization of Φ and consider the associated hybrid representation $HR=HR_{H,\mu}$. By Theorem 33 HR is a representation of Ψ_Φ. By Lemma 12 there exists a reachable hybrid representation HR_r of Ψ_Φ such that $\dim HR_r \leq \dim HR$, the equality being equivalent to reachability of HR. Then $(H_r, \mu_r) = (H_{HR_r}, \mu_{HR_r})$ is a realization of Φ by Theorem 33 and it is semi-reachable by Theorem 34. Moreover, $\dim H_r = \dim HR_r \leq \dim HR = \dim H$ and equality holds only if HR is reachable and thus H is semi-reachable .

Notice that both H and $HR_{H,\mu}$ have the same state-space. It is easy to see that the following holds.

Lemma 35. *Let (H_i, μ_i),$i = 1, 2$ be a two linear hybrid system realizations, The map $T : (H_1, \mu_1) \to (H_2, \mu_2)$ is a linear hybrid morphism, then T is also a $T : HR_{H_1,\mu_1} \to HR_{H_2,\mu_2}$ hybrid representation morphism. Conversely, if $T : HR_1 \to HR_2$ is a a hybrid representation morphism then T can be viewed as a $T : (H_{HR_1}, \mu_{HR_1}) \to (HR_2, \mu_{HR_2})$ linear hybrid morphism. The map T is a surjective, injective , isomorphism as a linear hybrid morphism if and only if T is surjective, injective, isomorphism as a hybrid representation morphism.*

Proof. Indeed, assume that

$$H_1 = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, B_q, C_q)_{q\in Q}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2, \gamma)\})$$

Then

$$HR_{H,\mu} = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j\in J_2, z\in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y\in X_2})_{q\in Q}, J, \mu)$$

with $X_1 = \{e\}$, $X_2 = \Gamma$, $J_1 = \Phi$, $J_2 = \Gamma$ and $A_{q,e} = A_q$, $q \in Q$. Assume that

$$H_2 = (\mathcal{A}', \mathcal{U}, \mathcal{Y}, (\mathcal{X}'_q, A'_q, B'_q, C'_q)_{q\in Q'}, \{M'_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q', \gamma \in \Gamma, q_1 = \delta'(q_2, \gamma)\})$$

and consequently

$$HR_{H_2,\mu_2} = (\mathcal{A}', \mathcal{Y}, (\mathcal{X}'_q, \{A'_{q,z}, B'_{q,z,j_2}\}_{j\in J_2, z\in X_1}, C'_q, \{M'_{\delta'(q,y),y,q}\}_{y\in X_2})_{q\in Q'}, J, \mu')$$

with $X_1 = \{e\}$, $X_2 = \Gamma$, $J_1 = \Phi$, $J_2 = \Gamma$, $A'_{q,e} = A'_q$, $q \in Q$. Then a pair of maps $T = (T_D, T_C)$, $T_D : Q \to Q'$ and $T_C : \bigoplus_{q\in Q} \mathcal{X}_q \to \mathcal{X}'_q$ defines a linear hybrid morphism if and only if $T_D : (\mathcal{A}, (\mu_1)_D) \to (\mathcal{A}', (\mu_2)_D)$ is an automaton morphism and the following conditions are satisfied for T_C: $T_C(\mathcal{X}_q) \subseteq \mathcal{X}'_{T_D(q)}$, $C_q = C'_{T_D(q)} T_C$, $T_C A_q = A'_{T_D(q)} T_C$, $T_C M_{\delta(q,\gamma),\gamma,q} = M'_{\delta'(T_D(q),\gamma),\gamma,T_D(q)} T_C$ and $T_C \circ (\mu_1)C = (\mu_2)_C$ for all $q \in Q$, $\gamma \in \Gamma$. But $A_q = A_{q,e}$ and $A'_{T_D(q)} = A'_{T_D(q),e}$, therefore the conditions above are precisely the conditions for $T = (T_D, T_C)$ to be a hybrid representation morphism $T : HR_{H_1,\mu_1} \to HR_{H_2,\mu_2}$. That is, T is a linear hybrid morphism if and only if it is a hybrid representation morphism.

In particular, if T is a linear hybrid morphism, it is also a hybrid representation morphism. The second part of the lemma follows from the observation above by noticing that $HR_i = HR_{H_{HR_i}, \mu_{HR_i}}$, $i = 1, 2$. $\square$

Notice that the lemma above implies Proposition 27. Indeed, if $T : (H_1, \mu_1) \rightarrow (H_2, \mu_2)$ is a linear hybrid morphism, then $T : HR_{H_1,\mu_1} \rightarrow HR_{H_2,\mu_2}$ is a hybrid representation morphism. Moreover, T is injective, surjective or isomorphism as a linear hybrid morphism if and only if it is injective, surjective or isomorphism as a hybrid representation morphism, and $\dim HR_{H_i,\mu_i} = \dim H_i$, $i = 1, 2$. Thus, applying Proposition 6 we easily get the statement of Proposition 27.

Recall from Section 3.3 the definitions of $H_{O,\Omega}$, $\mathcal{D}_\Omega$ and Ω_D for an indexed set of hybrid formal power series Ω. Let $H_{O,\Phi} = H_{O,\Psi_\Phi}$, $\Phi_D = (\Psi_\Phi)_D$. From the discussion above, using the results on theory of hybrid formal power series, namely Theorem 7 and Theorem 8, we can derive the following result.

Theorem 35 (Realization of input/output map). *Let $\Phi \subseteq F(PC(T, \mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$. The following are equivalent.*

(i) *Φ has a realization by a linear hybrid system,*

(ii) *Φ has a hybrid kernel representation, Ψ_Φ is rational*

(iii) *Φ has a hybrid kernel representation,* $\mathrm{rank}\, H_\Phi < +\infty$*, $card(W_{\Phi_D}) < +\infty$ and $card(H_{\Phi,O}) < +\infty$.*

Proof. Assume that (H, μ) is a linear hybrid system realization of Φ. Then by Theorem 33 Φ has a hybrid kernel representation and $HR_{H,\mu}$ is a representation of Ψ_Φ, thus Ψ_Φ is rational. Conversely, if Φ has hybrid kernel representation and Ψ_Φ is rational, i.e., there exists a hybrid representation HR of Ψ_Φ, then by Theorem 33 (H_{HR}, μ_{HR}) is a realization of Φ. Thus, (i) $\iff$ (ii). The second part, (ii) $\iff$ (iii) follows from Theorem 7 $\square$

We can also characterise minimal linear hybrid realizations.

Theorem 36 (Minimal realization). *If Φ has a linear hybrid system realization, then it has a minimal linear hybrid system realization. If (H, μ) is a realization of Φ, then the following are equivalent.*

(i) *(H, μ) is minimal,*

(ii) *(H, μ) is semi-reachable and it is observable,*

(iii) *For each (H', μ') semi-reachable linear hybrid system realization of Φ there exists a surjective linear hybrid morphism $T : (H', \mu') \rightarrow (H, \mu)$. In particular, all minimal hybrid linear systems realizing Φ are isomorphic.*

Proof. If Φ has a linear hybrid realization, then it has a hybrid kernel representation.

First of all, (H, μ) is a minimal realization of Φ if and only if $HR_{H,\mu}$ is a minimal representation of Ψ_Φ. Indeed, assume that (H, μ) is a minimal linear hybrid realization of Φ. Then by Theorem 33 for any hybrid representation $HR^{'}$, $(H_{HR'}, \mu_{HR'})$ is a realization of Φ, thus $\dim HR_{H,\mu} = \dim H \leq \dim H_{HR'} = \dim HR^{'}$, i.e., $HR_{H,\mu}$ is a minimal hybrid representation of Φ. Conversely, assume that $HR_{H,\mu}$ is a minimal representation of Φ. Then for any linear hybrid system realization of $(H^{'}, \mu^{'})$ of Φ, $HR_{H^{'},\mu^{'}}$ is a hybrid representation of Ψ_Φ by Theorem 33, thus $\dim H = \dim HR_{H,\mu} \leq \dim HR_{H^{'},\mu^{'}} = \dim H^{'}$. That is, (H, μ) is indeed a minimal realization of Φ.

Assume that Φ has a linear hybrid realization. Then by Theorem 12 Φ has a hybrid kernel representation and Ψ_Φ is rational. Thus, by Theorem 8 Ψ_Φ admits a minimal hybrid representation HR_m. Then by discussion above, $(H_m, \mu_m) = (H_{HR_m}, \mu_{HR_m})$ is a minimal realization of Φ, since $HR_{H_m,\mu_m} = HR_m$. Thus, if Φ has a realization by a linear hybrid system, then it also has a minimal linear hybrid system realization.

The linear hybrid system realization (H, μ) is minimal if and only if $HR_{H,\mu}$ is a minimal hybrid representation of Φ. By Theorem 8, $HR_{H,\mu}$ is minimal if and only if it is reachable and observable, which by Theorem 34 is equivalent to (H, μ) being semi-reachable and observable. Thus, (i) is equivalent to (ii). Similarly, by Theorem 8, $HR_{H,\mu}$ is a minimal hybrid representation of Ψ_Φ if and only if for any reachable representation $HR^{'}$ of Ψ_Φ there exists a surjective hybrid representation morphism $T : HR^{'} \rightarrow HR_{H,\mu}$. The latter is equivalent to part (iii) of the current theorem.

Indeed, assume that for any reachable hybrid representation $HR^{'}$ of Ψ_Φ there exists a surjective morphism $T : HR^{'} \rightarrow HR_{H,\mu}$. Then for any semi-reachable linear hybrid realization $(H^{'}, \mu^{'})$of Φ the hybrid representation $HR_{H^{'},\mu^{'}}$ is a reachable representation of Ψ_Φ and thus there exists a surjective hybrid representation morphism $T : HR_{H^{'},\mu^{'}} \rightarrow HR_{H,\mu}$. By Lemma 35 $T : (H^{'}, \mu^{'}) \rightarrow (H, \mu)$ is surjective linear hybrid morphism too. Conversely, assume that for any semi-reachable linear hybrid realization $(H^{'}, \mu^{'})$ of Φ there exists a surjective linear hybrid morphism $T : (H^{'}, \mu^{'}) \rightarrow (H, \mu)$. Then for any reachable hybrid representation $HR^{'}$, the linear hybrid realization $(H_{HR'}, \mu_{HR'})$ is a reachable realization of Φ. Thus, there exists a surjective linear hybrid morphism $T : (H_{HR'}, \mu_{HR'}) \rightarrow (H, \mu)$. But by Lemma 35 we get that $T : HR^{'} \rightarrow HR_{H,\mu}$ is a surjective hybrid representation morphism too.

Thus, we just have shown that (H, μ) is minimal if and only if condition (iii) of the current theorem holds. The last statement of the theorem, that is that all minimal linear hybrid realizations are isomorphic can be proven as follows. By Theorem 8 all minimal hybrid representations of the same family of hybrid formal power series are isomorphic. But (H,μ) is a minimal linear hybrid system realization of Φ if and only if $HR_{H,\mu}$ is a minimal hybrid representations of Ψ_{Φ}. Consequently, if (H_i,μ_i), $i=1,2$ are two minimal linear hybrid system realizations of Φ, then there exists a hybrid representation isomorphism $T : HR_{H_1,\mu_1} \to HR_{H_2,\mu_2}$, which yields a linear hybrid isomorphism $T : (H_1,\mu_1) \to (H_2,\mu_2)$. □

7.2 Realization Theory for Bilinear Hybrid Systems

In this section realization theory of bilinear hybrid systems will be presented. As it was mentioned in the introduction to this chapter, the main tool will be the theory of hybrid formal power series from Section 3.3.

Realization theory of bilinear hybrid systems can be developed without the use of hybrid formal power series, as it was done in [48]. However, such a direct construction has very little additional value, in fact it mimics the constructions from theory of hybrid formal power series.

The structure of the section is the following. Subsection 7.2.1 presents the necessary definitions and some basic properties of bilinear hybrid systems. Subsection 7.2.2 discusses the structure of input-output maps of bilinear hybrid systems and it introduces the notion of hybrid Fliess-series expansion. Finally, in Subsection 7.2.3 we develop realization theory for bilinear hybrid systems.

7.2.1 Definition and Basic Properties

Recall from Section 2.3 the definition of bilinear hybrid systems. Similarly to ordinary bilinear systems, the trajectory of a hybrid bilinear system admits a representation by an absolutely convergent series of iterated integrals.

Before giving the precise formulation of such a representation some additional notation has to be introduced.

Let $H = (\mathcal{A},\mathcal{U},\mathcal{Y},(\mathcal{X}_q,A_q,\{B_{q,j}\}_{j=1,\ldots,m},C_q)_{q\in Q},\{M_{\delta(q,\gamma),\gamma,q} \mid q\in Q,\gamma\in\Gamma\})$ a bilinear hybrid system. For each $q\in Q$ and $w=j_1\cdots j_k, k\geq 0, j_1,\cdots j_k\in \mathbb{Z}_m$ let us introduce the following notation $B_{q,0}:=A_q$, $B_{q,\epsilon}:=Id_{\mathcal{X}_q}$, ,
$B_{q,w}:=B_{q,j_k}B_{q,j_{k-1}}\cdots B_{q,j_1}$. Recall from Section 2.6 the notion of iterated integral

$$V_{w_1,\ldots,w_k}[u](t_1,\ldots,t_k)$$

of u at $t_1,\ldots,t_k$ with respect to $w_1,\ldots,w_k$. With the notation above the following holds.

Proposition 30. *For each $h_0 \in \mathcal{H}$, $u \in PC(T,\mathcal{U})$, $s = (\gamma_1,t_1)(\gamma_2,t_k)\cdots(\gamma_k,t_k) \in (\Gamma \times T)^*, t \in T$, $x_H(h_0,u,s,t)$ and $y_H(h_0,u,s,t) = \Pi_{\mathcal{Y}} \circ \upsilon_H(h,u,s,t)$ are equal to the following absolutely convergent series*

$$x_H(h_0,u,s,t) = \sum_{w_1,\ldots,w_{k+1} \in \mathbb{Z}_m^*} (B_{q_k,w_{k+1}} M_{q_k,\gamma_k,q_{k-1}} B_{q_{k-1},w_k} \cdots \cdots M_{q_1,\gamma_1,q_0} B_{q_0,w_1} x_0) V_{w_1,\ldots,w_{k+1}}[u](t_1,\ldots,t_{k+1}) \tag{7.12}$$

$$y_H(h_0,u,s,t) = \sum_{w_1,\ldots,w_{k+1} \in \mathbb{Z}_m^*} (C_{q_k} B_{q_k,w_{k+1}} M_{q_k,\gamma_k,q_{k-1}} B_{q_{k-1},w_k} \cdots \cdots M_{q_1,\gamma_1,q_0} B_{q_0,w_1} x_0) V_{w_1,\ldots,w_{k+1}}[u](t_1,\ldots,t_{k+1}) \tag{7.13}$$

where $t_{k+1} = t$, $q_{i+1} = \delta(q_i,\gamma_{i+1})$, $h_0 = (q_0,x_0)$ and $0 \le i \le k$.

Proof. First we have to show that the series in the right hand side of (7.12) and (7.13) are absolutely convergent. Consider the notion of hybrid convergent generating series described in Section 7.2.2. It is easy to see that the vector space $\bigoplus_{q \in Q} \mathcal{X}_q$ can be identified with the space $\mathbb{R}^{\sum_{q\in Q} n_q}$. For each $h = (q,x) \in \mathcal{H}$ define the series $d_{q,x} : \widetilde{\Gamma}^* \to \bigoplus_{q \in q} \mathcal{X}_q$ and $c_{q,x} : \widetilde{\Gamma}^* \to \mathcal{Y}$ as follows. For each $w_1,\ldots,w_{k+1} \in \mathbb{Z}_m^*$, $\gamma_1,\ldots,\gamma_k \in \Gamma$, $k \ge 0$ let

$$d_{q,x}(w_1\gamma_1 w_2 \cdots \gamma_k w_{k+1}) = B_{q_k,w_{k+1}} M_{q_k,\gamma_k,q_{k-1}} B_{q_{k-1},w_k} \cdots M_{q_1,\gamma_1,q_0} B_{q_0,w_1} x$$

$$c_{q,x}(w_1\gamma_1 w_2 \cdots \gamma_k w_{k+1}) = C_{q_k} B_{q_k,w_{k+1}} M_{q_k,\gamma_k,q_{k-1}} B_{q_{k-1},w_k} \cdots M_{q_1,\gamma_1,q_0} B_{q_0,w_1} x$$

where $q_i = \delta(q,\gamma_1\cdots\gamma_i)$, $i = 0,\ldots,k$. It is easy to see that the maps $c_{q,x}$ and $d_{q,x}$ are hybrid convergent generating series. Indeed, let $M = \max\{||B_{q,j}||, ||M_{\delta(q,\gamma),\gamma,q}|| \mid q \in Q, \gamma \in \Gamma, j = 0,1,\ldots,m\}$. Notice that for all $w \in \mathbb{Z}_m^*$, $w = j_1\cdots j_k, j_1,\ldots,j_k \in \mathbb{Z}_m$, $k \ge 0$, $q \in Q$,

$$||B_{q,w}|| = ||B_{q,j_k} B_{q,j_{k-1}} \cdots B_{q,j_1}|| \le \le ||B_{q,j_k}|| \cdot ||B_{q,j_{k-1}}|| \cdots ||B_{q,j_1}|| \le M^{|w|}$$

Let $K_2 = ||x|| \cdot \max\{||C_q|| \mid q \in Q\}$ and let $K_1 = ||x||$. Then it is immediate from the definition that

$$||d_{q,x}(w_1\gamma_1 w_2 \cdots \gamma_k w_{k+1})|| = = ||B_{q_k,w_k} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} B_{q_0,w_1} x|| \le \le ||B_{q_k,w_k}|| \cdot ||M_{q_k,\gamma_k,q_{k-1}}|| \cdots ||M_{q_1,\gamma_1,q_0}|| \cdot ||x|| \le K_2 M^{k+\sum_{j=1}^{k+1}|w_j|}$$

$$||c_{q,x}(w_1\gamma_1 w_2 \cdots \gamma_k w_{k+1})|| = = ||C_{q_k} B_{q_k,w_k} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} B_{q_0,w_1} x|| \le \le ||C_{q_k}|| \cdot ||B_{q_k,w_k}|| \cdot ||M_{q_k,\gamma_k,q_{k-1}}|| \cdots ||M_{q_1,\gamma_1,q_0}|| \cdot ||x|| \le K_1 M^{k+\sum_{j=1}^{k+1}|w_j|}$$

Thus, $c_{q,x}$ and $d_{q,x}$ are indeed hybrid convergent generating series and thus the series

$$F_{d_{q,x}}(u,s,t) =$$
$$= \sum_{w_1,\ldots,w_{k+1}\mathbb{Z}_m^*} d_{q,x}(w_1\gamma_1\cdots\gamma_k w_{k+1})V_{w_1,\ldots,w_{k+1}}[u](t_1,\ldots,t_{k+1}) =$$
$$= \sum_{w_1,\ldots,w_{k+1}\in\mathbb{Z}_m^*} (B_{q_k,w_{k+1}}M_{q_k,\gamma_k,q_{k-1}}\cdots M_{q_1,\gamma_1,q_0}x)V_{w_1,\ldots,w_{k+1}}[u](t_1,\ldots,t_{k+1})$$

and

$$F_{c_{q,x}}(u,s,t) =$$
$$= \sum_{w_1,\ldots,w_{k+1}\mathbb{Z}_m^*} c_{q,x}(w_1\gamma_1\cdots\gamma_k w_{k+1})V_{w_1,\ldots,w_{k+1}}[u](t_1,\ldots,t_{k+1}) =$$
$$= \sum_{w_1,\ldots,w_{k+1}\in\mathbb{Z}_m^*} (C_qB_{q_k,w_{k+1}}M_{q_k,\gamma_k,q_{k-1}}\cdots M_{q_1,\gamma_1,q_0}x)V_{w_1,\ldots,w_{k+1}}[u](t_1,\ldots,t_{k+1})$$

are absolutely convergent for all $u \in PC(T,\mathcal{U})$, $s = (\gamma_1,t_1)(\gamma_2,t_2)\cdots(\gamma_k,t_k) \in (\Gamma\times T)^*$, $t \in T$.

Next, we will proceed with the proof of equalities (7.12) and (7.13). We will proceed by induction on k. If $k = 0$, then $x_H(h,\epsilon,t)$ and $y_H(h,\epsilon,t)$ is the state- respectively the output-trajectory of the bilinear control system $\frac{d}{dt}x(t) = A_qx(t) + \sum_{j=1}^m (B_{q,j}x(t))u_j(t)$, $y(t) = C_qx(t)$ induced by the initial state x. Thus, by the classical result on trajectories of bilinear control systems ([32, 33]) we get that

$$x_H(h,\epsilon,t) = \sum_{w\in\mathbb{Z}_m^*} (B_{q,w}x)V_w[u](t)$$

and

$$y_H(h,\epsilon,t) = \sum_{w\in\mathbb{Z}_m^*} (C_qB_{q,w}x)V_w[u](t)$$

Assume that the statement of the proposition is true for all $k \le n$. Let $s = (\gamma_1,t_1)\cdots(\gamma_{n+1},t_{n+1}) \in (\Gamma\times T)^*$ and $t_{n+2} \in T$. Consider $x_H(h,u,s,t_{n+2})$. From definition of $x_H(h,u,s,t_{n+2})$ it follows that $x_H(h,u,s,t_{n+2}) = x(t_{n+2})$, where $x(0) = M_{q_{n+1},\gamma_{n+1},q_n}x_H(h,u,(\gamma_1,t_1)(\gamma_2,t_2)\cdots(\gamma_n,t_n),t_{n+1})$ and

$$\frac{d}{dt}x(t) = A_{q_{n+1}}x(t) + \sum_{j=1}^m u_j(t+\sum_{j=1}^{n+1}t_j)(B_{q_{n+1},j}x(t)$$

$q_i = \delta(q,\gamma_1\cdots\gamma_i)$, $i = 0,\ldots,n+1$. Then from the induction hypothesis it follows that

$$x(0) = \sum_{w_1,\ldots,w_{n+1}\in\mathbb{Z}_m^*} M_{q_{n+1},\gamma_{n+1},q_n}B_{q_n,w_{n+1}}M_{q_n,\gamma_n,q_{n-1}}\cdots$$
$$\cdots M_{q_1,\gamma_1,q_0}B_{q_0,w_1}xV_{w_1,\ldots,w_{n+1}}[u](t_1,\ldots,t_{n+1})$$

On the other hand, $x(t)$ is a state-trajectory of the bilinear control system $\frac{d}{dt}x(t) = A_{q_{n+1}}x(t) + \sum_{j=1}^{m} u_j(t + \sum_{j=1}^{n+1} t_j)(B_{q_{n+1},j}x(t),\ y(t) = C_{q_{n+1}}x(t)$. Thus, by the classical result on state trajectories of bilinear control systems we get that $x(t) = \sum_{w_{n+2}\in \mathbb{Z}_m^*}(B_{q_{n+1},w_{n+2}}x(0))V_{w_{n+2}}[\mathrm{Shift}_{T_{n+1}}u](t)$ where $T_{n+1} = \sum_{j=1}^{n+1} t_j$. Taking into account the expression for $x(0)$ and that all the series involved are absolutely convergent, we get after substitution

$$x_H(h,s,t_{n+2}) = x(t_{n+2}) = \sum_{w_{n+2}\in \mathbb{Z}_m^*} \sum_{w_1,\dots,w_{n+1}\in \mathbb{Z}_m^*} (B_{q_{n+1},w_{n+2}}M_{q_{n+1},\gamma_{n+1},q_n}\times$$

$$\times B_{q_n,w_{n+1}}\cdots M_{q_1,\gamma_1,q_0}B_{q_0,w_1}x)V_{w_1,\dots,w_{n+1}}[u](t_1,\dots,t_{n+1})V_{w_{n+2}}[\mathrm{Shift}_{T_{n+1}}u](t_{n+2}) =$$

$$= \sum_{w_1,\dots,w_{n+2}\in \mathbb{Z}_m^*} (B_{q_{n+1},w_{n+2}}M_{q_{n+1},\gamma_{n+1},q_n}\cdots$$

$$\cdots M_{q_1,\gamma_1,q_0}B_{q_0,w_1}xV_{w_1,\dots,w_{n+2}}[u](t_1,\dots,t_{n+2})$$

In the last step we used the equality

$$V_{w_1,\dots,w_{n+1}}[u](t_1,\dots,t_{n+1})V_{w_{n+2}}[\mathrm{Shift}_{T_{n+1}}u](t_{n+2}) = V_{w_1,\dots,w_{n+2}}[u](t_1,\dots,t_{n+2})$$

Thus, (7.12) holds for $k = n+2$. Taking into account that $y_H(h,u,s,t_{n+2}) = C_{q_{n+1}}x_H(h,u,s,t_{n+2})$ we get that (7.13) holds too. $\square$

Let $H = (\mathcal{A},\mathcal{U},\mathcal{Y},(\mathcal{X}_q,A_q,\{B_{q,j}\}_{j=1,\dots,m},C_q)_{q\in Q},\{M_{\delta(q,\gamma),\gamma,q} \mid q\in Q,\gamma\in\Gamma\})$ be a bilinear hybrid system. Notice that $\bigcup_{q\in Q}\mathcal{X}_q$ can be naturally viewed as a subset of $\bigoplus_{q\in Q}\mathcal{X}_q$. Let $\mathcal{H}_0 \subseteq \mathcal{H}$ be a set of states. Recall that $Reach(H,\mathcal{H}_0) \subseteq \bigcup_{q\in Q}\mathcal{X}_q$ and thus $Reach(H,\mathcal{H}_0)$ can be viewed as a subspace of $\bigoplus_{q\in Q}\mathcal{X}_q$. We will say that H is *semi-reachable* from $\mathcal{H}_0$ if $\bigoplus_{q\in Q}\mathcal{X}_q$ contains no proper vector subspace containing $Reach(H,\mathcal{H}_0)$ and the automaton $\mathcal{A}_H$ is reachable from $\Pi_Q(\mathcal{H}_0)$. In other words, (H,μ) is semi-reachablefrom $\mathcal{H}_0$ if $\mathcal{A}_H$ is reachable from $\mathcal{H}_0$ and $\mathrm{Span}\{x \mid x\in Reach(H,\mathcal{H}_0)\} = \bigoplus_{q\in Q}\mathcal{X}_q$.

Consider two hybrid bilinear system realizations (H,μ) and $(H^{'},\mu^{'})$, where

$$\begin{aligned}
H &= (\mathcal{A},\mathcal{U},\mathcal{Y},(\mathcal{X}_q,A_q,\{B_{q,j}\}_{j=1,\dots,m},C_q)_{q\in Q},\{M_{\delta(q,\gamma),\gamma,q} \mid q\in Q,\gamma\in\Gamma\}) \\
H^{'} &= (\mathcal{A}^{'},\mathcal{U},\mathcal{Y},(\mathcal{X}_q^{'},A_q^{'},\{B_{q,j}^{'}\}_{j=1,\dots,m},C_q^{'})_{q\in Q^{'}},\{M_{\delta^{'}(q,\gamma),\gamma,q}^{'} \mid q\in Q^{'},\gamma\in\Gamma\})
\end{aligned}$$

$\mathcal{A} = (Q,\Gamma,O,\delta,\lambda)$ and $\mathcal{A}^{'} = (Q^{'},\Gamma,O,\delta^{'},\lambda^{'})$. A pair $T = (T_D,T_C)$ is called a *bilinear hybrid morphism* from (H,μ) to $(H^{'},\mu^{'})$, denoted by $T:(H,\mu)\to(H^{'},\mu^{'})$ if the the following holds.

$$T_D : (\mathcal{A},\mu_D) \to (\mathcal{A}^{'},\mu_D^{'})$$

where $\mu_D(f) = \Pi_Q(\mu_D(f))$, $\mu^{'}_D(f) = \Pi_{Q'}(\mu^{'}_D(f))$, is an automaton morphism and

$$T_C : \bigoplus_{q \in Q} \mathcal{X}_q \to \bigoplus_{q \in Q'} \mathcal{X}^{'}_q$$

is a linear morphism, such that

(a) $\forall q \in Q : T_C(\mathcal{X}_q) \subseteq \mathcal{X}^{'}_{T_D(q)}$,

(b) $T_C A_q = A^{'}_{T_D(q)} T_C$, $T_C B_{q,j} = B^{'}_{T_D(q)} T_C$, $C_q = C^{'}_{T_D(q)} T_C$, for all $q \in Q, j = 1, \ldots, m$,

(c) $T_C M_{q_1,\gamma,q_2} = M^{'}_{T_D(q_1),\gamma,T_D(q_2)} T_C$, $\forall q_1, q_2 \in Q, \gamma \in \Gamma, \delta(q_2, \gamma) = q_1$,

(d) $T_C(\Pi_{\mathcal{X}_q}(\mu(f))) = \Pi_{\mathcal{X}^{'}_{T_D(q)}}(\mu^{'}(f))$ for each $q = \mu_D(f), f \in \Phi$.

The bilinear hybrid morphism T is said to be injective, surjective, or bijective if both T_D and T_C are respectively injective, surjective, or bijective. Bijective bilinear hybrid morphisms are called bilinear hybrid isomorphisms. Two bilinear hybrid system realizations are *isomorphic* if there exists a bilinear hybrid isomorphism between them.

It is easy to see that the map $T_C : \bigoplus_{q \in Q} \mathcal{X}_q \to \bigoplus_{q \in Q'} \mathcal{X}^{'}_q$ is completely determined by its restriction to $\bigcup_{q \in Q} \mathcal{X}_q$. We will denote this restriction by $M(T)$. Notice that $M(T) : \bigcup_{q \in Q} \mathcal{X}_q \to \bigcup_{q \in Q'} \mathcal{X}^{'}_q$.

Recall the concept of hybrid system morphism from Section 2.3. The following proposition clarifies the relationship between morphisms of bilinear hybrid systems and hybrid system morphisms.

Proposition 31. *If the pair $T = (T_D, T_C)$ defines a bilinear hybrid morphism $T : (H_1, \mu_1) \to (H_2, \mu_2)$, then $\psi(T) = (T_D, M(T))$ defines a hybrid system morphism $H(T) : (H_1, \mu_1) \to (H_2, \mu_2)$ in sense of Section 2.3. Moreover, $H(T)$ is a hybrid isomorphism if and only if T is a bilinear hybrid isomorphism.*

7.2.2 Input-output Maps of Bilinear Hybrid Systems

This subsection reviews the notion of hybrid Fliess-series expansion and its connection to input-output maps of bilinear hybrid systems.

Let $\widetilde{\Gamma} = \Gamma \cup \mathrm{Z}_m$. Then any $w \in \widetilde{\Gamma}$ is of the form $w = w_1\gamma_1 \cdots w_k\gamma_k w_{k+1}$, $\gamma_1, \ldots, \gamma_k \in \Gamma$, $w_1, \ldots, w_{k+1} \in \mathrm{Z}^*_m$, $k \geq 0$. A map $c : \widetilde{\Gamma}^* \to \mathcal{Y}$ is called a *hybrid generating convergent series on* $\widetilde{\Gamma}^*$ if there exists $K, M > 0$, $K, M \in \mathbb{R}$ such that for each $w \in \widetilde{\Gamma}^*$,

$$||c(w)|| < KM^{|w|}$$

where $||.||$ is some norm in $\mathcal{Y} = \mathbb{R}^p$. The notion of generating convergent series is related to the notion of convergent power series from [32]. Let $c : \widetilde{\Gamma}^* \to \mathcal{Y}$ be a hybrid generating convergent series. For each $u \in PC(T,\mathcal{U})$ and $s = (\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*, t_{k+1} \in T$ define the series

$$F_c(u, s, t_{k+1}) = \sum_{w_1,\dots,w_{k+1} \in \mathrm{Z}_m^*} c(w_1\gamma_1 \cdots \gamma_k w_{k+1}) V_{w_1,\dots,w_{k+1}}[u](t_1, \dots, t_{k+1})$$

Later in this section we will show that $F_c(u, s, t)$ is an absolutely convergent series and thus we can define the function $F_c \in F(PC(T,\mathcal{U}) \times (\Gamma \times T)^*, \mathcal{Y})$ as $F_c : (u, w, t) \mapsto F_c(u, w, t)$.

It is easy to see that there is a one-to-one correspondence between hybrid generating convergent series and abstract globally convergent generating series on $I = \bigcup_{k=1}^{\infty} I_k \times (\mathrm{Z}_m)^*)^k$, where $I_k = \Gamma^{k-1}$ for all $k \geq 1$. The correspondence cab be defined as follows. Define the map $\phi : I \to \widetilde{\Gamma}^*$ by $\phi((\gamma_1, \dots, \gamma_k), (w_1, \dots, w_{k+1})) = w_1\gamma_1 w_2 \cdots \gamma_k w_{k+1}$. It is easy to see that this map is a bijection.

If $c : \widetilde{\Gamma}^* \to \mathcal{Y}$ is a hybrid generating convergent series, then it is easy to see that the map

$$c_{abs} : I \ni s \mapsto c(\phi(s)) \in \mathcal{Y}$$

is an abstract globally convergent generating series. Indeed, for any $i = (\gamma_1, \dots, \gamma_k) \in I_{k+1}$, $k \geq 0$ let $K_i = KM^k$. Then for any $w_1, \dots, w_{k+1} \in \mathrm{Z}_m$ we get that

$$\begin{aligned} ||c_{abs}(i, (w_1, \dots, w_{k+1}))|| = ||c(w_1\gamma_1 w_2 \cdots \gamma_k w_{k+1})|| < KM^{k+\sum_{j=1}^{k+1}|w_j|} = \\ = KM^k M^{|w_1|} M^{|w_2|} \cdots M^{|w_{k+1}|} = K_i M^{|w_1|} M^{|w_2|} \cdots M^{|w_{k+1}|} \end{aligned}$$

thus, c_{abs} is indeed an abstract globally convergent generating series. It is also clear that the correspondence $c \mapsto c_{abs}$ is one-to-one. If d, c are hybrid convergent generating series such that $c_{abs} = d_{abs}$, then it is easy to see that $c = d$.

Define the map $\phi^T : (\Gamma \times T)^* \times T \to I^T$ by $\phi^T((\gamma_1, t_1)(\gamma_2, t_2) \cdots (\gamma_k, t_k)), t_{k+1}) = (\gamma_1, \dots, \gamma_k, (t_1, \dots, t_{k+1}))$ Then it is easy to see that ϕ^T is a bijection and

$$F_c(u, w, t) = F_{c_{abs}}(u, \phi^T(w, t))$$

Thus, from Lemma 1 and Lemma 3 we get the following

Lemma 36. *Let $c : \widetilde{\Gamma} \to \mathcal{Y}$ be a hybrid generating convergent series. Then for each $u \in PC(T,\mathcal{U})$, $w \in (\Gamma \times T)^*$, $t \in T$, the series $F_c(u, w, t)$ is absolutely convergent. Thus, the map*

$$F_c : PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T \ni (u, w, t) \mapsto F_c(u, w, t) \in \mathcal{Y}$$

is well-defined. The hybrid convergent generating series c determines the map F_c uniquely, that is, if for some hybrid convergent generating series d $F_c = F_d$, then $c = d$.

Now we are ready to define the concept of hybrid Fliess-series representation of a set of input/output maps, which is related to the concept of Fliess-series expansion in [32]. For any map $f \in F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$, define $f_C = \Pi_{\mathcal{Y}} \circ f$, $f_D = \Pi_O \circ f$. Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$.

Definition 17 (Hybrid Fliess-series expansion). Φ *is said to admit* a hybrid Fliess-series expansion *if*

(1) For each $f \in \Phi$ there exists a generating convergent series $c_f : \widetilde{\Gamma}^* \to \mathcal{Y}$ such that $F_{c_f} = f_C$

(2) For each $f \in \Phi$ the map f_D depends only on Γ^*, that is, for each $w \in \Gamma^*$,

$$\forall u_1, u_2 \in PC(T,\mathcal{U}), \tau_1, \tau_2 \in T^{|w|}, t_1, t_2 \in T : \\ f_D(u_1, (w, \tau_1), t_1) = f_D(u_2, (w, \tau_2), t_2)$$

We will regard f_D as a function $f_D : \Gamma^* \to O$.

The notion of hybrid Fliess-series representation is an extension of the notion of Fliess-series for input-output maps of non-linear systems, see [32]. The following proposition gives a description of the hybrid Fliess-series expansion of Φ in the case when Φ is realized by a bilinear hybrid system.

Proposition 32. *(H, μ) is a bilinear hybrid system realization of Φ if and only if Φ has a hybrid Fliess-series expansion such that for each $f \in \Phi, w_1\gamma_1 \cdots \gamma_k w_{k+1} \in \widetilde{\Gamma}^*$, $\gamma_1, \ldots, \gamma_k \in \Gamma$, $w_1, \ldots, w_{k+1} \in \mathbb{Z}_m^*$, $k \geq 0$*

$$\begin{aligned} & c_f(w_1\gamma_1 \cdots \gamma_k w_{k+1}) = C_{q_k} B_{q_k,w_{k+1}} M_{q_k,\gamma_k,q_{k-1}} B_{q_{k-1},w_k} \cdots \\ & \cdots M_{q_1,\gamma_1,q_0} B_{q_0,w_1} \mu_C(f) \\ & f_D(\gamma_1 \cdots \gamma_k) = \lambda(q_0, \gamma_1 \cdots \gamma_k) \end{aligned} \tag{7.14}$$

where $\mu(f) = (q_0, \mu_C(f))$ and $q_i = \delta(q_0, \gamma_1 \cdots \gamma_i)$, $i = 0, \ldots, k$.

Proof. Assume that (H, μ) is a realization of Φ. Then for each $f \in \Phi$, $u \in PC(T,\mathcal{U})$, $w = (\gamma_1, t_1)(\gamma_2, t_2) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*$, $k \geq 0$, $t_{k+1} \in T$,

$$f(u, w, t_{k+1}) = \upsilon_H(\mu(f), u, w, t_{k+1})$$

That is,

$$f_D(u, w, t_{k+1}) = \lambda(\mu_D(f), \gamma_1 \cdots \gamma_k) \tag{7.15}$$

and

$$f_C(u, w, t_{k+1}) = y_H(\mu(f), u, w, t_{k+1})$$

Assume that $\mu(f) = (q_f, x_f) \in \mathcal{H}$. Recall from the proof Proposition 30 the definition of c_{q_f,x_f}. It follows from the proof of Proposition 30 that c_{q_f,x_f} is a hybrid convergent generating series and $y_H(\mu(f), .) = F_{c_{q_f,x_f}}$.

Thus, we get that $f_D(u, w, t_{k+1})$ depends only on $\gamma_1 \cdots \gamma_k$, i.e. $f_D : \Gamma^* \to O$ and $f_C = F_{c_{q_f,x_f}}$. Thus, Φ indeed admits a hybrid Fliess-series expansion. By Lemma 36, if $f_C = F_{c_f} = F_{c_{q_f,x_f}}$ for some hybrid convergent generating series c_f, then $c_f = c_{q_f,x_f}$. From the definition of c_{q_f,x_f} it follows that $c_f = c_{q_f,x_f}$ is equivalent to the first equation in (7.14). The second equation in (7.14) is the same as (7.15).

Conversely, assume that Φ has a hybrid Fliess-series expansion and (7.14) holds. The first equation of (7.14) implies that $c_f = c_{q_f,x_f}$ and thus $f = F_{c_f} = F_{c_f,x_f} = y_H(\mu(f), .)$ for all $f \in \Phi$. The second equation of (7.14) is equivalent to

$$\forall s \in \Gamma^* : f_D(s) = \lambda(q_f, s)$$

Thus, we get that for each $f \in \Phi$, $\mu(f) = (q_f, x_f)$,

$$f(u, w, t) = (f_D(\gamma_1 \cdots \gamma_k), f_C(u, w, t)) = (\lambda(q_f, \gamma_1 \cdots \gamma_k), y_H((q_f, x_f), u, w, t)) = \upsilon_H((q_f, x_f), u, w, t)$$

for all $u \in PC(T, \mathcal{U})$, $w = (\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*$, $k \geq 0$, $t \in T$. The last equation means that (H, μ) is a realization of Φ. $\square$

7.2.3 Realization of Input-output Maps by Bilinear Hybrid Systems

In this section the solution to the realization problem for bilinear hybrid systems will be presented. In addition, characterisation of minimal bilinear hybrid systems realizing the specified set of input-output maps will be given. We will use the theory of hybrid formal power series developed in Section 3.3.

Let us recall the characterisation of semi-reachability and observability for bilinear hybrid systems presented in [48, 54]. Using the notation of Definition 4, the following holds.

Theorem 37. *The bilinear hybrid system H is observable if and only if*

(i)

$\mathcal{A}_H = \mathcal{A}$ is observable, and

(ii) *For each* $q \in Q$,

$$O_{H,q} =$$

$$\bigcap_{\gamma_1,\ldots,\gamma_k \in \Gamma, k \geq 0} \bigcap_{w_1,\ldots,w_{k+1} \in \mathbb{Z}_m^*} \ker C_{q_k} B_{q_k,w_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} B_{q_0,w_1} =$$

$$= \{0\}$$

where $q_l = \delta(q, \gamma_1 \cdots \gamma_l), 0 \leq l \leq k, k \geq 0, q = q_0$.

Notice that part (i) of the theorem above is equivalent to

$$\upsilon_H((q_1, 0), .) = \upsilon_H(q_2, 0), .) \iff q_1 = q_1, \forall q_1, q_2 \in Q$$

Part (ii) of the theorem says that for each $q \in Q$:

$$\upsilon_H((q, x_1), .) = \upsilon_H((q, x_2), .) \iff x_1 = x_2, , \forall x_1, x_2 \in \mathcal{X}_q$$

Proof of Theorem 37. We will start with stating a number of relatively simple observations.

`Observation 1`

Assume that $q_1, q_2 \in Q$. Then

$$\upsilon_H((q_1, 0), .) = \upsilon_H((q_2, 0), .) \iff (\forall w \in \Gamma^* : \lambda(q_1, w) = \lambda(q_2, w))$$

Indeed, $\upsilon_H((q, 0), u, (w, \tau), t) = (\lambda(q, w), 0)$, and thus

$$\upsilon_H((q_1, 0), u, (w, \tau), t) = \upsilon_H((q_2, 0), u, (w, \tau), t) \iff \lambda(q_1, w) = \lambda(q_2, w)$$

`Observation 2`

Let $(q_1, x_1), (q_2, x_2) \in \mathcal{H}$.

$$\upsilon_H((q_1, x_1), .) = \upsilon_H((q_2, x_2), .) \implies (\forall w \in \Gamma^* : \lambda(q_1, w) = \lambda(q_2, w))$$

Indeed, for $i = 1, 2$, $\Pi_O \circ \upsilon_H((q_i, x_i), 0, (w, \underbrace{00 \cdots 0}_{|w|--times}), 0) = \lambda(q_i, w)$, and thus the implication above follows.

`Observation 3`

Let $q \in Q$, $x_1, x_2 \in \mathcal{X}_q$. Then

$$\upsilon_H((q, x_1), .) = \upsilon_H((q, x_2), .) \iff x_1 - x_2 \in O_{H,q}$$

Indeed, $\upsilon_H((q, x_i), u, (w, \tau), t) = (\lambda(q, w), y_H((q, x_i), u, (w, \tau), t))$, thus $\upsilon_H((q, x_1), .) = \upsilon_H((q, x_2), .)$ is equivalent to $y_H((q, x_1), .) = y_H((q, x_2), .)$. Recall from the proof of Proposition 30 the definition of the series c_{q,x_i}, $i = 1, 2$ and recall that $y_H((q, x_i), .) =$

$F_{c_{q,x_i}}$, $i=1,2$. Thus, $\upsilon_H((q,x_1),.) = \upsilon_H((q,x_2),.)$ is equivalent to $F_{c_{q,x_1}} = F_{c_{q,x_2}}$. By Lemma 36 the latter is equivalent to $c_{q,x_1} = c_{q,x_2}$, or, in other words, for all $\gamma_1,\ldots,\gamma_k \in \Gamma$, $w_1,\ldots,w_{k+1} \in \mathbb{Z}_m^*$, $k \geq 0$,

$$C_{q_k} B_{q_k,w_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} B_{q_0,w_1}(x_1 - x_2) = 0$$

where $q_i = \delta(q, \gamma_1 \cdots \gamma_i)$, $i = 0,\ldots,k$. Thus,

$$x_1 - x_2 \in \ker C_{q_k} B_{q_k,w_k} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} B_{q_0,w_1}$$

for all $w_1,\ldots,w_{k+1} \in \mathbb{Z}_m^*$, $\gamma_1,\ldots,\gamma_k \in \Gamma$, $k \geq 0$. That is, $c_{q,x_1} = c_{q,x_2}$ is equivalent to $x_1 - x_2 \in O_{H,q}$.

Now we will prove the statement of the theorem. Assume that (H,μ) is observable. Suppose $\mathcal{A}_H$ is not observable. Then there exists $q_1,q_2 \in Q$, $q_1 \neq q_2$ such that $\forall w \in \Gamma^* : \lambda(q_1,w) = \lambda(q_2,w)$. By Observation 1 it is equivalent to $\upsilon_H((q_1,0),.) = \upsilon_H((q_2,0),.)$, which by observability of H implies $q_1 = q_2$, a contradiction. Thus, $\mathcal{A}_H$ is indeed observable. Assume now that $O_{H,q} \neq \{0\}$, for some $q \in Q$, that is, there exists $0 \neq x \in O_{H,q}$. Then by Observation 3 we get that $\upsilon_H((q,x),.) = \upsilon_H((q,0),.)$ which by observability of H implies $x = 0$, a contradiction. Thus, $O_{H,q} = \{0\}$ for all $q \in Q$. That is, we showed that conditions (i) and (ii) of the theorem are necessary.

Assume that condition (i) and (ii) of the theorem holds. Assume that (q_1,x_1) and (q_2,x_2) are indistinguishable, that is, $\upsilon_H((q_1,x_1),.) = \upsilon_H((q_2,x_2),.)$ Then by Observation 2 we get that q_1 and q_2 are indistinguishable in $\mathcal{A}_H$. Thus, by observability of $\mathcal{A}_H$ it follows that $q_1 = q_2 = q$. By Observation 3, $\upsilon_H((q,x_1),.) = \upsilon_H((q,x_2),.)$ implies that $x_1 - x_2 \in O_{H,q}$. But condition (ii) implies that $O_{H,q} = \{0\}$, thus $x_1 = x_2$. That is, we get that $(q_1,x_1) = (q_2,x_2))$. Thus, it follows that H is observable. □

Theorem 38. *(H,μ) is semi-reachable if and only if (A_H,μ_D), $\mu_D = \Pi_Q \circ \mu$, is reachable and* $\dim W_H = \sum_{q\in Q} \dim \mathcal{X}_q$, *where*

$$W_H = \mathrm{Span}\{B_{q_k,w_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} B_{q_0,w_1} x_f, \mid (q_f,x_f) = \mu(f),$$
$$f \in \Phi, w_1,\ldots,w_{k+1} \in \mathbb{Z}_m^*, q_j = \delta(q_f,\gamma_1\cdots\gamma_j), 0 \leq j \leq k, k \geq 0\}$$

Proof. We will prove that W_H is the smallest vector space containing $Reach(H,\mathrm{Im}\mu) = \{x_H(\mu(f),u,s,t) \mid u \in PC(T,\mathcal{U}), s \in (\Gamma \times T)^*, t \in T\}$.

Recall the definition of the hybrid generating series $d_{q,x}$ from the proof of Proposition 30. Notice that $W_H = \mathrm{Span}\{d_{(q,x)}(s) \mid s \in \widetilde{\Gamma}^*, (q,x) \in \mathrm{Im}\mu\}$

First we will show that $Reach(H,\mathrm{Im}\mu) \subseteq W_H$. Indeed,

$$x_H(\mu(f),u,(\gamma_1,t_1)\cdots(\gamma_k,t_k),t) = \sum_{w_1,\ldots,w_{k+1}\in\mathbb{Z}_m^*} d_{\mu(f)}(w_1\gamma_1\cdots\gamma_k w_{k+1}) \times$$
$$\times V_{w_1,\ldots,w_{k+1}}[u](t_1,\ldots,t_k,t)$$

Since W_H is a finite-dimensional vector space and thus closed and every element

$$d_{\mu(f)}(w_1\gamma_1\cdots\gamma_k w_{k+1})V_{w_1,\dots,w_{k+1}}[u](t_1,\dots,t_k,t) \in W_H$$

we get that $x_H(\mu(f),(\gamma_1,t_1)\cdots(\gamma_k,t_k),t) \in W_H$. Thus, $Reach(H,\mathrm{Im}\mu) \subseteq W_H$.

Let U be any subspace of $\bigoplus_{q\in q}\mathcal{X}_q$ containing $Reach(H,\mathrm{Im}\mu)$. Recall that we can associate with $d_{\mu(f)}$ an abstract absolutely convergent generating series $d_{abs,\mu(f)} = (d_{\mu(f)})_{abs}$ defined on $I = \bigcup_{k=1}^{\infty} I_k \times (\mathrm{Z}_m^*)^k$, where $I_k = \Gamma^{k-1}$, $k \geq 1$, as follows

$$d_{abs,\mu(f)}((\gamma_1,\dots,\gamma_k),(w_1,\dots,w_{k+1})) = d_{\mu(f)}(w_1\gamma_1\cdots\gamma_k w_{k+1})$$

Recall that

$$F_{d_{abs,\mu(f)}}(u,((\gamma_1,\dots,\gamma_k),(t_1,\dots,t_{k+1}))) = F_{d_{\mu(f)}}(u,(\gamma_1,t_1)\cdots(\gamma_k,t_k),t_{k+1})$$

Recall from the proof of Lemma 3 that for all $\eta = (\gamma_1,\dots,\gamma_k) \in I_{k+1}$, $w_1,\dots,w_{k+1} \in \mathrm{Z}_m$ there exists an analytic map $g_\eta : W \times V \to \bigoplus_{q\in Q}\mathcal{X}_q$ such that $W \subseteq \mathcal{U}^N$, $V \subseteq T^N$, $N = \sum_{j=1}^{k+1}|w_j|$, $\mathrm{Im}g_\eta \subseteq \mathrm{Im}F_{d_{abs,\mu(f)}} = \mathrm{Im}F_{d_{\mu(f)}}$. Moreover, for suitable high-order differential operators, which were denoted by D_{w_i}, $i = 1,\dots,k+1$ it holds

$$D_{w_1}D_{w_2}\cdots D_{w_{k+1}}g_\eta(u)|_{u=0} = d_{abs,\mu(f)}(\eta,(w_1,\dots,w_{k+1}))$$

Since $F_{d_{\mu(f)}} = x_H(\mu(f),.)$, we get that $\mathrm{Im}F_{d_{\mu(f)}} \subseteq Reach(H,\mathrm{Im}\mu) \subseteq U$ and thus $g_\eta : W \times V \to U$. Since U is a finite dimensional vector space, we get that $D_{w_1}D_{w_2}\cdots D_{w_{k+1}}g_\eta(u)|_{u=0} \in U$ and thus

$$d_{abs,\mu(f)}(\eta,(w_1,\dots,w_{k+1})) = d_{\mu(f)}(w_1\gamma_1\cdots\gamma_k w_{k+1}) \in U$$

That is, $d_{\mu(f)}(s) \in U$ for all $f \in \Phi$ and $s \in \widetilde{\Gamma}^*$. Thus, $W_H \subseteq U$. That is, W_H is indeed the smallest vector subspace of $\bigoplus_{q\in Q}\mathcal{X}_q$ containing $Reach(H,\mathrm{Im}\mu)$.

We will proceed with the proof of the statement of the theorem. (H,μ) is semi-reachable if and only if $(\mathcal{A}_H,\mu_D)$ is reachable and $\bigoplus_{q\in Q}\mathcal{X}_q$ contains no proper subspace containing $Reach(H,\mathrm{Im}\ mu)$. But W_H is the smallest vector space containing $Reach(H,\mathrm{Im}\mu)$. Thus, $\bigoplus_{q\in Q}\mathcal{X}_q$ has no proper subspaces containing $Reach(H,\mathrm{Im}\mu)$ if and only if $W_H = \bigoplus_{q\in Q}\mathcal{X}_q$, or, in other words, $\dim W_H = \dim\bigoplus_{q\in Q}\mathcal{X}_q = \sum_{q\in Q}\dim\mathcal{X}_q$.

□

Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ be a set of input-output maps. Assume that Φ has a hybrid Fliess-series expansion. Then Proposition 32 allows us to reformulate the realization problem in terms of rationality of certain hybrid formal power series. Recall that $\widetilde{\Gamma} = \Gamma \cup \mathrm{Z}_m$. Let $J = \Phi$ and for each $f \in \Phi$ define the hybrid formal power series $T_f \in \mathbb{R}^p \ll \widetilde{\Gamma}^* \gg \times F(\Gamma, O)$ by

$$(T_f)_C = c_f \text{ and } (T_f)_D = f_D$$

It is easy to see that J is a hybrid power series index set with $J_1 = J = \Phi$ and $J_2 = \emptyset$. Define the indexed set of *hybrid formal power series associated with* Φ by

$$\Psi_\Phi = \{T_f \in \mathbb{R}^p \ll \widetilde{\Gamma}^* \gg \times F(\Gamma^*, O) \mid f \in \Phi\}$$

It is easy to see that Ψ_Φ is a well-posed indexed set of hybrid formal power series with the index set J. Define the *Hankel-matrix* H_Φ of Φ as $H_\Phi = H_{\Psi_\Phi}$. Notice that if Φ is finite, then Ψ_Φ is a finite set. Let

$$H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, \{B_{q,j}\}_{j=1,\dots,m}, C_q)_{q \in Q}, \{M_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\})$$

(H, μ) be a bilinear hybrid system realization with $\mu : \Phi \to \bigcup_{q \in Q} \{q\} \times \mathcal{X}_q$. Define the *hybrid representation* $HR_{H,\mu}$ *associated with* (H, μ) by

$$HR_{H,\mu} = (\mathcal{A}, (\mathcal{X}_q, \{A_{q,z}\}_{z \in X_1}, C_q)_{q \in Q}, \{M_{\delta(q,y),y,q} \mid q \in Q, y \in X_2\}, J, \mu)$$

where $J = J_1 = \Phi$, $J_2 = \emptyset$, $X = \widetilde{\Gamma}$, $X_1 = \mathrm{Z}_m$, $X_2 = \Gamma$ and for each $q \in Q$, $j = 1, \dots, m$

$$A_{q,0} = A_q \text{ and } A_{q,j} = B_{q,j}$$

Conversely, let $HR = (\mathcal{A}, (\mathcal{X}_q, \{A_{q,z}\}_{z \in X_1}, C_q)_{q \in Q}, \{M_{\delta(q,y),y,q} \mid q \in Q, y \in X_2\}, J, \mu)$ be a hybrid representation with index set $J = \Phi$ such that $X_1 = \mathrm{Z}_m$, $X_2 = \Gamma$, $J_1 = \Phi$, $J_2 = \emptyset$, $X = \widetilde{\Gamma}$. Define the *bilinear hybrid realization* (H_{HR}, μ_{HR}) *associated with* HR as follows

$$H_{HR} = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, \{B_{q,j}\}_{j=1,\dots,m}, C_q)_{q \in Q}, \{M_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\})$$

and $\mu_{HR} = \mu$, where for each $q \in Q$, $j = 1, \dots, m$,

$$A_q = A_{q,0} \text{ and } B_{q,j} = A_{q,j}$$

It is easy to see that $(H_{HR_{H,\mu}}, \mu_{HR_{H,\mu}}) = (H, \mu)$ and $HR_{H_{HR},\mu_{HR}} = HR$ for any hybrid representation HR and bilinear hybrid realization (H, μ). It is also easy to see that $\dim H = \dim HR_{H,\mu}$.

The following theorem follows easily from Proposition 32 and plays a crucial role in realization theory of bilinear hybrid system.

Theorem 39. *A bilinear hybrid system (H, μ) is a realization of Φ if and only if Φ has a hybrid Fliess-series expansion and $HR_{H,\mu}$ is a hybrid representation of Ψ_Φ. Conversely, if Φ has a hybrid Fliess-series expansion and HR is a hybrid representation of Ψ_Φ then (H_{HR}, μ_{HR}) is a bilinear hybrid system realization of Φ.*

Proof. Assume that (H,μ) is a bilinear hybrid system. Let

$$HR_{H,\mu} = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j\in J_2, z\in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y\in X_2})_{q\in Q}, J, \mu)$$

Then by Proposition 32 (H,μ) is a realization of Φ if and only if it admits a hybrid Fliess-series expansion and

$$\begin{aligned}&(T_f)_C)(w_1\gamma_1\cdots\gamma_k w_{k+1}) = c_f(w_1\gamma_1\cdots\gamma_k w_{k+1}) = C_{q_k}B_{q_k,w_{k+1}}M_{q_k,\gamma_k,q_{k-1}}B_{q_{k-1},w_k}\\ &\cdots M_{q_1,\gamma_1,q_0}B_{q_0,w_1}\mu_C(f) = C_{q_k}A_{q_k,w_{k+1}}M_{q_k,\gamma_k,q_{k-1}}\cdots M_{q_1,\gamma_1,q_0}A_{q_0,w_1}\mu_C(f)\\ &(T_f)_D(\gamma_1\cdots\gamma_k) = f_D(\gamma_1\cdots\gamma_k) = \lambda(q_0,\gamma_1\cdots\gamma_k)\end{aligned}$$

holds for all $\gamma_1,\dots,\gamma_k \in \Gamma$, $w_1,\dots,w_{k+1} \in Z_m^*$, $k \geq 0$, $f \in \Phi$. It is easy to see that the equation above is equivalent to $HR_{H,\mu}$ being a hybrid representation of Ψ_Φ. Thus, we get that (H,μ) is a realization of Φ if and only if Φ has a hybrid Fliess-series expansion and $HR_{H,\mu}$ is a representation of Ψ_Φ. The second part of the theorem follows from the first part and the observation that $HR_{H_{HR},\mu_{HR}} = HR$. □

The theorem above allows us to reduce the realization problem for bilinear hybrid systems to existence of a hybrid representation of a indexed set of hybrid formal power series. Moreover, Theorem 37 and Theorem 38 allow us to relate observability and semi-reachability of bilinear hybrid systems to observability and reachability of hybrid representations.

Theorem 40. *A bilinear hybrid system realization (H,μ) is observable if and only if $HR_{H,\mu}$ is observable. A bilinear hybrid system realization (H,μ) is semi-reachable if and only if $HR_{H,\mu}$ is reachable.*

Proof. Let $(H,\mu) = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, \{B_{q,j}\}_{j=1,\dots,m}, C_q)_{q\in Q}, \{M_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\})$ and let

$$HR = HR_{H,\mu} = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j\in J_2, z\in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y\in X_2})_{q\in Q}, J, \mu)$$

Notice that

$$\begin{aligned}A_{q_k,w_{k+1}}M_{q_k,\gamma_k,q_{k-1}}A_{q_{k-1},w_k}\cdots M_{q_1,\gamma_1,q_0}A_{q_0,w_1}x =\\ B_{q_k,w_{k+1}}M_{q_k,\gamma_k,q_{k-1}}B_{q_{k-1},w_k}\cdots M_{q_1,\gamma_1,q_0}B_{q_0,w_1}x\end{aligned}$$

for all $\gamma_1,\dots,\gamma_k \in \Gamma$, $k \geq 0$, $w_1,\dots,w_{k+1} \in Z_m^*$, $k \geq 0$, $q \in Q$, $q_i = \delta(q,\gamma_1\cdots\gamma_i)$, $i = 0,\dots,k$. Thus, it follows that $W_H = W_{HR}$ and $O_{H,q} = O_{HR,q}$ for all $q \in Q$.

By Theorem 37 H is observable, if and only if $\mathcal{A}$ is observable and for all $q \in Q$, $O_{H,q} = \{0\}$. By Proposition 4, taking into account that $J_2 = \emptyset$ and $O_{H,q} = O_{HR,q}$, this is equivalent to HR being observable.

By Theorem 38 (H,μ) is semi-reachable if and only if $(\mathcal{A},\zeta)$ is reachable and $W_{HR} = W_H = \bigoplus_{q\in Q} \mathcal{X}_q$. By Proposition 3 that is equivalent to HR being reachable. $\square$

Notice that both H and $HR_{H,\mu}$ have the same state-space. It is easy to see that the following holds.

Lemma 37. *Let (H_i,μ_i),$i=1,2$ be two bilinear hybrid systems. If $T:(H_1,\mu_1)\rightarrow(H_2,\mu_2)$ is a bilinear hybrid morphism, then T is also a $T: HR_{H_1,\mu_1}\rightarrow HR_{H_2,\mu_2}$ hybrid representation morphism. Conversely, if HR_i, $i=1,2$ are two hybrid representations with hybrid power series index set $J=\Phi$ and $T: HR_1\rightarrow HR_2$ is a a hybrid representation morphism then T can be viewed as a $T:(H_{HR_1},\mu_{HR_1})\rightarrow(HR_2,\mu_{HR_2})$ bilinear hybrid morphism. The map T is a surjective, injective , isomorphism as a hybrid bilinear morphism if and only if T is surjective, injective, isomorphism as a hybrid representation morphism.*

Proof. Assume that

$$H_i = (\mathcal{A}^i,\mathcal{U},\mathcal{Y},(\mathcal{X}^i_q,A^i_q,\{B^i_{q,j}\}_{j=1,\ldots,m},C^i_q)_{q\in Q^i},\{M^i_{\delta^i(q,\gamma),\gamma,q}\mid q\in Q^i,\gamma\in\Gamma\})$$

$i=1,2$. We will show that a pair of maps $T=(T_D,T_C)$, where $T_D:Q^1\rightarrow Q^2$, $T_C:\bigoplus_{q\in Q^1}\mathcal{X}^1_q\rightarrow\bigoplus_{q\in Q^2}\mathcal{X}^2_q$ is a linear map, defines a bilinear morphism $T:(H_1,\mu_1)\rightarrow(H_2,\mu_2)$ if and only if it defines a hybrid representation morphism $T:HR_{H_1,\mu_1}\rightarrow HR_{H_2,\mu_2}$.

The pair T is a bilinear morphism if and only if $T_D:(\mathcal{A}^1,(\mu_1)_D)\rightarrow(\mathcal{A}^2,(\mu_2)_D)$ is an automaton morphism and for the linear map T_C the following holds:

1. $T_C(\mathcal{X}^1_q)\subseteq\mathcal{X}^2_{T_D(q)}$,
2. $T_C B^1_{q,j} = B^2_{T_D(q),j}T_C$,
3. $T_C M^1_{q_2,\gamma,q_1} = M^2_{T_D(q_2),\gamma,T_D(q_1)}T_C$ and
4. $T_C(\mu_1)_C(f) = (\mu_2)_C(f)$, where $f\in\Phi, q,q_1,q_2\in Q^1$, $j\in \mathrm{Z}_m$, $\gamma\in\Gamma$.

Notice that

$$HR_{H_i,\mu_i} = (\mathcal{A}^i,\mathcal{Y},(\mathcal{X}^i_q,\{A^i_{q,z},B^i_{q,z,j_2}\}_{j\in J_2,z\in X_1},C^i_q,\{M^i_{\delta^i(q,y),y,q}\}_{y\in X_2})_{q\in Q^i},J,\mu^i)$$

for $i=1,2$, where $A^i_{q,j}=B^i_{q,j}$ for all $q\in Q^i$, $j\in\mathrm{Z}_m$ and thus the conditions above are exactly equivalent to $T=(T_D,T_C)$ being a hybrid representation morphism $T:HR_{H_1,\mu_1}\rightarrow HR_{H_2,\mu_2}$. The bilinear hybrid morphism T is injective, surjective, bijective if both the maps T_D and T_C are injective, surjective, bijective respectively, which is equivalent to T being respectively a injective, surjective, bijective hybrid representation morphism.

If HR_i,$i = 1, 2$ are two hybrid representations, then $HR_{H_{HR_i},\mu_{HR_i}} = HR_i$. Thus by the first part of the lemma T is a bilinear hybrid morphism $T : (H_{HR_1}, \mu_{HR_1}) \rightarrow (H_{HR_2}, \mu_{HR_2})$ if and only if T is a hybrid representation morphism $T : HR_1 \rightarrow HR_2$. □

Let $\Phi_D = (\Psi_\Phi)_D$. From the discussion above, using the results on theory of hybrid formal power series (Theorem 7 and Theorem 8 and Corollary 5) we can derive the following theorem, which was already published in [48].

Theorem 41 (Realization of input/output map). *Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ be a set of input-output maps. The following are equivalent.*

(i) *Φ has a realization by a bilinear hybrid system,*

(ii) *Φ has a hybrid Fliess-series expansion, Ψ_Φ is rational indexed set of hybrid formal power series*

(iii) *Φ has a hybrid Fliess-series expansion,* rank $H_\Phi < +\infty$ *and Φ_D has a realization by a finite Moore-automaton, i.e. $card(W_{\Phi_D}) < +\infty$.*

Proof. `(i)` $\implies$ `(ii)`
Assume that (H, μ) is a realization of Φ. By Theorem 39 it implies that Φ has a hybrid Fliess-series expansion and $HR_{H,\mu}$ is a representation of Ψ_Φ, thus Ψ_Φ is a rational family of hybrid formal power series.

`(ii)` $\implies$ `(i)`
Assume that Φ has a hybrid Fliess-series expansion and HR is a hybrid representation of Ψ_Φ. Then by Theorem 39 we get that (H_{HR}, μ_{HR}) is a realization of Φ.

`(ii)` $\iff$ `(iii)` follows from Corollary 6. □

Below we will give a characterisation of minimal bilinear hybrid systems.

Theorem 42 (Minimal realization). *If Φ has a bilinear hybrid system realization, then Φ has a minimal bilinear hybrid system realization. If (H, μ) is a bilinear hybrid system realization of Φ, then the following are equivalent.*

(i) *(H, μ) is minimal,*

(ii) *(H, μ) is semi-reachable and it is observable,*

(iii) *For each $(H^{'}, \mu^{'})$ semi-reachable bilinear hybrid realization of Φ there exists a surjective bilinear hybrid morphism $T : (H^{'}, \mu^{'}) \rightarrow (H, \mu)$. In particular, all minimal hybrid bilinear systems realizing Φ are isomorphic.*

Proof. By Theorem 41, if Φ has a realization by a bilinear hybrid system, then Ψ_Φ has a hybrid representation and Φ admits a hybrid Fliess-series expansion. But the by Theorem 8 Ψ_Φ admits a minimal hybrid representation HR_m. From Theorem 39 it follows that $(H_m, \mu_m) = (H_{HR_m}, \mu_{HR_m})$ is a bilinear hybrid realization of Φ. We will argue that (H_m, μ_m) is a minimal bilinear hybrid realization of Φ. Indeed, let (H, μ) be a bilinear hybrid realization of Φ. Then by Theorem 39 $HR_{H,\mu}$ is a hybrid representation of Ψ_Φ. Thus by minimality of HR_m, $\dim H_m = \dim HR_m \leq \dim HR_{H,\mu} = \dim H$. That is, (H_m, μ_m) is indeed a minimal bilinear hybrid realization of Φ. Thus, if Φ has a realization by a bilinear hybrid system it also has a minimal bilinear hybrid system realization. The argument above also demonstrates that (H, μ) is a minimal bilinear hybrid realization of Φ if and only if $HR_{H,\mu}$ is a minimal hybrid representation of Ψ_Φ.

We proceed with the proof of `(i)` $\iff$ `(ii)` $\iff$ `(iii)`. A bilinear hybrid realization (H, μ) of Φ is minimal if and only if $HR_{H,\mu}$ is a minimal hybrid representation of Ψ_Φ. By Theorem 8 the latter is equivalent to $HR_{H,\mu}$ being reachable and observable. But by Theorem 40 the latter is equivalent to (H, μ) being semi-reachable and observable. Thus, `(i)` $\iff$ `(ii)`. From Theorem 8 it also follows that $HR_{H,\mu}$ is minimal if and only if for any reachable hybrid representation HR of Ψ_Φ there exists a surjective hybrid representation morphism $T : HR \to HR_{H,\mu}$, which by Lemma 37 is equivalent to (iii), i.e. that for each bilinear hybrid realization $(H^{'}, \mu^{'})$ there exists a surjective bilinear morphism $T : (H^{'}, \mu^{'}) \to (H, \mu)$. Indeed, assume that (iii) holds. Then by Theorem 40 for any reachable representation $HR^{'}$ of Ψ_Φ, $(H_{HR'}, \mu_{HR'})$ is a semi-reachable bilinear hybrid realization of Φ and thus there exists a surjective bilinear morphism $T : (H_{HR'}, \mu_{HR'}) \to (H, \mu)$. But by Lemma 37 the latter implies that $T : HR^{'} \to HR_{H,\mu}$ is a surjective hybrid representation morphism. Conversely, assume that for any reachable hybrid representation $HR^{'}$ of Ψ_Φ there exists a surjective hybrid morphism $T : HR^{'} \to HR_{H,\mu}$. For any semi-reachable bilinear hybrid realization $(H^{'}, \mu^{'})$ of Φ, Theorem 40 implies that $HR_{H^{'},\mu^{'}}$ is a reachable hybrid representation of Ψ_Φ, thus by the assumption there exists a surjective hybrid representation morphism $T : HR_{H^{'},\mu^{'}} \to HR_{H,\mu}$. But from Lemma 37 we get that $T : (H^{'}, \mu^{'}) \to (H, \mu)$ is a surjective morphism. Thus, we showed that `(i)` $\iff$ `(iii)`.

Finally, isomorphism of minimal bilinear hybrid system realizations follows from isomorphism of minimal hybrid power series representations. Indeed, if (H_i, μ_i),$i = 1, 2$ are two minimal bilinear hybrid system realizations of Φ, then $HR_i = HR_{H_i,\mu_i}$, $i = 1, 2$ are two minimal hybrid representations of Ψ_Φ. Thus, there exists a hybrid representation isomorphism $S : HR_1 \to HR_2$, which implies that $S : (H_1, \mu_1) \to (H_2, \mu_2)$ is a bilinear hybrid isomorphism. $\square$

Chapter 8

Realization Theory of Nonlinear Hybrid Systems Without Guards

8.1 Introduction

In this chapter we will address the following question. Consider an input-output map and formulate conditions for existence of a realization the class of nonlinear hybrid system without guards.

The problem as it is stated above is quite difficult, therefore we will adopt a number of simplifications. First of all we will restrict ourselves to *analytic hybrid systems* , i.e. hybrid systems such that the underlying continuous control systems are analytic and the reset maps are analytic. To simplify the problem further, we will look only at local and formal realization. That is, we will try to find conditions with respect to which the input-output map coincides with the input-output map of a hybrid system locally, i.e. for small times. To facilitate the transition from global to the local problem we will introduce the concept of the hybrid Fliess-series expansion. Roughly speaking, an input-output map admits a hybrid Fliess-series expansion if its continuous-valued part can be represented as infinite series of iterated integrals of the continuous inputs. The coefficients of these iterated integrals form a sequence which completely determines the input-output map locally. We will refer to this sequence as the *hybrid generating series associated with the input-output map.* Existence of a hybrid Fliess-series expansion is a necessary condition for existence of a local realization by an analytic hybrid system. The associated hybrid generating series can be thought of as a collection of high-order derivatives of the input-output map. It turns out that a necessary condition for existence of a hybrid system realization for an input-output map is that the corresponding generating series admits a representation of the following form. There exists a finite collection of rings of formal power series in finitely many commuting variables and a finite collection of continuous derivations and algebra homomorphisms on these rings such that the following holds. Each value of the generating series can be represented as evaluation at zero of a formal power series, obtained by applying consecutively the specified derivations and algebra homomorphisms to a formal power series from a finite collection of formal power series belonging to the specified rings.

To be more precise, since the hybrid systems considered are analytic, we can associate with each underlying continuous system a formal power series ring, a finite family of continuous derivations and a formal power series. The formal power series ring corresponds to the ring of germs of analytic functions around a point, the derivations are just the Taylor-series expansion of the vector fields of the system and the formal power series is just the Taylor series expansion of the readout map of the system. In the context of the transformation described above the analytic reset maps become continuous homomorphisms on formal power series rings, by taking the Taylor series expansion of each reset map around a suitably chosen point.

In this manner we get a construct which we will call a *formal hybrid system*. A formal hybrid system consists of a Moore-automaton and a family of rings of formal power series in finitely many commuting variables. With each discrete state of the automaton we associate a ring of formal power series from the family. On each ring we define a finite family of continuous derivations on that ring. The elements of these families of formal vector fields are indexed by the same set of *inputs*. We define formal power series ring homomorphism for each discrete state transition, such that the homomorphism acts between the rings belonging to the old and to the new discrete states respectively. We will call these maps reset maps. With each discrete state we associate an element of the ring associated with that discrete state. This element will be called the *readout map associated with the discrete state*.

The concept of formal hybrid system allows us to reformulate the necessary condition for existence of a hybrid system realization mentioned above. Namely, it turns out that existence of a realization by an analytic hybrid system implies that the generating series associated with the hybrid Fliess-series expansion of the input-output map *has a realization by a formal hybrid system*. Conversely, if we have a formal hybrid system such that the vector fields, reset maps and readout maps are in fact convergent formal power series, it will immediately yield us a hybrid system. This hybrid system is obtained from the formal hybrid system as follows. The automaton of the hybrid system is the same as the automaton of the formal hybrid system. The continuous state space of the hybrid system which is associated with the discrete state q is $\mathbb{R}^{n_q}$, where n_q is the number of variables in the formal power series ring associated with the same discrete state q in the formal hybrid system. Each vector field of the hybrid system has the property, that its Taylor-series expansion coincides with the corresponding formal vector field of the formal hybrid system. The Taylor-series expansions of the readout and reset maps of the hybrid system coincide with the corresponding formal power series in the formal system.

In fact, most of the chapter is devoted to the realization problem for formal hybrid systems. That is, consider a map mapping sequences of discrete and continuous inputs symbols to discrete and continuous outputs. We would like to find necessary and sufficient conditions for existence of a formal hybrid system realizing this map. We will be able to present some necessary conditions and some results which indicate that these necessary conditions are very close to being sufficient ones.

The approach to realization theory of analytic hybrid systems sketched above is very similar to the classical approach to local realization theory of analytic nonlinear systems, [36, 21]. The classical solution to local nonlinear realization problem starts with associating with each nonlinear system a formal system defined as a follows. We associate with each vector field of the nonlinear systems a derivation on the ring of formal power series. The derivations are obtained by taking the Taylor-series expansion of each vector field around the initial point. The solution to the local realization problem is reduced to finding a formal system realization for a map, which maps sequences of input symbols to continuous outputs. There are many ways to solve the problem of existence of a formal realization. One of them is to use the theory of Sweedler-type coalgebras and bialgebras [29, 27]. The other one gives a direct construction of a realization, using theory of Lie-algebras [36, 21].

In this chapter we will use the theory of Sweedler-type coalgebras. Note that Sweedler-type coalgebras are not identical to coalgebras used by Jan Rutten ([59]). Although Sweedler-type coalgebra are a special case of the category theoretical coalgebras, they have much more structure. Roughly speaking a Sweedler-type coalgebra is a vector space on which a so called comultiplication and counit are defined. We will show that existence of a formal hybrid system realization is equivalent to existence of a realization by an abstract system of a certain type, which we will call *CCPI hybrid coalgebra systems*. Roughly speaking such a system is a system, state space of which is a coalgebra satisfying certain properties. Our efforts will be directed towards finding conditions for existence of such a hybrid coalgebra realization.

This chapter is not the first attempt to use coalgebras for hybrid system. Already the paper by [28] advocated an approach based on coalgebras, and this chapter uses similar ideas. Although the stated goal of the paper by Grossman and Larson was to use coalgebra theory for developing realization theory for hybrid systems, it just presented some reformulation of the already known results for finite-state automata and nonlinear control systems. It did not contain any new results for hybrid systems. The main contribution of the current chapter when compared to the paper by Grossman and Larson is that it does present conditions for existence of a realization by hybrid systems. Moreover, the class of hybrid systems studied in this chapter is more general and closer to what is generally understood as hybrid systems than the one in Grossman's and Larson's paper.

The approach to realization theory adopted in this chapter bears more resemblance to [29]. In particular, the general theory of coalgebra systems, as it is presented in Subsection 8.6.1 of this chapter is very similar to what was presented in [29], except that there the input space was assumed to be a Hopf-algebra, as opposed to our framework where the inputs are simply bialgebras (the latter is more general). The latter difference is not a very important one, most of the constructions can be done in a similar way. Note however that [29] dealt only with abstract systems corresponding to nonlinear systems.

Note that linear and bilinear hybrid systems are special cases of analytic hybrid systems studied in this chapter. The conditions for existence of a (bi)linear hybrid system realization presented in Chapter 7 imply the conditions derived in this chapter, thus the results of the current chapter are consistent with the previous ones.

Let us present an informal summary of the main results of the chapter.

- An input-output map has a realization by a hybrid system if and only if it has a hybrid Fliess-series expansion and the corresponding convergent generating series has a realization by a formal hybrid system such that all the readout maps and vector fields are convergent.

- A convergent generating series is a map, which maps sequences of discrete events and input symbols to continuous and discrete outputs. Such a map has a realization by a formal hybrid system, if it has a realization by a hybrid coalgebra system of a certain type (CCPI hybrid coalgebra system).

- We define the *Lie-rank* and *strong Lie-rank* of a map mapping input sequences to outputs. We will prove that if a map has a CCPI hybrid coalgebra system realization (equivalently it has a formal hybrid system realization), then its Lie-rank is finite. If its strong Lie-rank is finite, then it has a hybrid coalgebra realization which is very similar to a CCPI hybrid coalgebra realization We will prove that an input-output map cannot have a CCPI hybrid coalgebra realization (formal hybrid system realization), dimension of which is smaller than the Lie-rank of the map. We will also present a hybrid system, which can not be realized by a system, dimension of which equals the Lie-rank of the input-output map.

The outline of the chapter is the following. Section 8.2 settles the notation and terminology used in the chapter. Section 8.3 presents the necessary results and terminology on formal power series and coalgebras. The reader might postpone reading this section until Section 8.5. Section 8.4 discusses the notion of hybrid Fliess-series expansion and characterises the input-output maps of hybrid systems in terms of Fliess-series expansion. Section 8.5 presents the relationship between local realization and formal realization problem. Section 8.6 presents the conditions for existence of a formal hybrid system realization.

The material of this chapter is based on [57] and it is a joint work of the author and Jean-Baptiste Pomet.

8.2 Notation and Terminology

For a finite set Σ denote by $\mathbb{R} < \Sigma^* >$ the set of all finite formal linear combinations of words over Σ. That is, a typical element of $\mathbb{R} < \Sigma^* >$ is of the form $\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_k w_k$, where $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ and $w_1, \dots, w_k \in \Sigma^*$. It is easy to see that $\mathbb{R} < \Sigma^* >$ is a vector space. Moreover, we can define a linear associative multiplication on $\mathbb{R} < \Sigma^* >$, by $(\sum_{i=1}^{N} \alpha_i w_i)(\sum_{j=1}^{M} \beta_j v_j) = \sum_{i=1}^{N} \sum_{j=1}^{M} \alpha_i \beta_j w_i v_j$. The element ϵ which we will identify with 1 is the neutral element with respect to multiplication. It is easy to see that $R < \Sigma^* >$ is an algebra with the multiplication defined above.

In this chapter we will deal only with realizations of *one single input-output map*. Therefore, we will use a special notation to denote Moore-automata and hybrid system realizations of a family of input-output maps consisting of one single input-output map.

Let Γ be a finite set, O be the set of discrete outputs. Let $f : \Gamma^* \to O$ be an input-output map. Assume that $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ is a Moore-automaton and $q_0 \in Q$. By abuse of terminology we will denote by $(\mathcal{A}, q_0)$ the Moore-automaton realization $(\mathcal{A}, \zeta_{q_0})$ such that $\zeta_{q_0} : f \mapsto q_0$ and $dom(\zeta_{q_0}) = \{f\}$. A map $S : (\mathcal{A}, q_0) \to (\mathcal{A}^{'}, q_0^{'})$ will denote the Moore-automaton morphism $S : (\mathcal{A}, \zeta_{q_0}) \to (\mathcal{A}^{'}, \zeta_{q_0^{'}})$ where $dom(\zeta_{q_0}) = dom(\zeta_{q_0^{'}}) = \{f\}$ for some $f : \Gamma^* \to O$.

Similarly, if $H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, f_q, h_q)_{q \in Q}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\})$ is a hybrid system without guards, $f \in F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ and $h_0 \in \mathcal{H}$, then the pair (H, h_0) will denote the hybrid system realization (H, μ_{h_0}) such that $\mu_{h_0} : f \mapsto h_0$ and $dom(h_0) = \{f\}$. We will denote by $T : (H, h_0) \rightarrow (H^{'}, h_0^{'})$ the hybrid system morphism $T : (H, \mu_{h_0}) \rightarrow (H^{'}, \mu_{h_0^{'}})$ where $dom(\mu_{h_0}) = dom(\mu_{h_0^{'}}) = \{f\}$ for some $f \in F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$.

In this paper we will be primarily concerned with the local realization problem of analytic hybrid systems without guards. As a further simplification we will restrict attention to the following class of analytic hybrid systems.

Definition 18. *We will call an analytic hybrid system*

$$H = (\mathcal{A},\mathcal{U},\mathcal{Y},(\mathcal{X}_q, f_q, h_q)_{q \in Q}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\})$$

nicely analytic input-affine hybrid system (abbreviated as NHS) *if the following holds.*

- $\mathcal{U} = \mathbb{R}^m$ *for some* $m \geq 0$
- $\mathcal{Y} = \mathbb{R}^p$ *for some* $p \geq 0$
- *For each* $q \in Q$*, the vector field* $f_q(x, u)$ *is of the form*

$$f_q(x, u) = g_{q,0}(x) + \sum_{j=1}^{m} g_{q,j}(x) u_j$$

  where $g_{q,j}$*,* $j = 0, \ldots, m$ *are analytic maps.*

- *There exists a collection* $\{x_q \in \mathcal{X}_q \mid q \in Q\}$ *of continuous states, such that for each* $q \in Q$

$$\forall \gamma \in \Gamma : R_{\delta(q,\gamma),\gamma,q}(x_q) = x_{\delta(q,\gamma)}$$

We will use the following short-hand notation for such systems

$H = (\mathcal{A}, (\mathcal{X}_q, g_{q,j}, h_{q,i})_{q \in Q, j=0,\ldots,m, i=1,\ldots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, \{x_q\}_{q \in Q})$ *where for each* $q \in Q$*,* $i = 1, \ldots, p$ *the maps* $h_{q,i} : \mathcal{X}_q \rightarrow \mathbb{R}$ *are the coordinate maps of* $h_q(x) = (h_{q,1}(x), \ldots, h_{q,p}(x))^T$*.*

Let

$$H_i = (\mathcal{A}^i, (\mathcal{X}^i_q, g^i_{q,j}, h^i_{q,i})_{q \in Q^i, j=0,\ldots,m, i=1,\ldots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q^i, \gamma \in \Gamma\}, \{x^i_q\}_{q \in Q})$$

$(i = 1, 2)$ be two NHS's. A NHS morphism from $T = (T_D, T_C) : H_1 \rightarrow H_2$ is a hybrid morphism $T : (H_1, \mu_1) \rightarrow (H_2, \mu_2)$ such that $\mu_i = (q_i, x_{q_i})$ and for all $q \in Q^1$, $T_C(x^1_q) = x^2_{T_D}(q)$.

Even for the case of NHS systems, the realization problem is still too difficult to solve. That is why we will be interested in the local realization problem. That is, we will be interested in finding a NHS which realizes the restriction of the specified input-output map to small enough times and small enough inputs.

Consider the set $PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T$ and define the topology generated by the following collection of open sets $\{V_K \mid K \in \mathbb{R}, K > 0\}$, where $V_K = \{(u,(\gamma_1,t_1)\cdots(\gamma_k,t_k),t_{k+1}) \mid (\sum_{j=1}^{k+1} t_j) \cdot ||u||_{\sum_{j=1}^{k+1} t_j,\infty} < K\}$. Notice that for any open subset U in this topology it holds that $(u,(\gamma_1,0)\cdots(\gamma_k,0),0) \in U$ for all $\gamma_1,\dots,\gamma_k \in \Gamma, k \geq 0$. In the rest of the chapter we will tacitly assume that all topological statements about the set $PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T$ refer to the topology defined above.

We will say that an NHS
$H = (\mathcal{A},(\mathcal{X}_q,g_{q,j},h_{q,i})_{q\in Q,j=0,\dots,m,i=1,\dots,p},\{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\},\{x_q\}_{q\in Q})$
is a *local realization* of an input-output map $f \in F(PC(T,\mathcal{U})\times(\Gamma\times T)^*\times T, \mathcal{Y}\times O)$ if there exist an open set $U \subseteq PC(T,\mathcal{U})\times(\Gamma\times T)^*\times T$ such that for some $(q,x_q) \in \mathcal{H}$,

$$\forall (u,w,t) \in \mathcal{U} : f(u,w,t) = \upsilon_H((q,x_q),u,w,t)$$

8.3 Algebraic Preliminaries

The goal of this section is to give a brief overview of the algebraic notions used in this chapter and to fix the notation and terminology. The material presented in this section is standard. The reader is strongly encouraged to consult the references provided in the text for further details. Subsection 8.3.1 presents a summary on formal power series in finitely many commuting variables. Subsection 8.3.2 presents the necessary preliminaries on Sweedler-type coalgebras. In this chapter in general, and throughout this section in particular we will assume that the reader is familiar with such basic algebraic notions as ring, algebra, ideal, module etc. The reader is referred to any textbook in this subject, for example [85].

8.3.1 Preliminaries on Formal Power Series

The goal of this subsection is to present a very short overview of the main properties of formal power series in commuting variables. For a more detailed exposition the reader should consult [85].

Consider the set $\mathbb{N}^n$ and define addition on this set as follows. If $\alpha = (\alpha_1,\dots,\alpha_n)$ and $\beta = (\beta_1,\dots,\beta_n)$, then let $\alpha+\beta = (\alpha_1+\beta_1,\alpha_2+\beta_2,\dots,\alpha_n+\beta_n)$. The ring of formal power series $\mathbb{R}[[X_1,\dots,X_n]]$ in commuting variables $X_1,X_2,\dots,X_n$ is defined as the $\mathbb{R}$ vector space of formal infinite sums $S = \sum_{\alpha\in\mathbb{N}^n} S_\alpha X^\alpha$, where $X^\alpha = X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_n^{\alpha_n}$ for $\alpha = (\alpha_1,\dots,\alpha_n)$. Addition, multiplication are defined by

$$(\sum_{\alpha\in\mathbb{N}^n} S_\alpha X^\alpha) + (\sum_{\alpha\in\mathbb{N}^n} T_\alpha X^\alpha) = \sum_{\gamma\in\mathbb{N}^n} (S_\gamma + T_\gamma) X^\gamma$$

and

$$(\sum_{\alpha\in\mathbb{N}^n} S_\alpha X^\alpha) \cdot (\sum_{\alpha\in\mathbb{N}^n} T_\alpha X^\alpha) = \sum_{\gamma\in\mathbb{N}^n} (\sum_{\alpha+\beta=\gamma} S_\alpha T_\beta) X^\gamma$$

Multiplication by scalar is defined as $a(\sum_{\alpha\in\mathbb{N}^n} S_\alpha X^\alpha) = \sum_{\alpha\in\mathbb{N}^n} aS_\alpha X^\alpha$. The neutral element for addition is $\sum_{\alpha\in\mathbb{N}^n} S_\alpha X^\alpha$, with $S_\alpha = 0$ for all $\alpha \in \mathbb{N}^n$. The neutral element for multiplication is $\sum_{\alpha\in\mathbb{N}^n} S_\alpha X^\alpha$ with $S_{(0,0,\ldots,0)} = 1$ and $S_\alpha = 0$ for all other $\alpha \in \mathbb{N}$. The latter element will be denoted simply by 1. It is easy to see that $\mathbb{R}[[X_1,\ldots,X_n]]$ forms an algebra with the operations above. For each $\alpha \in \mathbb{N}^n$ let $deg(\alpha) = \sum_{j=1}^n \alpha_i$. For each $n \in \mathbb{N}$ define the ideal $I_n = \{\sum_{\alpha\in\mathbb{N}^n} S_\alpha X^\alpha \mid S_\alpha = 0 \text{ for all } \alpha \in \mathbb{N}^n, deg(\alpha) \leq n\}$. We define the Zariski topology on $\mathbb{R}[[X_1,\ldots,X_n]]$ as the topology generated by the open sets $f + I_n$ for $f \in \mathbb{R}[[X_1,\ldots,X_n]]$ and $n \in \mathbb{N}$. A map $D : \mathbb{R}[[X_1,\ldots,X_n]] \to \mathbb{R}[[Y_1,\ldots,Y_m]]$ is said to be continuous if it it continuous with respect to the Zariski topology. A map $D : \mathbb{R}[[X_1,\ldots,X_n]] \to \mathbb{R}$ is said to be continuous, if it is continuous as a map between topological spaces, where $\mathbb{R}[[X_1,\ldots,X_n]]$ is considered with the Zariski topology and $\mathbb{R}$ is considered with the discrete topology Recall that if A, B are two $\mathbb{R}$ algebras, then a linear map $f : A \to B$ is called a *derivation*, if the Leibniz-rule holds. That is, $f(ab) = af(b) + bf(a)$. If $f(ab) = f(a)f(b)$, then we will call f an algebra morphism.

Denote by A the ring $A = \mathbb{R}[[X_1,\ldots,X_n]]$. Denote by D_i, $i = 1,\ldots,n$ the continuous derivations $D_i : A \to \mathbb{R}$ such that $D_i(X_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. Denote by 1_A^* the map $1^* : A \to \mathbb{R}$ such that $1_A^*(\sum_{\alpha\in\mathbb{N}^n} a_\alpha X^\alpha) = a_{(0,0,\ldots,0)}$. It is well-known ([85]) that 1_A^* is a continuous algebra morphism. Denote by $\frac{d}{dX_i}$, $i = 1,\ldots,n$ the ith partial derivative of the ring $A = \mathbb{R}[[X_1,\ldots,X_n]]$. That is, $\frac{d}{dX_i} : A \to A$ is a continuous derivation such that $\frac{d}{dX_i}(X_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$ The set of all continuous derivations $A \to A$ forms an A module and any continuous derivation $D : A \to A$ can be written as $D = \sum_{j=1}^n S_i \frac{d}{dX_i}$, where $S_i \in A$. Notice that for any continuous derivation $D : A \to A$ the map $1_A^* \circ D : A \to A$ defines a continuous derivation to $\mathbb{R}$. It is also well-known that $D_i = 1_A^* \circ \frac{d}{dX_i}$ for all $i = 1,\ldots,n$. For each $k \in \mathbb{N}$ denote by $\frac{d^k}{dX_i^k}$ the maps $\underbrace{\frac{d}{dX_i} \circ \cdots \circ \frac{d}{dX_i}}_{k-times} : A \to A$, If $k = 0$ the we assume that $\frac{d}{dX_i}^0(h) = h$, i.e., $\frac{d}{dX_i}^0$ is the identity map. For each $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n$ define the map $\frac{d}{dX}^\alpha$ as $\frac{d}{dX_1}^{\alpha_1} \circ \frac{d}{dX_2}^{\alpha_2} \circ \cdots \circ \frac{d}{dX_n}^{\alpha_n} : A \to A$.

8.3.2 Preliminaries on Sweedler-type Coalgebras

The goal of this subsection is to give a very short introduction to the field of coalgebras, bialgebras. Readers for whom this is the first encounter with the field are strongly encouraged to consult the book [74].

Let k be a field of characteristic 0, for our purposes the reader can assume that $k = \mathbb{R}$. Recall the notion of *a tensor product of two vector spaces* and recall that the tensor product of A and B is denoted by $A \otimes B$. A tuple (C, δ, ϵ) is called a *coalgebra* if C is a k-vector space, $\delta : C \to C \otimes C$ and $\epsilon : C \to k$ are k-linear maps such that a number of properties hold. Before describing these properties we will have to introduce some additional notation. Notice that for each $c \in C$, $\delta(c) = \sum_{i=1}^{m} c_{i,1} \otimes c_{i,2}$ such that $c_{i,1}, c_{i,2} \in C$, $i = 1, \ldots, m$. We will use the following notation $\delta(c) = \sum c_{(1)} \otimes c_{(2)}$ to denote the situation above. If we refer to $c_{(1)}$ or to $c_{(2)}$ we will always mean $c_{1,i}$ or $c_{2,i}$ respectively. Although this notation is definitively confusing at the first sight, this is the convention widely adopted in the field of coalgebras and in fact it does help to write and read formal statements concerning coalgebras. We require the following conditions to hold for coalgebras. For each $c \in C$, if $\delta(c) = \sum_{i=1}^{m} c_{i,1} \otimes c_{i,2}$, then

$$\sum_{i=1}^{m} c_{i,1} \otimes \delta(c_{i,2}) = \sum_{i=1}^{m} \delta(c_{i,1}) \otimes c_{i,2} \in C \otimes C \otimes C$$

and

$$c = \sum_{i=1}^{m} \epsilon(c_{i,1}) c_{i,2} = \sum_{i=1}^{m} \epsilon(c_{i,2}) c_{i,1}$$

The first condition is referred to as *coasscociativity.* The second condition says that ϵ has the *counit property.* If in addition, for each $c \in C$,

$$\delta(c) = \sum_{i=1}^{m} c_{i,1} \otimes c_{i,2} = \sum_{i=1}^{m} c_{i,2} \otimes c_{i,1}$$

then we will say that (C, δ, ϵ) is *cocommutative.* The map δ will be referred to as the *comultiplication* and the map ϵ will be referred to as the *counit.*

A map T is said to be a coalgebra map from coalgebra (C, δ, ϵ) to coalgebra $(B, \delta^{'}, \epsilon^{'})$ if $T : C \to B$ is a linear map such that $\epsilon^{'} = \epsilon \circ T$ and $(T \otimes T) \circ \delta = \delta^{'} \circ T$, where $T \otimes T : C \otimes C \ni c_1 \otimes c_2 \mapsto T(c_1) \otimes T(c_2)$.

In the sequel we will denote a coalgebra (C, δ, ϵ) simply by C and if T is a coalgebra map from (C, δ, ϵ) to (B, δ, ϵ) we will write $T : C \to B$ and we will state that T is a coalgebra map.

Recall that a k-vector space A with k-linear maps $M : A \otimes A \to A$ and $u : k \to A$ is called an *algebra* if M defines an associative multiplication and $u(1)$ defines the unit element. That is, for each $a, b, c \in A$, $M(a, M(b, c)) = M(M(a, b), c)$ and $M(a, u(1)) = M(u(1), a) = a$. If in addition M defines a commutative multiplication, that is, $M(a, b) = M(b, a)$ for all $a, b \in A$, then we will say that A is a *commutative algebra*. As usual in mathematics, we will write ab instead of $M(a, b)$ and 1 instead of $u(1)$ if the maps M and u are clear from the context. All the notions we are going to use for algebras such as ideals, maximal ideals, etc. are the standard ones, the reader can consult [85].

For any k-vector space V denote by V^* the linear dual of it, that is, $V^* = \{f : V \to k \mid f \text{ is a linear map}\}$.

It is easy to see that if C is a coalgebra, then the vector space C^* is an algebra with the multiplication and unit defined as follows. For each $c_1^*, c_2^* \in C^*$ let

$$M(c_1^*, c_2^*)(c) = \sum_{i=1}^{m} c_1^*(c_{i,1}) c_2^*(c_{i,2})$$

where $\delta(c) = \sum_{i=1}^{m} c_{i,1} \otimes c_{i,2}$. Just to let the reader appreciate the usefulness of the notation for the result of comultiplication the expression for the multiplication above can be written as $M(c_1^*, c_2^*)(c) = \sum c_1^*(c_1) c_2^*(c_2)$. Going back to defining the algebra structure on C^*, we will define the unit u as follows. For each $s \in k$ let $u(s)(c) = s\epsilon(c)$. It is not difficult to see that u can be identified with ϵ^* and $M = \delta^* \circ i$, where $i : C^* \otimes C^* \to (C \otimes C)^*$ is the natural inclusion defined by $i(c_1^* \otimes c_2^*)(c) = c_1^*(c) c_2^*(c)$ for all $c_1^*, c_2^* \in C^*$, $c \in C$. If C is a cocommutative coalgebra, then C^* is a commutative algebra. If $f : C \to D$ is a coalgebra map, then $f^* : D^* \to C^*$ is an algebra map, where $f^*(d^*)(c) = d^*(f(c))$ for all $d^* \in D^*$ and $c \in C$. That is, f^* is the usual dual map of f, as it is usually defined in linear algebra.

Notice that if $(C, \delta_C, \epsilon_D)$ and $(D, \delta_D, \epsilon_D)$ are coalgebras, then $C \otimes D$ has a natural coalgebra structure $(C \otimes D, \delta^{'}, \epsilon^{'})$, where $\delta^{'}(c \otimes d) = \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{i,1} \otimes d_{j,1}) \otimes (c_{i,2} \otimes d_{j,2}) \in (C \otimes D) \otimes (C \otimes D)$ and $\epsilon^{'}(c \otimes d) = \epsilon_C(c)\epsilon_D(d)$. with the assumption that $c, d \in C$, $\delta_C(c) = \sum_{i=1}^{m} c_{i,1} \otimes c_{i,2}$ and $\delta_D(d) = \sum_{j=1}^{n} d_{j,1} \otimes d_{j,2}$.

Similarly, if A is an algebra, then $A \otimes A$ has a natural algebra structure $(A \otimes A, M^{'}, u^{'})$ where $M^{'}((a \otimes b), (a^{'} \otimes b^{'}) = (aa^{'} \otimes bb^{'})$ and $u^{'}(1) = u(1) \otimes u(1)$. It is easy to see that the ground field k has a natural algebra and coalgebra structure.

We will say that $(C, \delta, \epsilon, M, u)$ is a *bialgebra* if (C, δ, ϵ) is a coalgebra, (C, M, u) is an algebra, δ, ϵ are algebra morphisms and M, u are coalgebra morphisms. Here, we assumed that $C \otimes C$ has the natural algebra and coalgebra structure inherited from C, see the discussion above.

If C is a coalgebra, then a subspace $J \subseteq C$ is called *coideal* if $\delta(J) = J \otimes C + C \otimes J$ and $J \subseteq \ker \epsilon$. A subspace $D \subseteq C$ is called *subcoalgebra* if $\delta(D) \subseteq D \otimes D$. If J is a coideal of C, the the quotient space C/J admits a natural coalgebra structure, such that the canonical projection $\pi : C \ni c \mapsto [c] \in C/J$ is a coalgebra map. Conversely, if $f : C \to D$ is a coalgebra map, then $\ker f$ is a coideal and $C/\ker f$ is isomorphic to $\mathrm{Im} f$ as a coalgebra.

Recall the duality between algebras and coalgebras. For any coalgebra C and any subspace $D \subseteq C$, denote by $D^{\perp}$ the annihilator $D^{\perp} = \{c^* \in C^* \mid \forall d \in D : c^*(d) = 0\} \subseteq C^*$. Conversely, for any subspace $A \subseteq C^*$ denote by $A^{\perp} = \{c \in C \mid \forall a \in A : a(c) = 0\}$. Then it follows that for any subspace $D \subseteq C$, $(D^{\perp})^{\perp} = D$. If D is a subcoalgebra of C, then $D^{\perp}$ is an ideal in C^*. If $A \subseteq C^*$, then $A^{\perp}$ is a coideal in C. It is also easy to see that $(C/A^{\perp})^*$ is isomorphic to $A \subseteq (A^{\perp})^{\perp}$.

For a coalgebra C an element $g \in C$ such that $\delta(g) = g \otimes g$ and $\epsilon(g) = 1$ will be called *of group-like element of* C. The set of all group-like elements of C will be denoted by $G(C)$. An element $p \in C$ will be called *primitive* if $\delta(p) = g \otimes p + p \otimes g$ for some group-like element $g \in G(C)$ and $\epsilon(p) = 0$. The set of all primitive elements will be denoted by $P(C)$. A subcoalgebra $D \subseteq C$ is called *simple* if D does not contain any proper subcoalgebra, i.e. if $S \subseteq D$ is a subcoalgebra, then either $S = \{0\}$ or $S = D$. The coalgebra C is called *pointed* if every simple coalgebra D of C is of dimension one. That is, C is pointed if every simple subcoalgebra D of C is of the form $D = \{\alpha g \mid \alpha \in k\}$ for some group-like element $g \in G(C)$. A coalgebra C is called *irreducible*, if for every pair of subcoalgebras $S, D \subseteq C$, $S \cap D \neq \{0\}$, unless either $S = \{0\}$ or $D = \{0\}$. If C is pointed irreducible, then it follows that C has a unique group-like element g, i.e. $G(C) = \{g\}$ and for any subcoalgebra $\{0\} \neq D \subseteq C$, $g \in D$. If C is cocommutative, then $C = \bigoplus_{i \in I} C_i$ such that C_i is an irreducible subcoalgebra of C and there is no irreducible subcoalgebra of C properly containing C_i. Such C_is will be called *irreducible components of* C. Thus, an irreducible component of a coalgebra C is a subcoalgebra $D \subseteq C$ such that for each irreducible subcoalgebra $S \subseteq C$, if $D \subseteq S$, then $S = D$. If $f : C \to D$ is a algebra morphism, then $f(G(C)) \subseteq G(D)$ and $f(P(C)) \subseteq P(D)$. Moreover, if f is surjective, then $f(G(C)) = G(D)$. It also holds that if C is pointed irreducible, then $f(C)$ is pointed irreducible too.

Let A, B be algebras and let C be a coalgebra and consider a linear map $\psi : C \otimes A \to B$. We will say that ψ is a *measuring* , if for all $c \in C$, $a, b \in A$, $\psi(c \otimes ab) = \sum_{i=1}^{n} \psi(c_{i,1} \otimes a)\psi(c_{i,2} \otimes b)$ where $\delta(c) = \sum_{i=1}^{n} c_{i,1} \otimes c_{i,2}$.

Let V be a k-vector space and define the *cofree commutative pointed irreducible coalgebra* $B(V)$ as the cocommutative pointed irreducible coalgebra for which the following holds.

- There exists a linear map $\pi : B(V) \to V$

- If C is a cocommutative pointed irreducible coalgebra, $C^{+} = \ker \epsilon$ and $f : C^{+} \to V$ is a linear map, then there exists a unique coalgebra map $F : C \to B(V)$ such that $\pi \circ F|_{C^{+}} = f$.

It is known that $B(V)$ exists for each vector space V and $P(B(V)) = V$. Moreover, for each cocommutative pointed irreducible coalgebra C there exists a unique injective coalgebra $\pi : C \to B(P(C))$ such that $\pi|_{P(C)} : P(C) \to P(C)$ is the identity map. It is also known that if $k = \mathbb{R}$ and $\dim V = n < +\infty$ then the dual $B(V)^{*}$ of V is isomorphic to the algebra of formal power series $\mathbb{R}[[X_1, \dots, X_n]]$ in n commuting variables (in fact, it holds for any field k of characteristic zero that $B(V)^{*} \cong k[[X_1, \dots, X_n]]$).

8.4 Input-output Maps of Nicely Nonlinear Hybrid Systems

Recall from classical nonlinear systems theory [32, 83] that state and output trajectories of nonlinear analytic input-affine control systems admit a representation in terms of iterated integrals. A similar statement remains true for hybrid systems too. In order to state the the existence of such a representation formally, we will need to introduce some additional notation and terminology.

We will start with defining the concept of hybrid convergent generating series and hybrid Fliess-series expansions. Notice, that we already defined a concept called hybrid convergent series and a concept called hybrid Fliess-series expansion in Section 7.2. The concepts which were defined in Section 7.2 are special cases of the concepts which we will define below. *In the rest of the chapter, unless stated otherwise, if we speak of hybrid convergent generating series and hybrid Fliess-series expansion, then we will always mean the objects defined below, not the objects defined in Section 7.2.*

8.4.1 Hybrid Convergent Generating Series

Recall from Section 2.6 the notions of abstract generating series and iterated integrals. That is, for each $u = (u_1, \dots, u_k) \in \mathcal{U}$ denote

$$d\zeta_j[u] = u_j, j = 1, 2, \dots, m, \quad d\zeta_0[u] = 1$$

Denote the set $\{0, 1, \dots, m\}$ by Z_m. For each $j_1 \cdots j_k \in \mathrm{Z}_m^{*}, j_1, \cdots, j_k \in \mathrm{Z}_m$, $k \geq 0, t \in T$, $u \in PC(T, \mathcal{U})$ define

$$V_{j_1 \cdots j_k}[u](t) = 1 \text{ if } k = 0$$

For all $k > 1$, let

$$V_{j_1 \cdots j_k}[u](t) = \int_0^t d\zeta_{j_k}[u(\tau)] V_{j_1,\ldots,j_{k-1}}[u](\tau) d\tau$$

For each $w_1, \ldots, w_k \in \mathrm{Z}_m^*$, $(t_1, \cdots, t_k) \in T^k$, $u \in PC(T, \mathcal{U})$ define

$$V_{w_1,\ldots,w_k}[u](t_1, \ldots, t_k) = V_{w_1}(t_1)[u]$$
$$V_{w_2}(t_2)[\mathrm{Shift}_1(u)] \cdots V_{w_k}[\mathrm{Shift}_{k-1}(u)](t_k)$$

where $\mathrm{Shift}_i(u) = \mathrm{Shift}_{\sum_1^i t_i}(u)$, $i = 1, 2, \ldots, k-1$.

Assume that Z_m and Γ are disjoint sets. Denote by $\widetilde{\Gamma}$ the set $\Gamma \cup \mathrm{Z}_m$. Then any $w \in \widetilde{\Gamma}^*$ is of the form $w = w_1\gamma_1 \cdots w_k\gamma_k w_{k+1}$, where $\gamma_1, \ldots, \gamma_k \in \Gamma$, $w_1, \ldots, w_{k+1} \in \mathrm{Z}_m^*$, $k \geq 0$.

Definition 19. *A map $c : \widetilde{\Gamma}^* \to \mathcal{Y}$ is called a* hybrid generating convergent series *on $\widetilde{\Gamma}^*$ if there exists $K, M > 0$, $K, M \in \mathbb{R}$ such that for each $w \in \widetilde{\Gamma}^*$,*

$$||c(w)|| < |w|!KM^{|w|}$$

where $||.||$ is some norm in $\mathcal{Y} = \mathbb{R}^p$.

The notion of generating convergent series is related to the notion of convergent power series from [32, 83].

Let $c : \widetilde{\Gamma}^* \to \mathcal{Y}$ be a generating convergent series. For each $u \in PC(T, \mathcal{U})$ and $s = (\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*, t_{k+1} \in T$ define the series

$$F_c(u, s, t_{k+1}) = \sum_{w_1,\ldots,w_{k+1} \in \mathrm{Z}_m^*} c(w_1\gamma_1 \cdots \gamma_k w_{k+1}) \times$$
$$\times V_{w_1,\ldots,w_{k+1}}[u](t_1, \ldots, t_{k+1})$$

It is easy to see that for small enough $t_1, \ldots, t_{k+1} \in T$, u the series above is absolutely convergent. More precisely, let $T_s = \sum_{j=1}^{k+1} t_j$ and $||u||_{S,\infty} = \sup\{||u(t)|| \mid t \in [0, S]\}$

Lemma 38. *If $T_s \cdot ||u||_{T_s,\infty} < (2M(1+m))^{-1}$, then $F_c(u, s, t_{k+1})$ is absolutely convergent.*

Define the set

$$\begin{aligned} &dom(F_c) = \{(u, s, t) \in PC(T, \mathcal{U}) \times (\Gamma \times T)^* \times T \mid \\ &s = (\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*, k \geq 0, \\ &(t + \sum_{j=1}^k t_j) \cdot ||u||_{t + \sum_{j=1}^k t_j, \infty} < (2M(1+m))^{-1}\} \end{aligned}$$

Then for each $(u, s, t) \in dom(F_c)$ the series $F_c(u, s, t)$ is absolutely convergent and thus we can define the map

$$F_c : dom(F_c) \ni (u, s, t) \mapsto F_c(u, s, t)$$

By an argument similar to the classical one, one could show that c defines F_c locally uniquely. Recall the definition of the topology of $PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T$ from Section 8.2. It is easy to see that for any hybrid convergent generating series c the set $dom(F_c)$ is open in that topology.

Lemma 39. *If there exists a non-empty open subset of $U \subseteq domF_c \cap domF_d$, such that $\forall s \in U : F_c(s) = F_d(s)$, i.e. $F_c = F_d$ on the open set U, then $c = d$.*

It is also easy to see that F_c is a *causal map* that is, if $u, v \in PC(T,\mathcal{U})$ and $u(s) = v(s)$ for all $s \in [0, S]$, then $F_c(u, w, t_{k+1}) = F_c(v, w, t_{k+1})$ for all $w = (\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*$, $t_{k+1} \in T$, $k \geq 0$ such that $\sum_{j=1}^{k+1} t_j \leq S$.

8.4.2 Input-output Maps of Nonlinear Hybrid Systems

Consider a NHS

$$H = (\mathcal{A}, (\mathcal{X}_q, g_{q,j}, h_{q,i})_{q \in Q, j=0,\ldots,m, i=1,\ldots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, \{x_q\}_{q \in Q}).$$

For each $q \in Q$ denote by A_q the algebra of real-valued real analytic functions of $\mathcal{X}_q$, i.e.

$$A_q = C^{\omega}(X_q) = \{f : \mathcal{X}_q \to \mathbb{R} \mid f \text{ is real analytic } \}$$

It is well-known that each vector field $X \in T\mathcal{X}_q$ induces a map $X : A_q \to A_q$, defined by $X(f)(x) = \sum_{j=1}^{n_q} X_j(x) \frac{df}{dx_j i}(x)$, where X is assume to be of the form $X = \sum_{j=1}^{n} X_j \frac{d}{dx_i}$. In particular, each vector field $g_{q,j}$, $j \in \mathbb{Z}_m$ induces a map

$$g_{q,j} : A_q \to A_q$$

Assume that $w = j_1 \cdots j_k \in \mathbb{Z}_m^*$, $j_1, \ldots, j_k \in \mathbb{Z}_m$, $k \geq 0$. Then define the map $g_{q,w} : A_q \to A_q$ by

$$g_{q,w} = g_{q,j_1} \circ g_{q,j_2} \circ \cdots \circ g_{q,j_k}$$

Notice that each reset map $R_{\delta(q,\gamma),\gamma,q}$ induces a map $R^*_{\delta(q,\gamma),\gamma,q} : A_{\delta(q,\gamma)} \to A_q$ defined by

$$R^*_{\delta(q,\gamma),\gamma,q}(f)(x) = f(R_{\delta(q,\gamma),\gamma,q}(x))$$

Thus, for any $s = w_1\gamma_1 \cdots \gamma_k w_{k+1} \in \widetilde{\Gamma}^*$, such that $w_1, \ldots, w_k \in \mathbb{Z}_m^*$, $\gamma_1, \ldots, \gamma_k \in \Gamma$, we get that the map

$$\begin{aligned} G_{H,q,s} &= g_{q_0,w_1} \circ R^*_{q_1,\gamma_1,q_0} g_{q_1,w_2} \circ \cdots \\ &\cdots \circ R^*_{q_{k+1},\gamma_k,q_k} \circ g_{q_k,w_{k+1}} : A_{q_k} \to A_q \end{aligned} \tag{8.1}$$

is well-defined, where $q_i = \delta(q, \gamma_1 \cdots \gamma_i)$, $i = 0, \dots, k$, $q_0 = q$. In particular, if $h \in A_{q_k}$, and $x \in \mathcal{X}_q$, then $G_{H,q,s}(h)(x) \in \mathbb{R}$.

More precisely, define for any $(q,x) \in \mathcal{H}$ define the generating series $c_{q,x} : \widetilde{\Gamma}^* \to \mathcal{Y}$, as follows

$$\begin{aligned} & c_{q,x}(s) = G_{H,q,s}(h_{q_k})(x) = \\ & g_{q_0,w_1} \circ R^*_{q_0,\gamma_1,q_0} g_{q_1,w_2} \circ \cdots \\ & \cdots \circ R^*_{q_{k+1},\gamma_k,q_k} \circ g_{q_k,w_{k+1}}(h_{q_k})(x) \end{aligned} \tag{8.2}$$

where $s = w_1\gamma_1 \cdots w_k\gamma_k w_{k+1}$, $w_1, \dots, w_{k+1} \in \mathrm{Z}_m^*$, $\gamma_1, \dots, \gamma_k \in \Gamma$, $\delta(q, \gamma_1 \cdots \gamma_i) = q_i$, $i = 0, \dots, k$. It is easy to see that $c_{q,x}$ is a generating convergent power series. Using arguments similar to the standard ones for nonlinear state affine systems, one gets that

Lemma 40. *Using the notation above, for each $(q,x) \in \mathcal{H}$, and for each $(u, (\gamma_1, t_1) \cdots (\gamma_k, t_k), t_{k+1}) \in dom(F_{c_{q,x}})$,*

$$\begin{aligned} & y_H((q,x), u, (\gamma_1, t_1) \cdots (\gamma_k, t_k), t_{k+1}) = \\ & \sum_{w_1, \dots, w_{k+1} \in \mathrm{Z}_m^*} c_{q,x}(w_1 z_1 \cdots w_k z_k w_{k+1}) \times \\ & \times V_{w_1, \dots w_k}[u](t_1, \dots, t_{k+1}) = \\ & = F_{c_{q,x}}(u, (\gamma_1, t_1) \cdots (\gamma_k, t_k), t_{k+1}) \end{aligned} \tag{8.3}$$

Let $f \in F(PC(T, \mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ be an input-output map. Denote by f_D the map $\Pi_O \circ f$ and denote by f_C the map $\Pi_{\mathcal{Y}} \circ f$.

Definition 20. *We will say that f admits a* local hybrid Fliess-series expansion*, if and only if*

- *The map f_D depends only on Γ^*, that is,*

  $$f_D(u, (s, \underline{t}), t) = f_D(v, (s, \underline{\tau}), \tau)$$

  for all $u, v \in PC(T, \mathcal{U})$, $\tau, t \in T$, $\underline{\tau}, \underline{t} \in T$, $s \in \Gamma^*$. Thus, the map f_D can be viewed as a map $f_D : \Gamma^* \to O$.

- *There exists a generating convergent series $c_f : \widetilde{\Gamma}^* \to \mathcal{Y}$ and an open subset $U \subseteq dom(F_{c_f})$ such that*

  $$\forall (u, w, t) \in U : f_C(u, w, t) = F_{c_f}(u, w, t)$$

It is clear that if $f \in F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ has a local realization by a NHS, then f admits a local hybrid Fliess-series expansion. Notice that the values of the corresponding convergent generating series can be directly obtained from f by feeding in piecewise-constant inputs and taking derivatives with respect to inputs and switching times of the input function.

Since a hybrid convergent generating series determines F_c uniquely, we get the following.

Theorem 43. *Let $H = (\mathcal{A}, (\mathcal{X}_q, g_{q,j}, h_{q,i})_{q \in Q, j=0,\ldots,m, i=1,\ldots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, \{x_q\}_{q \in Q})$ be a NHS and let $f \in F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ be an input-output map. Then H is a local realization of f if and only if f has a hybrid Fliess-series expansion and there exists $q \in Q$ such that*

- $\forall w \in \Gamma^* : f_D(w) = \lambda(q, w)$
- *For all $s = w_1\gamma_1 \cdots \gamma_k w_{k+1} \in \widetilde{\Gamma}^*$, $\gamma_1, \ldots, \gamma_k \in \Gamma$, $w_1, \ldots, w_{k+1} \in Z_m^*$, $k \geq 0$*

$$\begin{aligned} & c_f(w_1\gamma_1 w_2 \cdots \gamma_k w_{k+1}) = \\ & g_{q_0,w_1} \circ R^*_{q_1,\gamma_1,q_0} \circ g_{q_1,w_2} \cdots \\ & \cdots \circ R^*_{q_k,\gamma_k,q_{k-1}} \circ g_{q_k,w_{k+1}}(h_{q_k})(x_q) \end{aligned} \tag{8.4}$$

where $q_i = \delta(q, \gamma_1 \cdots \gamma_i)$, $i = 0, \ldots, k$.

8.5 Formal Realization Problem For Hybrid Systems

As it was seen in the previous section, the local realization problem for nonlinear hybrid systems is equivalent to finding a particular representation for the hybrid convergent generating series corresponding to the input-output map. Notice that this representation was formulated completely in terms of reset maps and vector fields around a point and it is completely determined by the formal power series expansion of the analytic maps and vector fields involved.

More precisely, consider a hybrid system

$$H = (\mathcal{A}, (\mathcal{X}_q, g_{q,j}, h_{q,i})_{q \in Q, j=0,\ldots,m, i=1,\ldots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, \{x_q\}_{q \in Q})$$

For each $q \in Q$, $j \in Z_m$ consider the formal power series expansion of $R_{\delta(q,\gamma),\gamma,q}$, $g_{q,j}$ and $h_{q,i}$. That is for each $q \in Q$ consider the ring of formal power series $A^f_q = \mathbb{R}[[X_1, \ldots, X_{n_q}]]$ in commuting variables $X_1, \ldots, X_{n_q}$. Then the formal power series expansion of

$$h_{q,i}(x) = \sum_{\alpha \in \mathbb{N}^{n_q}} h_{q,i,\alpha}(x - x_q)^\alpha$$

$i = 1, \ldots, p$ around x_q results in a formal power series $h^f_{q,i} \in \mathbb{R}[[X_1, \ldots, X_{n_q}]]$, defined by $h^f_{q,i} = \sum_{\alpha \in \mathbb{N}^{n_q}} h_{q,i,\alpha} X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_{n_q}^{\alpha_{n_q}}$. Similarly, if $g_{q,j} = \sum_{i=1}^{n_q} g_{q,j,i} \frac{d}{dx_i}$, then take the Taylor-series expansion of each $g_{q,j,i}$ around x_q, i.e.

$$g_{q,j,i}(x) = \sum_{\alpha \in \mathbb{N}} g_{q,j,i,\alpha}(x - x_q)^\alpha$$

and define the following continuous derivation on $\mathbb{R}[[X_1, \ldots, X_{n_q}]]$,

$$g^f_{q,j} = \sum_{i=1}^{n_q} g_{q,j,i} \frac{d}{dX_i}$$

where

$$g_{q,j,i} = \sum_{\alpha \in \mathbb{N}^{n_q}} g_{q,j,i,\alpha} X^\alpha$$

Finally, assume that $R_{\delta(q,\gamma),\gamma,q} - x_{\delta(q,\gamma)}$ is of the form

$$R_{\delta(q,\gamma),\gamma,q} - x_{\delta(q,\gamma)} = (R_{\delta(q,\gamma),\gamma,q,1}, \ldots, R_{\delta(q,\gamma),\gamma,q,n_{\delta(q,\gamma)}})^T$$

Each map $R_{\delta(q,\gamma),\gamma,q,i}$, $i = 1, \ldots, n_{\delta(q,n)}$ is an analytic map with values in $\mathbb{R}$ and thus around x_q it admits a Taylor series expansion of the form

$$R_{\delta(q,\gamma),\gamma,q,i}(x) = \sum_{\alpha \in \mathbb{N}^{n_q}} r_{\delta(q,\gamma),\gamma,q,i,\alpha}(x - x_q)^\alpha$$

Notice that $R_{\delta(q,\gamma),\gamma,q}(x_q) - x_{\delta(q,\gamma)} = 0$ and thus $r_{\delta(q,\gamma),\gamma,q,i,(0,0,\ldots,0)} = R_{\delta(q,\gamma),\gamma,q,i}(x_q) = 0$.

Define the formal power series

$$R^f_{\delta(q,\gamma),\gamma,q,i} = \sum_{\alpha \in \mathbb{N}^{n_q}} r_{\delta(q,\gamma),\gamma,q,i,\alpha} X^\alpha$$

Let $r = \delta(q, \gamma)$ and $A_r = \mathbb{R}[[X_1, \ldots, X_{n_r}]]$ and define the continuous algebraic map

$$R^{f,*}_{r,\gamma,q} : A^f_r \to A^f_q$$

by $R^{f,*}_{r,\gamma,q}(X_i) = R^f_{r,\gamma,q,i}$ for all $i = 1, \ldots, n_r$, and for all

$$S = \sum_{\alpha \in \mathbb{N}^{n_r}} S_\alpha X^\alpha \in A^f_r = \mathbb{R}[[X_1, \ldots, X_{n_r}]]$$

let

$$R^{f,*}_{r,\gamma,q}(S) = \sum_{\alpha \in \mathbb{N}^{n_r}} S_\alpha (R^f_{r,\gamma,q,1})^{\alpha_1} (R^f_{r,\gamma,q,2})^{\alpha_2} \cdot \cdots \cdot (R^f_{r,\gamma,q,n_r})^{\alpha_{n_r}} \in \mathbb{R}[[X_1, \ldots, X_{n_q}]]$$

It is easy to see that $R^{f,*}_{r,\gamma,q}$ is indeed an algebra morphism.

It is easy to see that with the notation above, the following holds

$$g_{q_0,w_1} \circ R^{*}_{q_0,\gamma_1,q_0} g_{q_1,w_2} \circ \cdots\cdots \circ R^{*}_{q_{k+1},\gamma_k,q_k} \circ g_{q_k,w_{k+1}}(h_{q_k})(x_{q_0}) =$$
$$g^f_{q_0,w_1} \circ R^{f,*}_{q_0,\gamma_1,q_0} g^f_{q_1,w_2} \circ \cdots\cdots \circ R^{f,*}_{q_{k+1},\gamma_k,q_k} \circ g^f_{q_k,w_{k+1}}(h^f_{q_k})(0)$$

for all $\gamma_1, \ldots, \gamma_k \in \Gamma$, $w_1, \ldots, w_{k+1} \in \mathrm{Z}^*_m$, $k \geq 0$, $q \in Q$. Here the following notation was used, $q_i = \delta(q_0, \gamma_1 \cdots \gamma_i)$, $i = 0, \ldots, k$, and for all $w = j_1 \cdots j_l, j_1, \ldots, j_l \in \mathrm{Z}^*_m$,

$$g^f_{q,w} = g^f_{q,j_1} \circ g^f_{q,j_2} \circ \cdots \circ g^f_{q,j_l} : A^f_q \to A^f_q$$

That is, a necessary condition for existence of a realization of an input-output map by hybrid systems is that the corresponding hybrid convergent generating series can be represented as composition of derivations and algebra maps on finitely many formal power series rings.

This observation, which will be discussed more formally on a later stage, motivates the introduction of the *formal realization problem*.

Definition 21 (Formal Hybrid System). *A tuple*

$$F = (\mathcal{A}, (A_q, g_{q,j}, h_{q,i})_{q \in Q, j=0,\ldots,m, i=1,\ldots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, q_0)$$

is called a formal hybrid system, where

- $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ *is a Moore-automaton*
- *For each* $q \in Q$, $A_q = \mathbb{R}[[X_1, \ldots, X_{n_q}]]$ *is the ring of formal power series in commuting variable* $X_q, \ldots, X_{n_q}$
- *For each* $q \in Q$, $j \in \mathcal{Z}_m$,
$$g_{q,j} : A_q \to A_q$$
defines a continuous derivation on A_q, *i.e.* $g_{q,j} = \sum_{i=1}^{n_q} g_{q,j,i} \frac{d}{dX_i}$, *where* $g_{q,j,i} \in A_q$, $i = 1, \ldots n_q$.
- *For each* $q \in Q$, $i = 1, \ldots, p$, $h_{q,i} \in A_q$
- *For each* $q \in Q$, $\gamma \in \Gamma$,
$$R_{\delta(q,\gamma),\gamma,q} : A_{\delta(q,\gamma)} \to A_q$$
is a continuous algebra morphism, i.e. it is uniquely defined by its values
$$R_{\delta(q,\gamma),\gamma,q}(X_i) \in A_q$$
and the free coefficient of $R_{\delta(q,\gamma),\gamma,q}(X_i)$ *is zero, i.e.*
$$1^*_{\mathbb{R}[[X_1,\ldots,X_{n_q}]]}(R_{\delta(q,\gamma),\gamma,q}(X_{i,\delta(q,\gamma)})) = 0$$

- $q_0 \in Q$ – *the initial state*

The *dimension* of the formal hybrid system F is defined as

$$\dim F = (card(Q), \sum_{q \in Q} n_q)$$

It is easy to see that ϕ_q is a continuous algebra morphism and $\phi_q(T) = T(0)$.

A morphism between two formal hybrid systems
$F_1 = (\mathcal{A}^1, (A^1_q, g^1_{q,j}, h^1_{q,i})_{q \in Q^1, j=0,\dots,m, i=1,\dots,p}, \{R^1_{\delta(q,\gamma),\gamma,q} \mid q \in Q^1, \gamma \in \Gamma\}, q_0)$
and
$F_2 = (\mathcal{A}^2, (A^2_q, g^2_{q,j}, h^2_{q,i})_{q \in Q^2, j=0,\dots,m, i=1,\dots,p}, \{R^2_{\delta(q,\gamma),\gamma,q} \mid q \in Q^2, \gamma \in \Gamma\}, q_0)$
is a pair $T = (T_D, (T_{C,q})_{q \in Q})$, where $T_D : \mathcal{A}_1 \to \mathcal{A}_2$ is an automaton morphism such that $T_D(q^1_0) = q^2_0$ and for each $q \in Q^1$, $T_{C,q} : A^2_{T_D(q)} \to A^1_q$ such that

- For all $q \in Q^1$, $j \in \mathbb{Z}_m$,

$$T_{C,q} \circ g^2_{T_D(q),j} = g^1_{T_D(q),j} \circ T_{C,q}$$

- For all $q \in Q^1$, $i = 1, \dots, p$,

$$h^1_{q,i} \circ T_{C,q} = h^2_{T_D(q),i}$$

- For all $q \in Q^1$, $\gamma \in \Gamma$,

$$T_{C,q} \circ R^2_{\delta^2(T_D(q),\gamma),\gamma,T_D(q)} = R^2_{\delta^1(q,\gamma),\gamma,q} \circ T_{C,\delta(q,\gamma)}$$

The pair T is said to be an formal hybrid system isomorphism, if T_D is an automaton isomorphism and for all $q \in Q^1$ $T_{C,q}$ is an algebra isomorphism. The fact that T is a formal hybrid system morphism from F_1 to F_2 will be denoted by $T : F_1 \to F_2$.

Let $F = (\mathcal{A}, (A_q, g_{q,j}, h_{q,i})_{q \in Q, j=0,\dots,m, i=1,\dots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, q_0)$ be a formal hybrid system. For each $q \in Q$, $w = j_1 j_2 \cdots j_l$, $j_1, \dots, j_l \in \mathrm{Z}_m$,$l \geq 0$, denote by $g_{q,w}$ the following map

$$g_{q,w} = g_{q,j_1} \circ g_{q,j_2} \circ \cdots \circ g_{q,j_k} : A_q \to A_q$$

For each $q \in Q$, $v = w_1 \gamma_1 w_2 \cdots \gamma_k w_{k+1} \in \widetilde{\Gamma}^*$, $k \geq 0$, $\gamma_1, \dots, \gamma_k \in \Gamma$, $w_1, \dots, w_{k+1} \in \mathrm{Z}^*_m$, denote by $G_{H,q,v}$ the map

$$\begin{aligned} G_{H,q,v} &= g_{q_0,w_1} \circ R_{q_1,\gamma,q_0} \circ g_{q_1,w_2} \circ \cdots \\ &\cdots \circ R_{q_k,\gamma_k,q_{k-1}} \circ g_{q_k,w_{k+1}} : A_{q_k} \to A_q \end{aligned}$$

where $q_i = \delta(q, \gamma_1 \cdots \gamma_i)$, $i = 0, \dots, k$.

Consider the maps $f_c : \widetilde{\Gamma}^* \rightarrow \mathbb{R}^p$ and $f_d : \Gamma^* \rightarrow O$. We will say the the formal hybrid system $F = (\mathcal{A}, (A_q, g_{q,j}, h_{q,i})_{q \in Q, j=0,\dots,m, i=1,\dots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, q_0)$ *is a realization of* (f_d, f_c) , if for all $s \in \widetilde{\Gamma}^*$:

$$\begin{aligned} \forall w \in \Gamma^* : f_d(w) = \lambda(q_0, w) \\ \forall v \in \widetilde{\Gamma}^* : f_c(v) = \phi_{q_0} \circ G_{H,q_0,v}(h_{q_e}) \end{aligned} \tag{8.5}$$

where $q_e = \delta(q_0, \gamma_1 \cdots \gamma_k)$ such that $v = w_1 \gamma_1 \cdots \gamma_k w_{k+1}$, $\gamma_1, \dots, \gamma_k \in \Gamma$, $w_1, \dots, w_{k+1} \in \mathrm{Z}_m^*$, $k \geq 0$.

Consider the hybrid system $H = (\mathcal{A}, (\mathcal{X}_q, g_{q,j}, h_{q,i})_{q \in Q, j=0,\dots,m, i=1,\dots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, \{x_q\}_{q \in Q})$. Let $q_0 \in Q$. Recall the discussion at the beginning of this section. Using the notation there, we define the *formal hybrid system associated with* $(H, (q_0, x_{q_0}))$ as follows

$$F_H = (\mathcal{A}, (A_q^f, g_{q,j}^f, h_{q,i}^f)_{q \in Q, j \in \mathrm{Z}_m, i=1,\dots,p}, \{R_{\delta(q,\gamma),\gamma,q}^{f,*} \mid q \in Q, \gamma \in \Gamma\}, q_0)$$

It is easy to see that F_H is indeed a formal hybrid system. Theorem 43 has the following easy consequence

Lemma 41. *Let $f \in F(PC(T, \mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ and assume that f has a hybrid Fliess-series expansion. Then $(H, (q_0, x_0))$ is a realization of f if and only if the formal hybrid system F_H is a realization of (f_D, c_f).*

8.6 Solution of the Formal Realization Problem

This section presents the conditions for existence of a hybrid formal power series realization. The outline of the section is the following. Recall from Section 8.3 the notion of coalgebra. Recall that there exists a natural duality between algebras and coalgebras. We will exploit this duality by looking at formal hybrid systems defined on algebras instead of coalgebras. Recall from Section 8.3 that rings of formal power series in commuting variables have a natural characterisation as duals of certain coalgebras with very special property. This observation will enable us to use coalgebra theory for finding necessary and sufficient conditions for existence of a formal hybrid system realization. It will also enable us to place our results in the wider context of nonlinear realization theory. The outline of the section is the following. Subsection 8.6.1 presents the notion of algebra and coalgebra systems, discusses duality between the two concepts and presents the basic results on realization theory of such systems. Subsection 8.6.2 presents the concept of hybrid coalgebra and algebra systems and presents the basic results on realization theory of hybrid coalgebra system. Subsection 8.6.4 discusses the relationship between hybrid coalgebra systems and formal hybrid systems. It states the equivalence between formal hybrid systems and the so called CCPI hybrid coalgebra systems. Finally, Subsection 8.6.5 discusses criteria for existence of a CCPI hybrid coalgebra realization. Because of the equivalence between formal hybrid systems and CCPI hybrid coalgebra systems these criteria are also criteria for existence of a formal hybrid system realization.

8.6.1 Algebra and Coalgebra Systems

In this subsection we will present the definition of a control system on algebra and coalgebra, and we will show that there exists a duality between the two concepts. The idea of such an abstract definition is not really new, it appeared earlier in other works [29, 64]. For example, Sontag's definition of a k-system is very closely related to what will be presented below. This abstract representation will enable us to present the results in a clear and conceptual way. First we will present the definition of algebra and coalgebra systems and discuss the duality between them. After that we will present some basic results on realization theory of coalgebra systems. The latter is very similar to the results presented in [29].

Definition of Algebra and Coalgebra Systems

Let H be a bialgebra, which will be referred to as the *bialgebra of inputs*.

Definition 22. *A tuple* $\Sigma_a = (A, H, \psi, \phi, J, \mu)$ *is called a* control system on an algebra *if*

- A *is a commutative algebra.*
- J *is an arbitrary set.*
- $\psi : A \otimes H \to A$ *is a measuring such that* $\psi(a \otimes h_1 h_2) = \psi(\psi(a \otimes h_2) \otimes h_1)$ *for all* $h_1, h_2 \in H$. *The map* ψ *will be called the dynamics of* Σ_a.
- $\phi : A \to \mathbb{R}$ *is a an algebra map, called the readout map.*
- $\mu : J \to A$ *specifies the initial state.*

We will say that a family of maps $\Psi = \{y_j : H \to \mathbb{R} \mid j \in J\}$ is *realized* by an algebra system Σ_a if

$$\forall h \in H, \forall j \in J : y_j(h) = \phi \circ \psi(\mu(j) \otimes h)$$

That is, one can think of Σ_a as an automaton, inputs of which are elements of H (which itself is a monoid) and the state-space A has an algebra structure. This point of view leads to the quite natural question of what if we took a coalgebra as a state-space instead of an algebra. Below we will do exactly that.

Definition 23. *A tuple $\Sigma_c = (C, H, \psi, \phi, J, \mu)$ is called a* control system on a coalgebra *if*

- *C is a cocommutative coalgebra.*
- *J is an arbitrary set.*
- *$\psi : C \otimes H \to C$ is a coalgebra map $\psi(a \otimes h_1 h_2) = \psi(\psi(a \otimes h_1) \otimes h_2)$ for all $h_1, h_2 \in H$. The map ψ will be called the dynamics of Σ_a.*
- *$\phi \in G(C)$, i.e. ϕ is a group-like element of C*
- *$\mu : J \to C^*$ is the family of readout maps.*

We say that Σ_c realizes a family of maps $\Psi = \{y_j : H \to \mathbb{R} \mid j \in J\}$ if

$$\forall h \in H, \forall j \in J : y_j(h) = \mu_j \circ \psi(\phi \otimes h)$$

Consider the tuple

$$\Sigma_c^* = (C^*, \psi^*, \phi^*, \mu^*)$$

where $\psi^* : C^* \otimes H \to C^*$, $\phi^* : C^* \to \mathbb{R}$ and $\mu^* : J \to C^*$ are defined as follows. The map ψ^* is defined as

$$\psi^*(c^* \otimes h)(c) = c^*(\psi(c \otimes h))$$

For all $j \in J$, $\mu^*(j) = \mu(j)$ and $\phi^*(c^*) = c^*(\phi)$ for all $c^* \in C^*$. It is easy to see that Σ_c^* is a control system defined on an algebra, moreover, for all $j \in J$, $h \in H$,

$$\phi^* \circ \psi^*(\mu^*(j) \otimes h) = \mu(j) \circ \psi(\phi \otimes h)$$

Thus, Σ_c^* is a realization of Φ if and only if Σ_c is a realization of Φ.

Realization Theory for Algebra and Coalgebra Systems

Let $\Sigma = (C, H, \psi, \phi, J, \mu)$ be a coalgebra system. Define the maps $R_\Sigma : H \to C$ by

$$R_\Sigma(h) = \psi(h \otimes \phi) \text{ for all } h \in H$$

It is easy to see that R_Σ is a coalgebra map. We will call C *reachable* if R_Σ is surjective.

For each $h \in H, j \in J$ consider the map $O_{h,j} : C \ni c \mapsto \mu_j \circ \psi(c \otimes h) \in \mathbb{R}$. Notice that $O_{h,j} \in C^*$. Define the set $L_\Sigma = \{O_{h,j} \mid j \in J, h \in H\} \subseteq C^*$ and let $A_\Sigma = Alg(L_\Sigma)$ be the subalgebra of C^* generated by L_Σ (i.e., A_Σ is the smallest subalgebra of C^* which contains L_Σ). We will call L_Σ the *set of observables of* Σ and A_Σ the *algebra of observables of* Σ. Let $A_\Sigma^\perp = \{c \in C \mid \forall f \in A_\Sigma : f(c) = 0\}$. It follows that $A_\Sigma^\perp$ is a coideal. We will call Σ *observable* if $A_\Sigma^\perp = \{0\}$.

Consider a coalgebra system $\Sigma = (C, H, \psi, \phi, J, \mu)$. Define the system $\Sigma_r = (\mathrm{Im}R_\Sigma, \psi_r, \phi_r, J, \mu_r)$ as follows. Let $\psi_r = \psi|_{\mathrm{Im}R_\Sigma \otimes H}$,i.e. ψ_r is the restriction of ψ to $\mathrm{Im}R_\Sigma \otimes H$, $\phi_r = \phi = \psi(\phi \otimes 1) \in \mathrm{Im}R_\Sigma$, $\mu_r(j) = \mu(j)|_{\mathrm{Im}R_\Sigma}$, i.e. $\mu_r(j)$ is the restriction of the map $\mu(j) : C \to \mathbb{R}$ to $\mathrm{Im}R_\Sigma$. It is easy to see that Σ_r is a well-defined coalgebra system, and it is reachable. Moreover, if Σ is a realization of $\Psi = \{f_j \mid j \in J\}$, then Σ_r is a realization of Ψ too.

Consider again a coalgebra system

$$\Sigma = (C, H, \psi, \phi, J, \mu)$$

Notice that if $c \in A_\Sigma^\perp$, then $\psi(h \otimes c) \in A_\Sigma^\perp$. Indeed, for all $d \in C$,

$$O_{h_1,j_1} O_{h_2,j_2} \cdots O_{h_k,j_k}(d) = \sum O_{h_1,j_1}(d_{(1)}) \cdots O_{h_k,j_k}(d_{(k)})$$

where $\delta^k(d) = \sum d_{(1)} \otimes \cdots \otimes d_{(k)}$. Taking into account that ψ is a coalgebra map and thus

$$\underbrace{\psi \otimes \cdots \psi}_{k-times}(\delta^k(c \otimes h)) = \delta^k(\psi(c \otimes h))$$

we can derive the following

$$\delta^k(\psi(h \otimes c) = \sum \psi(x_{(1)} \otimes h_{(1)}) \otimes \cdots \otimes \psi(c_{(k)} \otimes h_{(k)}).$$

Hence we get that

$$O_{h_1,j_1} \cdots O_{h_k,j_k}(\psi(c \otimes h) = \sum O_{h_1 h_{(1)},j_1}(c_{(1)}) \cdots O_{h_k h_{(k)},j_k}(c_{(k)}).$$

Notice that $A_\Sigma^\perp$ is a coideal, thus for each

$$\delta^k(c) = \sum c_{(1)} \otimes c_{(2)} \otimes \cdots \otimes \otimes c_{(k)} \in$$
$$\in \sum_{j=1}^{k} \underbrace{C \otimes \cdots \otimes C}_{j-1\text{–times}} \otimes A_\Sigma^\perp \otimes \underbrace{C \otimes \cdots \otimes C}_{k-j+1\text{–times}}$$

Thus, it follows that for each term of the form $c_{(1)} \otimes \cdots \otimes c_{(k)}$ there exists a $i = 1, \ldots, k$, such that $c_{(i)} \in A_\Sigma^\perp$. Then it follows that $O_{h_1,j_1} \cdots O_{h_k,j_k}(\psi(h \otimes c)) = 0$ for all $h_1, \ldots, h_k \in H, j_1, \ldots, j_k \in J$. Thus, it follows that $\psi(h \otimes c) \in A_\Sigma^\perp$.

Define the coalgebra system $\Sigma_O = (C_O, H, \psi_O, \phi_O, J, \mu_O)$ as follows. Let $C_O = C/A_\Sigma^\perp$, and for each $c \in C$ denote by $[c]$ the equivalence class generated by taking the quotient, i.e. $[c] = [d] \iff c - d \in A_\Sigma^\perp$. Define the map $\psi_O : H \otimes C_O \to C_O$ by $\psi_O([c] \otimes h) = [\psi_O(c \otimes h)]$. If $c - d \in A_\Sigma^\perp$, then $\psi(h \otimes (c-d)) = \psi(c \otimes h) - \psi(d \otimes h) \in A_\Sigma^\perp$. Thus, ψ_O is well-defined. For each $j \in J$ let $\mu_O(j)([c]) = \mu(j)(c)$. Notice that for each $j \in J$, $\mu_j = O_{1,j}$ and thus $A_\Sigma^\perp \subseteq \ker \mu_j$. Hence, $\mu_O : J \to C_O^*$ is well-defined. Finally, let $\phi_O = [\phi]$. It is easy to see that Σ_O is a well-defined coalgebra system,and it is observable. Moreover, if Σ is a realization of $\Psi = \{f_j \in H^* \mid j \in J\}$, then Σ_O is a realization of Ψ too. If Σ is reachable, then Σ_O is a reachable too.

We will call a coalgebra system Σ_m realizing Ψ *a minimal realization* if for any reachable coalgebra system Σ realizing Ψ there exists a surjective coalgebra system morphism $T : \Sigma \to \Sigma_m$.

Let $\Sigma_a = (\mathcal{A}, H, \psi, \phi, J, \mu)$ be an algebra system. Define the map $O_{\Sigma_a} : A \to H^*$ as follows. For each $h \in H$ let $O_{\Sigma_a}(a)(h) = (\phi \circ \psi(a \otimes h))$. It is easy to see that O_{Σ_a} is an algebra map. We will say that Σ_a is *observable*, if O_{Σ_a} is injective.

Define the algebra R_{Σ_a} as the subalgebra of A generated by the set $L_{\Sigma_a} = \{\psi(h \otimes \mu(j)) \mid h \in H, j \in J\}$. We will call Σ_a *reachable* if $R_{\Sigma_a} = A$.

Consider a coalgebra system Σ_c. It is easy to see that the dual $R^*_{\Sigma_c} : C^* \to H^*$ of R_{Σ_c} equals $O_{\Sigma_c^*}$. It is also easy to see that $A_{\Sigma_c} = R^*_{\Sigma_c}$. It follows that if Σ_c is reachable then Σ_c^* is observable, and if Σ_c^* is reachable, then Σ_c is observable.

Denote by M the multiplication map on H. That is, $M : H \otimes H \to H$, $M(s \otimes v) = sv$. Since H is a bialgebra, the map M is a coalgebra map, moreover, $M(v, M(s, x)) = M(v, sx)$. Let $\Psi = \{f_j \in H^* \mid j \in J\}$ be an indexed set of elements of H^*. Define the map $\mu_\Psi : J \to H^*$ by $\mu_\Psi(j) = f_j$. Define the coalgebra control system

$$\Sigma_\Psi = (H, H, M, 1, J, \mu_\Psi)$$

It is easy to see that Σ_Ψ is indeed a coalgebra system, moreover, Σ_Ψ is a realization of Ψ, since $f_j(h) = f_j(M(1 \otimes h)) = \mu_\Psi(j) \circ M(1 \otimes h)$ for all $j \in J$. We will call Σ_Ψ the *cofree realization* of Ψ. We will denote the algebra of observables of Σ_Ψ by A_Ψ. That is, $A_{\Sigma_\Psi} = A_\Psi$. Notice that $A_\Psi \subseteq H^*$. It is easy to see that for Σ_Ψ the maps $O_{h,j}$ are of the form $O_{h,j}(v) = f_j(vh) = R_h f_j$. If $\Sigma = (C, H, \psi, \phi, J, \mu)$ is a realization of Ψ, then it is easy to see that $T_\Sigma : H \to C$, $T_\Sigma(h) = \psi(\phi \otimes h)$ defines a coalgebra system morphism $T_\Sigma : \Sigma \to \Sigma_\Psi$. Notice that $T_\Sigma = R_\Sigma$, i.e., T_Σ equals the reachability map.

Dually, the algebra system Σ_Ψ^* will be called the *free realization of f*, and if $\Sigma = \Sigma_c^*$ where Σ_c is a coalgebra system realizing Φ, then $T^*_{\Sigma_c^*} = T^*_\Sigma$ defines an algebra system morphism $T^*_\Sigma : \Sigma \to \Sigma^*_\Psi$. Notice that the realization Σ_Ψ is reachable and thus Σ^*_Ψ is observable.

The results discussed below will play an important role in the construction of a minimal coalgebra system realization. Let $\Sigma_i = (C_i, H, \psi_i, \phi_i, J, \mu_i)$, $i = 1, 2$ be two coalgebra systems and assume that $T : \Sigma_1 \to \Sigma_2$ is a coalgebra map. Define the system $\Sigma_T = (C_1/\ker T, H, \psi_T, \phi_T, J, \mu_T)$, where, $\psi_T([x] \otimes h) = [hx]$, $\phi_T = [\phi_1]$ and $\mu_T(j)([x]) = \mu_1(j)(x)$ for all $x \in C_1$, where $[x]$ denotes the equivalence class generated by taking the quotient by $\ker T$, i.e., $[x] = [y] \iff x - y \in \ker T$. It is easy to see that $\ker T \subseteq \ker \mu_1(j)$, for all $j \in J$ and $\ker T \subseteq \ker \psi_1(h, .)$ for each $h \in H$, where $\psi_1(h, .) : C_1 \ni x \mapsto \psi_1(x \otimes h)$. Thus, Σ_T is well defined. Moreover, $T_m : \Sigma_T \to \Sigma_2$ and $T_s : \Sigma_1 \to \Sigma_T$ are injective and surjective coalgebra system morphisms, where $T_m : C_1/\ker T \ni [x] \mapsto Tx \in C_2$ and $T_s : C_1 \ni x \mapsto [x] \in C_1/\ker T$. If T is surjective, then T_m is an isomorphism and thus $T_m^{-1} : \Sigma_2 \to \Sigma_T$ is a well-defined coalgebra system isomorphism.

Below we will state and prove that any set of input/output maps Ψ admits a minimal coalgebra realization.

Theorem 44. *Let $\Psi = \{f_j \in H^* \mid j \in J\}$. Then there always exists a minimal coalgebra system realization of Ψ. A coalgebra system realizing Ψ is minimal if and only if it is reachable and observable.*

Proof. We will sketch the (easy) proof in order to present some constructions, which will be very useful later on. Take the cofree realization Σ_Ψ of Ψ. It is easy to see that Σ_Ψ is reachable. Consider the system $\Sigma_m = (\Sigma_\Psi)_O$. That is, $\Sigma_m = (H/A_\Psi^\perp, H, \widetilde{M}, [1], J, \widetilde{\mu}_\Psi)$ where $\widetilde{M}(h \times [k]) = [hk]$ and $\widetilde{\mu}_\Psi(j)([h]) = f_j(h)$, and $[h]$ denote the equivalence class generated by h with respect to the relation $[h] = [d] \iff h - d \in A_\Psi^\perp$. In fact, $A_\Psi^\perp$ is also an ideal of H. Indeed, if $h \in A_\Psi^\perp$, then for all $k \in H$, $M(k \otimes h) \in A_\Psi^\perp$, since M is the state-transition map of Σ_Ψ. But is means precisely that $A_\Psi^\perp$ is an ideal. Thus, $H/A_\Psi^\perp$ is a bialgebra. Let $\Sigma = (C, H, \psi, \phi, J, \mu)$ be a reachable coalgebra system realization of Ψ. Recall that there exists a coalgebra system morphism $T_\Sigma : \Sigma_\Psi \to \Sigma$, defined as $T_\Sigma = R_\Sigma : H \to C$, i.e., $T_\Sigma(h) = R_\Sigma(h) = \psi(\phi \otimes h)$. It is easy to see that $\ker T_\Sigma \subseteq A_\Psi^\perp$. Indeed, $T_\Sigma(h) = \psi(\phi \otimes h) = 0$ implies that for all $k \in H$, $\mu(j) \circ T_\Sigma(kh) = \mu(j) \circ \psi(\phi \otimes kh) = f_j(kh) = R_h f_j(k) = 0$. Thus, $R_h f_j = 0$, which implies that $O_{k,j}(h) = 0$ for all $k \in H, j \in J$. Since $\ker T_\Sigma$ is a coideal and thus

$$\delta^m(h) = \sum h_{(1)} \otimes \cdots \otimes h_{(m)} \subseteq \sum_{j=1}^{m} H \otimes \cdots \otimes \ker R_\Sigma \otimes \cdots \otimes H$$

, it follows that $O_{k_1,j_1} \cdots O_{k_m,j_m}(h) = 0$ for all $k_1, \ldots, k_m \in H$, $j_1, \ldots, j_m \in J$.

Thus, $H/A_\Psi^\perp = (H/\ker T_\Sigma)/(A_\Psi^\perp/\ker T_\Sigma)$. Thus there exists a surjective coalgebra map $S : H/\ker T_\Sigma \mapsto H/A_\Psi^\perp$ defined by $S(h + T_\Sigma) = h + A_\Psi^\perp$. Recall that there exists a coalgebra system $\Sigma_{T_\Sigma} = (H/\ker T_\Sigma, H, \psi_{T_\Sigma}, \phi_{T_\Sigma}, J, \mu_{T_\Sigma})$ such that $T_m : \Sigma_{T_\Sigma} \to \Sigma$ and $T_s : \Sigma_\Psi \to \Sigma_{T_\Sigma}$ are injective and surjective coalgebra system morphisms respectively and $T_\Sigma = T_m \circ T_s$. If Σ is reachable, then $T_\Sigma = R_\Sigma$ is surjective and thus T_m is a coalgebra system isomorphism.

It is easy to see that S defines a surjective coalgebra system morphism $S : \Sigma_{T_\Sigma} \to \Sigma_m$. In fact, Σ_m is the result of observability reduction of Σ_{T_Σ}. Thus we get that $S \circ T_m^{-1}$ defines a surjective coalgebra system morphism $S \circ T_m^{-1} : \Sigma \to \Sigma_m$. It is easy to see that Σ_m is reachable and observable.

Assume that the coalgebra system Σ is minimal. Then it has to be reachable. Indeed, Σ_m above is reachable and since Σ is minimal, then there exists a surjective $T : \Sigma_m \to \Sigma$. But then $R_\Sigma = T \circ R_{\Sigma_m}$ and since both T and R_{Σ_m} are surjective it follows that R_Σ is surjective, which implies that Σ is reachable. We will argue that T is an isomorphism and thus Σ is also observable. Indeed, notice that $\ker T \subseteq A^{\perp}_{\Sigma_m} = \{0\}$, that is, T is an isomorphism. It implies that Σ is observable, since Σ_m is observable. Thus, we have shown that any minimal realization is reachable and observable and it is isomorphic to Σ_m. Hence, any two minimal coalgebra realizations of Φ are isomorphic. It is left to show that any reachable and observable coalgebra system is minimal. Let Σ be a reachable and observable coalgebra system realizing Ψ. Then there exists a surjective coalgebra system morphism $Z : \Sigma \to \Sigma_m$. Since $\ker Z \subseteq A^{\perp}_{\Sigma} = \{0\}$ we get that Z is a coalgebra system isomorphism and thus Σ is minimal. □

We will call the minimal realization Σ_m from the above proof *canonical minimal realization* and we will denote it by $\Sigma_{\Psi,m}$.

8.6.2 Hybrid Algebra and Coalgebra Systems

The goal of this subsection is to present the notion of hybrid coalgebra and algebra systems. We will start with defining the concept of hybrid (co)algebra systems. After that we will proceed with presenting realization theory for hybrid coalgebra systems.

Definition of Hybrid Algebra Systems and Hybrid Coalgebra Systems

Recall the notation from Section 8.2. Consider the set $\widetilde{\Gamma} = \Gamma \cup \mathrm{Z}_m$. The set $H = \mathbb{R} < \widetilde{\Gamma}^* >$ of all formal linear combinations words over $\widetilde{\Gamma}$ has a natural bialgebra structure defined by

$$\begin{aligned} \delta(\gamma) &= \gamma \otimes \gamma \text{ for all } \gamma \in \Gamma \cup \{1\} \\ \delta(x) &= 1 \otimes x + x \otimes 1 \text{ for all } x \in \mathrm{Z}_m \end{aligned} \tag{8.6}$$

$$
\begin{aligned}
&\delta(w_1 w_2 \cdots w_k) = \delta(w_1)\delta(w_2)\cdots\delta(w_k) \\
&\text{ for all } w_1, \dots, w_k \in \widetilde{\Gamma} \\
&\epsilon(x) = \begin{cases} 1 & \text{if } x \in \Gamma \cup \{1\} \\ 0 & \text{if } x \in \mathrm{Z}_m \end{cases} \\
&\epsilon(w_1 w_2 \cdots w_k) = \epsilon(w_1)\epsilon(w_2)\cdots\epsilon(w_k) \\
&\text{ for all } w_1, \dots, w_k \in \widetilde{\Gamma}, k \geq 0
\end{aligned}
\tag{8.7}
$$

Although H is a bialgebra, it is not a Hopf-algebra. H as a coalgebra is cocommutative pointed coalgebra, but it is not irreducible. It is also easy to see that $G(H) = \gamma \in \Gamma \cup \{1\}$ is the set of group-like elements, and in fact

$$H = \bigoplus_{w \in \Gamma^*} H_w$$

where for all $w = w_1 \cdots w_k$, $k \geq 0$, $w_1, \dots, w_k \in \Gamma$,

$$H_w = \mathrm{Span}\{s_1 w_1 s_2 \cdots w_k s_{k+1} \mid s_1, \dots, s_{k+1} \in \mathrm{Z}_m^*\}$$

It is easy to see that for each $w \in \Gamma^*$ the linear space H_w is in fact a subcoalgebra of H, moreover, H_w is pointed irreducible and cocommutative. It is also easy to see that the map $\psi : H_w \otimes \mathbb{R} < \mathrm{Z}_m^* > \to H_w$, $\psi(v \otimes s) = vs$, $s \in \mathrm{Z}_m^*, v \in H_w$ is well-defined and it is a coalgebra map. Similarly, for each $\gamma \in \Gamma$ the map $\psi_\gamma : H_w \ni s \mapsto s\gamma \in H_{w\gamma}$ is a well-defined coalgebra map.

Consider the pair of maps $f = (f_D, f_C)$, where $f_D : \Gamma^* \to O$ and $f_C : \widetilde{\Gamma}^* \to \mathbb{R}^p$. Consider the maps $f_{C,i} : \widetilde{\Gamma}^* \to \mathbb{R}$, where $f(w) = (f_{C,1}(w), f_{C,2}(w), \dots, f_{C,p}(w))^T$ for each $w \in \widetilde{\Gamma}^*$. Notice that each map $f_{C,i}$ can be uniquely extended to a linear map $\widetilde{f}_{C,i} : H \to \mathbb{R}$. In the sequel we will identify maps $f_{C,i}$ and linear maps $\widetilde{f}_{C,i}$ and we will denote both of them by $f_{C,i}$. Define the *family of input-output maps associated with* f as the following indexed set of maps $\Psi_f = \{f_{C,i} : H \to \mathbb{R} \mid i = 1, \dots, p\}$.

A *hybrid coalgebra system* is a tuple $HC = (\mathcal{A}, \Sigma, q_0)$, where

- $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ is a Moore-automata
- $q_0 \in Q$,
- $\Sigma = (C, H, \psi, \phi, \mu)$ is a coalgebra system, such that
  - $C = \bigoplus_{q \in Q} C_q$, where C_q is a subcoalgebra of C for each $q \in Q$ and C_q is pointed irreducible.
  - $\phi \in C_{q_0}$
  - For each $q \in Q$, $\forall w \in \mathrm{Z}_m^*, \forall z \in C_q : \psi(z \otimes w) \in C_q$ and $\forall \gamma \in \Gamma, \forall z \in C_q :$ $\psi(z \otimes \gamma) \in C_{\delta(q,\gamma)}$

Since for each $q \in Q$, the coalgebra C_q is pointed irreducible, it has a unique group like element which we will denote by ϕ_q. It follows that $\phi = \phi_{q_0}$ and for each $w \in \Gamma$, $q \in Q$, $\phi(w \otimes \phi_q) = \phi_{\delta(q,w)}$. It also follows that C_q precisely coincides with the irreducible component of ϕ_q in C. We know that C is a direct sum of its irreducible components and it follows that C is pointed. Thus, it follows that there is a bijection between irreducible components of C and the coalgebras C_q, $q \in Q$.

A pair of maps $T = (T_D, T_C) : HC_1 \to HC_2$ with $HC_i = (\mathcal{A}_i, \Sigma_i, q_{0,i})$ is called a *hybrid coalgebra system morphism* if $T_D : \mathcal{A}_1 \to \mathcal{A}_2$ is a automata morphism, $T_D(q_{0,1}) = q_{0,2}$, $T_C : \Sigma_1 \to \Sigma_2$ is a coalgebra system morphism such that $T_C(C^1_q) \subseteq C^2_{T_D(q)}$ for all $q \in Q^1$.

A pair of maps $f = (f_D, f_C)$, where $f_D : \Gamma^* \to O$ and $f_C : \widetilde{\Gamma}^* \to \mathbb{R}^p$ is said to be *realized by a hybrid coalgebra system* $HC = (\mathcal{A}, \Sigma, q_0)$ if $(\mathcal{A}, q_0)$ is a realization of f_D and Σ is a realization of Ψ_f.

We will call the hybrid coalgebra system $HC = (\mathcal{A}, \Sigma, q_0)$ *reachable* if $(\mathcal{A}, q_0)$ is reachable and Σ is reachable.

We will say that a hybrid coalgebra system HC which realizes f is *a minimal realization of* f if for any reachable hybrid coalgebra system HC_r such that HC_r realizes f, there exists a surjective $T = (T_D, T_C)$ hybrid coalgebra map $T : HC_r \to HC$.

A *hybrid algebra system* is a tuple $HA = (\mathcal{A}, \Sigma, q_0)$ where

- $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ is a Moore automaton
- $\Sigma = (A, H, \psi, \phi, \mu)$ is an algebra system such that
  - $A = \Pi_{q \in Q} A_q$ and for each $q \in Q$, A_q is a commutative algebra.
  - There exists $\psi_{q,h}, q \in Q$, $h \in \widetilde{\Gamma}$ such that $\forall \gamma \in \Gamma : \psi_{q,\gamma} : A_{\delta(q,\gamma)} \to A_q$ is an algebra map, $\forall j \in \mathrm{Z}_m : \psi_{q,j} : A_q \to A_q$ is a derivation, and for each $\gamma \in \Gamma$,
  $$\psi((a_q)_{q \in Q} \otimes \gamma) = (\psi_q(a_{\delta(q,\gamma)})_{q \in Q})$$
  for each $j \in \mathrm{Z}_m$
  $$\psi((a_q)_{q \in Q} \otimes j) = (\psi_{q,j}(a_q))_{q \in Q}$$

A pair of maps $T = (T_D, T_C)$ is said to be a hybrid algebra system morphism $T : HA_1 \to HA_2$, where $HA_i = (\mathcal{A}_i, \Sigma_i, q_{0,i}), i = 1, 2$, if $T_D : (\mathcal{A}_1, q_{0,1}) \to (\mathcal{A}_2, q_{0,2})$ is an automaton morphism, $T_C : \Sigma_2 \to \Sigma_1$ is an algebra system morphism such that there exists algebra morphisms $T_{C,q} : A^2_{T_D(q)} \to A^1_q$, $q \in Q^1$ such that $T_C((a_q)_{q \in Q^2}) = (T_{C,q}(a_{T_D(q)}))_{q \in Q^1}$.

A pair of map $f = (f_D, f_C)$, where $f_D : \Gamma^* \to O$ and $f_C : \widetilde{\Gamma}^* \to \mathbb{R}^p$ is said to be realized by a a hybrid algebra system $HA = (\mathcal{A}, \Sigma, q_0)$ if $\mathcal{A}$ realizes f_D from initial state q_0 and Σ is a realization of Ψ_f.

It is easy to see that if $HC = (\mathcal{A}, \Sigma_c, q_0)$ is a hybrid coalgebra system, then the dual system $HC^* = (\mathcal{A}, \Sigma_c^*, q_0)$ is a hybrid algebra system. Moreover, HC is a realization of f if and only if HC^* is a realization of f. Moreover, if $T : HC_1 \to HC_2$ is a hybrid coalgebra system morphism, then $T^* : HC_1^* \to HC_2^*$ is a hybrid algebra system morphism.

Notice that a formal hybrid system
$HF = (\mathcal{A}, (A_q, g_{q,j}, h_{q,i})_{q \in Q, j=0,\ldots,m, i=1,\ldots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, q_0)$
can be viewed as a hybrid algebra system $HA_{HF} = (\mathcal{A}, \Sigma_{HF}, q_0)$ such that

$$\Sigma_{HF} = (\Pi_{q \in Q} A_q, H, \psi, \phi, \{1, \ldots, p\}, \mu)$$

where for all $x \in \mathrm{Z}_m$, $\psi_x((a_q)_{q \in Q}) = (f_{q,x}(a_q))_{q \in Q})$, for all $\gamma \in \Gamma$, $\psi_\gamma((a_q)_{q \in Q}) = (b_q)_{q \in Q}$, with $R_{\delta(q,\gamma),\gamma,q)}(a_{\delta(q,\gamma)}) = b_q$, $q \in Q$ and $\psi : (a_q)_{q \in Q} \mapsto 1^*_{A_{q_0}}(a_{q_0})$, $\mu(i) = (h_{q,i})_{q \in Q}$, $i = 1, \ldots p$. It is easy to see that the correspondence $HF \mapsto HA_{HR}$ is one to one. Moreover, $T : HF_1 \to HF_2$ is a formal hybrid system morphism if and only if $T : HA_{HF_1} \to HA_{HF_2}$ is a hybrid algebra morphism.

8.6.3 Realization of hybrid coalgebra systems

The aim of this subsection is to present conditions on existence of a realization by hybrid coalgebra systems. We will look at the realization by a fairly general class of hybrid coalgebra systems. This more abstract approach will enable us to disregard certain irrelevant details. We will also characterise minimal hybrid coalgebra realizations in terms of reachability and observability. We will use the results of this subsection to give necessary and sufficient conditions for existence of a realization by a CCPI hybrid coalgebra system.

Consider a pair of maps $f = (f_D, f_C)$, with $f_D : \Gamma^* \to O$ and $f_C : \widetilde{\Gamma}^* \to \mathbb{R}^p$. Recall the definition of the set $\Psi_f = \{f_{C,i} : H \to \mathbb{R} \mid i = 1, \ldots, p\}$ such that $f_C = (f_{C,1}, \ldots, f_{C,p})^T$. Recall that the maps $f_{C,i}$ are linear and thus belong to the dual H^* of H. Since $\Gamma \subseteq H$, we can define the map $L_w g$ for all $g \in H^*$ by

$$L_w g(h) = g(wh)$$

Define the map $d_f : \Gamma^* \to O \times (H^*)^p$ as follows

$$\forall w \in \Gamma^* : d_f(w) = (f_D(w), (L_w f_{C,i})_{i=1,\ldots p})$$

Denote by $\bar{O}$ the set

$$\bar{O} = O \times (H^*)^p$$

Assume that $HC = (\mathcal{A}, \Sigma, q_0)$ is a hybrid coalgebra system and assume that $\Sigma = (C, H, \psi, \phi, \{1, \ldots, p\}, \mu)$ and $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$. Define the automaton $\bar{\mathcal{A}}_{HC} = (Q, \Gamma, \bar{O}, \delta, \bar{\lambda})$ as follows. Let $\bar{\lambda}(q) = (\lambda(q), (T_{q,j})_{j=1,\ldots p})$, where $T_{q,j} \in H^*$ and $T_{q,j}(h) = \mu(j) \circ \psi(\phi_q \otimes h)$. Here ϕ_q denotes the unique group like element of C_q.

We get the following theorem, which gives a necessary and sufficient condition for HC to be a realization of f.

Theorem 45. *The hybrid coalgebra system $HC = (\mathcal{A}, \Sigma, q_0)$ is a realization of f if and only if $(\bar{\mathcal{A}}_{HC}, q_0)$ is a realization of d_f and Σ is a realization of Ψ_f.*

Sketch of the proof. Assume that HC is a realization of f. Then Σ_f is a realization of Ψ_f and for all $w \in \Gamma^*$, $\lambda(q, w) = f_D(w)$. Notice that $\psi(\phi_q \otimes w) \in C_{\delta(q,w)}$ and that $\psi(\phi_q \otimes w)$ has to be a group-like element, since ψ is a coalgebra morphism. Since $C_{\delta(q,w)}$ has only one group-like element, and that is $\phi_{\delta(q,w)}$, we get that $\psi(\phi_q \otimes w) = \phi_{\delta(q,w)}$. Then it follows that for all $w \in \Gamma^*$, $i = 1, \ldots, p$, $L_w f_{C,i}(h) = f_{C,i}(wh) = \mu(i) \circ \psi(\phi_{q_0} \otimes wh) = \psi(\phi_{\delta(q,w)} \otimes h) = T_{q,i}$, where ϕ_q denotes the unique group-like element of C_q. Thus, $(\bar{\mathcal{A}}_{HR}, q_0)$ is a realization of d_f.

Assume that Σ is a realization of Ψ_f and $(\bar{\mathcal{A}}_{HC}, q_0)$ is a realization of d_f. But then it is easy to see that $(\mathcal{A}, q_0)$ is a realization of f_D and thus HC is a realization of f. □

Above we associated with each hybrid coalgebra system a Moore-automaton and a coalgebra system and we showed that the hybrid coalgebra system is a realization of f if and only if the associated Moore-automaton is a realization of d_f and the associated coalgebra system is a realization of Ψ_f. Below we will show that the converse is also true. Namely, if we have a coalgebra system of a certain type which realizes Ψ_f and a reachable Moore-automaton realization of d_f we will construct a hybrid coalgebra system realizing f. The construction goes as follows.

Let $\Sigma = (C, H, \psi, \phi, \{1, \ldots, p\}, \mu)$ be a coalgebra system such that C is pointed. We will say that Σ is *point-observable*, if $A_{\Sigma}^{\perp} \cap C_0 = \{0\}$, that is, if for some $g, h \in G(C)$, $g - h \in A_{\Sigma}^{\perp}$, then $g = h$. That is, the states belonging to $G(C)$ are distinguishable (observable). In particular, if Σ is observable, then it is point-observable.

Let $\Sigma = (C, H, \psi, \phi, \{1, \ldots, p\}, \mu)$ be a point-observable coalgebra realization of Ψ_f, such that C is pointed. Let $\bar{\mathcal{A}} = (Q, \Gamma, \bar{O}, \delta, \bar{\lambda})$ be a Moore-automaton such that $(\bar{\mathcal{A}}, q_0)$ is a reachable realization of d_f. Define the *hybrid coalgebra system* $HC_{\bar{\mathcal{A}},\Sigma,q_0}$ *associated with* Σ, $(\bar{\mathcal{A}}, q_0)$ as follows.

$$HC_{\bar{\mathcal{A}},\Sigma,q_0} = (\mathcal{A}, \widetilde{\Sigma}, q_0)$$

where

- $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ where $\lambda(q) = o$ if $\bar{\lambda}(q) = (o, \bar{o})$.
- $\widetilde{\Sigma} = (\widetilde{C}, H, \widetilde{\psi}, \widetilde{\phi}, \{1, \dots, p\}, \widetilde{\mu})$ where
  - $\widetilde{C} = \bigoplus_{q \in Q} \widetilde{C}_q$, where for each $q \in Q$, $\widetilde{C}_q$ is defined as follows. Let $H_{\mathcal{A},q} = \bigoplus_{w \in \Gamma^*, \delta(q_0,w)=q} H_w$ and let $\widetilde{C}_q = C_q \cap R_\Sigma(H_{\mathcal{A},q})$ where C_q is the (isomorphic copy of the) irreducible component of C with the unique group-like element ϕ_q defined by $\phi_q = \psi(\phi \otimes w)$, where $w \in \Gamma^*$ such that $\delta(q_0, w) = q$. That is, for each $q \in Q$, $\widetilde{C}_q$ is a subcoalgebra of C_q.
  - With the notation above $\widetilde{\phi} = \phi_{q_0}$
  - The map $\widetilde{\psi} : H \otimes \widetilde{C} \to \widetilde{C}$ is defined as follows. For each $q \in Q$,$c \in \widetilde{C}_q$, $x \in \mathrm{Z}_m$, $\widetilde{\psi}(c \otimes x) = \psi(c \otimes x) \in \widetilde{C}_q$. For each $q \in Q$, $c \in \widetilde{C}_q$, $\gamma \in \Gamma$, $\widetilde{\psi}(c \otimes \gamma) = \psi(c \otimes \gamma) \in \widetilde{C}_{\delta(q,\gamma)}$.
  - For all $j \in J$, the map $\widetilde{\mu}(j) \in \widetilde{C}^*$ is such that for all $q \in Q$, $c \in \widetilde{C}_q$, $\widetilde{\mu}(j)(c_q) = \mu(j)(c_q)$.

Below we will argue that the construction above indeed yields a hybrid coalgebra system.

Lemma 42. *With the notation and assumptions above $HC = HC_{\bar{\mathcal{A}},\Sigma,q_0}$ is a well-defined reachable hybrid coalgebra system which realizes f.*

It is easy to see that with the notation above there exists a coalgebra system morphism $T : \widetilde{\Sigma} \to \Sigma$ such that $T|_{\widetilde{C}_q}(c) = c$ for all $c \in \widetilde{C}_q$, $q \in Q$. Assume that $(\bar{\mathcal{A}}, q_0)$ is a reachable realization of d_f and Σ is a point-observable realization of Ψ_f. Assume that $HC^{'} = (\mathcal{A}^{'}, \Sigma^{'}, q_0^{'})$ is a reachable hybrid coalgebra system realization of f and there exists an automaton morphism $\phi : (\bar{\mathcal{A}}^{'}_{HC^{'}}, q_0^{'}) \to (\bar{\mathcal{A}}, q_0)$ and a coalgebra system morphism $T : \Sigma^{'} \to \Sigma$. Then it follows that there exists a surjective hybrid coalgebra morphism $(\phi, S) : HC^{'} \to HC_{\bar{\mathcal{A}},\Sigma,q_0}$. such that if $\Sigma^{'} = (C^{'}, H, \psi^{'}, \phi^{'}, \{1, \dots, p\}, \mu^{'})$ and $\mathcal{A}^{'} = (Q^{'}, \Gamma, O, \delta^{'}, \lambda^{'})$ then $S|_{C^{'}_q}(c) = T(c) \in C_{\phi(q)}$ for all $q \in Q^{'}$, $c \in C^{'}_q$. Here we assumed that $C^{'} = \bigoplus_{q \in Q^{'}} C^{'}_q$, each $C^{'}_q$ is the coalgebra belonging to the discrete state $q \in Q^{'}$ and $C_{\phi(q)}$ is the coalgebra belonging to the discrete state $\phi(q)$ in $HC_{\bar{\mathcal{A}},\Sigma,q_0}$.

Let Σ be a minimal coalgebra system realization of Ψ_f and let $(\bar{\mathcal{A}}, q_0)$ be a minimal realization of d_f. Assume that $\Sigma = (C, H, \psi, \phi, \{1, \dots, p\}, \mu)$ is such that C is pointed. Then it follows that Σ is observable and $(\bar{\mathcal{A}}, q_0)$ is reachable, thus $HC_{\bar{\mathcal{A}},\Sigma,q_0}$ is well defined. Moreover, if $HC^{'} = (\mathcal{A}^{'}, \Sigma^{'}, q_0^{'})$ is a reachable hybrid coalgebra realization of f, then there exists surjective maps $\phi : (\bar{\mathcal{A}}_{HC^{'}}, q_0^{'}) \to (\bar{\mathcal{A}}, q_0)$ and $T : \Sigma^{'} \to \Sigma$ and thus there exists a surjective hybrid coalgebra map $S : HC^{'} \to HC_{\bar{\mathcal{A}},\Sigma,q_0}$. Thus, we get the following

Lemma 43. *If $(\bar{\mathcal{A}}, q_0)$ is a minimal realization of d_f and Σ is a minimal realization of Ψ_f, then $HC_{\bar{\mathcal{A}},\Sigma,q_0}$ is a minimal realization of f.*

It follows from the standard theory of Moore-automata that d_f had a Moore-automaton realization if and only if $W_{d_f} = \{w \circ d_f \mid w \in \Gamma^*\}$ is a finite set. Define the sets $D_f = \{w \circ f_D \mid w \in \Gamma^*\}$ and $K_f = \{(L_w f_{C,j})_{j=1,\dots,p} \in (H^*)^p \mid w \in \Gamma^*\}$.

Lemma 44. *With the notation above W_{d_f} is finite if and only if K_f is finite and D_f is finite. That is, d_f has a realization by a Moore-automaton if and only if f_D has a realization by a Moore-automaton and K_f is finite.*

Assume that K_f is finite, more precisely, let $K_f = \{q_i = (L_{w_i} f_{C,j})_{j=1,\dots,p} \mid i = 1, \dots N\}$. For each $q_i \in K_f$ define the set $H_{q_i} = \bigoplus_{w\in\Gamma^*,(L_w f_{C,j})_{j=1,\dots,p}=q_i} H_w$. It is easy to see that $H = \bigoplus_{i=1}^N H_{q_i}$. Consider the cofree realization Σ_{Ψ_f} and the minimal coalgebra realization $\Sigma_{\Psi_f,m} = (D, H, \psi, \phi, \{1, \dots, p\}, \mu)$ of Ψ_f where $D = H/A^{\perp}_{\Psi_f}$. There exists a canonical morphism $\pi : H \to D$ which defines a coalgebra system morphism $\pi : \Sigma_{\Psi_f} \to \Sigma_{\Psi_f,m}$. Since π is surjective and H is pointed, it follows that D is pointed. Moreover, it follows that $\Sigma_{\Psi_f,m}$ is observable. In fact, the following holds.

Lemma 45. *With the notation above $D = \bigoplus_{i=1}^N \pi(H_{q_i})$ and $\pi(H_{q_i})$ is pointed irreducible.*

That is, if $(\bar{\mathcal{A}}, q_0)$ is a minimal realization d_f and $\Sigma_{\Psi_f,m}$ is the canonical minimal realization of Ψ_f, then $HC_{\bar{\mathcal{A}},\Sigma_{\Psi_f,m},q_0}$ is a well-defined hybrid coalgebra system realization. Moreover, $HC_{\bar{\mathcal{A}},\Sigma_{\Psi_f,m},q_0}$ is a minimal realization of f.

That is, we can formulate the following theorem.

Theorem 46. *The pair $f = (f_C, f_D)$, $f_C : \widetilde{\Gamma}^* \to \mathbb{R}^p$ and $f_D : \Gamma^* \to O$ has a realization by a hybrid coalgebra system, if and only if $card(K_f) < +\infty$ and $card(D_f) < +\infty$. If f has a realization by a hybrid coalgebra system, it also has a minimal hybrid coalgebra system realization. If $(\bar{\mathcal{A}}, q_0)$ is a minimal Moore-automaton realization of d_f and $\Sigma_{\Psi,m}$ is the canonical minimal coalgebra system realization of Ψ_f, then $HC_{f,m} = HC_{\bar{\mathcal{A}},\Sigma_{\Psi_f,m},q_0}$ is a minimal hybrid coalgebra system realization of f.*

We will call the hybrid coalgebra system $HC_{f,m}$ the *canonical minimal hybrid coalgebra realization of f*. Strictly speaking $HC_{f,m}$ is not uniquely defined, since it depends on the choice of a minimal Moore-automaton realization $(\bar{\mathcal{A}}, q_0)$ of d_f. But all minimal Moore-automaton realizations are isomorphic, thus we will identify all hybrid systems obtained by choosing some minimal Moore-automaton realization.

We will call the hybrid coalgebra system $HC = (\mathcal{A}, \Sigma, q_0)$ *observable* , if

(i) $\bar{\mathcal{A}}_{HC}$ is observable

(ii) For each $q \in Q$, $A_{\Sigma}^{\perp} \cap C_q = \{0\}$

It is also easy to see that if $(\bar{\mathcal{A}}, q_0)$ is reachable and observable and Σ is observable, then $HC_{\bar{\mathcal{A}},\Sigma,q_0}$ is observable.

The discussion above can be summed up in the following theorem.

Theorem 47. *Let $f = (f_C, f_D)$, $f_C : \widetilde{\Gamma}^* \to \mathbb{R}^p$ and $f_D : \Gamma^* \to O$. A hybrid coalgebra system is a minimal realization of f if and only if it is reachable and observable. Minimal hybrid coalgebra system realizations of the same map f are isomorphic.*

8.6.4 Formal Hybrid Systems as Duals of Hybrid Coalgebra Systems

Recall from Subsection 8.3.2 that the ring of formal power series $\mathbb{R}[[X_1, \dots, X_n]]$ is isomorphic to the dual of of the cofree pointed irreducible cocommutative coalgebra $B(V)$, where V is any n-dimensional vector space. That is, $B(V)^* \cong \mathbb{R}[[X_1, \dots, X_n]]$. Below we will choose a particular V. Denote by A the ring $A = \mathbb{R}[[X_1, \dots, X_n]]$. Recall from Subsection 8.3.1 the definition and properties of continuous derivations on formal power series rings. Define the map $D_\alpha = 1_A \circ \frac{d}{dX}^{\alpha}$ for all $\alpha \in \mathbb{N}^n$. Define the set $\mathcal{D}_A^{\infty} = \mathrm{Span}\{D_\alpha \mid \alpha \in \mathbb{N}^n\}$. Notice that $\phi = D_{(0,0,\dots,0)} = 1_A^* \in \mathcal{D}_A^{\infty}$. Let $\mathcal{D}_A = \mathrm{Span}\{D_i \mid i = 1, \dots, n\}$. Define the linear maps $\epsilon : \mathcal{D}_A^{\infty} \to \mathbb{R}$ and $\delta : \mathcal{D}_A^{\infty} \to \mathcal{D}_A^{\infty} \otimes \mathcal{D}_A^{\infty}$ by

$$\epsilon(\phi) = 1 \text{ and } \epsilon(D_\alpha) = 0 \text{ if } \alpha \in \mathbb{N}^n, \alpha \neq (0,0,\dots,0)$$

For each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ let

$$\delta(D_\alpha) = \sum_{\beta,\gamma \in \mathbb{N}^n, \beta+\gamma=\alpha} D_\beta \otimes D_\gamma$$

where $\beta + \gamma = (\beta_1 + \gamma_1, \beta_2 + \gamma_2, \dots, \beta_n + \gamma_n)$, $\beta = (\beta_1, \dots, \beta_n)$, $\gamma = (\gamma_1, \dots, \gamma_n)$. Define the multiplication $M : \mathcal{D}_A^{\infty} \otimes \mathcal{D}_A^{\infty} \to \mathcal{D}_A^{\infty}$ by $M(D_\alpha \otimes D_\beta) = D_{\alpha+\beta}$. Define the map $u : \mathbb{R} \to \mathcal{D}_A^{\infty}$ by $u(x) = x\phi$.

With the notation above the following holds.

Lemma 46. *The tuple $(\mathcal{D}_A^{\infty}, \delta, \epsilon, M, u)$ is a bialgebra, moreover $\mathcal{D}_A^{\infty}$ is isomorphic as a bialgebra to the cofree pointed irreducible cocommutative coalgebra $B(\mathcal{D}_A)$ generated by $\mathcal{D}_A$.*

The lemma above implies that $(\mathcal{D}_A^\infty)^*$ is isomorphic to A. This algebra isomorphism is defined by

$$\psi_A : (\mathcal{D}_A^\infty)^* \ni S \mapsto \sum_{\alpha \in \mathbb{N}^n} S(\frac{1}{\alpha_1 !}\frac{1}{\alpha_2 !}\cdots\frac{1}{\alpha_n !}D_\alpha)X^\alpha$$

The following lemma relates measuring of A and coalgebra maps of $\mathcal{D}_A^\infty$.

Lemma 47. *Let C be an coalgebra, let $A = \mathbb{R}[[X_1, \dots, X_n]]$ and $B = \mathbb{R}[[X_1, \dots, X_n]]$. Assume that $\psi : C \otimes A \to B$ is a measuring such that for each $c \in C$, the map $\psi_c : A \ni a \mapsto \psi(c \otimes a) \in B$ is a continuous map. Then $\eta_\psi : C \otimes \mathcal{D}_B^\infty \to \mathcal{D}_A^\infty$ is a coalgebra map, where $\eta_\psi(c \otimes D_\alpha)(a) = D_\alpha \circ \psi_c(a)$ for all $a \in A$.*

Conversely, assume that $\eta : C \otimes \mathcal{D}_B^\infty \to \mathcal{D}_A^\infty$ is a coalgebra map. Consider the map $\psi_\eta : C \otimes A \to B$, defined by $\psi_B^{-1} \circ \psi_\eta(c \times a)(D) = \eta(c \otimes D)(\psi_A^{-1}(a))$, for all $a \in A, c \in C$, $D \in \mathcal{D}_B^\infty$.Here ψ_A^{-1} and ψ_B^{-1} are the inverses of the algebra isomorphisms $\psi_A : (\mathcal{D}_A)^* \to A$ and $\psi_B : (\mathcal{D}_B)^* \to B$ respectively. Then ψ_η is a measuring such that for each $c \in C$ the map $\psi_{\eta,c} : A \ni a \mapsto \psi_\eta(c \otimes a) \in B$ is a continuous map.

In the sequel we will identify $\mathcal{D}_A^\infty$ and $B(\mathcal{D}_A)$ and we will identify their respective duals $(\mathcal{D}_A^\infty)^*$, $B(\mathcal{D}_A)^*$ with A. We will also identify $(B(V))^*$ with $A_V = \mathbb{R}[[X_1, \dots, X_n]]$ if $\dim V = n$.

Using Lemma 46 and Lemma 47 we can associate with each formal hybrid system a hybrid coalgebra system of a certain type and conversely, with each hybrid coalgebra system of a suitable type we can associate a formal hybrid system. Let HF be formal hybrid system of the form

$$HF = (\mathcal{A}, (A_q, g_{q,j}, h_{q,i})_{q \in Q, j=0,\dots,m, i=1,\dots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, q_0)$$

and consider the associated hybrid algebra system $HA_{HF} = (\mathcal{A}, \Sigma_{HF}, q_0)$, where $\Sigma = (A, H, \psi, \phi, J, \mu)$. Define the hybrid coalgebra system HC_{HF} *associated with* HF as follows. $HC_{HF} = (\mathcal{A}, \Sigma_{HC}, q_0)$, where $\Sigma_{HC} = (C, H, \widetilde{\psi}, \widetilde{\phi}, \{1, \dots, p\}, \widetilde{\mu})$ such that

- For all $q \in Q$, $C_q = B(\mathcal{D}_{A_q})$.
- $\widetilde{\psi} = \eta_\psi$
- $\widetilde{\phi} = 1_{q_0}$ where 1_{q_0} is the unique group-like element of C_q. Notice that $1_{q_0} = 1^*_{A_{q_0}}$ viewed as a map $A_q \to \mathbb{R}$.
- $\widetilde{\mu}(j)(D) = D(\mu(j))$ for all $j = 1, \dots p$.

It is an easy consequence of Lemma 46 and Lemma 47 that HC_{HF} is well-defined and $HC_{HF}^{*} = HA_{HF}$.

Conversely, let $HC = (\mathcal{A}, \Sigma, q_0)$ be a hybrid coalgebra system such that $\Sigma = (C, H, \psi, \phi, \{1, \dots, p\}, \mu)$, $C = \bigoplus_{q \in Q} C_q$, $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ and $C_q = B(V_q)$, $\dim V_q = n_q$ for all $q \in Q$. We will call such hybrid coalgebra systems *CCPI hybrid coalgebra systems* (CCPI stands for cofree cocommutative pointed irreducible). Then using Lemma 46 and Lemma 47 and the conventions discussed after Lemma 47 we get that

$$HF_{HC} = (\mathcal{A}, (A_q, g_{q,j}, h_{q,i})_{q \in Q, j=0,\dots,m, i=1,\dots,p}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}, q_0)$$

is a well-defined formal hybrid system, where for all $q \in Q$, $A_q = C_q^{*}$, for all $j \in \mathrm{Z}_m$, $g_{q,j} = \psi(1_q \otimes j)$, 1_q being the unique group-like element of C_q, $h_{q,i} \in A_q$ are such that $(h_{q,i})_{q \in Q} = \mu(i)$ for all $i = 1, \dots, p$, and $R_{\delta(q,y),y,q}$, $y \in \Gamma$ are such that $R_{\delta(q,y),y,q}(h)(D) = \psi(D \otimes y)(h)$ for all $D \in C_q$. It is also easy to see that $HC^{*} = HA_{HF_{HC}}$.

Combining the results above we arrive to the following important characterisation of existence of a formal hybrid system realization of a pair of maps $f = (f_D, f_C)$, where $f_D : \Gamma^{*} \to O$ and $f_C : \widetilde{\Gamma}^{*} \to \mathbb{R}^p$.

Theorem 48. *A pair of maps $f = (f_D, f_C)$, $f_D : \Gamma^{*} \to O$, $f_C : \widetilde{\Gamma}^{*} \to \mathbb{R}^p$ has a realization by a formal hybrid system if and only if it has a CCPI hybrid coalgebra system realization.*

8.6.5 Realization by CCPI Hybrid Coalgebra Systems

In this section we will discuss criteria for existence of a realization by a hybrid coalgebra system, such that the coalgebras associated with each discrete state of the automaton are cofree cocommutative pointed irreducible with finite dimensional space of primitive elements. We will give a necessary condition and a condition which is an "almost" sufficient one. More precisely, the "almost" sufficient condition implies existence of a hybrid coalgebra system realization such that each coalgebra associated with some discrete state is pointed cocommutative irreducible with finite dimensional space of primitive elements. Such a hybrid coalgebra system is indeed very close to a CCPI hybrid coalgebra system. In fact, we conjecture that any such hybrid coalgebra system gives rise to a CCPI hybrid coalgebra system.

From Theorem 48 it follows that these criteria will give necessary and sufficient conditions for existence of a formal hybrid realization.

Let us recall the results of Subsection 8.6.3. We will call a coalgebra system $\Sigma = (C, H, \psi, \phi, \{1, \dots, p\}, \mu)$ *a CCPI coalgebra system* if $C = \bigoplus_{i \in I} C_i$ such that

I is finite, and for all $i \in I$, $C_i \cong B(V_i)$, $\dim V_i < +\infty$. Consequently, C is pointed and $G(C) = \{g_i \mid i \in I\}$, where g_i is the unique group-like element of C_i.

It is easy to see that Theorem 45 implies the following.

Theorem 49. *The pair $f = (f_D, f_C)$ admits a CCPI hybrid coalgebra system realization, only if d_f admits a Moore-automaton realization and Ψ_f admits a CCPI coalgebra system realization.*

We can also prove a result which is in some sense the converse of the theorem above.

Let $\Sigma = (C, H, \psi, \phi, \{1, \dots, p\}, \mu)$ be a point-observable coalgebra realization of Ψ_f, such that C is pointed. Let $\bar{\mathcal{A}} = (Q, \Gamma, \bar{O}, \delta, \bar{\lambda})$ be a Moore-automaton such that $(\bar{\mathcal{A}}, q_0)$ is a reachable realization of d_f. Recall the definition of the hybrid coalgebra system $HC_{\bar{\mathcal{A}},\Sigma,q_0}$ associated with $(\bar{\mathcal{A}}, q_0)$ and Σ. Recall that the hybrid coalgebra system $HC_{\bar{\mathcal{A}},\Sigma,q_0}$ is reachable. We can associate a hybrid coalgebra system $HC^n_{\bar{\mathcal{A}},\Sigma,q_0}$ with $(\bar{\mathcal{A}}, q_0)$ and Σ in alternative way, so that $HC^n_{\bar{\mathcal{A}},\Sigma,q_0}$ is not reachable but preserves more of the structure of Σ. The construction goes as follows.

$$HC^n_{\bar{\mathcal{A}},\Sigma,q_0} = (\mathcal{A}, \widetilde{\Sigma}, q_0)$$

where

- $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ where $\lambda(q) = o$ if $\bar{\lambda}(q) = (o, \bar{o})$.
- $\widetilde{\Sigma} = (\widetilde{C}, H, \widetilde{\psi}, \widetilde{\phi}, \{1, \dots, p\}, \widetilde{\mu})$ where
  - $\widetilde{C} = \bigoplus_{q \in Q} C_q$, where for each $q \in Q$, C_q is the irreducible component of C with the unique group-like element ϕ_q defined by $\phi_q = \psi(w \otimes \phi)$, where $w \in \Gamma^*$ such that $\delta(q_0, w) = q$.
  - With the notation above $\widetilde{\phi} = \phi_{q_0}$
  - The map $\widetilde{\psi} : \widetilde{C} \otimes H \to \widetilde{C}$ is defined as follows. For each $q \in Q$,$c \in C_q$, $x \in \mathrm{Z}_m$, $\widetilde{\psi}(c \otimes x) = \psi(c \otimes x) \in C_q$. For each $q \in Q$, $c \in C_q$, $\gamma \in \Gamma$, $\widetilde{\psi}(c \otimes \gamma) = \psi(c \otimes \gamma) \in C_{\delta(q,\gamma)}$.
  - For all $j \in J$, the map $\widetilde{\mu}(j) \in \widetilde{C}^*$ is such that for all $q \in Q$, $c \in C_q$, $\widetilde{\mu}(j)(c_q) = \mu(j)(c_q)$.

Lemma 48. *With the notation and assumptions above $HC = HC_{\bar{\mathcal{A}},\Sigma,q_0}$ is a well-defined hybrid coalgebra system which realizes f. If Σ is a CCPI coalgebra system then HC is a CCPI hybrid coalgebra system.*

Thus, we get the following characterisation of existence of a realization by a CCPI hybrid coalgebra system

Theorem 50. *The pair $f = (f_D, f_C)$ admits a CCPI hybrid coalgebra system realization, if d_f admits a Moore-automaton realization and Ψ_f admits a point-observable CCPI coalgebra system realization.*

Below we will formulate necessary conditions for existence of a realization by a hybrid coalgebra systems. These conditions will involve finiteness requirements. That is, they will require that a certain infinite matrix has a finite rank and that certain sets are finite. Although such conditions are difficult to check, yet they are more informative than requiring that there exists a realization by a coalgebra system of a certain class. The obtained rank condition is similar to the classical Lie-rank condition for existence of a realization by a nonlinear system [32, 21, 36].

Define the set $P(H) \subseteq H$ by $P(H) = \mathrm{Span}\{wPv \mid w, v \in \Gamma^*, P \in Lie < \mathrm{Z}_m^* >\}$, where $Lie < \mathrm{Z}_m^* >$ denotes the set of all Lie-polynomials over Z_m. That is, $Lie < \mathrm{Z}_m^* >$ is the smallest subset of the set of all polynomials $\mathbb{R} < \mathrm{Z}_m^* >$ such that

- For all $x \in \mathrm{Z}_m$, $x \in Lie < \mathrm{Z}_m^* >$
- If $P_1, P_2 \in Lie < \mathrm{Z}_m^* >$, then $P_1P_2 - P_2P_1 \in Lie < \mathrm{Z}_m^* >$.

Let $\Sigma = (C, H, \psi, \phi, \{1, \dots, p\}, \mu)$ be a CCPI coalgebra realization of Ψ_f. Assume that $C = \bigoplus_{i\in I} B(V_i)$, where I is finite. Define the set $P(C) = \bigoplus_{i\in I} V_i$. It is easy to see that $P(C)$ is finite dimensional. Consider the coalgebra map $T_\Sigma : H \to C$. It is easy to see that $T_\Sigma(P(H)) \subseteq P(C)$ and $P(H)/P(H) \cap \ker T_\Sigma \cong T_\Sigma(P(H))$.

Recall that $\ker T_\Sigma \subseteq A_{\Psi_f}^{\perp}$, where A_{Ψ_f} is the algebra generated by $R_h f$, $h \in H$ and $A_{\Psi_f}^{\perp} = \{h \in H \mid \forall g \in A_{\Psi_f}, g(h) = 0\}$. Since $P(H) \cap \ker T_\Sigma \subseteq A_{\Psi_f}^{\perp} \cap P(H)$ we get that $+\infty > \dim P(C) \geq \dim P(H)/P(H) \cap \ker T_\Sigma \geq \dim P(H)/P(H) \cap A_{\Psi_f}^{\perp}$.

Define the *Lie-rank of* f as

$$\mathrm{rank}\,_L f = \dim P(H)/A_{\Psi_f}^{\perp} \cap P(H)$$

Let $HC = (\mathcal{A}, \Sigma, q_0)$ be a CCPI hybrid coalgebra system, Assume that $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$, $\Sigma = (C, H, \psi, \phi, \{1, \dots, p\}, \mu)$ and $C = \bigoplus_{q\in Q} C_q$. Define the dimension of HC as $\dim HC = (card(Q), \sum_{q\in Q} \dim P(C_q))$. It is easy to see that if HF is a formal hybrid system realization of f, then $\dim HC_{HF} = \dim HF$. Conversely, if HF_{HC} is the formal hybrid system associated with HC, then $\dim HF_{HC} = \dim HC$.

From the discussion above we get the following necessary condition for existence of a CCPI hybrid coalgebra system realization

Theorem 51. *The pair $f = (f_D, f_C)$, $f_D : \Gamma^* \to O$, $f_C : \widetilde{\Gamma}^* \to \mathbb{R}^p$ has a realization by a CCPI hybrid coalgebra system only if* $\mathrm{rank}\,_L f < +\infty$, *$card(K_f) < +\infty$ and $card(D_f) < +\infty$. For any CCPI hybrid coalgebra system realization HC of f,*

$(card(W_{d_f}, \text{rank }_L f) \leq \dim HC$. *That is, if* $\dim HC = (p,q)$, *then* $card(W_{d_f}) \leq p$ *and* $\text{rank }_L f \leq q$.

Taking into account that f has a realization by a CCPI hybrid coalgebra system if and only if it has a realization by a formal hybrid system we get the main result of the chapter.

Theorem 52. *The pair* $f = (f_D, f_C)$, $f_D : \Gamma^* \to O$, $f_C : \widetilde{\Gamma}^* \to \mathbb{R}^p$ *has a realization by a formal hybrid system only if* $\text{rank }_L f < +\infty$, $card(K_f) < +\infty$ *and* $card(D_f) < +\infty$. *For any formal hybrid system realization* HF *of* f, $(card(W_{d_f}), \text{rank }_L f) \leq \dim HF$.

That is, $\text{rank }_L f$ *gives a lower bound on the dimension of the continuous state space (number of variables) for each formal hybrid realization of* f.

The conditions above are almost sufficient. That is, if the conditions above hold, then we can prove existence of a hybrid coalgebra system which is very close to a CCPI hybrid coalgebra system.

Consider the canonical minimal hybrid coalgebra system realization $HC_{f,m}$ of f. Recall that $HC_{f,m} = HC_{\bar{\mathcal{A}},\Sigma_{\Psi_f,m},q_0}$ where $(\bar{\mathcal{A}}, q_0)$ is some minimal Moore-automaton realization of d_f and $\Sigma_{\Psi_f,m}$ is the canonical minimal realization of f. Recall from Subsection 8.6.1 that $\Sigma_{\Psi_f} = (D, H, \psi, \phi, \{1,\ldots,m\}, \mu)$ where $D = H/A^{\perp}_{\Psi_f}$. Recall from Lemma 45 that $D = \bigoplus_{i=1}^{N} \pi(H_{q_i})$ where $H_{q_i} = \bigoplus_{(L_w f_{C,j})_{j=1,\ldots,p}=q_i} H_w$, $K_f = \{q_1,\ldots,q_N\}$ and π is the canonical projection map $\pi : H \to D = H/A^{\perp}_{\Psi_f}$. That each, each irreducible component of D is of the form $\pi(H_{q_i})$ for some $q_i \in K$. It is easy to see that $P(D) = \sum_{i=1}^{N} P(\pi(H_{q_i}))$. It is also easy to see that

$$P(H)/P(H) \cap A^f_\Psi \cong \pi(P(H)) \subseteq P(D)$$

Thus,

$$\dim P(D) < +\infty \implies \text{rank }_L f < +\infty$$

Recall the construction of $HC_{f,m} = HC_{\bar{\mathcal{A}},\Sigma_{\Psi,m},q_0}$ Recall $HC_{f,m} = (\mathcal{A}, \widetilde{\Sigma}, q_0)$, such that $\bar{\mathcal{A}} = (Q, \Gamma, \bar{O}, \delta, \bar{\lambda})$, $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$,

$$\widetilde{\Sigma} = (\widetilde{C}, H, \widetilde{\psi}, \widetilde{\phi}, \{1,\ldots,p\}, \widetilde{\mu})$$

such that $\widetilde{C} = \bigoplus_{q\in Q} \widetilde{C}_q$ and $\widetilde{C}_q$ is a subcoalgebra of $\pi(H_{q_i})$ such that $q_i = \Pi_{(H^*)^p}(\bar{\lambda}(q))$.

That is, if $\dim P(\pi(H_{q_i})) < +\infty$, then $\dim P(\widetilde{C}_q) \leq \dim P(\pi(H_{q_i} < +\infty$. Thus, if $\dim P(D) < +\infty$, then for each $q \in Q$, $\dim P(\widetilde{C}_q) < +\infty$.

From the discussion above we get the following results.

Theorem 53. *With the notation above the following holds.*

(a) *If* $card(K_f) < +\infty$, $card(D_f) < +\infty$ *and* $\mathrm{rank}\ _L f < +\infty$, *then there exists a hybrid coalgebra system realization* HC *of* f *such that* $HC = (\mathcal{A}, \Sigma, q_0)$, $\Sigma = (C, H, \psi, \phi, J, \mu)$, $C = \bigoplus_{q \in Q} C_q$ *and for each* $q \in Q$, C_q *is pointed irreducible and* $\dim T_{\Sigma}(P(H)) \cap P(C_q) = T(\Sigma)(H_{q_i}) < +\infty$,*where* $q_i = L_{w_i} f \in K_f$, $\delta(q_0, w_i) = q$ *and* $T_{\Sigma} : H \ni h \mapsto \psi(\phi_{q_0} \otimes h)$ *is the canonical map* $T_{\Sigma} : \Sigma_{\Psi_f} \to \Sigma$.

(b) *If* $\dim P(H/A^{\perp}_{\Psi_f}) < +\infty$, *then* f *has a realization by a hybrid coalgebra system* $HC = (\mathcal{A}, \Sigma, q_0)$ *such that* $\Sigma = (C, H, \psi, \phi, J, \mu)$, $C = \bigoplus_{q \in Q} C_q$ *and for each* $q \in Q$ C_q *is pointed irreducible and* $\dim P(C_q) < +\infty$.

Sketch of the proof. In both cases let $HC = HC_{\mathcal{A}, \Sigma_{\Psi_f}, q_0}$ where $(\mathcal{A}, q_0)$ is a minimal Moore-automaton realization of d_f and $\Sigma_{\Psi_f, m}$ is the canonical minimal coalgebra system realization of Ψ_f. □

Let us try to find interpretation of the results of the theorem above. Part (a) of the theorem above says that the subspace of each C_q spanned by the elements of *Lie* $< \mathbb{Z}^*_m >$ and their translates by $\psi(. \otimes \gamma) : C \ni c \mapsto \psi(c \otimes \gamma)$, $\gamma \in \Gamma$ is finite dimensional.

Part (b) implies that for each $q \in Q$, C_q is pointed, irreducible and $n_q = \dim P(C_q) < +\infty$. But this implies that for each q, there exists an injective $S_q : C_q \to B(V_q)$, where $V_q = P(C_q)$. That is, there exists an algebra map

$$S^*_q : \mathbb{R}[[X_1, \dots, X_{n_q}]] \to C^*_q$$

such that $(\mathrm{Im} S^*_q)^{\perp} = \{0\}$, i.e. for all $c \in C_q$ and $g \in C^*_q$ there exists some $Z \in \mathbb{R}[[X_1, \dots, X_{n_q}]]$ such that $S^*_q(Z)(c) = g(c)$. That is, S^*_q is "almost" surjective. Thus, $\dim P(D) < +\infty$ implies existence of an "almost" formal hybrid system realization. This observation prompts us to define the *strong Lie-rank of* f as

$$\mathrm{rank}\ _{L,S} f = \dim P(H/A^{\perp}_{\Psi_f}) = \dim P(D)$$

As we have already remarked,

$$\mathrm{rank}\ _L f \leq \mathrm{rank}\ _{L,S} f$$

Thus, finiteness of $\mathrm{rank}\ _{L,S} f$ is a stronger requirement than finiteness of $\mathrm{rank}\ _L f$. As we have seen, if $\mathrm{rank}\ _{L,S} f < +\infty$, then there exists an "almost CCPI" realization of f, i.e. f can be realized by a *hybrid system with finite state space of some sort.*

In fact, we can give the following sufficient condition for finiteness of $\mathrm{rank}\ _L f$. Define the following space

$$H_{L,f} = \{(L_P f_{C,i})_{i=1,\dots,p} \mid P \in P(H)\}$$

Consider the map $T : P(H) \ni P \mapsto (L_P f_{C,i})_{i=1,\ldots,p}$. Then $P(H) \cap A^{\perp}_{\Psi_f} \subseteq \ker T$ and thus

rank $_L f = \dim(P(H)/kerT)/(kerT/P(H) \cap A^{\perp}_{\Psi_f}) \leq$
$\leq \dim P(H)/KerT$

That is, the following holds.

Lemma 49. *With the notation above, the following relationship holds*

$$\dim H_{L,f} < +\infty \implies \text{rank }_L f < +\infty$$

and

$$\text{rank }_{L,S} f < +\infty \implies \text{rank }_L f < +\infty$$

Below we will present an example, which demonstrates that the Lie-rank might simply be not enough to capture all the necessary dimensions. Consider the following hybrid system $H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, f_q, h_q)_{q \in Q}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\})$ such that

- $\Gamma = \{\gamma\}$, $\mathcal{A} = (\{q_1, q_2\}, \{\gamma\}, \{o\}, \delta, \lambda)$, where $\delta(q_1, \gamma) = q_2$, $\delta(q_2, \gamma) = q_2$, $\lambda(q_i) = o$, $i = 1, 2$.
- $U = \mathbb{R}$, $\mathcal{Y} = \mathbb{R}$,
- $\mathcal{X}_{q_1} = \mathcal{X}_{q_2} = \mathbb{R}$,
- $f_{q_1}(x, u) = u$, $h_{q_1}(x) = 0$ and $R_{q_2,\gamma,q_2}(x) = x^2$, for all $x \in X_{q_1}, u \in \mathcal{U}$,
- $h_{q_2}(x) = 0$, $f_{q_2}(x, u) = 0$ and $R_{q_2,\gamma,q_2}(x) = x$ for all $x \in X_{q_2}$, $u \in \mathcal{U}$.

Consider the input-output map $f = \upsilon_H((q_1, 0), .)$. It is easy to see that H is a NHS system, thus f has a hybrid Fliess-series expansion. Consider the pair $\widetilde{f} = (f_D, c_f)$, where c_f is a generating convergent series such that $F_{c_f} = f_C$. It is easy to see that rank $_L \widetilde{f} = 1$. But $card(K_{\widetilde{f}}) = 2 = card(W_{d_{\widetilde{f}}})$, thus one needs at least two discrete states to realize f thus no realization can be of dimension smaller than $(2, 2)$. Notice that the discrete input-output map f_D is constant, i.e. $f_D(w) = o$ for all $w \in \Gamma^*$. Thus, the problem above is inherent to the continuous dynamics.

Chapter 9

Piecewise-affine Hybrid Systems in Discrete-time

In this chapter realization theory for *discrete-time autonomous piecewise affine hybrid systems* will be investigated. A piecewise-affine hybrid system is a discrete-time system such that the state-transition and the readout maps are piecewise-affine. By a piecewise-affine function we mean a function the domain of which is covered by polyhedral sets and on each such polyhedral set the function is affine. The class of discrete-time piecewise-affine hybrid systems was studied in several papers, see [9, 80, 45, 3].

In this chapter we will investigate the following problem. For a specified output trajectory, i.e., for a specified sequence of output values, find a discrete-time autonomous piecewise-affine hybrid system realizing it. We will not address the issue of minimality in this chapter.

We will present the following results.

- An output trajectory has a realization by an autonomous discrete-time piecewise-affine hybrid system if and only if it has a realization by a discrete-time linear switched system. That is, any switching sequence can be generated by a piecewise-affine hybrid system.
- An output trajectory has a realization by an autonomous discrete-time piecewise-affine hybrid system with *almost-periodic dynamics* if and only if it has a realization by a discrete-time linear system. By almost-periodic dynamics we will mean that the shift invariant set generated by the sequence of polyhedral regions visited by the state-trajectory starting from the initial state is finite.

- An output trajectory has a realization by a discrete-time piecewise-affine system such that
  - The polyhedrons of the system are indexed by elements of a set specified in advance
  - The system has almost-periodic dynamics
  - The sequence of indexes of polyhedrons visited by the state-trajectory coincides with an infinite sequence specified in advance

  if and only if the Hankel-matrix of y has finite rank. Here by Hankel-matrix we mean an infinite matrix constructed from the values of y in a special way. Note that in the preceding paragraph we were looking for a realization by a system with arbitrary indexing of polyhedral regions and with the restriction that the symbolic dynamics is almost-periodical.

- An output trajectory has a realization by an autonomous discrete-time piecewise-affine hybrid system if and only if the shifts invariant space generated by the output trajectory is contained in a finitely generated module over a certain algebra. This condition is a counterpart of the usual finite-rank Hankel-matrix condition for the linear case.

One of the most important observations of the current chapter is that a discrete-time piecewise-affine hybrid system can generate arbitrary symbolic dynamics. That is, if one specifies a finite alphabet and an infinite sequence of symbols over this alphabet, then it is always possible to construct a discrete-time piecewise-affine hybrid system such that the following holds. The polyhedral regions of the system are indexed by the elements of the alphabet. The sequence of indexes of the polyhedral regions visited by the state-trajectory which starts from the initial state coincides the specified infinite sequence. In fact, such a system can be constructed on the state-space $[0,1]$. That is, *the switching mechanism of a piecewise-affine hybrid system is as general as any other switching mechanism.* Thus, any switching sequence can be generated by a discrete-time piecewise-affine hybrid system. Moreover, if the switching is nice, more precisely, if *the switching sequence is a trace of a finite automaton, then the expressive power of a piecewise-affine hybrid system is not greater than the expressive power of a linear system.* The observation above has the following important consequence. Any piecewise-affine hybrid system is output equivalent to a piecewise-affine hybrid system which is a composition of a linear switched system and a piecewise-affine system on $[0,1]$. The linear switched system generates the observable output, the piecewise-affine system on $[0,1]$ generates the required switching sequence, but does not contribute to the output. The conclusions above might be an indication that discrete-time piecewise-affine hybrid systems might be a too general class of hybrid systems.

In [80] identifiability and realisability of the so called jump-linear systems was investigated. Discrete-time linear switched systems and jump-linear systems are closely related. In [80] only identifiability and realisability of finite output trajectories were treated. That is, in [80] the authors aimed at finding a state-space realization, such that this state-space realization generates the specified output trajectory up to some time step T. Whether the computed state-space realization generates the specified output trajectory after time T was not investigated. In contrast, the current chapter investigates existence of a realization of an infinite output trajectory. Studying infinite trajectories might seem unreasonable, as it can not yield algorithms for computing a realization. But as development of realization theory for other classes of systems has demonstrated, realization theory for infinite trajectories may yield *partial realization theory*. That is, it can lead to an algorithm which computes a realization of the *whole infinite trajectory* from a *finite part of this trajectory*. In fact, partial realization theory for other classes of hybrid systems exists, see [52, 53, 54]. The hope is that the results of the current chapter will eventually lead to a similar partial realization theory for piecewise-affine hybrid systems.

The solution of the realization problem presented in this chapter uses methods related to time-varying linear systems and linear systems over rings.

The chapter is organised as follows. Section 9.1 presents the necessary notation and terminology. It also presents the definition and some elementary properties of discrete-time piecewise-affine and discrete-time linear switched systems. Section 9.2 discusses the relationship between discrete-time piecewise-affine hybrid systems and discrete-time linear switched systems. It also introduces a canonical representation for discrete-time piecewise-affine hybrid systems as a interconnection of a linear switched system and a piecewise-affine hybrid system. The former generates the output, the latter generates the switching signal. Section 9.3 deals with realization theory of piecewise-affine hybrid systems with almost periodic dynamics. Section 9.3.2 investigates the realization problem for piecewise-affine hybrid systems with arbitrary symbolic dynamics.

9.1 Discrete-time Linear Switched Systems

Below we will introduce a class of discrete-time switched systems which will play an important role in realization theory of DTAPA systems. A *discrete-time autonomous linear switched system (DTALS)* is a tuple $H = (\mathcal{X}, \mathcal{Y}, Q, \{A_q, C_q\}_{q \in Q}, x_0)$. Again,

$\mathcal{X} = \mathbb{R}^n$ will be called the state-space, $\mathcal{Y} = \mathbb{R}^p$ will be called the output space of H. The vector x_0 will be called the initial state of H. The inputs of a linear switched system are finite sequences of elements of Q. The state-trajectory of such a system can be described as a map $x_H : \mathcal{X} \times Q^* \to X$ defined as follows

$$x_H(x, wq) = A_q x_H(x, w), \quad x_H(x, \epsilon) = x$$

for each $w \in Q^*, q \in Q$. The output trajectory can be thought of as a map $y_H : \mathcal{X} \times Q^+ \to \mathcal{Y}$ defined as follows

$$y_H(x, w) = C_{w_k} x_H(x, w_1 w_2 \cdots w_{k-1})$$

where $w = w_1 \cdots w_k$, $w_1, \ldots, w_k \in Q$, $k > 0$. A map $y : Q^+ \to \mathcal{Y}$ is said to be *realized* by a DTALS system H if

$$\forall w \in Q^+ : y_H(x_0, w) = y(w)$$

Similarly, if $L \subseteq Q^+$ and $y : L \to \mathcal{Y}$ then y is said to be realized by a DTALS system H if

$$\forall w \in L : y_H(x_0, w) = y(w)$$

Let $y : \mathbb{N} \to \mathcal{Y}$ be an output trajectory. For each $w = w_0 w_1 \cdots w_k \cdots \in Q^\omega$, $w_1, w_2, \ldots \in Q$ define the set $L_w = \{w_0 \cdots w_k \in Q^+ \mid k > 0\}$. It is easy to see that $s \in L_w \iff s = w_0 \cdots w_{|s|-1}$. Define the map $y_w : L_w \ni s \mapsto y(|s| - 1)$. It is easy to see that the map y_w is well defined. We will define types of realization problems for DTALS systems

Classical realization problem For a specified $y : L \to \mathcal{Y}$, $L \subseteq Q^+$ find a DTALS system which realizes y.

Weak realization problem for DTALS systems For a specified $y : \mathbb{N} \to \mathcal{Y}$ and $w \in Q^\omega$ find a DTALS system $H = (\mathcal{X}, \mathcal{Y}, Q, (A_q, C_q)_{q \in Q}, x_0)$ such that H realizes y_w.

Strong realization problem for DTALS systems For a specified $y : \mathbb{N} \to \mathcal{Y}$ find a set of discrete modes Q, an infinite sequence $w \in Q^\omega$ and a DTALS system $H = (\mathcal{X}, \mathcal{Y}, Q, (A_q, C_q)_{q \in Q}, x_0)$ such that H realizes y_w.

We can associate with each DTAPA system Σ a DTALS system H_Σ defined as follows. Let $\Sigma_l = (\mathcal{X}, \mathcal{Y}, Q, (\mathcal{X}_q, A_q, C_q)_{q \in Q}, (q_0, x_0))$ be the linearised DTAPA associated with Σ and define H_Σ by

$$H_\Sigma = (\mathcal{X}, \mathcal{Y}, Q, (A_q, C_q)_{q \in Q}, x_0)$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ was assumed. We will call H_Σ *the DTALS system associated with* Σ. Notice that if $\phi(x_0) = w \in Q^\omega$ and Σ is a realization of a map $y : \mathbb{N} \to \mathcal{Y}$, then H_Σ is a realization of the map $y_w : L_w \to \mathcal{Y}$.

9.2 Canonical Form of DTAPA Systems

In the section a canonical form for state-space realization of DTAPA systems will be discussed. It will be shown that any DTAPA system can be transformed into a equivalent DTAPA system in canonical form.

Recall from [9] the following encoding of any infinite sequence $w \in Q^{\omega}$ into a real number in $[0,1]$. Assume $card(Q) = d$ and $Q = \{q_1, \ldots, q_d\}$. Identify each q_i with the natural number $\eta(q_i) = i-1$ for each $i = 1, \ldots, d$. Thus, we get a map $\eta : Q \to \{0, \ldots, d-1\}$.

Assume that $w = w_1 w_2 \ldots w_k \ldots \in Q^{\omega}$. Define the following series

$$\psi(w) = \sum_{k=1}^{\infty} \frac{\eta(w_k)}{(2d)^k}$$

It is easy to see that $\frac{\eta(w_k)}{2d^k} \leq \frac{1}{2^k}$, thus the series above is absolutely convergent and $0 \leq \psi(w) \leq 1$. Recall that from [9] that piecewise-affine operations on $[0,1]$ can be used to retrieve the first element of the sequence w and to compute $\psi(S(w))$, where S is the shift operator on sequences. That is, $S : Q^{\omega} \to Q^{\omega}$ and for each $w = w_0 w_1 w_2 \cdots$, $S(w) = w_1 w_2 \cdots$. These operations can be described as follows. Define the map $H : [0,1] \to \mathbb{R}$ as follows. For each $z \in [0,1]$,

$$H(z) = \begin{cases} 0 & \text{if } 0 \leq 2dz < 1 \\ 1 & \text{if } 1 \leq 2dz < 2 \\ \cdots & \cdots \\ i & \text{if } i \leq 2dz < i+1 \\ \cdots & \cdots \\ d-1 & \text{if } d-1 \leq 2dz < d \\ d & \text{otherwise} \end{cases}$$

It is easy to see that $H(\psi(w)) = i-1$ if $w_0 = q_i$. Define the map $M : [0,1] \to [0,1]$ by

$$M(z) = \begin{cases} 2dz & \text{if } 0 \leq 2dz < 1 \\ 2dz-1 & \text{if } 1 \leq 2dz < 2 \\ \cdots & \cdots \\ 2dz-i & \text{if } i \leq 2dz < i+1 \\ \cdots & \cdots \\ 2dz-(d-1) & \text{if } d-1 \leq 2dz < d \\ z & \text{otherwise} \end{cases}$$

It is easy to see that H and M are well defined maps and $M(\psi(w)) = \psi(S(w))$.

Consider a DTAPA $\Sigma = (\mathcal{X}, \mathcal{Y}, Q, (\mathcal{X}_q, A_q, a_q, C_q, c_q)_{q \in Q}, (q_0, x_0))$. We say that Σ is in *canonical form* if the following holds.

- $Q = F \cup \{s\}$, $s \notin F$,
- $\mathcal{X} \subseteq \mathbb{R}^n \oplus \mathbb{R}$, $\mathcal{X} = \bigcup_{q \in Q} \mathcal{X}_q$.
- For each $q \in F$, $\mathcal{X}_q = \mathbb{R}^n \times Z_q$, where

$$Z_q = \{z \in [0,1] \mid H(z) = \eta(q)\} \subseteq [0,1]$$

  That is, $Z_q = \{z \in [0,1] \mid \eta(q) \leq 2dz < \eta(q)+1\}$. It is easy to see that Z_q and thus $\mathcal{X}_q$ are polyhedral sets. Let $\mathcal{X}_s = \mathbb{R}^n \times ([0,1] \setminus (\bigcup_{q \in F} Z_q))$.

- For each $q \in F$ the maps $C_q x + c_q$ and $A_q x + a_q$ are of the following form

$$A_q = \begin{bmatrix} \widetilde{A}_q & 0 \\ 0 & 2d \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)} \text{ and } a_q = \begin{bmatrix} 0 \\ -\eta(q) \end{bmatrix} \in \mathbb{R}^{n+1}$$

$$C_q = \begin{bmatrix} \widetilde{C}_q & 0 \end{bmatrix} \in \mathbb{R}^{p\times(n+1)} \text{ and } c_q = 0$$

  The maps $C_s x + c_s$ and $A_s x + a_s$ are of the following form

$$A_s = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)},\ a_s = 0,\ c_s = 0, \text{ and } C_s = 0$$

  That is, the map $x \mapsto A_s x + a_s$ is the identity map and the map $x \mapsto C_s x + c_s$ is the constant zero map.

- The initial state is of the form $x_0 = (\widetilde{x}_0, z_0)^T$.

Notice that a DTAPA in canonical form can be viewed as a discrete-time linear switched system

$$\widetilde{x}(k+1) = \widetilde{A}_{q_k}\widetilde{x}(k), \quad y(k) = \widetilde{C}_{q_k}\widetilde{x}(k),\ \widetilde{x}(0) = \widetilde{x}_0$$

such that the switching sequence $w = q_1 \cdots q_k \cdots$ is generated by the following system

$$z(k+1) = M(z(k)), \quad z(0) = z_0, \quad q_k = \eta^{-1}(H(z_k))$$

We can state the following theorem.

Theorem 54 (Existence of a canonical form). *Let Σ be an arbitrary DTAPA system. Then there exists a DTAPA system Σ_{can} in canonical form and an injective DTAPA morphism $T : \Sigma \to \Sigma_{can}$. In particular, Σ_{can} and Σ are equivalent DTAPA systems.*

Sketch of the proof. By the discussion in Section 9.1 we can assume that Σ is a linearised DTAPA. If not, then we can take the linearised DTAPA Σ_l associated with Σ. Notice that there exists Σ_l such that $S : \Sigma \rightarrow \Sigma_l$ is a DTAPA isomorphism. If we show existence of a canonical form $(\Sigma_l)_{can}$ and an injective morphism $\widetilde{T} : \Sigma_l \rightarrow (\Sigma_l)_{can}$, then by taking $\Sigma_{can} = (\Sigma_l)_{can}$ and $T = \widetilde{T} \circ S$ the statement of the theorem follows.

Thus, let $\Sigma = (\mathcal{X}, \mathcal{Y}, Q, (\mathcal{X}_q, A_q, C_q)_{q\in Q}, (q_0, x_0))$. Assume that $\mathcal{X} \subseteq \mathbb{R}^n$. Define

$$\widetilde{\Sigma} = (\mathcal{X} \oplus \mathbb{R}, \widetilde{Q}, \mathcal{Y}, (\widetilde{X}_q, \widetilde{A_q}, \widetilde{a}_q, \widetilde{C}_q, 0)_{q\in\widetilde{Q}}, (q_0, x_0))$$

as follows. Let $\widetilde{Q} = Q \cup \{q_e\}$, $q_e \notin Q$. Let $\widetilde{X}_q = \mathcal{X} \times Z_q$ for each $q \in Q$, where $Z_q = \{z \in [0,1] \mid H(z) = \eta(q)\}$. Let $\widetilde{X}_{q_e} = \mathcal{X} \bigoplus (\mathbb{R} \setminus \bigcup_{q\in Q} Z_q)$. For each $q \in Q$ define $\widetilde{A}_q$, $\widetilde{a}_q$, $\widetilde{C}_q$ by

$$\widetilde{A}_q = \begin{bmatrix} A_q & 0 \\ 0 & 2d \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)},\ \widetilde{a}_q = \begin{bmatrix} 0 \\ -\eta(q) \end{bmatrix} \in \mathbb{R}^{n+1}, \quad \widetilde{C}_q = \begin{bmatrix} C_q & 0 \end{bmatrix} \in \mathbb{R}^{p\times(n+1)}$$

Define $\widetilde{A}_{q_e}$, $\widetilde{a}_{q_e}$, $\widetilde{C}_{q_e}$ by

$$\widetilde{A}_{q_e} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)},\ \widetilde{a_{q_e}} = 0, \quad \widetilde{C}_{q_e} = 0$$

It is easy to see that $\widetilde{\Sigma}$ is well defined and

$$f_{\widetilde{\Sigma}}((x^T, z)^T) = \begin{cases} \begin{bmatrix} A_q x \\ M(z) \end{bmatrix} & \text{if } H(z) = \eta(q) \text{ for some } q \in Q \\ \\ (x^T, z^T)^T & \text{otherwise} \end{cases}$$

$$h_{\widetilde{\Sigma}}((x^T, z)^T) = \begin{cases} C_q x & \text{if } H(z) = \eta(q) \text{ for some } q \in Q \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that $\widetilde{\Sigma}$ is in canonical form. Define the map $T : \mathcal{X} \rightarrow \widetilde{\mathcal{X}}$ by $T(x) = \begin{bmatrix} x \\ \phi(x) \end{bmatrix}$. It is clear that for each $q \in Q$, $T(\mathcal{X}_q) \subseteq \mathcal{X} \times Z_q$. Moreover, for all $x \in \mathcal{X}_q$, $f_{\widetilde{\Sigma}}(T(x)) = ((A_q x)^T, M(z))^T = (f_\Sigma(x)^T, \phi(f_\Sigma(x)))^T)^T = T(f_\Sigma(x))$ and $h_{\widetilde{\Sigma}}(T(x)) = C_q x = h_\Sigma(x)$. Thus, T is a DTAPA morphism. It is easy to see that T is injective too. $\square$

The theorem above has the following important consequence. The realization problem for DTAPA systems is equivalent to the realization problem for discrete-time autonomous linear switched systems. More precisely, both the strong and weak realization problems for DTAPA are equivalent to respectively the strong and weak realization problems for DTALS systems. Consider a map $y : \mathbb{N} \to \mathcal{Y}$ and let Q be a finite set. Let $w \in Q^{\omega}$ be an infinite word over Q. Recall the definition of $y_w : L_w \ni w \mapsto y(|w|-1) \in \mathcal{Y}$, $L_w = \{w_0 \cdots w_k \in Q^+ \mid k \geq 0\}$. With this notation the following theorem holds.

Theorem 55 (Equivalence of DTAPA and DTALS systems). *Consider a map $y : \mathbb{N} \to \mathcal{Y}$.*

(i) The map y has a realization by a DTAPA system if and only if there exists a set of discrete modes Q, an infinite word $w \in Q^{\omega}$ such that the map y_w has a realization by a DTALS system.

(ii) The map y has a realization by a DTAPA system

$$\Sigma = (\mathcal{X}, \mathcal{Y}, Q, (\mathcal{X}_q, A_q, a_q, C_q, c_q)_{q \in Q}, (q_0, x_0))$$

with set of discrete modes Q such that $\phi(x_0) = w \in Q^{\omega}$ if and only if y_w has a realization by a DTALS system.

(iii) The strong realization problem for DTAPA systems is equivalent to the strong realization problem for DTALS systems. The weak realization problem for DTAPA systems is equivalent to the weak realization problem for DTALS systems.

Sketch of the proof. Notice that if $H = (\mathcal{X}, \mathcal{Y}, Q, (A_q, C_q)_{q \in Q}, x_0)$ and $w = w_0 w_1 w_2 \cdots \in Q^{\omega}$ then we can construct a DTAPA system $\Sigma_{H,w}$ *associated with H and w* such that $\Sigma_{H,w}$ is a realization of the map $y : \mathbb{N} \to \mathcal{Y}$, $y(k) = y_H(x_0, w_0 w_1 \cdots w_k)$. Define $\Sigma_{H,w}$ as follows.

$$\Sigma_H = (\widetilde{X}, \mathcal{Y}, \widetilde{Q}, (\widetilde{X}_q, \widetilde{A}_q, \widetilde{a}_q, \widetilde{C}_q, 0)_{q \in \widetilde{Q}}, (w_0, \widetilde{x}_0))$$

such that

- $\widetilde{Q} = Q \cup \{q_e\}$, $q_e \notin Q$.
- Assume that $\mathcal{X} = \mathbb{R}^n$. For each $q \in Q$, $\widetilde{\mathcal{X}}_q = \mathbb{R}^n \times Z_q$, where $Z_q = \{z \in [0,1] \mid H(z) = \eta(q)\}$
- $\widetilde{\mathcal{X}}_{q_e} = \mathbb{R}^n \times ([0,1] \setminus (\bigcup_{q \in Q} Z_q))$
- $\widetilde{\mathcal{X}} = \bigcup_{q \in \widetilde{Q}} \widetilde{\mathcal{X}}_q$

- For each $q \in Q$,

$$\widetilde{A}_q = \begin{bmatrix} A_q & 0 \\ 0 & 2d \end{bmatrix}, \ \widetilde{a}_q = \begin{bmatrix} 0 \\ -\eta(q) \end{bmatrix} \text{ and } \widetilde{C}_q = \begin{bmatrix} C_q & 0 \end{bmatrix}$$

-

$$\widetilde{A}_{q_e} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \ \widetilde{a}_{q_e} = 0, \ \widetilde{C}_{q_e} = 0 \text{ and } \widetilde{c}_{q_e} = 0$$

- The initial state is of the form $\widetilde{x}_0 = (x_0^T, \phi(w))^T$

Notice that

$$\widetilde{C}_{w_k}\widetilde{A}_{w_{k-1}} \cdots \widetilde{A}_{w_0}\widetilde{x}_0 = C_{w_k}A_{w_{k-1}} \cdots A_{w_0}x_0$$

and $\widetilde{A}_{w_k} \cdots \widetilde{A}_{w_0}\widetilde{x}_0 \in \widetilde{\mathcal{X}}_{w_k}$ for all $k \geq 0$. Thus $y_{H,w}$ is realized by $\Sigma_{H,w}$. Moreover, it is also easy to see that $\Sigma_{H,w}$ is in canonical form and for any DTAPA system Σ the canonical form Σ_{can} coincides with $\Sigma_{H_{\Sigma_l},\phi(x_0)}$.

Conversely, assume that the DTAPA Σ realizes a map $y : \mathbb{N} \to \mathcal{Y}$. Then the DTAPA $\Sigma_l = (\mathcal{X}, \mathcal{Y}, Q, (\mathcal{X}_q, A_q, C_q)_{q \in Q}, (q_0, x_0))$ realizes y too. Let $w = \phi(x_0) \in Q^\omega$. Then it is easy to see that the DTALS $H_\Sigma = (\mathcal{X}, \mathcal{Y}, Q, (A_q, C_q)_{q \in Q}, x_0)$ realizes y_w.

From the discussion above the statements of the theorem follow easily. □

9.3 Realization Theory of DTAPA Systems with Almost-periodical Dynamics

In this section realization theory for DTAPA systems with almost-periodical dynamics will be discussed. By Theorem 55 existence of a realization by a DTAPA system is equivalent to existence of a realization by a DTALS system. If Σ is a DTAPA system with almost-periodical dynamics and $w = \phi(x_0) = w_0w_1w_2 \cdots$, then $L_w = \{w_0w_1 \cdots w_k \mid k \geq 0\}$ is a regular language. Recall that $y : \mathbb{N} \to \mathcal{Y}$ is realized by Σ if $y_w : L_w \to \mathcal{Y}$ is realized by H_Σ. That is why we will first study realization of maps of the form $y : L \to \mathcal{Y}$, L is a regular language, by a DTALS system. In order to study realization by DTALS systems of the maps described above we will use theory of rational formal power series. We will then apply the obtained results to DTAPA systems with almost-periodic dynamics. The outline of the section is the following. Subsection 9.3.1 presents results on realization theory of DTALS systems. Subsection 9.3.2 presents the solution of the realization problem for DTAPA systems with almost-periodic dynamics.

9.3.1 Realization of DTALS Systems: Regular Case

Recall from Section 3.1 the results on theory of formal power series. In this section we will be interested in rational families of formal power series consisting of one single series. In the rest of the section we will tacitly use the notation and terminology of Section 3.1.

Let Q be a finite set and consider a subset $L \subseteq Q^+$. In this subsection we will investigate the problem of finding a realization for a map $y : L \to \mathcal{Y}$, $\mathcal{Y} = \mathbb{R}^p$ by a DTALS system. We proceed as follows. Define the languages $L_q = \{w \in Q^* \mid wq \in L\}$ for all $q \in Q$. Assume that $Q = \{q_1, \ldots, q_N\}$. For each $q \in Q$ define the formal power series $S_{y,q} \in \mathbb{R}^p \ll Q^* \gg$ by

$$\forall w \in Q^* : S_{y,q}(w) = \begin{cases} y(wq) & \text{if } w \in L_q \\ 0 & \text{otherwise} \end{cases}$$

Define the *formal power series* $S_y \in \mathbb{R}^{Np} \ll Q^* \gg$ *associated with* y by

$$\forall w \in Q^* : S_y(w) = \begin{bmatrix} S_{y,q_1}(w) \\ S_{y,q_2}(w) \\ \vdots \\ S_{y,q_N}(w) \end{bmatrix}$$

Define the Hankel-matrix of y by $H_y = H_{S_y}$. Notice that H_y is an infinite matrix which can be constructed from the values of y. Let $H = (\mathcal{X}, \mathcal{Y}, Q, (A_q, C_q)_{q \in Q}, x_0)$ be a DTALS system such that $\mathcal{Y} = \mathbb{R}^p$. Define the *representation* R_H *associated with* H by

$$R_H = (\mathcal{X}, \{A_q\}_{q \in Q}, x_0, \widetilde{C})$$

where $\widetilde{C}x = \begin{bmatrix} C_{q_1}x \\ C_{q_2}x \\ \vdots \\ C_{q_N}x \end{bmatrix}$ for each $x \in \mathcal{X}$. Conversely, let $R = (\mathcal{X}, \{A_q\}_{q \in Q}, x_0\widetilde{C})$ be a representation with $\widetilde{C} : \mathcal{X} \to \mathbb{R}^{pN}$. Define the *DTALS system* H_R *associated with* R by

$$H_R = (\mathcal{X}, \mathcal{Y}, Q, (A_q, C_q)_{q \in Q}, x_0)$$

where $\mathcal{Y} = \mathbb{R}^p$ and $\widetilde{C}x = \begin{bmatrix} C_{q_1}x \\ C_{q_2}x \\ \vdots \\ C_{q_N}x \end{bmatrix}$ for each $x \in \mathcal{X}$.

It is easy to see that $R_{H_R} = R$. The following theorem is an easy consequence of the definition of realization by a DTALS and the definition of a representation.

Theorem 56. *Let $y : \mathbb{N} \to \mathcal{Y}$. If R is a representation of S_y, then H_R is a DTALS realization of y. If $L = Q^+$, then H is a DTALS realization of y if and only if H_R is a representation of S_y.*

Corollary 17. *A map $y : Q^+ \to \mathcal{Y}$ has a realization by a DTALS system if and only if S_y is rational.*

Consider the following formal power series $Z_q \in \mathbb{R}^p \ll Q^* \gg$

$$\forall w \in Q^* : Z_q = \begin{cases} (1,1,\dots,1)^T \in \mathbb{R}^p & \text{if } w \in L_q \\ 0 & \text{otherwise} \end{cases}$$

Define $Z = \begin{bmatrix} Z_{q_1} \\ Z_{q_2} \\ \vdots \\ Z_{q_N} \end{bmatrix} \in \mathbb{R}^p \ll Q^* \gg$. That is, $Z(w) = \begin{bmatrix} Z_{q_1}(w) \\ Z_{q_2}(w) \\ \vdots \\ Z_{q_N}(w) \end{bmatrix}$. Notice that Z is rational if L is regular. Let $H = (\mathcal{X}, \mathcal{Y}, Q, (A_q, C_q)_{q\in Q}, x_0)$ be a DTALS and define $y_H : Q^* \ni w \mapsto y_H(x_0, w)$. It is easy to see that the following theorem holds

Theorem 57. *With the notation above, H is a DTALS realization of $y : L \to \mathcal{Y}$ if and only if*

$$S_y = S_{y_H} \odot Z$$

Notice that S_{y_H} is a rational formal power series, by Theorem 56. We arrive to the following important theorem.

Theorem 58. *Assume that L is regular. Then an input-output map $y : L \to \mathcal{Y}$ has a realization by a DTALS system if and only if S_y is rational, or equivalently* rank $H_y < +\infty$.

Sketch of the proof. If H is a DTALS realization of y then $S_y = S_{y_H} \odot Z$. If L is regular then Z is rational. By Corollary 17 above S_{y_H} is rational, thus $S_y = S_{y_H} \odot Z$ is rational too. Conversely, assume that S_y is rational. Then there exists a representation R of S_y and thus H_R is a DTALS realization of y. □

Thus, if L is regular, then the theorem above allows to construct a realization of y by using the theory of rational formal power series Recall the results on partial realization of rational formal power series from [54, 53]. If L is regular and the number of states of the minimal automaton recognising L is n_L, then rank $H_Z \leq n_L$.

If it is known that y has DTALS realization of state-space dimension at most M, then a representation R of S_y can be constructed from the $|Q|^{M \cdot n \cdot p} \times |Q|^{M \cdot n}$ left upper block of H_y and the construction can be implemented by a numerical algorithm. It is easy to see that the construction of H_R from R can be implemented by a numerical algorithm and H_R is a realization of y.

Let $\widetilde{y} : \mathbb{N} \to \mathbb{R}^p$. Let Q be a a set of discrete modes, let $w \in Q^\omega$ be an infinite word . Recall the definitions of $L_w = \{w_0 \cdots w_k \mid k \geq 0\}$ and $y = \widetilde{y}_w : L_w \ni w_0 \cdots w_k \mapsto \widetilde{y}(k)$ Assume that L_w is regular. The following theorem holds.

Theorem 59. *The map $\widetilde{y}_w : L_w \to \mathbb{R}^p$ has a realization by a DTALS system if and only if y has a realization by a linear discrete-time system, i.e., by a system of the form*

$$x(k+1) = Ax(k) \text{ and } y(k) = Cx(k),\ k \in \mathbb{N} \tag{9.1}$$

where $A \in \mathbb{R}^{n \times n}, C \in \mathbb{R}^{p \times n}, x(k) \in \mathbb{R}^n$.

Sketch of the proof . If y has a realization by a system of the form (9.1), then define the DTALS $H = (\mathcal{X}, \mathcal{Y}, Q, (A_q, C_q)_{q \in Q}, x_0)$ by $\mathcal{X} = \mathbb{R}^n$, $A_q = A, C_q = C, x_0 = x(0)$. It is then clear that

$$C_{w_k} A_{w_{k-1}} A_{w_{k-2}} \cdots A_{w_0} x_0 = CA^k x(0) = Cx(k) = y(k) = \widetilde{y}(w_0 \cdots w_k)$$

and thus H is a realization of $\widetilde{y}_w$. Conversely, assume that

$$H = (\mathcal{X}, \mathcal{Y}, Q, (A_q, C_q)_{q \in Q}, x_0)$$

is a realization of $\widetilde{y}_w$. Let $\mathcal{A} = (S, Q, \delta, F, s_0)$ be a minimal finite-state automaton accepting L_w with the set of accepting states $F \subseteq S$. Here we used the notation of [17, 24]. Due to the very special structure of L_w the automaton $\mathcal{A}$ has a number of remarkable properties. Let $\widetilde{F} = F \cup \{s_0\}$. The automaton $\mathcal{A}$ can be chosen such that $S \setminus \widetilde{F} = \{s_f\}$ and for each $s \in \widetilde{F}$ there exists a unique $q \in Q$ for which $s^{'} = \delta(s, q) \in F$. For each $s \in \widetilde{F}$ define $\mathcal{X}_s = \mathcal{X}$ and let $\widetilde{\mathcal{X}} = \bigoplus_{s \in \widetilde{F}} \mathcal{X}_s$. Define the map $\widetilde{A} : \widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}$ as follows. For each $s \in \widetilde{F}$, $z \in \mathcal{X}_s$ let $\widetilde{A}z = A_q z \in \mathcal{X}_{\delta(s,q)}$ where $q \in Q$ is the unique element of Q such that $\delta(s, q) \in F$. Define the map $\widetilde{C} : \widetilde{\mathcal{X}} \to \mathbb{R}^p$ as follows. For each $s \in \widetilde{F}, z \in \mathcal{X}_s$ define $\widetilde{C}z = C_q z$ where $q \in Q$ is such that $\delta(s, q) \in F$. Define the initial state $x(0) = x_0 \in \mathcal{X}_{s_0}$. Then it is easy to see that $\widetilde{C}\widetilde{A}^k x(0) = C_{w_k} A_{w_{k-1}} \cdots A_{w_0} x_0$. and thus

$$x(k+1) = \widetilde{A}x(k),\ y(k) = \widetilde{C}x(k),\ x(0) = x_0$$

is indeed a linear system realizing y. $\square$

9.3.2 Realization of DTAPA Systems: Almost-periodical Dynamics

Consider a DTAPA system Σ. Assume that Σ has an almost periodical dynamics, i.e., $card(\{S^k(\phi(x_0)) \mid k \geq 0\}) < +\infty$, where $S^0 = id$, $S^{k+1} = S^k \circ S$, $k \geq 0$ and $S(w_0 w_1 \cdots) = w_1 w_2 \cdots$, that is, S is the shift operator on infinite sequences. It is easy to see that Σ has an almost periodic dynamics if and only if $\Sigma_l = (\mathcal{X}, \mathcal{Y}, Q, (\mathcal{X}_q, A_q, C_q)_{q \in Q}, (q_0, x_0))$ has an almost-periodic dynamics. It is easy to see that $card(\{S^k(\phi(x_0)) \mid k \geq 0\}) < +\infty$ holds if and only if $L_{\phi(x_0)}$ is a regular language. That is, Σ is almost-periodic if and only if $L_{\phi(x_0)}$ is a regular language. Using the results from the previous subsection and recalling Theorem 55 we get the following result

Theorem 60. *Consider an input-output map $y : \mathbb{N} \to \mathbb{R}^p$.*

(i) The map y has a realization by a DTAPA system with almost-periodic dynamics if and only if y has a realization by autonomous discrete-time linear system of the form

$$x(k+1) = Ax(k) \text{ and } y(k) = Cx(k),\ k \in \mathbb{N} \tag{9.2}$$

(ii) Let $\widetilde{Q}$ be a finite set and let $w \in \widetilde{Q}^\omega$. The map y has a realization by a DTAPA system

$$\Sigma = (\mathcal{X}, \mathcal{Y}, Q, (\mathcal{X}_q, A_q, a_q, C_q, c_q)_{q \in Q}, (q_0, x_0))$$

such that $\phi(x_0) = w \in \widetilde{Q}^\omega$, $\widetilde{Q} \subseteq Q$ and Σ has almost periodic dynamics if and only if rank $H_{y_w} < +\infty$.

9.4 Realization of General DTAPA Systems

In this section we will study the realization problem for DTAPA systems with not necessarily almost-periodic dynamics. By Theorem 55 the realization problem for DTAPA systems is equivalent to the realization problem for DTALS systems. Thus, we will study the weak and strong realization problems for DTALS systems. More precisely, we will start with solving the following problem

Weak realization problem for DTALS For a specified map $y : \mathbb{N} \to \mathcal{Y}$, for a specified set of discrete modes Q and infinite word $w \in Q^\omega$ find a DTALS system H such that H is a realization of $y_w : L_w \to \mathcal{Y}$.

Strong realization problem for DTALS For a specified map $y : \mathbb{N} \to \mathcal{Y}$ find a set of discrete modes Q, an infinite word $w \in Q^\omega$ and a DTALS system H such that H realizes $y_w : L_w \to \mathcal{Y}$.

Unlike in the previous section, in the current section we do not assume that L_w is regular. We will use the solution of the problem above to solve the weak and strong realization problems for DTAPA systems. The outline of the section is the following. Subsection 9.4.1 discusses the weak and strong realization problems for DTALS systems. Subsection 9.4.2 presents results on the weak and strong realization problem for DTAPA systems.

9.4.1 Realization of DTALS Systems

We will study the weak and the strong realization problems of DTALS systems. We will adopt an abstract approach, similar to realization theory of linear systems over rings and realization theory of time-varying systems, see [63, 42].

For any function $h : C \to D$ denote the range of the function by $R(h) = \{h(c) \mid c \in C\} \subseteq D$ Define the following sets.

$$\mathcal{A} = \{g : \mathbb{N} \to \mathbb{R}\}$$

$$\mathcal{A}_f = \{g : \mathbb{N} \to \mathbb{R} \mid R(g) \text{is finite, i.e.,} \, card(R(g)) < +\infty\}$$

For each finite set Q and each infinite word $w \in Q^\omega$ define the set

$$\mathcal{A}_w = \{g : \mathbb{N} \to \mathcal{R} \mid \forall i, j \in \mathbb{N} : w_i = w_j \implies g(i) = g(j)\}$$

Define the shift map $\sigma : \mathcal{A} \to \mathcal{A}$ by $\sigma(f)(n) = f(n+1)$. It is easy to see that

$$\mathcal{A}_w \subseteq \mathcal{A}_f \subseteq \mathcal{A}$$

It is also easy to see that $\mathcal{A}$ is an algebra with point-wise multiplication, point-wise addition and point-wise multiplication by a scalar. That is, $(g+f)(n) = f(n)+g(n)$, $(gf)(n) = g(n)f(n)$, $(\alpha g)(n) = \alpha g(n)$. With the operations above $\mathcal{A}_f$ is a sub-algebra of $\mathcal{A}$ and $\mathcal{A}_w$ is a sub-algebra of $\mathcal{A}_f$. Notice that σ becomes an algebra homomorphism. It is also easy to see that $\sigma(\mathcal{A}_f) \subseteq \mathcal{A}_f$ and $\sigma(A_w) \subseteq \mathcal{A}_{S(w)}$, where $S : Q^\omega \to Q^\omega$ is the shift map on infinite sequences. Let $\mathcal{A}_{S,w}$ be the smallest sub-algebra of $\mathcal{A}_f$ generated by algebras $\mathcal{A}_{S^k(w)}$,$k \geq 0$. Define the kth iterate of the shift by $\sigma^0 = id$,i.e. $\sigma^0(g) = g$ and $\sigma^{k+1} = \sigma \circ \sigma^{k+1}$ for all $k \in \mathbb{N}$.

Let $y : \mathbb{N} \to \mathbb{R}^p$ be a input-output map. Define the maps $y_i : \mathbb{N} \to \mathbb{R}$, $i = 1, \ldots, p$ by $y(k) = (y_1(k), \ldots, y_p(k))^T$, i.e., y_i are the coordinate functions of y. Define the set $W_y = \{\sigma^k(y_i) \mid k \in \mathbb{N}, i = 1, \ldots, p\}$. We will call W_y the *Hankel-matrix* of y. The following theorem holds.

Theorem 61. *Consider a map $y : \mathbb{N} \to \mathcal{Y}$. Let Q be a finite set and let $w = w_0 w_1 \cdots \in Q^\omega$ be an infinite word. There exists a DTALS*

$$H = (\mathcal{X}, \mathcal{Y}, Q, (A_q, C_q)_{q \in Q}, x_0)$$

such that $y_H(x_0, w_0 \cdots w_k) = y_w(w_0 \cdots w_k) = y(k)$, $k \in \mathbb{N}$, i.e. Σ is a realization of y_w if and only if there exists a finitely generated $A_{S,w}$ submodule $Z \subseteq \mathcal{A}$ such that

- $W_y \subseteq Z$
- $\sigma(Z) \subseteq Z$
- *There exists elements $z_1, \ldots, z_d \in Z$ such that*

$$y_1, \ldots, y_p, \sigma(z_1), \sigma(z_2), \ldots, \sigma(z_d) \in \{\sum_{j=1}^{d} \alpha_j z_j \mid \alpha_j \in A_w, j = 1, \ldots, d\}$$

Sketch of the proof. `"only if part"`
Let $\Sigma = (\mathcal{X}, \mathcal{Y}, Q, (A_q, C_q)_{q \in Q}, x_0)$ such that $y_\Sigma(x_0, w_0 \cdots w_k) = y(k)$ for all $k \in \mathbb{N}$. Without loss of generality we can assume that $\mathcal{X} = \mathbb{R}^n$. Define the maps $z_i : \mathbb{N} \to \mathbb{R}$ by $z_i(k) = e_i^T A_{w_{k-1}} \cdots A_{w_0} x_0$ for all $i = 1, \ldots, n$, $k \geq 0$, where e_i is the ith unit vector of $\mathbb{R}^n$. Define the maps $A : \mathbb{N} \to \mathbb{R}^{n \times n}$ and $C : \mathbb{N} \to \mathbb{R}^{p \times n}$ by $A(k) = A_{w_k}$ and $C(k) = C_{w_k}$. Then it is easy to see that $z_i(k+1) = \sum_{j=1}^{n} (A(k))_{i,j} z_j(k)$ and $y_i(k) = \sum_{j=1}^{n} (C(k))_{i,j} z_j(k)$. That is, $\sigma(z_i) = \sum_{j=1}^{n} A_{i,j} z_j$ and $y_i = \sum_{j=1}^{n} C_i z_i$, where $A_{i,j}(k) = (A(k))_{i,j}$ and $C_i(k) = (C(k))_i$. Define $Z = \mathrm{Span}_{A_{S,w}}\{z_1, \ldots, z_n\}$. and $V = \mathrm{Span}_{A_w}\{z_1, \ldots, z_n\}$. Then it is easy to see that Z is a finite $A_{S,w}$ module, $W_y \subseteq Z$, $y_1, \ldots, y_p, \sigma(z_1), \ldots, \sigma(z_n) \in V$.

`"if part"`
Assume that $\sigma(z_i) = \sum_{j=1}^{d} a_{i,j} z_i$ and $y_i = \sum_{j=1}^{d} c_{i,j} z_j$. Let $A_q = (a_{i,j}(k))_{i,j=1,d}$ if $w_k = q$ for some $k \in \mathbb{N}$ and A_q arbitrary otherwise. Let $C_q = (c_{i,j}(k))_{i,j=1,d}$ if $w_k = q$ for some $k \in \mathbb{N}$ and arbitrary otherwise. Let $\mathcal{X} = \mathbb{R}^d$ and $x_0 = (z_1(0), \ldots, z_d(0))^T$. Then $\Sigma = (\mathcal{X}, \mathcal{Y}, Q, (A_q, C_q)_{q \in Q}, x_0)$ is a DTALS realization of y_w. $\square$

The following is an easy corollary of the theorem above.

Corollary 18. *Withe the assumptions of the theorem above the following holds. If $\mathcal{A}_{S,w}$ is a Noethierian ring, then y_w has a realization by a DTALS if and only if $Z = \mathrm{Span}_{A_{S,w}}\{z \in \mathcal{A} \mid z \in W_y\}$ is a finitely generated $\mathcal{A}_{S,w}$ module and there exists $z_1, \ldots, z_d \in Z$ such that $y_i, \sigma(z_j) \in \mathrm{Span}_{A_w}\{z_1, \ldots, z_d\}$ for each $i = 1, \ldots, p, j = 1, \ldots, d$.*

Next we turn to the strong realization problem. We get the following theorem. Denote by $\mathrm{Im}W_y = \mathrm{Span}_{\mathcal{A}_f}\{z \in W_y\} = \{\sum_{j=1}^{N} \alpha_j z_j \mid N \geq 0, \alpha_j \in \mathcal{A}_f, z_j \in W_y, j = 1, \ldots, N\}$.

Theorem 62. *Let $y : \mathbb{N} \to \mathbb{R}^p$. There exists a set of discrete modes Q, an infinite word $w \in Q^\omega$ and a DTALS H such that H is a realization of y_w if and only if there exists a finitely generated $\mathcal{A}_f$ submodule $Z \subseteq \mathcal{A}$ of $\mathcal{A}$ such that*

- $\sigma(Z) \subseteq Z$
- $W_y \subseteq Z$

Sketch of the proof. The only if part is clear from Theorem 61 by noticing that if Z is a finitely generated $\mathcal{A}_{S,w}$ module, then $\widetilde{Z} = \mathrm{Span}_{\mathcal{A}_f}\{z \in Z\} = \{\sum_{j=1}^{K} \alpha_j z_j \mid K \geq 0, \alpha_j \in \mathcal{A}_f, z_j \in Z, j = 1, \ldots, K\}$ is a finitely generated $\mathcal{A}_f$ module.

Assume that Z is a finitely generated $\mathcal{A}_f$ submodule of $\mathcal{A}$ satisfying the condition of the theorem. Assume that $z_1, \ldots, z_n$ is a basis of Z. Assume that $\sigma(z_i) = \sum_{j=1}^{n} a_{i,j} z_j$ and $y_i = \sum_{j=1}^{n} c_{i,j} z_j$. Let $A(k) = (a_{i,j}(k))_{i,j=1,\ldots,n}$ and $C(k) = (c_{i,j}(k))_{i=1,\ldots,p,j=1,\ldots,n}$ for all $k \geq 0$. Define $Q = \{(A(k), C(k)) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{p \times n} \mid k \geq 0\}$. Since $a_{i,j}, c_{l,j} \in A_f$ for all $i, j = 1, \ldots, n, l = 1, \ldots, p$ we get that Q is finite. Define $w = w_0 \cdots w_k \cdots \in Q^\omega$ such that $w_i = (A(i), C(i))$ for all $i \in \mathbb{N}$. Let $\mathcal{X} = \mathbb{R}^n$ and for each $q = (A(k), C(k)) \in Q$ let $A_q = A(k)$ and $C_q = C(k)$. Let $x_0 = (z_1(0), \ldots, z_n(0))^T$. Then it is easy to see that $H = (\mathcal{X}, \mathcal{Y}, Q, (A_q, C_q)_{q \in Q}, x_0)$ is a realization of y_w. $\square$

Corollary 19. *Let $y : \mathbb{N} \to \mathbb{R}^p$. If $\mathrm{Im}W_y$ is a finitely generated $\mathcal{A}_f$ module, then there exists a finite set Q, an infinite word $w \in Q^\omega$ and DTALS H realizing y_w.*

Corollary 20. *Assume that there exists a finite collection of real number $\{\alpha_{i,j} \in \mathbb{R} \mid i = 1, \ldots, M, j = 1, \ldots, K\}$ such that for each $l \in \mathbb{N}$ there exists a $i_l \in \{1, \ldots, M\}$ such that*

$$y(K + l) = \sum_{j=1}^{K} \alpha_{i_l,j} y(l + j)$$

Then y can be realized by a DTALS system in the strong sense, that is, there exists a finite set Q, an infinite word $w \in Q^\omega$ and a DTALS Σ such that Σ realizes y_w.

9.4.2 Realization of DTAPA Systems

By Theorem 55 the strong and weak realization problems for DTAPA systems and DTALS systems are equivalent. That is, if y is realized by a DTALS H with an infinite word $w \in Q^\omega$, i.e., H is a realization of y_w, then the DTAPA system $\Sigma_{H,w}$

associated with Σ (see proof of Theorem 55), is a realization of y. Conversely, if Σ is a DTAPA system realizing y, then H_Σ is a DTALS system realizing y_w, where $w = \phi(x_0)$. Combining these results with Theorem 61 and Theorem 62 we get the following results, which in some sense are the main results of the paper.

Theorem 63 (Main result). *Let $y : \mathbb{N} \to \mathbb{R}^p$. The following holds.*

- *There exists a DTAPA system realizing y if and only if W_y is contained in a finitely generated shift-invariant $\mathcal{A}_f$ submodule of $\mathcal{A}$, i.e. there exists a finitely generated $\mathcal{A}_f$ submodule $Z \subseteq \mathcal{A}$ such that*

$$W_y \subseteq Z \text{ and } \sigma(Z) \subseteq Z$$

- *Let $\widetilde{Q}$ be a set of discrete modes and let $w \in \widetilde{Q}^\infty$ be an infinite word. There exists a DTAPA $\Sigma = (\mathcal{X}, \mathcal{Y}, Q, (\mathcal{X}_q, A_q, a_q, C_q, c_q)_{q \in Q}, (q_0, x_0))$ such that $\phi(x_0) = w$, $\widetilde{Q} \subseteq Q$, if and only if there exists a finitely generated $\mathcal{A}_{S,w}$ submodule Z of $\mathcal{A}$ such that*

  - $W_y \subseteq Z$
  - $\sigma(Z) \subseteq Z$
  - *There exists elements $z_1, \ldots, z_d \in Z$ such that*

$$y_1, \ldots, y_p, \sigma(z_1), \sigma(z_2), \ldots, \sigma(z_d) \in \{\sum_{j=1}^{d} \alpha_j z_j \mid \alpha_j \in A_w, j = 1, \ldots, d\}$$

We can easily restate the corollaries from the end of the previous section in terms of DTAPA realizations. Note that the DTAPA realizations existence of which is stated in the theorem above can be constructed as follows. Using the proofs of Theorem 61 or Theorem 62 (depending on which theorem can be applied) construct the DTALS H system realizing y_w and then construct the DTAPA system $\Sigma_{H,w}$ associated with H.

Corollary 21. *Let $y : \mathbb{N} \to \mathbb{R}^p$. If $\mathrm{Im} W_y$ is a finitely generated $\mathcal{A}_f$ module, then there exists a DTAPA system realizing y.*

Corollary 22. *Assume that there exists a finite collection of real number $\{\alpha_{i,j} \in \mathbb{R} \mid i = 1, \ldots, M, j = 1, \ldots, K\}$ such that for each $l \in \mathbb{N}$ there exists a $i_l \in \{1, \ldots, M\}$ such that*

$$y(K + l) = \sum_{j=1}^{K} \alpha_{i_l,j} y(l + j)$$

Then y can be realized by a DTAPA system,

Chapter 10

Computational Issues and Partial Realization

The goal of the present chapter is to present partial realization theory for a number of classes of hybrid systems and to discuss the algorithmic aspects of realization theory for these classes of hybrid systems. The classes of hybrid systems discussed in this chapter are the following: linear and bilinear switched systems and linear and bilinear hybrid systems. We will discuss the following issues concerning hybrid systems

Partial realization theory

Computation of a minimal realization

Checking observability, semi-reachability and minimality

In the previous chapters we gave necessary and sufficient conditions for existence of a realization by a hybrid system belonging to one of the classes mentioned above.The common feature of the proof of these conditions is that they all involve a procedure for construction of a hybrid system realization of suitable class from data which can be directly extracted from the input-output maps. Unfortunately the procedures described in the proofs use infinite number of data and thus can not be implemented. Partial realization theory aims at solving this problem. Its goal is to formulate algorithms which compute a realization of a set of input-output maps from finite data. Of course, the available data has to be rich enough to contain all the necessary information about the input-output maps. Therefore, formulating a partial realization theory for a class of systems also involves specifying conditions under which the data is rich enough to construct a realization for the whole set of input-output maps. Partial realization theory also serves as a theoretical basis for system identification. If an algorithm is available for (re)constructing a realization of the input-output behaviour from finite data, then it is enough to concentrate on obtaining the necessary data in order to reconstruct the state-space representation of the system.

Another issue which will be addressed in this chapter is computation of minimal realizations. That is, we will present algorithms for computing a minimal hybrid system realization of a set of input-output maps from arbitrary hybrid system realizations. We will also present algorithms for checking observability and semi-reachability for a number of classes of hybrid systems. In the previous chapters we already presented linear algebraic conditions for observability and semi-reachability. In this chapter we will show that these conditions can be checked by numerical algorithms involving standard linear algebraic operations.

Recall that realization theory of linear and bilinear hybrid and switched systems relies on theory of hybrid formal power series and classical formal power series respectively. More precisely, existence of a realization by a hybrid system belonging to one of the classes mentioned above is equivalent to existence of a rational formal power series representation or a hybrid representation of a family of classical or hybrid formal power series. Minimality, observability or semi-reachability of hybrid systems of the above type can also be reformulated as minimality, observability and reachability of certain classical rational or hybrid representations. Thus, it is enough to formulate a partial realization theory and algorithms for hybrid and classical formal power series representations. The obtained theory and algorithms can be then directly applied to hybrid systems of the above type.

As we already mentioned several times in previous chapters, the theory of formal power series and their representations is a classical one, see [64, 65, 32, 4, 43, 20, 22]. In this chapter we will use the extension of this theory developed in Section 3.1. That is, instead of dealing with a *single formal power series* we work with *families of formal power series* and their representations. Many of the algorithms for formal power series representations which are presented in this paper have already been formulated for the classical case of a single formal power series. In particular, results on partial realization theory can be found in [25], where partial realization theory for bilinear discrete-time systems was discussed. In view of the well-known correspondence between bilinear systems and representations of rational formal power series ([64, 65, 32], for instance), the theory formulated in [25] can be easily adapted to the case of rational formal power series. Unfortunately the results of [25] can not be directly used for the general framework adopted in this paper. Besides, the author failed to find a paper containing the proofs of the results from [25]. There are works on subspace identification for discrete-time bilinear systems [19, 7]. The paper [19, 7] also contains a SVD decomposition algorithm for computing discrete-time bilinear system realizations. Again, the results presented in [19, 7] can not be applied directly to the framework adopted in this paper. Therefore we felt compelled to present all the results in detail again. However, the presented algorithms for formal power series are indeed very similar to the already known ones.

The partial realization theory and the presented algorithms for hybrid formal power series are, to our best knowledge, new. Recall that the problem of finding a hybrid representation for a family of hybrid formal power series can be decomposed into two subproblems. One subproblem is finding a rational representation for a family of formal power series, the other subproblem is finding a realization by a Moore-automaton for a family of discrete-valued input-output maps. Thus, partial realization theory of hybrid formal power series can be based on partial realization theory for formal power series and partial realization theory for Moore-automata. Both theories are to large extend classical, although in the current context we will have to extend the classical theories to accommodate the use of families of formal power series and input-output maps. But the necessary extension of the classical theory is quite straightforward. Recall that deciding minimality, observability and reachability of a hybrid representation can be reduced to deciding observability, reachability of a certain formal power series representation and a certain Moore-automaton. Thus, we can use the almost classical algorithms available for Moore-automata and formal power series representations to decide minimality, observability and reachability of hybrid representations.

The outline of the chapter is the following. Section 10.1 deals with partial realization theory of formal power series representations. It also presents algorithms for computing a minimal rational representation, for checking observability, reachability and minimality of rational representations. Section 10.2 presents partial realization theory for Moore-automata along with algorithms for computing a minimal Moore-automaton realization and checking reachability, observability and minimality. The material of Section 10.2 is a simple extension of the classical results. Section 10.3 presents partial realization theory and algorithms for hybrid formal power series and rational hybrid representations. It also presents algorithms for computing a minimal hybrid representation and for checking reachability, observability and minimality of hybrid representations. Section 10.4 presents partial realization and the algorithms for minimality reduction and deciding reachability, observability and minimality for linear and bilinear switched systems. Section 10.5 presents realization theory and and the corresponding algorithms for linear and bilinear hybrid systems.

10.1 Formal Power Series

The current section discusses partial realization theory and the corresponding algorithms for rational formal power series representations. The outline of the section is the following.

Subsection 10.1.1 presents partial realization theory for formal power series. Subsection 10.1.2 presents an algorithm for computing a minimal rational representation of a family of formal power series from a finite sub-matrix of the Hankel-matrix of the family. The algorithm employs a matrix factorization step, and it is very similar to classical subspace indetification like algorithms. Subsection 10.1.3 contains algorithms for computing a representation for a family of formal power series, for checking observability, reachability of a representation and for transforming a representation to a reachable (observable) one. Subsection 10.1.3 is in fact the backbone of the paper, the results from this section will be heavily used in the rest of the chapter.

10.1.1 Partial Realization Theory

Consider the Hankel matrix of H_Ψ of Ψ. Assume that $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$. Denote by $\mathbb{R}^\infty$ the set of infinite sequences of real numbers, that is, $\mathbb{R}^\infty = \{(\alpha_n) \mid \alpha_n \in \mathbb{R}, n \in \mathbb{N}\}$. It is easy to see that $\mathbb{R}^\infty$ is a vector space with respect to element-wise addition and element-wise multiplication by scalar. That is, $(\alpha_n) + (\beta_n) = (\alpha_n + \beta_n)$ and $b(\alpha_n) = (b\alpha_n)$, $b \in \mathbb{R}$. It is easy to see that if J is countable, then $K = X^* \times J$ is countable. The set $L = X^* \times \{1, \dots, p\}$ is always countable. That is, there exists maps $\psi_1 : L \to \mathbb{N}$ and $\psi_2 : K \to \mathbb{N}$ such that ψ_1 and ψ_2 are bijections and the following holds. For each $(u,i), (v,j) \in L$, if $|u| < |v|$, or $u = v$ and $i < j$ then $\psi_1((u,i)) \le \psi((v,j))$. For each $(u,j_1), (v,j_2) \in K$, if $|u| < |v|$, then $\psi_2((u,j_1)) < \psi_2((v,j_2))$. Then the Hankel-matrix H_Ψ can be viewed as a matrix $H_\Psi \in \mathbb{R}^{\infty \times \infty}$ such that

$$(H_\Psi)_{k,l} = (S_j(uv))_i \text{ if } \psi_1((v,i)) = k \text{ and } \psi_2((u,j)) = l$$

It is clear that the column space $\mathrm{Im} H_\Psi$ is a subspace of $\mathbb{R}^\infty$. Define the map $\phi : \mathbb{R}^p \ll X^* \gg \to \mathbb{R}^\infty$ by

$$\phi(T)_k = (T(u))_i \text{ if } \psi_1((u,i)) = k$$

It is clear that ϕ is a linear isomorphism. Moreover, it is also easy to see that $\phi(W_\Psi) = \mathrm{Im} H_\Psi$.

Below we will present conditions, under which a representation of Ψ can be constructed from finite data. The approach is similar to [25]. For each $S \in \mathbb{R}^p \ll X^* \gg$

let S_N denote the restriction of S to the set $X^{<N} = \{w \in X^* \mid |w| < N\}$, that is

$$S_N(w) = S|_{X^{<N}}(w) = S(w) \text{for all } w \in X^*, |w| < N$$

Denote by $\mathbb{R}^p \ll X^{<N} \gg$ the set of functions $S : X^{<N} \to \mathbb{R}^p$. It is clear that $\mathbb{R}^p \ll X^{<N} \gg$ forms a vector space with point-wise addition and point-wise multiplication by scalar. Moreover, for any $S \in \mathbb{R}^p \ll X^{<N} \gg$ there exists a $T \in \mathbb{R}^p \ll X^* \gg$ such that $S = T_N$. One can also define the map $\eta_N : \mathbb{R}^p \ll X^* \gg \to \mathbb{R}^p \ll X^{<N} \gg$ by

$$\eta_N(T) = T_N, \text{ for all } T \in \mathbb{R}^p \ll X^* \gg$$

It is easy to see that η_N is a surjective linear map.

Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ be an indexed set of formal power series. A representation $R = (\mathcal{X}, \{A_x\}_{x \in X}, C, B)$, $B = \{B_j \in \mathcal{X} \mid j \in J\}$ is said to be an *N-partial representation of* Ψ if for each $j \in J$ and $w \in X^*, |w| < N$

$$S_j(w) = CA_wB_j$$

Define the vector space $W_{\Psi,N,M}$ by

$$W_{\Psi,M,N} = \{(w \circ S_j)_M \mid w \in X^*, |w| < N, j \in J\}$$

Define the sets $I_M = \{(v,i) \mid v \in X^*, |v| < M, i = 1, \ldots, p\}$ and $J_N = \{(u,j) \mid j \in J, u \in X^*, |u| < N\}$. Define the maps $\psi_{1,M} : I_M \to \mathbb{N}$ and $\psi_{2,N} : J_M \to \mathbb{N}\}$ such that $\psi_{1,M}((u,i)) = \psi_1((u,i)), (u,i) \in I_M$ and $\psi_{2,N}((v,j)) = \psi_2((v,j)), (v,j) \in J_N$. It is easy to see that the following holds. For any $(u,i),(v,j) \in I_M$, if $|u| < |v|$ or $u = v$ and $i < j$, then $\psi_{1,M}((u,i)) < \psi_{1,M}((v,j))$. For any $(u,j_1),(v,j_2) \in J_N$, if $|u| < |v|$, then $\psi_{2,N}((u,j_1)) < \psi_{2,N}((v,j_2))$. It is also easy to see that $\psi_{1,M}(I_M) = \psi_1(I_M) = \{1, \ldots, |I|_M\}$. Indeed, since I_M is finite and ψ_1 is a bijection, $\psi_1(I_M)$ is finite and has $|I_M|$ elements. Let G be the maximal element of $\psi_1(I_M)$. Let $j \in \{1,2,\ldots,G\}$ such that $j \notin \psi_1(I_M)$. Assume that $j = \psi_1((u,i))$ and $G = \psi_1((v,l))$. Since $(v,l) \in I_M$, we get that $|v| \le M$. If $(u,i) \notin I_M$, then $|u|leM > |v|$. But then $G = \psi_1((v,l)) < \psi_1((u,i))$, which is a contradiction. Thus, $(u,i) \in I_M$, and thus $j \in \psi_1(I_M)$. That is, $\psi_1(I_M) = \{1,2,\ldots,G\}$. Since $\psi_1(I_M)$ is of cardinality $|I_M|$ we get that $G = |I_M|$. Similarly, we can show that $\psi_2(J_N) = \{1, \ldots, |J_N|\}$ if J is finite.

From now one we assume that J is a finite. Define $H_{\Psi,M,N} \in \mathbb{R}^{|I_M| \times |J_N|}$ by

$$(H_{\Psi,M,N})_{k,l} = (H_\Psi)_{k,l} = (S_j(uv))_i \text{ if } \psi_{1,M}((v,i)) = k, \psi_{2,N}((u,j)) = l$$

That is, $H_{\Psi,N,M}$ is the left upper corner $|I_N| \times |J_N|$ block matrix of H_Ψ. Notice that if J is finite, then $|J_N| < +\infty$, that is, $H_{\Psi,N,M}$ is a *finite matrix*. Define the map $\psi_N : \mathbb{R}^p \ll X^{<N} \gg \to \mathbb{R}^{I_N}$ by

$$(\psi_N(T))_k = (T(u))_i \text{ if } \psi_{1,N}((u,i)) = k$$

It is easy to see that ψ_N is a linear isomorphism. Moreover, $\mathrm{Im} H_{\Psi,M,N} = \psi_M(W_{\Psi,M,N})$. That is, $\dim W_{\Psi,M,N} = \text{rank } H_{\Psi,M,N}$.

It turns out that under certain circumstances partial representations not only exist but they also yield a minimal representation of the whole indexed set of formal power series. Moreover, such partial representations can be constructed from finite data.

Theorem 64 (Partial representation). *With the notation above the following holds.*

(i) If R is a representation of Ψ, $\dim R \leq N$, then

$$\text{rank } H_{\Psi} = \text{rank } H_{\Psi,N,N} = \text{rank } H_{\Psi,N+1,N} = \text{rank } H_{\Psi,N,N+1}$$

(ii) If

$$\text{rank } H_{\Psi,N,N} = \text{rank } H_{\Psi,N,N+1} = \text{rank } H_{\Psi,N+1,N}$$

then there exists an N-partial representation $R_N = (W_{\Psi,N,N}, \{A_x\}_{x\in X}, B, C)$ of Ψ, such that $A_x((w \circ S_j)_N) = (wx \circ S_j)_N$, $C(T) = T(\epsilon)$, $B_j = (S_j)_N, j \in J$,

(iii) If

$$\text{rank } H_{\Psi,N,N} = \text{rank } H_{\Psi}$$

then

$$\text{rank } H_{\Psi,N+1,N} = \text{rank } H_{\Psi,N,N} = \text{rank } H_{\Psi,N,N+1}$$

and R_N is a minimal representation of Ψ.

(iv) If Ψ has a representation R such that $N \geq \dim R$, then the representation R_N is a minimal representation of Ψ.

Proof. The proof of the theorem relies on a number of lemmas, which will be stated and proven after the proof of the theorem. Below we will proceed with the proof of the theorem.

`Part (i)`

Define $W_{\Psi,.,N} = \mathrm{Span}\{(w \circ S_j) \mid j \in J, |w| < N\}$. Then Lemma 50 implies that $W_{\Psi,.,N} = W_{\Psi}$. Notice that $\eta_N(W_{\Psi,N,.}) = W_{\Psi,N,N}$ and Lemma 53 implies that η_N is a linear isomorphism, that is $\dim W_{\Psi} = \dim W_{\Psi,.,N} = \dim W_{\Psi,N,N}$. That is, $\text{rank } H_{\Psi} = \dim W_{\Psi} = \dim W_{\Psi,N,N} = \text{rank } H_{\Psi,N,N}$, since $W_{\Psi,N,N}$ and the column space of $H_{\Psi,N,N}$ are isomorphic. Since $W_{\Psi,.,N} \subseteq W_{\Psi,.,N+1}$ we get that $W_{\Psi,.,N+1} = W_{\Psi,.,N}$. From Lemma 53 it follows that $\eta_{N+1}|_{W_{\Psi}}$ is an isomorphism. Thus, $\dim W_{\Psi} = \dim \eta_{N+1}(W_{\Psi,.,N}) = \dim W_{\Psi,N+1,N}$. Since $\eta_N|_{W_{\Psi}}$ is an isomorphism too, we get that $\dim W_{\Psi} = \dim \eta_N(W_{\Psi,.,N+1}) = \dim W_{\Psi,N+1,N}$ Thus, we get that rank $H_{\Psi,N+1,N} = \dim W_{\Psi,N+1,N} =$ rank H_Ψ and rank $H_{\Psi,N,N+1} = \dim W_{\Psi,N,N+1} =$ rank H_Ψ.

`Part (ii)`

rank $H_{\Psi,N,N} =$ rank $H_{\Psi,N+1,N} =$ rank $H_{\Psi,N,N+1}$ implies that $\dim W_{\Psi,N,N} = \dim W_{\Psi,N+1,N} = \dim W_{\Psi,N,N+1}$. Since $W_{\Psi,N,N} \subset W_{\Psi,N,N+1}$ we get that $W_{\Psi,N,N} = W_{\Psi,N,N+1}$. Define the map $\widetilde{\eta}_N : W_{\Psi,N+1,N} \ni S \mapsto S_N$, where $S_N = S|_{\{w \in X^* || w| < N\}}$. It is easy to see that $\widetilde{\eta}_N$ is linear, and since $(w \circ S_j)_N = \widetilde{\eta}_N((w \circ S_j)_{N+1})$ for all $|w| < N$, $j \in J$ we get that $\widetilde{\eta}_N$ is surjective. Since $\dim W_{\Psi,N+1,N} = \dim W_{\Psi,N,N}$ we get that $\widetilde{\eta}_N$ is a linear isomorphism. For each $x \in X$, consider the map $T_x : W_{\Psi,N+1,N} \to W_{\Psi,N,N}$ defined by $T_x(S)(w) = S(xw)$, $w \in X^*, |w| < N$. It is easy to see that T_x is a linear map. Let $A_x = T_x \circ \widetilde{\eta}_N^{-1}$, i.e. $A_x(Z) = T_x(\widetilde{\eta}_N^{-1}(Z))$. It is easy to see that A_x is linear and for all $w \in X^*, |w| < N$, $A_x((w \circ S_j)_N)(v) = S_j(wxv)$ for all $v \in X^*, |v| < N$. That is, $A_x((w \circ S_j)_N) = (wx \circ S_j)_N$ is satisfied for each $|w| \leq N, w \in X^*$. Define $C : W_{\Psi,N,N} \ni T \mapsto T(\epsilon)$ and $B_j = (S_j)_N$. It is easy to see that C is a linear map. That is, $R_N = (W_{\Psi,N,N}, \{A_x\}_{x \in X}, B, C)$ is indeed a well-defined representation. It is easy to see that $w \in X^*, |w| < N$ it holds that $A_w B_j = A_w(S_j)_N) = (w \circ S_j)_N$, which implies that $C A_w B_j = (w \circ S_j)_N(\epsilon) = S_j(w)$ for each $|w| < N, w \in X^*$. That is, R_N is an N-representation of Ψ.

`Part(iii)`

It is easy to see that rank $H_{\Psi,N,N} \leq$ rank $H_{\Psi,N+1,N} \leq$ rank H_Ψ and rank $H_{\Psi,N,N} \leq$ rank $H_{\Psi,N,N+1} \leq H_\Psi$. Thus, if rank $H_{\Psi,N,N} =$ rank H_Ψ then rank $H_{\Psi,N,N} =$ rank $H_{\Psi,N+1,N} =$ rank $H_{\Psi,N,N+1}$. We also get that $\dim W_{\Psi,N,N} = \dim W_{\Psi,N+1,N} = \dim W_{\Psi,N,N+1} = \dim W_\Psi$. Consider the map $\eta_N : W_\Psi \to W_{\Psi,N,.}$, where $W_{\Psi,N,.} = \mathrm{Span}\{(w \circ S_j)_N \mid j \in J, w \in X^*\}$. The map η_N is clearly surjective. It is easy that $W_{\Psi,N,M} \subseteq W_{\Psi,N,.}$ for all $M \leq 0$ and $\dim W_{\Psi,N,.} = \dim \eta_N(W_\Psi) \leq \dim W_\Psi$. Since $\dim W_{\Psi,N,N} = \dim W_\Psi$ we get that $\dim W_{\Psi,N,.} = \dim W_\Psi$. Thus, η_N is injective too, i.e., η_N is a linear isomorphism. Since $\eta_N(W_{\Psi,.,N}) = W_{\Psi,N,N}$, we get that $\dim W_{\Psi,.,N} = \dim W_{\Psi,N,N} = \dim W_\Psi$, thus $W_{\Psi,.,N} = W_\Psi$. That is, for each $w \in X^*$, $|w| \geq N, j \in J$ there exists $K > 0$ $v_i \in X^{<N}, j_i \in J, \alpha_i$, $i = 1, \dots, K$, such that $w \circ S_j = \sum_{i=1}^K \alpha_i (v_i \circ S_{j_i})$.

If we could show that R_N is a representation of Ψ, then Theorem 2 would yield that R_N is a minimal representation of Ψ. We will show that $A_w B_j = (w \circ S_j)_N$ holds for each $|w| \geq N$. In Part (ii) we showed that $A_w B_j = (w \circ S_j)_N$ for $|w| < N$. We also know that if $|w| < N$, then $A_x((w \circ S_j)_N) = (wx \circ S_j)_N$. We proceed by induction on $|w| - N$. Assume that $A_w B_j = (w \circ S_j)_N$ holds for all $|w| < N + n$. Let $z = wx$, $|w| = n + N - 1$. Then there exist $\alpha_i \in \mathbb{R}, v_i \in X^{<N}, j_i \in J$ $i = 1, \dots, K$ such that $w \circ S_j = \sum_{i=1}^K \alpha_i v_i \circ S_{j_i}$. It implies that $z \circ S_j = \sum_{i=1}^K \alpha_i v_i x \circ S_{j_i}$, thus $(z \circ S_j)_N = \sum_{i=1}^K \alpha_i (v_i x \circ S_{j_i})_N = \sum_{i=1}^K \alpha_i A_x((v_i \circ S_{j_i})_N) = A_x((w \circ S_j)_N)$. We

used the fact that $(w \circ S_j) = \sum_{i=1}^{K} \alpha_i (v_i \circ S_{j_i})_N$ and the induction hypothesis for $n = 0$ and the linearity of A_x. By induction hypothesis $(w \circ S_j)_N = A_w B_j$, thus we get $(z \circ S_j)_N = A_x A_w B_j = A_z B_j$. That is, we get that for any $j \in J, w \in X^*$, $A_w B_j = (w \circ S_j)_N$, which implies that $C A_w B_j = S_j(w)$,that is, R_N is a representation of Ψ.

`Part (iv)`

If $\dim R \leq N$, then rank $H_\Psi \leq N$, thus rank $H_{\Psi,N,N} = \dim W_{\Psi,N,N} =$ rank H_Ψ. That is, we can apply Part (iii) of the Theorem.

□

The proof of the above theorem relies on the following lemmas

Lemma 50. *Assume that* rank $H_\Psi \leq N$. *For any* $T \in W_\Psi$, $w \in X^*$ *it holds that there exists* $\alpha_{w,v} \in \mathbb{R}$, $v \in X^*, |v| < N$ *such that* $w \circ T = \sum_{v \in X^*, |v| < N} \alpha_{w,v} (v \circ T)$.

Proof. We will use the fact that $\dim W_\Psi =$ rank $H_\Psi \leq N$. Consider the free representation $R = (W_\Psi, \{A_x\}_{x \in X}, B, C)$ of Ψ defined in Theorem 1. Apply Lemma 51 to W_Ψ and $A_x : W_\Psi \to W_\Psi$, $x \in X$. We get that for each $w \in X^*$ there exist $\alpha_{T,w,v} \in \mathbb{R}$, $\in X^*, |v| < N$ such that $A_w T = w \circ T = \sum_{v \in X^*, |v| < N} \alpha_{w,v} A_v T = \sum_{v \in X^*, |v| < N} \alpha_{w,v} v \circ T$. □

Lemma 51. *Let $\mathcal{X}$ be finite-dimensional vector space,* $\dim \mathcal{X} \leq N$. *Let* $A_x : \mathcal{X} \to \mathcal{X}$, $x \in X$ *be a family of linear maps. Then for each* $y \in \mathcal{X}$, *for each* $w \in X^*$ *there exists* $\alpha_{y,w,v} \in \mathbb{R}$, $v \in X^*, |v| < N$ *such that*

$$A_w y = \sum_{v \in X^*, |v| < N} \alpha_{y,w,v} A_v y$$

Proof. If $|w| < N$, then choose $\alpha_{y,w,w} = 1$ and $\alpha_{y,w,v} = 0$, $v \neq w, |v| < N$. First we prove the lemma for $|w| = N$. Assume that $w = w_1 \cdots w_N$, $w_1, \ldots, w_N \in X$. Consider the elements $A_{w_1 \cdots w_i} y$, $i = 0, \ldots, N$. Since every $N+1$ elements of $\mathcal{X}$ are linearly dependent, we get that there exist $\beta_i \in \mathbb{R}$, $i = 0, \ldots, N$ such that $\sum_{i=0}^{N} \alpha_i A_{w_1 \cdots w_i} y = 0$. Let $l = \max\{j \mid \alpha_j \neq 0\}$. Then $A_{w_1 \cdots w_l} y = \sum_{i=0}^{l-1} \beta_i A_{w_1 \cdots w_i} y$, where $\beta_i = \alpha_i / \alpha_l$. Then we get that $A_w y = A_{w_{l+1} \cdots w_N} (A_{w_1 \cdots w_l} y) = \sum_{i=0}^{l-1} \beta_i A_{v_i} y$, where $v_i = w_1 \cdots w_i w_{l+1} \cdots w_N$, $i = 0, \ldots, l-1$. It is clear that $|v_i| < N$, $i = 0, \ldots, l-1$. To prove the lemma for arbitrary $|w| \geq N$ we proceed by induction on $|w| - N$. The case of $|w| = N$ we proved above. Assume that the statement of the lemma holds for $|w| \leq n + N$. Let be $z = wx$, $|w| = n + N$, $x \in X$. Then there exist $\alpha_{y,w,v} \in \mathbb{R}, |v| < N, v \in X^*$ such that $A_w y = \sum_{v \in X^*, |v| < N} \alpha_{y,w,v} A_v y$. Since $A_z x = A_x(A_w y)$ we get that $A_z y = \sum_{v \in X^*, |v| < N-1} \alpha_{y,w,v} A_{vx} y + \sum_{v \in X^*, |v| = N-1} \alpha_{y,w,v} A_{vx} y$ By the statement of the lemma for words of length N, we get that $A_{vx} y = \sum_{s \in X^*, |s| < N} \alpha_{y,vx,s} A_s y$.

for each $|v| = N-1, v \in X^*$. For each $s \in X^*, |s| < N$ define $\alpha_{y,z,s} = \alpha_{y,w,s'} + \sum_{v \in X^*, |v|=N-1} \alpha_{y,vx,s}$ if $s = s^{'}x$ and $\alpha_{y,z,s} = \sum_{v \in X^*, |v|=N-1} \alpha_{y,vx,s}$ otherwise. Then it is easy to see that $A_z y = \sum_{v \in X^*, |v|<N} \alpha_{y,z,v} A_v y$. □

Lemma 52. *Let $R = (\mathcal{X}, \{A_x\}_{x \in X}, B, C)$ be a representation of Ψ and assume that $\dim R < N$. Then for each $w \in X^*$ there exists $\alpha_{w,,v,j} \in \mathbb{R}$, $v \in X^*, |v| < N$, $j = 1, \ldots, p$ such that*

$$\forall x \in \mathcal{X} : C_j A_w x = \sum_{v \in X^*, |v|<N} \alpha_{w,v,j} C_j A_v x$$

where $C_j = e_j^T C$, e_j is the jth unit vector of $\mathbb{R}^p$.

Proof. `Part (i)` Consider the set of all linear homomorphisms $\mathcal{X}^* = Hom(\mathcal{X}, \mathbb{R})$. It is well known that $\dim \mathcal{X}^* = \dim \mathcal{X} \leq N$. It is easy to see that for each $x \in X$ the linear map $A_x : \mathcal{X} \to \mathcal{X}$ induces a dual map $A_x^* : \mathcal{X}^* \to \mathcal{X}^*$ defined by $A_x^*(f)(y) = f(A_x y)$ for each $f \in \mathcal{X}^*$, $y \in \mathcal{X}$. Let $w = w_1 \cdots w_k, w_1, \ldots, w_k \in X$ and let $\overleftarrow{w} = w_k w_{k-1} \cdots w_1$ be the mirror image of w. It is easy to see that $C_j A_w = A^*_{\overleftarrow{w}} C_j$. Applying Lemma 51 to $\mathcal{X}^*$ and A_x^* we get that there exist $\alpha_{j,w,v} \in \mathbb{R}$, $v \in X^*, |v| < N$ such that $C_j A_w = A^*_{\overleftarrow{w}} C_j = \sum_{v \in X^*, |v|<N} \alpha_{j,w,v} A^*_{\overleftarrow{v}} C_j = \sum_{v \in X^*, |v|<N} \alpha_{j,w,v} C_j A_v$. □

Corollary 23. *Consider a representation $R = (\mathcal{X}, \{A_x\}_{x \in X}, B, C)$. Assume that $\dim R \leq N$. Then $O_R = \bigcup_{v \in X^*, |v|<N} CA_v$ and $W_R = \mathrm{Span}\{A_v B_j \mid j \in J, v \in X^*, |v| < N\}$.*

Proof. It is clear that $O_R \subseteq \bigcap_{v \in X^*, |v|<N} CA_v$ If $C_j A_v x = 0$ for all $v \in X^*, |v| < N$, $j = 1, \ldots, p$, then by Lemma 52 for each $w \in X^*$,

$$C_j A_w x = \sum_{v \in X^*, |v|<N} \alpha_{w,v,j} C_j A_v x = 0$$

so $x \in O_R$. Similarly, by Lemma 51, for each $w \in X^*$,

$$A_w B_j = \sum_{v \in X^*, |v|<N} \alpha_{B_j,w,v} A_v B_j \in \mathrm{Span}\{A_v B_j \mid j \in J, v \in X^*, |v| < N\}$$

which implies the statement of the corollary. □

Recall the definition of the space $W_{\Psi,N,.} = \{S_N \in \mathbb{R}^p \ll X^{<N} \gg \mid S \in W_\Psi\}$. It is easy to see that $W_{\Psi,N,.}$ is a linear subspace of $\mathbb{R}^p \ll X^{<N} \gg$.

Lemma 53. *Consider the the mapping $\eta_N : \mathbb{R}^p \ll X^* \gg \ni T \mapsto T_N \in \mathbb{R}^p \ll X^{<N} \gg$. Assume that* rank $H_\Psi \leq N$. *Then $\eta_N|_{W_\Psi} : W_\Psi \to W_{\Psi,.,N}$ is a linear isomorphism.*

Proof. It is easy to see that η_N is a surjective linear map. Consider the free realization $R_f = (W_\Psi, \{A_x\}_{x\in X}, B, C)$ of Ψ as defined in Theorem 1 [55]. From Theorem 2 we know that R_f is minimal and therefore it is reachable and observable, i.e. $O_{R_f} = \{0\}$. From Corollary 23 we also know that if $\dim R_f = \dim H_\Psi = \text{rank } H_\Psi \leq N$, then $O_{R_f} = \bigcap_{v\in X^*,|v|<N} \ker CA_v$. Consider the kernel of η_N. $\eta_N(T) = 0$ if and only if $T(w) = 0$ for all $|w| < N$. That is, $CA_wT = 0$ for each $|w| < N$, i.e. $T \in O_{R_f} = \{0\}$. Thus, η_N is injective. That is, η_N is a linear isomorphism. □

The theorem above implies that if J is finite and we know that Ψ has a representation of dimension at most N, then a minimal representation of Ψ can be computed from finite data.

10.1.2 Construction of a Minimal Representation

The technique of Hankel-matrix factorization has been used in realization theory and systems identification for several decades. It forms the theoretical basis of algorithms for subspace identification, see for example [19, 7].

`ComputePartialRepresentation(`$H_{\Psi,N+1,N}$`)`

1. Compute a decomposition of $H_{\Psi,N+1,N}$

$$H_{\Psi,N+1,N} = OR$$

$O \in \mathbb{R}^{I_{N+1}\times r}$, $R \in \mathbb{R}^{r\times J_N}$, rank R = rank $O = r$

2. Let $\widetilde{C} = \begin{bmatrix} O_{\psi_1((\epsilon,1)),.} \\ O_{\psi_1((\epsilon,2)),.} \\ \cdots \\ O_{\psi_1((\epsilon,p)),.} \end{bmatrix}$ where $O_{k,.}$ denotes the kth row of O.

3. Let $\widetilde{B}_j = R_{.,\psi_2((\epsilon,j))}$, where $R_{.,\psi_2((\epsilon))}$ stands for the $\psi_2((\epsilon, j))$th column of R. Let $\widetilde{B} = \{\widetilde{B}_j \mid j \in J\}$.

4. For each $x \in X$ let $\widetilde{A}_x$ be the solution of

$$\bar{\Gamma}\widetilde{A}_x = \bar{\Gamma}_x \tag{10.1}$$

where $\bar{\Gamma}, \bar{\Gamma}_x \in \mathbb{R}^{|I_N|\times r}$,

$$\bar{\Gamma}_{i,j} = O,\ i = 1, \ldots, |I|_N, j = 1, \ldots, r$$

and

$$(\bar{\Gamma}_x)_{i,j} = O_{k,j}$$

where $i = 1, \ldots, |I|_N, \psi_1((u, l)) = i, k = \psi_1((xu, l)), u \in X^*, j = 1, \ldots, r$

5. If there no solution to (10.1) then return $NoRepresentation$.

6. Return $\widetilde{R}_N = (\mathbb{R}^r, \{\widetilde{A}_x\}_{x\in X}, \widetilde{C}, \widetilde{B})$.

Notice that the algorithm above requires the existence of a solution to the linear equation (10.1) at step 4. The algorithm above may return two different types of data. It returns a formal power series representation if (10.1) has a solution and the symbol $NoRepresentation$ otherwise.

Remark In step 1 of the algorithm above one can use any algorithm for computing a decomposition. For example, one could use SVD decomposition, in which case $H_{\Psi,N+1,N} = U\Sigma V^T$, and $O = U(\Sigma^{1/2})$, $R = (\Sigma^{1/2})V^T$ is a valid choice for decomposition. Perhaps the algorithm above may give rise to model reduction and identification methods similar to those for linear systems.

The following theorem characterises the outcome $\widetilde{R}_N$ of the algorithm above.

Theorem 65. *With the notation above the following holds.*

(i) If `ComputePartialRepresentation` *returns a formal power series representation $\widetilde{R}_N$, then the representation $\widetilde{R}_N$ is an $N+1$ partial representation of Ψ. The representation $\widetilde{R}_N$ is reachable and observable.*

(ii) Assume that $\text{rank } H_{\Psi,N,N+1} = \text{rank } H_{\Psi,N+1,N} = \text{rank } H_{\Psi,N,N}$. *Then the algorithm* `ComputePartialRepresentation` *always returns a formal power series representation. Consider the representation R_N from Theorem 64. Then $\xi : \mathbb{R}^r \to W_{\Psi,N,N}$, $\xi = \eta_{N+1} \circ \psi_{N+1}^{-1} \circ O$, is a linear isomorphism such that $\xi : \widetilde{R}_N \to R_N$ is a representation morphism, where η_{N+1}, ψ_{N+1} are the linear maps defined in Subsection 10.1.1.*

(iii) If $\text{rank } H_{\Psi,N,N} = \text{rank } H_\Psi$ *then*

$$\text{rank } H_{\Psi,N,N} = \text{rank } H_{\Psi,N+1,N} = \text{rank } H_{\Psi,N,N+1}$$

and $\widetilde{R}_N$ is a minimal representation of Ψ.

(iv) Assume $\text{rank } H_\Psi \leq N$, *or , equivalently, there exists a representation R of Ψ, such that* $\dim R \leq N$. *Then the representation $\widetilde{R}_N$ is a minimal representation of Ψ.*

Proof. `Part (i)`
First we will show that $\widetilde{R}_N$ is indeed a $N+1$-partial representation of Ψ. From the definition of $\widetilde{A}_x$, $x \in X$ it follows that $\bar{\Gamma}_{k,.}\widetilde{A}_w = O_{k,.}\widetilde{\mathcal{A}}_w = O_{l,.}$, where $k = \psi_1(\epsilon, i)$

and $l = \psi_1(w,i)$, $i = 1,\ldots,p$, $w \in X^{<N+1}$. Thus, we get that

$$\widetilde{C}\widetilde{A}_w = \begin{bmatrix}(O)^T_{\psi_1(\epsilon,1),.}, & (O)^T_{\psi_1(\epsilon,2),.}, & \cdots & (O)^T_{\psi_1(\epsilon,p),.}\end{bmatrix}^T \widetilde{A}_w = \begin{bmatrix}(O_{\psi_1(w,1),.})^T, & (O_{\psi_1(w,2),.})^T, & \cdots & (O_{\psi_1(w,p),.})^T\end{bmatrix}^T$$

Thus, we get that for each $i = 1,\ldots,p$,

$$e_i^T\widetilde{C}\widetilde{A}_w\widetilde{B}_j = O_{\psi_1(w,i),.}(R)_{.,\psi_2(\epsilon,j)} = = (H_{\Psi,N+1,N})_{\psi_1(w,i),\psi_2(\epsilon,j)} = e_i^T S_j(w)$$

That is, $\widetilde{C}\widetilde{A}_w\widetilde{B}_j = S_j(w)$. Thus, $\widetilde{R}_N$ is indeed an $N+1$-partial representation.

Next we will show that $\widetilde{R}_N$ is observable. Assume that there exists $x \in \mathbb{R}^r$ such that $\widetilde{C}\widetilde{A}_w x = 0$ for all $w \in X^*$. Assume that $x = Ry$ for some $y \in \mathbb{R}^{|J_N|}$. Since rank $R = r$, such a y always exists. Thus, we get that for each $w \in X^{<N+1}$, $i = 1,\ldots,p$

$$0 = e_i^T\widetilde{C}\widetilde{A}_w x = O_{\psi_1(w,i),.}Ry = (H_{\Psi,N+1,N}y)_{\psi_1(w,i)}$$

That is, $H_{\Psi,N+1,N}y = 0$. That is, $Ox = ORy = 0$, i.e. $x \in \ker O$. But rank $O = r$, thus $\ker O = \{0\}$, that is $x = 0$. We will show that $\widetilde{R}_N$ is reachable. Indeed,

$$\mathbb{R}^r = \mathrm{Im}R = \mathrm{Span}\{R_{.,\psi_2(w,j)} \mid j \in J, w \in X^{<N}\} \tag{10.2}$$

But

$$e_i^T\widetilde{C}\widetilde{A}_v R_{.,\psi_2(w,j)} = O_{\psi_1(v,i),.}R_{.,\psi_2(w,j)} = (H_{\Psi,N+1,N})_{\psi_1(v,i),\psi_2(w,j)} = (H_{\Psi,N+1,N})_{\psi_1(wv,1),\psi_2(\epsilon,j)} = e_i^T\widetilde{C}\widetilde{A}_{vw}\widetilde{B}_j = e_i^T\widetilde{C}\widetilde{A}_v(\widetilde{A}_w\widetilde{B}_j)$$

for each $i = 1,\ldots,p$, $v \in X^{<N+1}$. Since $\widetilde{R}_N$ is observable we get that $\widetilde{A}_w\widetilde{B}_j = (R)_{.,\psi_2(w,j)}$. Thus,

$$\mathbb{R}^r = \mathrm{Span}\{\widetilde{A}_w\widetilde{B}_j \mid j \in J, w \in X^{<N}\}$$

that is $\widetilde{R}_N$ is observable.

`Part (ii)`

It is clear that ξ is well defined. Indeed, $\mathrm{Im}O = \mathrm{Im}H_{\Psi,N+1,N}$ by definition of matrix factorization. Moreover, $O : \mathbb{R}^r \to \mathbb{R}^{|I_{N+1}|}$ is a injective and

$$\psi_{N+1} : W_{\Psi,N+1,N} \to \mathrm{Im}H_{\Psi,N+1,N}$$

is a linear isomorphism. Moreover, since

$$\text{rank } H_{\Psi,N+1,N} = \dim W_{\Psi,N+1,N} = \text{rank } H_{\Psi,N,N} = W_{\Psi,N,N}$$

we get that $\eta_{N+1} : W_{\Psi,N+1,N} \to W_{\Psi,N,N}$ is a linear isomorphism. Thus, $\xi = \eta_{N+1} \circ \psi_{N+1}^{-1} \circ O : \mathbb{R}^r \to W_{\Psi,N,N}$ is a well defined linear isomorphism. It is left to show that ξ is a representation morphism. Consider the representation $R_N = (W_{\Psi,N,N}, \{A_z\}_{z\in X}, C, B)$.

It is easy to see that for each $x \in \mathbb{R}^r$ there exists $y \in \mathbb{R}^{|J_N|}$ such that $x = Ry$. Then it is easy to see that

$$\xi(x) = \eta_{N+1} \circ \psi_{N+1}^{-1}(ORy) = \eta_{N+1} \circ \psi_{N+1}^{-1}(H_{\Psi,N+1,N}y) = T_{N+1} \tag{10.3}$$

where $T_{N+1} \in \mathbb{R}^p \ll X^{<N+1} \gg$ and for each $i = 1,\dots,p$, $w \in X^{<N}$ it holds that $e_i^T T_{N+1}(w) = (H_{\Psi,N+1,N}y)_{\psi_1(w,i)}$. Thus

$$\begin{gathered} e_i^T T_{N+1}(\epsilon) = e_i^T C(T_{N+1}) = e_i^T C(\xi(x)) = \\ (H_{\Psi,N+1,N}y)_{\psi_1(i,\epsilon)} = (O)_{\psi_1(\epsilon,i),.}(Ry) = e_i^T \widetilde{C}x \end{gathered}$$

Thus, $C\xi = \widetilde{C}$. It is also easy to see that $\xi(\widetilde{B}_j) = \xi(Re_{\psi_2(\epsilon,j)}) = T_{N+1}$, such that

$$\begin{aligned} e_i^T T_{N+1}(w) = (H_{\Psi,N+1,N}e_{\psi_2(\epsilon,j)})_{\psi_1(i,w)} = \\ = (H_{\Psi,N+1,N})_{\psi_1(i,w),\psi_2(\epsilon,j)} = e_i^T S_j(w) \end{aligned}$$

Thus, $\xi(\widetilde{B}_j) = (S_j)_N = B_j$. It is left to show that a solution to equation (10.1) exists and $A_x\xi = \xi\widetilde{A}_x$, for all $x \in X$. First, notice that $\bar{\Gamma}R = H_{\Psi,N,N}$. Thus, rank $\bar{\Gamma}R =$ rank $H_{\Psi,N,N} = r$. Thus, rank $\bar{\Gamma} = r$. That is, if solution (10.1) exists, then this solution is unique. Thus, if we show that $\bar{A}_x = \xi^{-1}A_x\xi$ is a solution to (10.1) then it follows that $\widetilde{A}_x = \bar{A}_x = \xi^{-1}A_x\xi$, $x \in X$ and thus ξ is a representation morphism. From (10.2) it is enough to prove that

$$\bar{\Gamma}\bar{A}_x R_{.,\psi_2(w,j)} = \widetilde{\Gamma}_x R_{.,\psi_2(w,j)} \tag{10.4}$$

for each $w \in X^{<N}, j \in J$. Using (10.3) we get

$$\begin{aligned} \bar{\Gamma}\bar{A}_x R_{.,\psi_2(w,j)} = \bar{\Gamma}\xi^{-1}(A_x\xi(R)_{.,\psi_2(w,j)}) = \\ \bar{\Gamma}\xi^{-1}(A_x(w \circ S_j)_N) = \bar{\Gamma}\xi^{-1}((wx \circ S_j)_N) = \\ = \bar{\Gamma}Ry = H_{\Psi,N,N}y \end{aligned}$$

where $y \in \mathbb{R}^{|J_N|}$ is such that $H_{\Psi,N,N}y = (H_{\Psi,N+1,N})_{.,\psi_2(wx,j)}$. On the other hand, notice that

$$\begin{aligned} e_{\psi_1(v,i)}^T \widetilde{\Gamma}_x R_{.,\psi_2(w,j)} = \\ (O)_{\psi_1(vx,i),.} R_{.,\psi_2(w,j)} = \\ (H_{\Psi,N+1,N})_{\psi_1(vx,i),\psi_2(w,j)} = e_i^T S_j(wxv) = (H_{\Psi,N+1,N})_{\psi_1(v,i),\psi_2(wx,j)} \end{aligned}$$

Thus, we get that (10.4) holds.

`Part(iii)`

From Theorem 64 it follows that if rank $H_{\Psi,N,N} =$ rank H_{Ψ} then rank $H_{\Psi,N+1,N} =$ rank $H_{\Psi,N,N+1} =$ rank $H_{\Psi,N,N}$ and R_N is a minimal representation of Ψ By Part (ii) of the theorem $\xi : \widetilde{R}_N \to R_N$ is a representation isomorphism and thus $\widetilde{R}_N$ is a minimal representation too.

`Part(iv)`

Again, from Theorem 64 it follows that R_N is a minimal representation. Since $\widetilde{R}_N$ is isomorphic to R_N we get that $\widetilde{R}_N$ is a minimal representation of Ψ too. □

10.1.3 Algorithmic Aspects

It was already mentioned that it is possible to transform a representation to an equivalent observable or reachable realization. Below we will give algorithms for carrying out these transformations.

Let R be a representation of some family of formal power series Ψ with finite index set J. Assume that $R = (\mathbb{R}^n, \{A_z\}_{z\in X}, C, B)$, where $B = \{B_j \mid j \in J\}$ and $J = \{j_1, \dots j_N\}$. Assume that $X = \{z_1, \dots, z_M\}$.

$R_R =$`ComputeReachabilityMatrix(`R`)`

1. $R_0 = \begin{bmatrix} B_{j_1} & \cdots & B_{j_N} \end{bmatrix} \mathbb{R}^{n\times N}$
2. $R_{k+1} = \begin{bmatrix} R_k & A_{z_1}R_k & A_{z_2}R_k \cdots & A_{z_M}R_k \end{bmatrix} \in \mathbb{R}^{n\times (M+1)^{k+1}N}$
3. If rank $R_{k+1} =$ rank R_k then $R_R = R_k$ else goto step 2

Proposition 33. *The algorithm* `ComputeReachabilityMatrix` *above always terminates and the matrix R_R computed by the algorithm is such that* $\mathrm{Im}R_R = W_R$

Proof. Notice that $\mathrm{Im}R_k \subseteq \mathrm{Im}R_{k+1}$ for all $k \in \mathbb{N} \cup \{0\}$. Notice that rank $R_k =$ rank R_{k+1} is equivalent to $\mathrm{Im}R_k = \mathrm{Im}R_{k+1}$. Assume that $\mathrm{Im}R_k = \mathrm{Im}R_{k+1}$ holds for some $k \geq 0$. Then $\mathrm{Im}R_{k+1} = \mathrm{Im}R_{k+2}$ holds too. Indeed, let $x \in \mathrm{Im}R_{k+2}$. Then $x = y_0 + \sum_{i=1}^{M} A_{z_i}y_i$, where $y_i \in \mathrm{Im}R_{k+1} = \mathrm{Im}R_k$, $i = 0, \dots, M$. Thus $A_{z_i}y_i \in \mathrm{Im}R_{k+1}$, $i = 1, \dots, M$ and therefore $x \in \mathrm{Im}R_{k+1}$. That is, rank $R_{k+1} =$ rank R_k implies that $\mathrm{Im}R_k = \mathrm{Im}R_{k+1} = \mathrm{Im}R_{k+2} = \cdots = \mathrm{Im}R_{k+l}$ for all $l \in \mathbb{N}$. It is easy to see that $\mathrm{Im}R_k = \mathrm{Span}\{A_wB_j \mid j \in J, w \in X^{<k+1}\}$. It implies that $W_R = \sum_1^k \mathrm{Im}R_k$. If $\mathrm{Im}R_k = \mathrm{Im}R_{k+1}$ then it follows that $W_R = \mathrm{Im}R_k$. That is, if the algorithm `ComputeReachabilityMatrix` stops, then it returns a matrix R_k such that $\mathrm{Im}R_k = W_R$. From Corollary 23 we get that $\mathrm{Im}R_n = W_R = \mathrm{Im}R_{n-1}$. That is, the algorithm stops after at most n steps. □

It is easy to see that the time complexity of the algorithm above is $O(n)$ but the storage complexity is $O((M+1)^n)$. The amount of required memory can be reduced by replacing at each step R_k with a matrix columns of which form a basis of $\mathrm{Im} R_k$.

The algorithm `ComputeReachabilityMatrix` allows us to give an algorithm for deciding the reachability of a representation R.

`IsReachable(R)`

1. $R_R =$ `ComputeReachabilityMatrix(R)`.

2. If rank $R_R = n$ then return True else return False

Now we can formulate an algorithm for computing a reachable representation R_r of Ψ from R.

`ReachableTransform(R)`

1. $R_R =$ `ComputeReachabilityMatrix(R)`

2. Compute $U \in \mathbb{R}^{n \times r}$, $r = \text{rank } R_R$ such that $\mathrm{Im} U = \mathrm{Im} R_R$ and $U^T U = I_r$.

3. Define $A^r_z = U^T A_z U$, $\forall z \in X$ $B^r_j = U^T B_j, \forall j \in J$, $C^r = CU$

4. Return $R_r = (\mathbb{R}^r, \{A^r_z\}_{z \in X}, \{B^r_j \mid j \in J\}, C)$

Proposition 34. *R_r is a reachable representation of Ψ*

Proof. From Proposition 33 $\mathrm{Im} R_R = W_R$. Then $B_j \in \mathrm{Im} U = \mathrm{Im} R_R$ and $A_w \mathrm{Im} U \subseteq \mathrm{Im} U$ for each $w \in X^*$. That is, for all $y \in \mathbb{R}^r$ there exists $z \in \mathbb{R}_r$ such that $A_w U y = U z$ for some $z \in \mathbb{R}^r$. Notice that $U U^T U z = U z$, thus, $U U^T A_w U y = U U^T U z = U z = A_w U y$. That is, $U U^T A_w U = A_w U$. Notice that for each $j \in J$ there exists $y \in \mathbb{R}^r$ such that $U y = B_j$, thus $U U^T B_j = U U^T U y = U y = B_j$. Then it follows that $A^r_w = U^T A_w U$, $A^r_w B^r_j = U^T A_w B_j$ and $C^r A^r_w B^r_j = C A_w B_j = S_j(w)$ for each $w \in X^*, j \in J$. Thus, R_r is a representation of Ψ. We will show that R_r is reachable. Indeed, $W_{R_r} = \mathrm{Span}\{A^r_w B^r_j \mid w \in X^*, j \in J\} = \mathrm{Span}\{U^T A_w B_j \mid w \in X^*, j \in J\} = U^T W_R = \mathrm{Im} U^T R_R$. Since rank $U^T =$ rank $R_R = r$, it follows that $\dim W_{R_r} = \text{rank } U^T R_R = r = \dim R_r$. □

Observability of representations can be treated in an algorithmic way too. Consider the following algorithm

`ComputeObservabilityMatrix(`R`)`

1. $O_0 = C$.

2. $O_{k+1} = \Big[O_k^T, \quad (O_k A_{z_1})^T, \quad (O_k A_{z_2})^T, \quad \cdots \quad (O_k A_{z_M})^T\Big]^T \in \mathbb{R}^{p(M+1)^k \times n}$

3. If rank $O_{k+1} =$ rank O_k then return O_k else goto step 2

Proposition 35. *The algorithm* $ComputeObservabilityMatrix(R)$ *always terminates in at most* n *steps and its return value* OB_R *has the property that* $\ker OB_R = O_R$.

Proof. Notice that $\ker O_R \subseteq \ker O_{k+1} \subseteq \ker O_k$ for all $k \geq 0$. Moreover, rank $O_k = n - \dim\ker O_k$, thus rank $O_{k+1} \leq$ rank O_k. Assume that rank $O_k =$ rank O_{k+1}. Then this is equivalent to $\dim\ker O_k = \dim\ker O_{k+1}$ which is equivalent to $\ker O_k = \ker O_{k+1}$. We will show that $\ker O_{k+1} = \ker O_{k+2}$ holds and thus rank $O_{k+2} = n - \dim\ker O_{k+2} = n - \dim\ker O_{k+1} =$ rank O_{k+1}. Indeed, assume that $x \in \ker O_{k+1} = \ker O_k$. Then for each $z \in X$, $O_k A_z x = 0$, that is, $A_z x \in \ker O_k = \ker O_{k+1}$. But it means that for each $z \in X$, $O_{k+1} A_z x = 0$, that is, $x \in \ker O_{k+2}$. Thus, $\ker O_{k+2} = \ker O_k$. That is, rank $O_k =$ rank O_{k+1} implies that $\ker O_k = \ker O_{k+l}$ for all $l > 0$. Notice that $\ker O_k = \bigcap_{w \in X^{<k+1}} \ker CA_w$. It follows that $\ker O_R = \bigcap_{k \geq 0} \ker O_k$. Thus, if rank $O_k =$ rank O_{k+1} then $\ker O_k = O_R$. On the other hand, by Corollary 23 we get that $\ker O_{n-1} = \ker O_n = \ker O_R$. Thus, rank $O_n =$ rank O_n and therefore the algorithm stops after at most n steps. □

The algorithm above can be used to decide whether R is observable or not.

`IsObservable`

1. $OB_R = ComputeObservabilityMatrix(R)$

2. If rank $OB_R = n$ then return True else False

We can also give an algorithm for computing an observable representation R_o of Φ.

`ComputeObservableRepresentation(R)`

1. $OB_R = ComputeObservabilityMatrix(R)$

2. Compute $U \in \mathbb{R}^{n \times r}$, $r =$ rank OB_R such that $\mathrm{Im} U = \mathrm{Im} OB_R^T$ and $U^T U = I_r$.

3. Define $A_z^o = U^T A_z U$, $\forall z \in X$, $B_j^o = U^T B_j$, $\forall j \in J$, $C^o = CU$.

4. Return $R_o = (\mathbb{R}^r, \{A_z^o\}_{z \in X}, \{B_j^o \mid j \in J\}, C^o)$

Proposition 36. R_o *is an observable representation of* Ψ. *Moreover, if* R *is reachable, then* R_o *is reachable too.*

Proof. From Proposition 35 it follows that $\ker OB_R = O_R$. From standard linear algebra it follows that $\mathrm{Im} OB_R^T \bigoplus \ker OB_R = \mathbb{R}^n$ and thus rank $OB_R = n - \dim OB_R = n - \dim O_R = r$. Notice that $A_w \ker O_R \subseteq \ker O_R$ for each $w \in X^*$. It is easy to see that $U^T(\ker O_R) = \{0\}$. Indeed, if $x \in O_R$, then for each $e_j, j = 1, \ldots, r$, $e_j^T U^T x = x^T U e_j = x^T OB_R^T s = s^T OB_R x = 0$, since $Ue_j = OB_R^T s$ for some s and $x \in \ker O_R = \ker OB_R$. We will show that $U^T A_w U U^T = U^T A_w$. Indeed, let $x \in \mathbb{R}^n$ such that $x = x_1 + x_2 = Uy + x_2$ such that $x_1 \in \mathrm{Im} OB_R = \mathrm{Im} U$ and $x_2 \in \ker O_R$. Then $U^T A_w U U^T x = U^T A_w U U^T U y = U^T A_w x_1$ and $U^T A_w x = U^T A_w x_1 + U^T A_w x_2 = U^T A_w x_1$, since $A_w x_2 \in \ker O_R$. Also notice that $CUU^T x = Cx_1 = Cx_1 + Cx_2 = Cx$, since $\ker O_R \subseteq \ker C$. Then it follows that $A_w^o = U^T A_w U$, $A_w^o B_j^o = U^T A_w B_j$ and $C^o A_w^o B_j^o = CA_w B_j = S_j$. That is, R_o is indeed a representation of Ψ. We will show that R_o is observable. Indeed, for each $x \in \mathbb{R}^r$, $w \in X^*$, $C^o A_w^o x = CA_w U x$. Thus $x \in O_{R_o}$ if and only if $Ux \in O_R = \ker OB_R$. But $\mathrm{Im} U \cap \ker OB_R = \{0\}$, thus $Ux = 0$ and since rank $U = r$ it follows that $x = 0$. That is, $O_{R_o} = \{0\}$.

It is easy to see that $W_{R_o} = U^T(W_R)$. Thus, $W_R = \mathbb{R}^n$ implies that $W_{R_o} = \mathrm{Im} U^T = \mathbb{R}^r$, since rank $U^T = r$. That is, if R is reachable then W_{R_o} is reachable too. □

Using the algorithm above it is straightforward to formulate an algorithm for computing a minimal representation of Ψ.

`ComputeMinimalRepresentation(R)`

1. $R_r = \mathit{ComputeReachableRepresentation}(R)$
2. $R_{min} = \mathit{ComputeObservableRepresentation}(R_r)$
3. Return R_{min}

It is easy to see that the algorithm above computes a minimal representation of Ψ. Note that if we first compute an observable representation and then we transform it to a reachable one, then the outcome is still a minimal representation, thus step 1 and step 2 of the algorithm above can be interchanged.

10.2 Moore-automata

The current section presents partial realization theory for Moore-automata. It also formulates algorithms for constructing a minimal Moore-automaton realization and for checking reachability, observability and minimality. The material of the section is an easy extension of already known results, although many of these results seem to be a folklore and it is difficult to trace the original publication. In any case, the results below are quite straightforward and are presented for the sake of completeness. The author does not claim that the results are original or new.

For $\phi : \Gamma^* \to O$ define $\phi_N = \phi|_{\{w \in \Gamma^* || w| < N\}}$. That is, $\phi_N : \{w \in \Gamma^* \mid |w| < N\} \to O$, $\phi_N(w) = \phi(w)$ for all $w \in \Gamma^*, |w| < N$.

Let $\mathcal{D} = \{\phi_j \in F(\Gamma^*, O) \mid j \in J\}$. Let $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda), \zeta : J \to Q$. The pair $(\mathcal{A}, \zeta)$ is said to be *N-partial realization of $\mathcal{D}$* if $\forall w \in \Gamma^*, |w| < N : \lambda(\zeta(j), w) = \phi_j(w)$. For each $N, M > 0$ define

$$W_{\mathcal{D},N,M} = \{(w \circ \phi_j)_M \mid j \in J, w \in \Gamma^*, |w| < N\}$$

Define the sets $W_{\mathcal{D},.,N} = \{\psi_N \mid \psi \in W_{\mathcal{D}}\}$ and $W_{\mathcal{D},N,.} = \{w \circ \phi_j \mid j \in J, w \in \Gamma^*, |w| < N\}$. Define the map $\eta_N : W_{\mathcal{D}} \to W_{\mathcal{D},.,N}$ by $\eta_N(\psi) = \psi_N$. The following holds.

Theorem 66 (Partial realization by automata). *(i) If $(\mathcal{A}, \zeta)$ is a realization of Φ and $card(\mathcal{A}) \leq N$, then*

$$card(W_{\mathcal{D},N,N}) = card(W_{\mathcal{D},N,N+1}) = card(W_{\mathcal{D},N+1,N}) = card(W_{\mathcal{D}})$$

(ii) If $card(W_{\mathcal{D},N,N+1}) = card(W_{\mathcal{D},N+1,N}) = card(W_{\mathcal{D},N,N})$, then $(\mathcal{A}_N, \zeta_N)$ is an N-partial realization of $\mathcal{D}$ where $\mathcal{A}_N = (W_{\mathcal{D},N,N}, \Gamma, O, \delta, \lambda)$ such that for each $w \in \Gamma^*, |w| < N, j \in J$, $\delta((w \circ \phi_j)_N, x) = (wx \circ \phi_j)_N$, $\forall f \in W_{\mathcal{D},N,N} : \lambda(f) = f(\epsilon)$, $\forall j \in J, \zeta(j) = (\phi_j)_N$,

(iii) If $card(W_{\mathcal{D},N,N}) = card(W_{\mathcal{D}})$, then $card(W_{\mathcal{D},N,N+1}) = card(W_{\mathcal{D},N+1,N}) = card(W_{\mathcal{D},N,N})$ and $(\mathcal{A}_N, \zeta_N)$ is a minimal realization of $\mathcal{D}$. In particular, if $\mathcal{D}$ has a realization $(\mathcal{A}, \zeta)$ such that $N \geq card(\mathcal{A})$, then $(\mathcal{A}_N, \zeta_N)$ is a minimal realization of $\mathcal{D}$.

Proof. `Part (i)`

From Lemma 54 it follows that $W_{\mathcal{D},N,.} = W_{\mathcal{D}}$. Notice that

$$W_{\mathcal{D},N,N} = \eta_N(W_{\mathcal{D},N,.}) = \eta_N(W_{\mathcal{D}})$$

From Lemma 55 it follows that $card(\eta_N(W_{\mathcal{D}})) = card(W_{\mathcal{D}})$, thus $card(W_{\mathcal{D},N,N}) = card(W_{\mathcal{D}})$.

`Part (ii)` It is easy to see that both ζ_N and λ are well defined. We will show that δ is well defined. Notice that $W_{\mathcal{D},N,N} \subseteq W_{\mathcal{D},N+1,N}$ and thus $W_{\mathcal{D},N,N} = W_{\mathcal{D},N+1,N}$. That is, for each $(w \circ \phi_j)_N \in W_{\mathcal{D},N,N}$, $|w| < N, j \in J$, the map $(wx \circ \phi_j)_N$ is an element of $W_{\mathcal{D},N,N+1} = W_{\mathcal{D},N,N}$. Assume that $(v \circ \phi_j)_N = (w \circ \phi_l)$ for some $j, l \in J, w, v \in \Gamma^*, |w|, |v| < N$. Define the map $\eta : W_{\mathcal{D},N,N+1} \to W_{\mathcal{D},N,N}$ by

$\eta(T) \mapsto T|_{\{w\in\Gamma^*,|w|<N\}}$. That is, $\eta(T)(w) = T(w)$ for all $w \in \Gamma^*, |w| < N$. It is easy to see that η is surjective. Since $card(W_{\mathcal{D},N,N}) = card(W_{\mathcal{D},N,N+1})$, we get that η is a bijection. Notice that $\eta((v \circ \phi_j)_{N+1}) = (v \circ \phi_j)_N = (w \circ \phi_k)_N = \eta((v \circ \phi_l)_{N+1})$. Thus, by bijectivity of η, $(v\circ\phi_j)_{N+1}) = (w\circ\phi_l)_{N+1}$. Thus, $(v\circ\phi_j)(xs) = (w\circ\phi_l)(xs)$ for all $x \in \Gamma$, $s \in \Gamma^*, |s| < N$. That is, $(vx \circ \phi_j)_N = (wx \circ \phi_l)_N$. Thus, δ is well defined. It is easy to see that $\delta((\phi_j)_N, w) = (w\circ\phi_j)_N$ for all $w \in \Gamma^*$, $|w| < N$. Thus, $\lambda(\zeta_N(j), w) = (w \circ \phi_j)_N(\epsilon) = \phi_j(w)$ for all $w \in \Gamma^*$, $|w| < N$. That is, $(\mathcal{A}_N, \zeta_N)$ is a N-partial realization of $\mathcal{D}$.

`Part (iii)`

Assume that $card(W_{\mathcal{D},N,N}) = card(W_{\mathcal{D}})$. Notice that $W_{\mathcal{D},N,N} \subseteq W_{\mathcal{D},N+1,N}$ and $\eta(W_{\mathcal{D},N,N+1}) = W_{\mathcal{D},N,N}$. Moreover, $W_{\mathcal{D},M,L} = \eta_L(W_{\mathcal{D},M,.}) \subseteq \eta_L(W_{\mathcal{D}})$, hence $card(W_{\mathcal{D}}) \geq card(W_{\mathcal{D},M,L})$ for each $M, L \in \mathbb{N}$. Thus $card(W_{\mathcal{D}}) \geq card(W_{\mathcal{D},N,N+1}) \geq card(W_{\mathcal{D},N,N})$ and $card(W_{\mathcal{D}}) \geq card(W_{\mathcal{D},N+1,N}) \geq card(W_{\mathcal{D},N,N})$, thus we get that $card(W_{\mathcal{D},N,N}) = card(W_{\mathcal{D},N+1,N}) = card(W_{\mathcal{D},N,N+1}) = card(W_{\mathcal{D}})$. It is easy to see that $\eta_N(W_{\mathcal{D},N,.}) = W_{\mathcal{D},N,N}$. Thus, $card(W_{\mathcal{D}}) \geq card(W_{\mathcal{D},N,.}) \geq card(W_{\mathcal{D},.,N})$. That is, $card(W_{\mathcal{D}}) = card(W_{\mathcal{D},N,.})$, i.e. $W_{\mathcal{D}} = W_{\mathcal{D},N,.}$ and η_N is injective and $\eta_N(W_{\mathcal{D}}) = W_{\mathcal{D},N,N} = W_{\mathcal{D},.,N}$. That is, $\eta_N : W_{\mathcal{D}} \to W_{\mathcal{D},.,N}$ is a bijective map.

First we show that $(\mathcal{A}_N, \zeta_N)$ is isomorphic to $(\mathcal{A}_{can}, \zeta_{can})$. Above it was shown that η_N is a bijection. It is left to show that η_N is an automaton morphism. $\eta(\zeta_{can}(j)) = \eta_N(\phi_j) = (\phi_j)_N = \zeta_N(j)$ and $\lambda(\eta_N(\psi)) = \psi_N(\epsilon) = \psi(\epsilon) = T(\psi)$ for all $\psi \in W_{\mathcal{D}}, j \in J$. For all $\psi \in W_{\mathcal{D}}$ there exists $w \in \Gamma^*$, $j \in J$, $|w| < N$ such that $\psi = w \circ \phi_j$. Then $L(\psi, x) = wx \circ \phi_j$, for all $x \in \Gamma$. But $\delta(\eta_N(\psi), x) = \delta((w \circ \phi_j)_N, x) = (wx \circ \phi_j)_N = \eta_N(L(\psi, x))$. That is, η_N is indeed an automaton morphism. Thus, $(\mathcal{A}_N, \zeta_N)$ and $(\mathcal{A}_{can}, \zeta_{can})$ are isomorphic. Since $(\mathcal{A}_{can}, \zeta_{can})$ is a minimal realization of $\mathcal{D}$, we get that $(\mathcal{A}_N, \zeta_N)$ is a minimal realization of $\mathcal{D}$ too. □

Lemma 54. *Assume that $card(W_{\mathcal{D}}) < N$. Then for every $\psi \in W_{\mathcal{D}}$ the following holds.*

$$\forall w \in \Gamma^* : \exists v_{w,\psi} \in \Gamma^*, |v_{w,\psi}| < N : w \circ \psi = v_{w,\psi} \circ \psi$$

Proof. If $|w| < N$, then choose $v_{w,\psi} = w$. We proceed by induction on $|w| - N$. Assume that $|w| = N$, that is, $w = w_1 \cdots w_N, w_1, \ldots, w_N \in \Gamma$. Consider the elements $w_1 \cdots w_i \circ \psi \in W_{\mathcal{D}}$, $i = 0, \ldots, N$. Since $card(W_{\mathcal{D}}) < N$, we get that there exists $0 \leq i < j \leq N$ such that $w_1 \cdots w_i \circ \psi = w_1 \cdots w_j \circ \psi$, which implies that $w_1 \cdots w_i w_{j+1} \cdots w_N \circ \psi = w \circ \psi$. Choose $v_{w,\psi} = w_1 \cdots w_i w_{j+1} \cdots w_N$. Notice that $|v_{w,\psi}| < N$. Then $w\circ\psi = v_{w,\psi}\circ\psi$. Assume the statement of the lemma is true for all $|w| \leq n + N$. Let $z = wx$ such that $x \in \Gamma, w \in \Gamma^*, |w| = n + N$. Then by the induction hypothesis, $w\circ\psi = v_{w,\psi}\circ\psi$, thus $z\circ\psi = x\circ(w\circ\psi) = x\circ(v_{w,\psi}\circ\psi) = v_{w,\psi}x\circ\psi$.

Since $|v_{w,\psi}| < N$, for $s = v_{w,\psi}x$, $|s| \leq N$, thus by induction hypothesis it holds that $s \circ \psi = v_{s,\psi} \circ \psi$. Let $v_{z,\psi} = v_{s,\psi}$. Then the statement of the lemma holds. □

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Lemma 55. *Assume that* $card(W_{\mathcal{D}}) < N$. *Then the map* $\eta_N : W_{\mathcal{D}} \to W_{\mathcal{D}}$ *defined above is a bijection.*

Proof. Define $R_M \subseteq W_{\mathcal{D}} \times W_{\mathcal{D}}$, $M = 1, 2, \ldots$, by

$$(S,T) \in R_M \iff S_M = T_M$$

It is easy to see that R_M is an equivalence relation and $R_{M+1} \subseteq R_M$. It is also easy to see that if $(S,T) \in R_{M+1}$ then for each $\gamma \in \Gamma$, $(\gamma \circ S, \gamma \circ T) \in R_M$. Indeed, for each $w \in \Gamma^*$, $|w| < M$ it holds that $|\gamma w| < M+1$ and thus $\gamma \circ S(w) = S(\gamma w) = T(\gamma w) = \gamma \circ T(w)$. Assume that $R_n = R_{n+1}$ for some $n > 0$. Then it holds that $R_{n+1} = R_{n+2} = \cdots = R_{n+k} = \cdots$. Indeed, assume that $(S,T) \in R_{n+1}$. Then for each $z = \gamma w \in \Gamma^*$, $\gamma \in \Gamma$, $w \in \Gamma^*$, $|w| = n+1$ it holds that $S(\gamma w) = \gamma \circ S(w)$ and $T(\gamma w) = \gamma \circ T(w)$. But $(\gamma \circ S, \gamma \circ T) \in R_n = R_{n+1}$, thus $(\gamma \circ S)_n = (\gamma \circ T)_n$, which implies that $\gamma \circ S(w) = \gamma \circ T(w)$ for all $w \in \Gamma^*, |w| < n$, that is, $S(z) = T(z)$ for all $|z| < n+2$. That is, $(S,T) \in R_{n+2}$. Since $R_{n+2} \subseteq R_{n+1}$ we get that $R_{n+1} = R_{n+2} = \cdots = R_{n+k}$ for any $k > 1$. Denote by $Z_i = W_{\mathcal{D}}/R_i$ the set of equivalence classes generated by R_i. It is easy to see that $card(W_{\mathcal{D}}/R_i) \leq card(W_{\mathcal{D}}/R_{i+1}) \leq N$ for all $i > 0$ and equality holds only if $R_i = R_{i+1}$. Assume that the strict inclusion $R_i \supset R_{i+1}$ holds for all $N \geq i \geq 1$. Then we get that $N \geq card(Z_{N+1}) > \cdots > card(Z_2) > card(Z_1)$. But then $card(Z_1) \leq -1$, which is a contradiction. That is, there exists $N \geq i \geq 1$ such that $R_i = R_{i+1} = \cdots = R_{N+k}$, $k > 0$. That is, $R_N = \bigcap_{i \leq 1} R_i = id$. That is, $S_N = T_N \iff S = T$. It implies that η_N is injective. It is straightforward to see that η_N is surjective. □

Theorem 66 above implies that if J is finite and we know that $\mathcal{D}$ has a realization with at most N states, then a minimal realization of $\mathcal{D}$ can be computed from finite data. In fact, the proof of the theorem above yields the following algorithm.

`ComputeAutomataRealization`$(W_{\mathcal{D},N,N+1})$

1. Assume that $W_{\mathcal{D},N,N+1} = \{S_1, \ldots, S_K\}$. Let $Q = \{1, \ldots, K\}$
2. Let $\delta(i,x) = j$ if $S_j(w) = S_i(xw)$ for all $w \in \Gamma^*$, $|w| < N$.
3. Let $\lambda(i) = S_i(\epsilon)$, $i = 1, 2, \ldots, K$.
4. Let $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$.

5. Let $\zeta(j) = i$ if $\psi_j(w) = S_i(w)$ for all $w \in \Gamma^*$, $|w| < N + 1$.

6. return $(\mathcal{A}, \zeta)$.

Proposition 37. *Assume that it is possible to check algorithmically if $o_1 = o_2$ for all $o_1, o_2 \in O$. The algorithm `ComputeAutomataRealization`$(W_{\mathcal{D},N,N+1})$ above always terminates if $card(W_{\mathcal{D},N,N}) = card(W_{\mathcal{D},N+1,N}) = card(W_{\mathcal{D},N,N+1})$ and it returns a N-partial realization $(\mathcal{A}, \zeta)$ of $\mathcal{D}$. The realization $(\mathcal{A}, \zeta)$ is isomorphic to $(\mathcal{A}_N, \zeta_N)$ from Theorem 66. If $card(W_{\mathcal{D},N,N}) = card(W_{\mathcal{D}})$ then the realization $(\mathcal{A}, \zeta)$ is a minimal realization of $\mathcal{D}$. In particular, if $\mathcal{D}$ has a realization $(\mathcal{A}^{'}, \zeta^{'})$ such that $card(\mathcal{A}^{'}) \leq N$, then $(\mathcal{A}, \zeta)$ is a minimal realization of $\mathcal{D}$*

Proof. Assume that $card(W_{\mathcal{D},N,N}) = card(W_{\mathcal{D},N+1,N}) = card(W_{\mathcal{D},N,N+1})$. Recall from the proof of Theorem 66 the map $\eta : W_{\mathcal{D},N,N+1} \to W_{\mathcal{D},N,N}$, $\eta(T)(w) = T(w)$ for all $w \in \Gamma^*, |w| < N$. Since $card(W_{\mathcal{D},N,N}) = card(W_{\mathcal{D},N,N+1})$, we get that η is bijective. Consider the map $\phi : W_{\mathcal{D},N,N+1} \to Q$, defined by $\phi(S_i) = i$, $i = 1, \ldots, K$. Define $\psi : W_{\mathcal{D},N,N} \to Q$ by $\psi = \phi \circ \eta^{-1}$. It is easy to see that ψ is a bijection. Consider the realization $(\mathcal{A}_N, \zeta_N)$ and denote $\mathcal{A}_N = (W_{\mathcal{D},N,N}, \Gamma, O, \delta_N, \lambda_N)$. Consider the map $f : (i, \gamma) \mapsto \psi(\delta_N(\psi^{-1}(i), \gamma))$. It is easy to see that $\psi^{-1}(i) = \eta(S_i)$. Thus, $\delta_N(\psi^{-1}(i), \gamma) = \delta_N(\eta(S_i), \gamma)$. Assume that $S_i = (w \circ \phi_j)_{N+1}$ for some $|w| < N$. Then $\delta_N(\eta(S_i), \gamma) = (w\gamma \circ \phi_j)_N = \eta(S_k)$ for some $k \in \{1, \ldots, K\}$, moreover such a k is unique due to bijectivity of η. That is, $\psi(\delta_N(\psi^{-1}(i), \gamma)) = k$ such that $S_k(v) = \eta(S_k)(v) = \phi_j(w\gamma v) = S_i(\gamma v)$ for all $v \in \Gamma^*$, $|v| < N + 1$. Thus, by taking $\delta = f$ we get that δ exists and it is well-defined.

It is easy to see that ζ and λ are both well-defined. Notice that $\zeta(j) = \psi(\zeta_N(j)) = \psi((\phi_j)_N)$ and $\lambda(\psi(T)) = T(\epsilon) = \lambda_N(T)$. Thus, we get that $(\mathcal{A}, \zeta)$ is well-defined and $\psi : (\mathcal{A}_N, \zeta_N) \to (\mathcal{A}, \zeta)$ is a automaton isomorphism. Since $(\mathcal{A}_N, \zeta_N)$ is a N-realization, we get that $(\mathcal{A}, \zeta)$ is a N-realization too.

If $card(W_{\mathcal{D},N,N}) = card(W_{\mathcal{D}})$, then

$$card(W_{\mathcal{D},N,N}) = card(W_{\mathcal{D},N+1,N}) = card(W_{\mathcal{D},N,N+1})$$

and $(\mathcal{A}_N, \mu_N)$ is a minimal realization of $\mathcal{D}$. Since $(\mathcal{A}, \mu)$ and $(\mathcal{A}_N, \zeta_N)$ are isomorphic we get that $(\mathcal{A}, \zeta)$ is a minimal realization of $\mathcal{D}$ too. □

It is very easy to give an algorithm for transforming an arbitrary Moore-automaton realization to a reachable and observable one. In fact, such algorithms are well-known for realization of a *single input-output map.* The adaptation of those algorithms are straightforward. We will present these algorithms in order to keep the paper self-contained.

Let $(\mathcal{A},\zeta)$, $\mathcal{A}=(Q,\Gamma,O,\delta,\lambda)$ be a Moore-automaton realization of $\mathcal{D}$. Assume that $card(Q)=n$. Define the set $Reach(\mathcal{A},\zeta)$ by $Reach(\mathcal{A},\zeta)=\{q\in Q \mid \exists j\in J, w\in \Gamma^{*}: \delta(\zeta(j),w)=q\}$. Below we present an algorithm for construction $Reach(\mathcal{A},\zeta)$, if J is finite.

`ComputeReachableSet(`$(\mathcal{A},\zeta)$`)`

1. $R_0=\{\zeta(j)\mid j\in J\}$
2. $R_{i+1}=R_i\cup\{\delta(q,x)\mid q\in R_i, x\in\Gamma\}$
3. If $R_{i+1}\neq R_i$, then goto Step 2 else return R_i.

Proposition 38. *The algorithm* ***ComputeReachableSet*** *terminates and it computes* $Reach(\mathcal{A},\zeta)$.

Proof. Consider the sets $R_0,R_1,\ldots$. It is easy to see by induction that $R_i=\{q\in Q\mid existsj\in J, w\in\Gamma^{*}, |w|\leq i: \delta(\zeta_j,w)=q\}$. Thus, $Reach(\mathcal{A},\zeta)=\bigcup_{i=1}^{+\infty}R_i$. On the other hand $R_i\subseteq R_{i+1}$ for each $i\in\mathbb{N}$. Assume that $R_i=R_{i+1}$. Then $R_{i+2}=R_{i+1}=R_i$. Indeed, let $q\in R_{i+2}$. Then either $q\in R_{i+1}$ or there exists $q^{'}\in R_{i+1}, x\in\Gamma$ such that $\delta(q^{'},x)=q$. But $q^{'}\in R_{i+1}=R_i$, thus $\delta(q^{'},x)=q\in R_{i+1}$. That is $R_{i+2}=R_{i+1}$. Thus, if the algorithm stops at Step 3 with $R_i=R_{i+1}$ then $R_i=R_{i+1}=R_{i+2}=\ldots=R_{i+k}$ for all $k\in\mathbb{N}$. But then $Reach(\mathcal{A},\zeta)=R_i$, since $R_j\subseteq R_i$ for all $j<i$. It is left to show that the algorithm always terminates. Assume that the algorithm does not terminate. Then $R_1\subset R_2\subset R_3\cdots\subset R_n$. But then $1\leq card(R_1)<card(R_2)<\ldots<card(R_n)\leq n$, which implies that $card(R_n)=n$ has to hold, that is, $R_n=Q$. Since $R_n\subseteq R_{n+1}$ we get that $R_n=R_{n+1}$ and the algorithm terminates after $n+1$ steps. ∎

Based on the algorithm above one can construct a reachable realization of $\mathcal{D}$ based on $(\mathcal{A},\zeta)$ as follows.

`ComputeReachableAutomata(`$(\mathcal{A},\zeta)$`)`

1. $Q_r=ComputeReachableSet(\mathcal{A},\zeta)$
2. Let $\delta_r(q,x)=\delta(q,x)$, for all $q\in Q_r, x\in\Gamma$
3. Let $\lambda_r(q)=\lambda(q)$ for all $q\in Q_r$
4. Let $\zeta_r(j)=\zeta_j$
5. Let $\mathcal{A}_r=(Q_r,\Gamma,O,\delta,\lambda)$.

It is easy to see that $(\mathcal{A}_r, \zeta_r)$ is a reachable realization of $\mathcal{D}$. It is also easy to see that the construction above gives an algorithm for computing $(\mathcal{A}_r, \zeta_r)$. The algorithm for constructing the set $Reach(\mathcal{A}, \zeta)$ can be used for checking reachability. The following procedure can be used to decide whether $(\mathcal{A}, \zeta)$ is reachable

`IsReachabelAutomata`$((\mathcal{A}, \zeta)$

1. $S = ComputeReachableSet(\mathcal{A}, \zeta)$.
2. If $card(S) = card(\mathcal{A})$ then return true else return false.

One can also formulate and algorithm for computing an observable realization of $\mathcal{D}$ based on $(\mathcal{A}, \zeta)$. In order to do so it must be possible to decide by an algorithm whether o_1 and o_2 are identical for any $o_1, o_2 \in O$. Define the indistinguishability relation $I \subseteq Q \times Q$ by $(q_1, q_2) \in I \iff (\forall w \in \Gamma^* : \lambda(q_1, w) = \lambda(q_2, w)$. It is easy to see that I is an equivalence relation. Moreover, $\mathcal{A}$ is observable if and only if $I = id$ or in other words $card(I) = n$, where $id = \{(q, q) \in (Q \times Q) \mid q \in Q\}$. The algorithm below computes the indistinguishability relation.

`ComputeIndistingRelation`$((\mathcal{A}, \zeta))$

1. $I_0 = \{(q_1, q_2) \in (Q \times Q) \mid \lambda(q_1) = \lambda(q_2)\}$
2. $I_{k+1} = I_k \cap \{(q_1, q_2) \in (Q \times Q) \mid (\delta(q_1, x), \delta(q_2, x)) \in I_k, \forall x \in \Gamma\}$
3. If $I_{k+1} \neq I_k$ the goto Step 2 else return I_k

Proposition 39. *The algorithm `ComputeIndistingRelation` always terminates and it computes the relation I.*

Proof. Notice that $I_k = \{(q_1, q_2) \in (Q \times Q) \mid \forall w \in \Gamma^*, |w| \leq k : \lambda(q_1, w) = \lambda(q_2, w)\}$. It is also easy to see that $I_{k+1} \subseteq I_k$. Notice that $I = \bigcap_{k=0}^{+\infty} I_k$. We will show that if $I_k = I_{k+1}$ then $I_{k+2} = I_{k+1}$. Indeed, assume that $(q_1, q_2) \in I_{k+1} = I_k$. Then for each $x \in \Gamma$ $(\delta(q_1, x), \delta(q_2, x) \in I_k = I_{k+1}$. Thus $(q_1, q_2) \in I_{k+2}$. Hence, if $I_k = I_{k+1}$ then $I_k = I_{k+1} = \ldots = I$. It is left to show that the algorithm terminates. Assume the contrary. Then $I_0 \supset I_2 \cdots \supset I_{n^2-n} \supseteq Id$. Thus, $n^2 \geq card(I_0) > card(I_1) > \cdots > card(I_{n^2-n}) \geq n$. But this implies that $card(I_{n^2-n}) = n$, thus, $I = I_{n^2-n} = Id = I_{n^2-n+1}$, thus the algorithm must terminate. ∎

The algorithm above can be used to check whether $(\mathcal{A}, \zeta)$ is observable.

`IsObservableAutomata`$((\mathcal{A}, \zeta))$

1. $I = ComputeIndistingRelation((\mathcal{A}, \zeta))$.

2. If $card(I) = card(\mathcal{A})$ then return true else return false.

Based on the algorithm above one can construct an observable realization as follows.

`ComputeObservableAutomata`$((\mathcal{A}, \zeta))$

1. $I = ComputeIndistingRelation((\mathcal{A}, \zeta))$
2. Compute the sets $[q] = \{s \in Q \mid (q, s) \in I\}$ for each $q \in Q$
3. Construct $Q_o = \{[q] \mid q \in Q\}$. Define $\lambda_o([q]) = \lambda(q)$, $\delta_o([q], x) = [\delta(q, x)]$ for each $q \in Q, x \in \Gamma$. Let $\zeta_o(j) = [\zeta(j)]$.
4. return $(\mathcal{A}_o, \zeta_o) = ((Q_o, \Gamma, O, \delta_o, \lambda_o), \zeta_o)$.

It is easy to see that the algorithm above indeed constructs an observable realization $(\mathcal{A}_o, \zeta_o)$ of $\mathcal{D}$. A minimal realization of $\mathcal{D}$ can be computed from $(\mathcal{A}, \zeta)$ as follows.

`ComputeMinimalAutomata`$((\mathcal{A}, \zeta))$

1. $(\mathcal{A}_r, \zeta_r) = ComputeReachableAutomata((\mathcal{A}, \zeta))$
2. $(\mathcal{A}_{min}, \zeta_{min}) = ComputeObservableAutomata((\mathcal{A}_r, \zeta_r))$
3. return $(\mathcal{A}_{min}, \zeta_{min})$

10.3 Hybrid Power Series

In this subsection the algorithmic aspects of hybrid formal power series will be discussed. That is, we will present a procedure for constructing a hybrid representation of a family of hybrid formal power series from finite data. We will also give algorithms for checking minimality, observability and reachability of hybrid representations and for construction of a minimal hybrid representation from a specified hybrid representation. Throughout the section we will assume that J_1 is finite, that is, we will study only *finite families of hybrid formal power series* .

Recall the results on partial realization by a Moore automaton from Section 10.2. Recall the results on partial representation of formal power series from Section 10.1.

Let $\Omega = \{Z_j \in \mathbb{R}^p \ll X^* \gg \times F(X_2^*, O) \mid j \in J\}$ be an indexed set of hybrid formal power series with $J = J_1 \cup (J_1 \times J_2)$. Assume that J_1 is a finite set.

Recall from Subsection 10.1.1 the definition of the map $\eta_N : \mathbb{R}^p \ll X^* \gg \to \mathbb{R}^p \ll X^{<N} \gg$, i.e. $\eta_N(T)$ is the restriction of T to the set of words over X of length less than N. For each $N \in \mathbb{N}$ define the set

$$\bar{O}_N = \{((\eta_N(S_j))_{j \in J_2} \mid \forall j \in J_2 : S_j \in \mathbb{R}^p \ll X^* \gg\}$$

Again, if $J_2 = \emptyset$ then we take $\bar{O}_N = \{\emptyset\} = \bar{O}$. For each $j_1 \in J_1$ define the map

$$\kappa_{j_1,N} : X_2^* \ni w \mapsto ((Z_{j_1})_D, (\eta_N(w \circ Z_{j_1,j_2})_C)_{j_2 \in J_2}),) \in O \times \bar{O}_N$$

Define the indexed set

$$\mathcal{D}_{\Omega,N} = \{\kappa_{j,N} \mid j \in J_1\}$$

Let $H_{\Omega,N,M} = H_{\Psi_\Omega,N,M}$ for each $M, N \in \mathbb{N}$. Recall from Subsection 10.1.1, Lemma 53, that if rank $H_{\Omega,N,N} =$ rank H_Ω, then the restriction of the map η_N to W_{Ψ_Ω} is a linear isomorphism. The discussion above yields the following.

Lemma 56. *Assume that $(\mathcal{A}, \zeta)$ is a reachable realization of $\mathcal{D}_{\Omega,N}$, where*

$$\mathcal{A} = (Q, X_2, O \times \bar{O}_N, \delta, \lambda)$$

Assume that rank $H_{\Omega,N,N}$ = rank H_Ω. *Consider the Moore-automaton realization $(\bar{\mathcal{A}}, \zeta)$ such that $\bar{\mathcal{A}} = (Q, X_2, O \times \bar{O}, \delta, \bar{\lambda})$ where*

$$\bar{\lambda}(q) = (o, (\eta_N^{-1}(T_j))_{j \in J_2}) \iff \lambda(q) = (o, (T_j)_{j \in J_2})$$

if $J_2 \neq \emptyset$, and $\bar{\lambda}(q) = \lambda(q)$ if $J_2 = \emptyset$. Then $(\bar{\mathcal{A}}, \zeta)$ is a realization of $\mathcal{D}_\Omega$. Moreover, $(\bar{\mathcal{A}}, \zeta)$ is reachable and if $(\mathcal{A}, \zeta)$ is observable, then $(\bar{\mathcal{A}}, \zeta)$ is observable too.

Proof. First of all, we have to show that $\bar{\mathcal{A}}$ is well-defined. For this, we have to show that $\bar{\lambda}(q)$ is well-defined. Notice that $(\mathcal{A}, \zeta)$ is a reachable realization of $\mathcal{D}_{\Omega,N}$. It means that for all $q \in Q$ there exists a $j \in J_1, w \in X_2^*$ such that $q = \delta(\zeta(j), w)$. Since $(\mathcal{A}, \zeta)$ is a realization of $\mathcal{D}_{\Omega,N}$, we get that $\lambda(q) = \kappa_{j,N}(w) = (Z_j(D)(w), (\eta_N((w \circ (Z_{j,j_2})_C))_{j_2 \in J_2})$. Notice that $\eta_N(w \circ (Z_{j,j_2})_C) \in \eta_N(W_{\Psi_\Omega}) = W_{\Psi_\Omega,.,N}$. Thus, we get that for all $q \in Q$, if $\lambda(q) = (o, (T_j)_{j \in J_2})$, then $T_j \in W_{\Psi_\Omega,.,N}$ for all $j \in J_2$ and thus $\eta_N^{-1}(T_j) \in W_{\Psi_\Omega}$ is uniquely defined for all $j \in J_2$. Thus, $\bar{\lambda}(q)$ is well defined.

Consider an arbitrary $f \in J_1$ and $w \in X_2^*$. If $(\mathcal{A}, \zeta)$ is a realization of $\mathcal{D}_{\Omega,N}$ then it holds that $\lambda(\delta(\zeta(f), w)) = \kappa_{f,N}(w)$. Let $q = \delta(\zeta(f), w)$. Thus, $\lambda(q) = \kappa_{f,N}(w)$. We will show that $\bar{\lambda}(q) = \kappa_f(w)$. Indeed, $\kappa_{f,N}(w) = ((Z_f)_D(w), (\eta_N(w \circ (Z_{f,j})_C))_{j \in J_2}$ Notice that $(\eta_N^{-1}(\eta_N(w \circ (Z_{f,j})_C)))_{j \in J_2} = (w \circ (Z_{f,j})_C)_{j \in J_2}$. Thus, $\bar{\lambda(q)} = ((Z_f)_D(w), (w \circ (Z_{f,j})_C)_{j \in J_2}) = \kappa_f(w)$.

Thus, $\bar{\lambda}(\delta(\zeta(f), w)) = \kappa_f(w)$ for any $w \in X_2^*$, $f \in J_1$. That is, $(\bar{\mathcal{A}}, \zeta)$ is a realization of $\mathcal{D}_\Omega$. If $(\mathcal{A}, \zeta)$ is reachable, then it is straightforward to see that $(\bar{\mathcal{A}}, \zeta)$ is reachable.

Assume that $(\mathcal{A}, \zeta)$ is observable. Let $q_1, q_2 \in Q$ and assume that q_1 and q_2 are indistinguishable in $\bar{\mathcal{A}}$. Fix a $w \in X_2^*$ and let $q_1^{'} = \delta(q_1, w)$ and $q_2^{'} = \delta(q_2, w)$. Then $(o^{'}, (\eta_N^{-1}(S)_j)_{j \in J_2}) = \bar{\lambda}(q_1^{'}) = \bar{\lambda}(q_2^{'}) = (o, (\eta_N^{-1}(T_j))_{j \in J_2})$ such that $\lambda(q_1^{'}) = (o^{'}, (S_j)_{j \in J_2})$ and $\lambda(q_2^{'}) = (o, (T_j)_{j \in J_2})$. Since $T_j, S_j \in W_{\Psi_\Omega,.,N}$ for all $j \in J_2$ and

η_N is bijective and $\eta_N^{-1}(T_j) = \eta_N^{-1}(S_j)$, we get that $S_j = T_j$ for all $j \in J_2$ and $o = o^{'}$. Thus we get that $\lambda(q_1, w) = \lambda(q_1^{'}) = \lambda(q_2^{'}) = \lambda(q_2, w)$. Thus, we get that $\lambda(q_1, w) = \lambda(q_2, w)$ for each $w \in X_2^*$. That is, q_1 and q_2 are indistinguishable in $\mathcal{A}$. By observability of $\mathcal{A}$ we get that $q_1 = q_2$. Hence $\bar{\mathcal{A}}$ is observable. ∎

Let R an observable representation of Ψ_Ω and assume that rank $H_{\Omega,N,N}$ = rank H_Ω. Let $(\mathcal{A}, \zeta)$ be a reachable realization of $\mathcal{D}_{\Omega,N}$. Then by the lemma above $(\bar{\mathcal{A}}, \zeta)$ is a reachable realization of $\mathcal{D}_\Omega$. Consider the hybrid representation $HR_{R,\bar{\mathcal{A}},\zeta}$. Notice $\mathcal{A}$ and $\bar{\mathcal{A}}$ have the same state-space and state-transition maps. Thus, all the information we need for the construction of $HR_{R,\bar{\mathcal{A}},\zeta}$ is already contained in R and $(\mathcal{A}, \zeta)$. In fact, if we know R and $(\mathcal{A}, \zeta)$, then the *construction of* $HR_{R,\bar{\mathcal{A}},\zeta}$ *can be carried out by a numerical computer algorithm.* Thus, denoting $HR_{R,\bar{\mathcal{A}},\zeta}$ simply by $HR_{R,\mathcal{A},\zeta}$ is justified in some sense. In the rest of the subsection we will use this abuse of notation and we will denote $HR_{R,\bar{\mathcal{A}},\zeta}$ by $HR_{R,\mathcal{A},\zeta}$

The following theorem is an easy consequence of Theorem 66 and Theorem 64.

Theorem 67. *Assume that* rank $H_{\Psi_\Omega,N,N}$ = rank $H_{\Psi_\Omega,N+1,N}$ = rank $H_{\Psi_\Omega,N,N+1}$ *and* $card(W_{\mathcal{D}_{\Omega,N},D,D}) = card(W_{\mathcal{D}_{\Omega,N},D+1,D}) = card(W_{\mathcal{D}_{\Omega,N},D,D+1})$. *Let* R_N *be the* N*-partial representation of* Ψ_Ω *from Theorem 64. Let* $(\mathcal{A}_D, \zeta_D)$ *be the* D*-partial realization of* $\mathcal{D}_{\Omega,N}$ *from Theorem 66. If* $card(W_{\mathcal{D}_{\Omega,N},D,D}) = card(W_{\mathcal{D}_{\Omega,N}})$ *and* rank $H_{\Omega,N,N}$ = rank H_Ω *then the hybrid representation*

$$HR_{N,D} = HR_{R_N,\bar{\mathcal{A}}_D,\zeta_D}, \mu_{R_N,\bar{\mathcal{A}}_D,\zeta_D}$$

is a minimal hybrid representation of Ω.

Proof. If rank $H_{\Psi_\Omega,N,N}$ = rank $H_{\Psi_\Omega,N+1,N}$ = rank $H_{\Psi_\Omega,N,N+1}$, then R_N is an N-partial representation of Ψ_Ω. If

$$card(W_{\mathcal{D}_{\Omega,N},D,D}) = card(W_{\mathcal{D}_{\Omega,N},D+1,D}) = card(W_{\mathcal{D}_{\Omega,N},D,D+1})$$

then $(\mathcal{A}_D, \zeta_D)$ is a D-partial realization of $\mathcal{D}_{\Omega,N}$. Assume that rank $H_{\Omega,N,N}$ = rank H_Ω. Then by Theorem 64 R_N is a minimal representation of Ψ_Ω. If

$$card(W_{\mathcal{D}_{\Omega,N},D,D} = card(W_{\mathcal{D}_{\Omega,N}})$$

then by Theorem 66 $(\mathcal{A}_D, \zeta_D)$ is a minimal realization of $\mathcal{D}_{\Omega,N}$. By Lemma 56 the condition rank $H_{\Omega,N,N}$ = rank H_Ω implies that $(\bar{\mathcal{A}}_D, \zeta_D)$ is a minimal realization of $\mathcal{D}_\Omega$. But if R_N is a minimal representation of Ψ_Ω and $(\bar{\mathcal{A}}_D, \zeta_D)$ is a minimal realization of $\mathcal{D}_\Omega$, then by Corollary 9 $HR_{N,D} = HR_{R_N,\bar{\mathcal{A}}_D,\zeta_D}$ is a minimal hybrid representation of Ω. ∎

Notice that R_N can be constructed from the columns of the finite matrix $H_{\Omega,N,N}$ and $(\mathcal{A}_D, \zeta_D)$ can be constructed from the finitely many data points of the (finite) set $W_{\mathcal{D}_{\Omega,N},D,D}$. Thus, $HR_{N,D}$ can be constructed from finitely many data and this data can be directly obtained from Ω. The following lemma is an easy consequence of Theorem 66 and Theorem 64.

Lemma 57. *If Ω has a hybrid representation HR such that $\dim HR \leq (q,p)$, then $\text{rank } H_{\Omega,M,M} = \text{rank } H_{\Omega}$ and $card(W_{\mathcal{D}_{\Omega,M},q,q}) = card(W_{\mathcal{D}_{\Omega,M}})$ where $M = q \cdot card(J_2) + p$ if $J_2 \neq \emptyset$ and $M = p$ otherwise. In particular, if $\dim HR = (q,p)$ and for some $N \in \mathbb{N}$*

$$N \geq \begin{cases} q \cdot card(J_2) + p & \text{if } J_2 \neq \emptyset \\ \max\{q,p\} & \text{if } J_2 = \emptyset \end{cases} \tag{10.5}$$

then $\text{rank } H_{\Omega,N,N} = \text{rank } H_{\Omega}$ and $card(W_{\mathcal{D}_{\Omega,N},N,N}) = card(W_{\mathcal{D}_{\Omega,N}})$.

Proof. Assume that $card(J_2) = m$. If HR is a hybrid representation of Ω, then R_{HR} is a representation of Ψ_Ω and $\dim R_{HR} \leq qm+p = M$. Thus, $\text{rank } H_\Omega \leq qm+p = M$. Hence, $\text{rank } H_{\Omega,M,M} = \text{rank } H_\Omega$.

Assume that $\bar{\mathcal{A}}_{HR} = (Q, X_2, O \times \bar{O}, \delta, \bar{\lambda})$. Define $\widetilde{\mathcal{A}}_{HR} = (Q, X_2, O \times \bar{O}_N, \delta, \widetilde{\lambda})$, such that $\widetilde{\lambda}(q) = (o, (\eta_N(T_j))_{j \in J_2})$ if $\bar{\lambda}(q) = (o, (T_j)_{j \in J_2})$. It is easy to see that $(\widetilde{\mathcal{A}}_{HR}, \mu_D)$ is a realization of $\mathcal{D}_{\Omega,N}$ and $card(\widetilde{\mathcal{A}}_{HR}) \leq q$. Thus, $card(W_{\mathcal{D}_{\Omega,N},q,q}) = card(W_{\mathcal{D}_{\Omega,N}})$.

Assume that $N \in \mathbb{N}$ is such that (10.5) holds. Notice that $N \geq q$ and $N \geq p$. Indeed, if $J_2 \neq \emptyset$, then $m > 1$ and thsu $N \geq qm + p \geq q + p \geq \max\{q,p\}$. If $J_2 = \emptyset$ then $N \geq \max\{q,p\}$ by definition. Thus, $\dim HR = (q,p) \leq (N,N)$ and the second statement of the lemma follows from the first one. $\square$

Corollary 24. *If Ω has a hybrid representation HR such that $\dim HR \leq (q,p)$ then for*

$$M = \begin{cases} q \cdot card(J_2) + p & \text{if } J_2 \neq \emptyset \\ p & \text{if } J_2 = \emptyset \end{cases}$$

$HR_{M,q}$ is a minimal representation of Ω. If

$$N \geq \begin{cases} q \cdot card(J_2) + p & \text{if } J_2 \neq \emptyset \\ \max\{q,p\} & \text{if } J_2 = \emptyset \end{cases}$$

then $HR_{N,N}$ is a minimal hybrid representation of Ω.

In particular, if Ω is a finite collection of hybrid formal power series it is known that Ω has a realization of dimension at most (p,q), then a minimal hybrid representation of Ω can be constructed from finite data.

The results above also allow us to check reachability and observability of hybrid representations algorithmically and to construct an equivalent minimal hybrid representation from a specified representation HR. Consider a hybrid representation.

$$HR = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j \in J_2, z \in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y \in X_2})_{q \in Q}, J, \mu)$$

where $\mathcal{A} = (Q, X_2, O, \delta, \lambda)$. Recall the definition of $\bar{A}_{HR}$ and recall the definition of the formal power series $T_{q,j}$, $q \in Q$, $j \in J_2$. For any $N \in \mathbb{N}$, $N > 0$ define the following Moore-automaton

$$\mathcal{A}_{HR,N} = (Q, X_2, O \times \bar{O}_N, \delta, \widetilde{\lambda}), \text{ and } \widetilde{\lambda}(q) = (\lambda(q), (\eta_N(T_{q,j}))_{j \in J_2})$$

That is, $\widetilde{\lambda}(q) = (o, (\eta_N(S_j))_{j \in J_2}$ if $\bar{\lambda}(q) = (o, (S_j)_{j \in J_2})$. Recall that for each $q \in Q$, $j \in J_2$, $y_1, \ldots, y_k \in X_2$, $k \geq 0$, $x_1, \ldots, x_{k+1} \in X_1^*$, $k + \sum_{z=1}^{k+1} x_z < N$

$$(T_{q,j})_N(x_1 y_1 \cdots x_k y_k x_{k+1}) = C_{q_k} A_{q_k, x_{k+1}} M_{q_k, y_k, q_{k-1}} \cdots M_{q_l, y_l, q_{l-1}} A_{q_{l-1}, s_l} B_{q_{l-1}, z_l, j}$$

where $l = \min\{z \mid |x_z| > 0\}$, $s_l \in X_1^*, z_l \in X_1, x_l = z_l s_l$ and $q_i = \delta(q, \gamma_1 \cdots \gamma_i)$, $i = 0, \ldots, k$.

Lemma 58. *Assume the notation above. If HR is a representation of Ω, then $(\mathcal{A}_{HR,N}, \mu_D)$ is a realization of $\mathcal{D}_{\Omega,N}$. The Moore-automaton $(\mathcal{A}_{HR,N}, \mu_D)$ is reachable if and only if $(\mathcal{A}, \mu_D)$ is reachable. Assume that $\dim HR = (q,p)$ and $N \geq q \cdot card(J_2) + p$, or, $\operatorname{rank} H_{\Omega,N,N} = \operatorname{rank} H_\Omega$ and $\mathcal{A}$ is reachable. Then $(\mathcal{A}_{HR,N}, \mu_D)$ is observable if and only if $(\bar{\mathcal{A}}_{HR}, \mu_D)$ is observable.*

Proof. Define the map $h_N : O \times \bar{O} \to O \times \bar{O}_N$ by $h_N : (o, (T_j)_{j \in J_2}) \mapsto (o, (\eta_N(T_j))_{j \in J_2})$. Notice that $\kappa_{f,N}(w) = h_N(\kappa_f(w))$, $f \in J_1$. Recall from Theorem 5 that if HR is a representation of Ω then $(\bar{\mathcal{A}}, \mu_D)$ is a realization of $\mathcal{D}_\Omega$. Thus, $\bar{\lambda}(\mu_D(f), w) = \kappa_f(w)$. But for each $q \in Q$, $\bar{\lambda}(q) = (\lambda(q), (T_{q,j})_{j \in J_2})$, thus $\widetilde{\lambda}(q) = h_N(\bar{\lambda}(q))$. That is, $\widetilde{\lambda}(\mu_D(f), w) = h_N(\bar{\lambda}(\mu_D(f), w) = h_N(\kappa_f(w)) = \kappa_{f,N}$. That is, $(\mathcal{A}_{HR,N}, \mu_D)$ is a realization of $\mathcal{D}_{\Omega,N}$. It follows from definition that $(\mathcal{A}_{HR,N}, \mu_D)$ is reachable if and only if $(\mathcal{A}, \mu_D)$ is reachable (the state-space and the state-transition maps of the two automata coincide).

First we will show that if $(\mathcal{A}, \mu_D)$ is reachable and $\operatorname{rank} H_{\Omega,N,N} = \operatorname{rank} H_\Omega$, or $N \geq q \cdot card(J_2) + p$, then the map $\eta_N : \mathbb{R}^p \ll X^* \gg \ni T \mapsto T|_{w \in X^*, |w| < N} \in \mathbb{R}^p \ll X^{<N} \gg$ is injective when restricted to $T_{q,j}$, $q \in Q$, $j \in J_2$.

Assume that $(\mathcal{A}, \mu_D)$ is reachable. Then $(\bar{A}, \mu_D)$ is reachable. It means that for all $q \in Q$ there exists $w \in X_2^*$, $j \in J_1$ such that $q = \delta(\mu_D(j), w)$. Since $(\bar{A}, \mu_D)$ is also a realization of $\mathcal{D}_\Omega$ we get that $\bar{\lambda}(q) = \kappa_j(w)$, that is, $\bar{\lambda}(q) = (\lambda(q), (T_{q,j_2})_{j_2 \in J_2}) = ((Z_j)_D(w), (w \circ (Z_{j,j_2})_C)_{j_2 \in J_2})$. Thus, $w \circ (Z_{j,j_2})_C = T_{q,j_2}$ for all $j_2 \in J_2$. That is,

$T_{q,j} \in W_{\Psi_\Omega}$ for all $j \in J_2$. Assume that rank $H_{\Omega,N,N}$ = rank H_Ω. Then $\eta_N : W_{\Psi_\Omega} \rightarrow \mathbb{R}^p \ll \widetilde{X}^{<N} \gg$ is injective. In particular, η_N is injective on $\{T_{q,j} \mid q \in Q, j \in J_2\}$.

Assume that $\dim HR = (d,n)$ and $n + dm \leq N$. Let $\widetilde{J}_1 = \mathcal{H}_{HR}$, $\widetilde{J} = \widetilde{J}_1 \cup (\widetilde{J}_1 \times J_2)$. Define the hybrid formal power series $S_{h,j}$ and S_h, $h \in \mathcal{H}_{HR}, j \in J_2$ as follows. Let $S_{h,j_2} = ((S_h)_D, T_{q,j_2})$, where $(S_h)_D(w) = \lambda(q,w)$, if $h = (q,x)$ and $w \in X_2^*$. Let $S_h = ((S_h)_D, (S_h)_C)$, where $S_h(w) = \Pi_{\mathcal{Y}} \circ \upsilon_{HR}(h,w)$. Define the following set of hybrid formal power series $\Theta = \{S_j \mid j \in \widetilde{J}\}$. It is easy to see that for all $j_1 \in J_1$, $Z_{j_1} = S_{\mu(j_1)}$ and $Z_{j_1,j_2} = S_{\mu(j_1),j_2}$, $j_2 \in J_2$. It is also easy to see that $T_{q,j} = (S_{(q,0),j})_C$ for all $j \in J_2$. Assume that $HR = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j \in J_2, z \in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y \in X_2})_{q \in Q}, J, \mu)$. Let

$$\widetilde{HR} = (\mathcal{A}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j}\}_{j \in J_2, z \in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y \in X_2})_{q \in Q}, \widetilde{J}, \widetilde{\mu})$$

where $\widetilde{\mu}((q,x)) = (q,x)$ be a hybrid representation. Notice that $\widetilde{HR}$ coincides with HR except $\widetilde{\mu}$ and $\mathcal{H}_{HR} = \mathcal{H}_{\widetilde{HR}}$, $\upsilon_{HR} = \upsilon_{\widetilde{HR}}$. It is easy to see that $\widetilde{HR}$ is a hybrid representation of Θ and thus by Lemma 57 rank $H_{\Theta,N,N}$ = rank H_Θ, thus η_N is injective on W_{Ψ_Θ}. Since $T_{q,j} = (S_{(q,0),j})_C \in W_{\Psi_\Theta}$ for all $j \in J_2, q \in Q$, we get that η_N is injective on the set $\{T_{q,j} \mid q \in Q, j \in J_2\}$.

That is, if either $\dim HR = (q,p)$ and $qm + p \leq N$ or rank $H_{\Omega,N,N}$ = rank H_Ω and $(\mathcal{A}, \mu_D)$ is reachable, then η_N is injective on the set $S = \{T_{q,j} \mid j \in J_2, q \in Q\}$. Thus, the map h_N is bijective on the set $O \times \{(T_{q,j})_{j \in J_2} \mid q \in Q\}$. Hence, for each $w \in X_2^*, q_1, q_2 \in Q$, $h_N(\bar{\lambda}(q_1,w)) = \widetilde{\lambda}(q_1,w) = \widetilde{\lambda}(q_2,w) = h_N(\bar{\lambda}(q_1,w))$ is equivalent to $\bar{\lambda}(q_1,w) = \bar{\lambda}(q_2,w)$. $(\bar{\mathcal{A}}, \mu_D)$ is observable if and only if $(\forall w \in X_2^* : \bar{\lambda}(q_1,w) = \bar{\lambda}(q_2,w)) \implies q_1 = q_2$. But $(\forall w \in X_2^* : \bar{\lambda}(q_1,w) = \bar{\lambda}(q_2,w))$ is equivalent to $(\forall w \in X_2^* : \widetilde{\lambda}(q_1,w) = \widetilde{\lambda}(q_2,w))$. Thus, observability of $(\bar{\mathcal{A}}, \mu_D)$ is equivalent to $(\forall w \in X_2^* : \widetilde{\lambda}(q_1,w) = \widetilde{\lambda}(q_2,w)) \implies q_1 = q_2)$, which is equivalent to $(\mathcal{A}_{HR,N}, \mu_D)$ being observable. □

Consider the following algorithm for computing $(\mathcal{A}_{HR,N}, \mu_D)$.

`ComputeMooreAutomata`(HR, N)

1. For each $q \in Q$, define $\widetilde{\lambda}(q) = (\lambda(q), ((T_{q,j})_N)_{j \in J_2}$, $(T_{q,j})_N \in \mathbb{R}^p \ll \widetilde{\Gamma}^{<N} \gg$, $j \in J_2$.

   $$(T_{q,j})_N(z_1\gamma_1 \cdots \gamma_k z_{k+1}) = C_{q_k} A_{q_k,z_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_l,\gamma_l,q_{l-1}} A_{q_{l-1},v_l} B_{q_{l-1},s,j}$$

   where $q_i = \delta(q, \gamma_1 \cdots \gamma_i)$, $l = \min\{h \mid |z_h| > 0\}$, $\sum_{i=1}^{k+1} |z_i| + k < N$, $i = 0, \ldots, k$, $\gamma_1, \ldots, \gamma_k \in X_2$, $z_1, \ldots, z_{k+1} \in X_1^*$, $k \geq 0$. $z_l = sv_l$, $s \in X_1$, $v_l \in X_1^*$,

2. return $(Q, X_2, O \times \bar{O}_N, \delta, \widetilde{\lambda}), \mu_D)$

Since

$$(T_{q,j})_N(z_1\gamma_1\cdots\gamma_k z_{k+1}) = C_{q_k}A_{q_k,z_{k+1}}M_{q_k,\gamma_k,q_{k-1}}\cdots M_{q_l,\gamma_l,q_{l-1}}A_{q_{l-1},v_l}B_{q_{l-1},s,j} = \\ = T_{q,j}(z_1\gamma_1\cdots\gamma_k z_{k+1})$$

for all $w = z_1\gamma_1\cdots\gamma_k z_{k+1} \in X^*$, $k \geq 0$, $z_1,\ldots,z_{k+1} \in X_1^*$, $\gamma_1,\ldots,\gamma_k \in X_2$, $z_1 = \cdots = z_{l-1} = \epsilon$, $z_l = sv_l$, $s \in X_1$, $|w| = k + \sum_{j=1}^{k+1}|z_j| < N$. It follows that `ComputeMooreAutomata(`$HR, N$`)` always terminates and returns $(\mathcal{A}_{HR,N}, \mu_D)$.

The following algorithm constructs R_{HR} from HR. Assume that

$$HR = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j\in J_2, z\in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y\in X_2})_{q\in Q}, J, \mu)$$

Assume that $Q = \{q_1,\ldots,q_d\}$, $card(J_2) = m$, $J_2 = \{j_1,\ldots,j_m\}$, $\mathcal{X}_q = \mathbb{R}^{n_q}$, $q \in Q$ and $n = n_{q_1} + n_{q_2} + \cdots + n_{q_d}$. Denote by $\mathbb{O}_{k,l} \in \mathbb{R}_{k\times l}$ the matrix, all entries of which are zero. We will represent the state-space of R_{HR} by $\mathbb{R}^{n+dm} \cong \mathbb{R}^n \bigoplus R^{dm}$. The first n_{q_1} coordinates correspond to the space $\mathcal{X}_{q_1}$, the second n_{q_2} coordinates correspond to the space $\mathcal{X}_{q_2}$ and so on. Thus, the coordinates from $n - n_{q_d}$ to n_d correspond to the space $\mathcal{X}_{q_d}$. The first m coordinates after the first n coordinates correspond the the space spanned by vectors $\{e_{q_1,j_1},\ldots,e_{q_1,j_m}\}$ taken in this order. That is, the first coordinate inside the block of m coordinates correspond to e_{q_1,j_1}, the second coordinate to e_{q_1,j_2} and so on. The subsequent block of m coordinates corresponds to the space spanned by $\{e_{q_2,j_1}\ldots,e_{q_2,j_m}\}$, where the first coordinate inside the block corresponds to e_{q_2,j_1}, the second coordinate to e_{q_2,j_2} and so on. That is, the lth coordinate in the ith block of m-coordinates corresponds to the vector e_{q_i,j_l}, for all $i = 1,\ldots,d$, $l = 1,\ldots,m$. Here we used the notation of the definition of R_{HR} in Subsection 3.3.2.

`ComputeRepresentation(`HR`)`

1. For all $z \in X_1$, define

$$M_{e,1,z} = \begin{bmatrix} A_{q_1,z} & 0 & 0 & \cdots & 0 \\ 0 & A_{q_2,z} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{q_d,z} \end{bmatrix}$$

$$M_{e,2,z} = \begin{bmatrix} \widetilde{B}_{q_1,z} & 0 & 0\cdots & 0 \\ 0 & \widetilde{B}_{q_2,z} & \cdots & 0 \\ 0 & 0 & \cdots & \widetilde{B}_{q_d} \end{bmatrix}$$

where $\widetilde{B}_{q,z} = \begin{bmatrix} B_{q,z,j_1} & B_{q,z,j_2} & \cdots & B_{q,z,j_m} \end{bmatrix} \in \mathbb{R}^{n_q\times m}$ for all $q \in Q$, $z \in X_1$.
Let $M_z = \begin{bmatrix} M_{e,1,z} & M_{e,2,z} \\ \mathbb{O}_{dm,n} & \mathbb{O}_{dm,dm} \end{bmatrix}$ for all $z \in X_1$.

2. For all $\gamma \in \Gamma$, define

$$M_{\gamma,1} = \begin{bmatrix} M_{q_1,\gamma,q_1} & M_{q_1,\gamma,q_2} & \cdots & M_{q_1,\gamma,q_d} \\ M_{q_2,\gamma,q_1} & M_{q_2,\gamma,q_2} & \cdots & M_{q_2,\gamma,q_d} \\ \vdots & \vdots & \vdots & \vdots \\ M_{q_d,\gamma,q_1} & M_{q_d,\gamma,q_2} & \cdots & M_{q_d,\gamma,q_d} \end{bmatrix}$$

$$M_{\gamma,2} = \begin{bmatrix} \delta_{q_1,\gamma,q_1} & \delta_{q_1,\gamma,q_2} & \cdots & \delta_{q_1,\gamma,q_d} \\ \delta_{q_2,\gamma,q_1} & \delta_{q_2,\gamma,q_2} & \cdots & \delta_{q_2,\gamma,q_d} \\ \vdots & \vdots & \vdots & \vdots \\ \delta_{q_d,\gamma,q_1} & \delta_{q_d,\gamma,q_2} & \cdots & \delta_{q_d,\gamma,q_d} \end{bmatrix}$$

, where $M_{q_1,\gamma,q_2} = 0$ if $\delta(q_2,\gamma) \neq q_1$ and

$$\delta_{q_1,\gamma,q_2} = \begin{cases} (1,1,\ldots,1) \in \mathbb{R}^{1\times m} & \text{if } \delta(q_2,\gamma) = q_1 \\ (0,0,\ldots,0) \in \mathbb{R}^{1\times m} & \text{otherwise} \end{cases}$$

Let $M_\gamma = \begin{bmatrix} M_{\gamma,1} & \mathbb{O}_{n,dm} \\ \mathbb{O}_{n,dm} & M_{\gamma,2} \end{bmatrix}$ for all $\gamma \in X_2$.

3. Define

$$\widetilde{C} = \begin{bmatrix} C_{q_1} & C_{q_2} & \cdots & C_{q_d} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

4. For all $f \in J_1$, $j_l \in J_2$, $l = 1,\ldots,m$, define $\widetilde{B}_{f,j_l} = e_k$, where $k = n+(i-1)m+l$, $\mu_D(f) = q_i$ and $e_k \in \mathbb{R}^{n+md}$.

5. For all $f \in J_1$, define $\widetilde{B}_f = \begin{bmatrix} \mathbb{O}_{k,1} \\ \mu_C(f) \\ \mathbb{O}_{n-k-n_{q_i},1} \\ \mathbb{O}_{dm,1} \end{bmatrix}$, where $\mu_D(f) = q_i$ and $k = \sum_{j=1}^{i-1} n_{q_j}$.

6. return $R = (\mathbb{R}^{n+dm}, \{M_z\}_{z\in X}, \widetilde{B}, \widetilde{C})$.

It is easy to see that the algorithm `ComputeRepresentation` returns a representation isomorphic to R_{HR}.

Let $R = (\mathbb{R}^n, \{M_z\}_{z\in X}, \widetilde{B}, \widetilde{C})$ be an observable representation of Ψ_Ω and assume that $(\mathcal{A},\zeta)$ is a reachable realization of $\mathcal{D}_{\Omega,N}$. The following algorithm constructs the linear hybrid representation $H_{R,\bar{\mathcal{A}},\zeta}$, where $\bar{\mathcal{A}}$ is constructed from $\mathcal{A}$ as described in Lemma 56.

`ComputeHybridRepresentation(`$R, \widetilde{\mathcal{A}}, \zeta$`)`

1. Assume $\widetilde{\mathcal{A}} = (Q, X_2, O \times \bar{O}_N, \delta, \widetilde{\lambda})$.

2. Let $\mathcal{A} = (Q, X_2, O, \delta, \lambda)$, $\lambda(q) = \Pi_O(\lambda(q))$, for all $q \in Q$.

3. Assume that $Q = \{q_1, \ldots, q_d\}$.
   Let $\begin{bmatrix} U_{q_1}, & U_{q_2}, & \cdots & , U_{q_d} \end{bmatrix} = ComputeStateSpace(R, \widetilde{\mathcal{A}}, \zeta)$
   where $U_q \in \mathbb{R}^{n \times n_q}$.

4. For each $q \in Q$, let $\widetilde{\mathcal{X}}_q = \mathbb{R}^{n_q}$, and $\widetilde{A}_{q,z} = U_q^T M_z U_q$, for all $z \in X_1$.

5. For each $q_1, q_2 \in Q, \gamma \in X_2$, $\delta(q_2, \gamma) = q_1$ let $\widetilde{M}_{q_1,\gamma,q_2} = U_{q_1}^T M_\gamma U_{q_2}$

6. Let $\widetilde{C}_q = C U_q$, for all $q \in Q$.

7. For each $q \in Q$, let $(w_q, f) = ComputePath(\widetilde{\mathcal{A}}, \zeta, q)$
   For all $j \in J_2$, $z \in X_1$ let $\widetilde{B}_{q,z,j} = U_q^T M_z M_{w_q} B_{f,j}$

8. For each $f \in J_1$ let $\widetilde{\mu}(f) = (\zeta(f), B_f)$.

9. Let $HR = (\mathcal{A}, (\widetilde{X}_q, \{\widetilde{A}_{q,z}, \widetilde{B}_{q,z,j}\}_{j \in J_2, z \in X_1}, \widetilde{C}_q, \{\widetilde{M}_{\delta(q,\gamma),\gamma,q}\}_{\gamma \in X_1})_{q \in Q}, J, \widetilde{\mu})$

10. return HR

We used the following algorithms

$$\texttt{ComputePath}(\mathcal{A}, \zeta, q)$$

1. $S_0 = \{(\epsilon, q)\}$

2. $S_{k+1} = \{(q, \gamma w) \in (Q \times X_2^*) \mid (\delta(q, \gamma), w) \in S_k\}$

3. if there exists $(q, w) \in S_k$ such that $q = \zeta(f)$, then return (w, f) else goto 2

Proposition 40. *If $(\mathcal{A}, \zeta)$ is reachable, then the algorithm* ***ComputePath***$(\mathcal{A}, \zeta, q)$ *terminates and it returns a pair (w, f) such that $\delta(\zeta(f), w) = q$.*

Proof. Assume that $w = \gamma_1 \cdots \gamma_k$ is such that $\delta(\zeta(f), w) = q$ for some $f \in J_1$. Let $q_i = \delta(\zeta(f), \gamma_1 \cdots \gamma_i)$. Then $(q_i, \gamma_{i+1}\gamma_{i+2} \cdots \gamma_k) \in S_{k-i}$. Indeed, by induction on $k - i$, if $k = i$, then $q_k = q$ and $S_0 = \{(\epsilon, q)\}$. Assume that the statement holds for $k - i \leq l$. Then $q_{l-1} = \delta(\zeta(f), \gamma_1 \cdots \gamma_{l-1})$ and $(q_l, \gamma_{l+1} \cdots \gamma_k) \in S_{k-l}$. But then $\delta(q_{l-1}, \gamma_l) = q_l$ and thus $(q_{l-1}, \gamma_l \gamma_{l+1} \cdots \gamma_k) \in S_{k-l+1}$. Thus, we get that $(\zeta(f), \gamma_1 \cdots \gamma_k) = S_k$. That is, after at most k steps the algorithm terminates

It is also easy to see by induction on k that $(s, w) \in S_k$ implies that $\delta(s, w) = q$. Hence, if the algorithm terminates, it returns (w, f) such that $\delta(\zeta(f), w) = q$. $\square$

$\texttt{ComputeStateSpace(}R, \mathcal{A}, \zeta\texttt{)}$

1. Assume that $\mathcal{A} = (Q, X_2, O, \delta, \lambda)$, and $Q = \{q_1, \ldots, q_d\}$. Assume that $R = (\mathbb{R}^n, \{M_z\}_{z\in X}, B, C)$. Assume $X_1 = \{z_1, \ldots, z_p\}$.

   For $i = 1, \ldots, d$,
$$(w_i, f_i) = ComputePath(\mathcal{A}, \zeta, q_i)$$
$$F_{q_i} = \{finJ_1 \mid \zeta(f) = q_i\}$$
   Assume $F_{q_i} = \{f_{i,1}, \ldots, f_{i,h_i}\}$ Let
$$BF_{q_i} = \begin{cases} \begin{bmatrix} B_{f_{i,1}}, & B_{f_{i,2}}, & \cdots & , B_{f_{i,h_i}} \end{bmatrix} & \text{if } F_{q_i} \neq \emptyset \\ 0 \in \mathbb{R}^n & \text{if } F_{q_i} = \emptyset \end{cases}$$
$$R_{q_i,0} = \begin{bmatrix} BF_{q_i}^T \\ (M_{z_1} M_{w_i} B_{f_i,j_1} \\ (M_{z_1} M_{w_i} B_{\zeta(f_i),j_2})^T \\ \cdots \\ (M_{z_1} M_{w_i} B_{f_i,j_m})^T \\ \cdots \\ (M_{z_1} M_{w_i} B_{f_i,j_1})^T \\ (M_{z_1} M_{w_i} B_{f_i,j_2})^T \\ \cdots \\ (M_{z_p} M_{w_i} B_{f_i,j_m})^T \end{bmatrix}^T$$

2. For each $i = 1, 2, \ldots, d$ compute the set $\{q_{i,1}, \ldots, q_{i,k_i}\}$ and $\{\gamma_{1,q_i}, \ldots, \gamma_{k_i,q_i}\}$ such that for each $q \in Q$, $\delta(q, \gamma) = q_i$ if and only if $q = q_{i,j}, \gamma = \gamma_{j,q_i}$ for some $j = 1, \ldots, k_i$.

3. For each $i = 1, \ldots, d$
$$A = \begin{bmatrix} R_{q_i,k}, & M_{z_1} R_{q_i,k}, & M_{z_2} R_{q_i,k}, & \cdots & M_{z_p} R_{q_i,k} \end{bmatrix}$$
$$B = \begin{bmatrix} M_{\gamma_{1,q_i}} R_{q_{i,1},k}, & M_{\gamma_{2,q_i}} R_{q_{i,2},k}, & \cdots M_{\gamma_{k_i,q_i}} R_{q_{i,k_i},k} \end{bmatrix}$$
$$R_{q_i,k+1} = \begin{bmatrix} A & B \end{bmatrix}$$

4. If for all $i = 1, \ldots d$, rank $R_{i,k+1} =$ rank $R_{i,k}$ then
   (a) Compute $U_{q_i} \in \mathbb{R}^{n \times n_{q_i}}$ such that $n_{q_i} =$ rank $R_{i,k}$, $U_{q_i}^T U_{q_i} = Id \in \mathbb{R}^{n_{q_i} \times n_{q_i}}$ and $\text{Im} U_{q_i} = R_{i,k}$.
   (b) return $\begin{bmatrix} U_{q_1} & U_{q_2} & \cdots & U_{q_d} \end{bmatrix}$

else repeat step 3

Recall the definition of a hybrid representation

$$HR_{R,\mathcal{A},\zeta} = (\mathcal{A}, \mathcal{Y}, (\mathcal{X}_q, \{A_{q,z}, B_{q,z,j_2}\}_{j \in J_2, z \in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y \in X_2})_{q \in Q}, J, \mu)$$

associated with the representation R and automata $(\mathcal{A}, \zeta)$ from Section 3.3. With the notation above the following holds.

Proposition 41. *The algorithm* `ComputeStateSpace(`$R, \mathcal{A}, \zeta$`)` *always terminates and it returns the matrix* $\begin{bmatrix} U_{q_1} & \dots & U_{q_d} \end{bmatrix}$ *such that* $\mathrm{Im} U_{q_i} = \mathcal{X}_{q_i}$ *and* $U_{qi}^T U_{q_i} = I$.

Proof. Recall the formula (3.15) from Section 3.3. Notice that $\mathrm{Im} R_{q,k} \subseteq \mathrm{Im} R_{q,k+1}$, i.e. rank $R_{q,k} \le$ rank $R_{q,k+1}$. By induction on k it follows that $\mathrm{Im} R_{q,k} \subseteq \mathcal{X}_q$ holds too. First we show that if rank $R_{q,k+1} =$ rank $R_{q,k}$ for all $q \in Q$, then rank $R_{q,k+2} =$ rank $R_{q,k+1}$ for all $q \in Q$. Equivalently, we will show that if $\mathrm{Im} R_{q,k+1} = \mathrm{Im} R_{q,k}$ then $\mathrm{Im} R_{q,k+2} = \mathrm{Im} R_{q,k+1}$ holds too. Indeed,

$$\begin{aligned}
&\mathrm{Im} R_{q_i,k+2} = \mathrm{Im} R_{q_i,k+1} + M_{z_1} \mathrm{Im} R_{q_i,k+1} + \cdots + M_{z_p} \mathrm{Im} R_{q_i,k+1} + \\
&+ M_{\gamma_{1,q_i}} \mathrm{Im} R_{q_{i,1},k+1} + \cdots + M_{\gamma_{k_i,q_i}} \mathrm{Im} R_{q_{i,k_i},k+1} = \mathrm{Im} R_{q_i,k} + \\
&+ M_{z_1} \mathrm{Im} R_{q_i,k} + \cdots + M_{z_k} \mathrm{Im} R_{q_i,k} + M_{\gamma_{1,q_i}} \mathrm{Im} R_{q_{i,1},k} + \cdots M_{\gamma_{k_i,q_i}} \mathrm{Im} R_{q_{i,k_i},k} = \\
&= \mathrm{Im} R_{q_i,k+1}
\end{aligned}$$

Assume that the algorithm never terminates, that is, for each $k \ge 0$ there exists q_k such that $\mathrm{Im} R_{q_k,k} \subsetneq \mathrm{Im} R_{q_k,k+1}$. Let $n_k = \sum_{i=1}^d \mathrm{rank}\, R_{q_i,k}$. Thus we get that $n_0 < n_1 < \cdots < n_k < \cdots$, which implies that $n_k \ge k$. But $n_k \le \sum_{i=1}^d \dim \mathcal{X}_{q_i} \le dn$, where $\dim R = n$. But $n_{dn+1} \ge dn + 1 > dn$, a contradiction. That is, the algorithm terminates and it terminates in at most nd steps. It is easy to see by induction on k that

$$\begin{aligned}
\mathrm{Im} R_{q,k} = &\mathrm{Span}\{M_{z_{r+1}} M_{\gamma_r} M_{z_r} \cdots M_{\gamma_l} M_{z_l} M_v M_{\gamma_{l-1}} \cdots M_{\gamma_2} M_{\gamma_1} \widetilde{B}_{f,j} \\
&\mid j \in J_2, \gamma_1, \dots, \gamma_r \in X_2, f \in J_1, 1 \le l \le r, z_{r+1}, \dots, z_l \in X_1^*, \\
&v \in X_1, k \ge 0, q = \delta(\zeta(f), \gamma_1 \cdots \gamma_r), r + \sum_{i=1}^{r+1} j_i \le k\} + \\
&+ \mathrm{Span}\{M_{z_{r+1}} M_{\gamma_r} M_{z_{r-1}} \cdots M_{\gamma_1} M_{z_1} \widetilde{B}_f \mid \gamma_r, \dots, \gamma_1 \in X_2, \\
&z_{r+1}, \dots, z_1 X_1^*, r \ge 0, q = \delta(\zeta(f), \gamma_1 \cdots \gamma_r), r + \sum_{i=1}^{r+1} j_i \le k\}
\end{aligned}$$

Thus, $\mathcal{X}_q = \sum_{k=1}^{\infty} \mathrm{Im} R_{q,k}$. If $\mathrm{Im} R_{q,k} = \mathrm{Im} R_{q,k+1}$ for all $q \in Q$ then $\mathrm{Im} R_{q,k} = \mathrm{Im} R_{q,k+l}$ for all $q \in Q$, $l \in \mathbb{N}$. Thus, if $\mathrm{Im} R_{q,k} = \mathrm{Im} R_{q,k+1}$ for all $q \in Q$ then $\mathcal{X}_q = \sum_{i=1}^k \mathrm{Im} R_{q,i} = \mathrm{Im} R_{q,k}$. Since $\mathrm{Im} U_q = \mathrm{Im} R_{q,k}$ and $U_q^T U_q = I$, we get the statement of the proposition. ∎

Now we are ready to show that $\texttt{ComputeHybridRepresentation(}R, \widetilde{\mathcal{A}}, \zeta\texttt{)}$ works correctly.

Proposition 42. *Assume that R is an observable representation of Ψ_Ω and $(\widetilde{\mathcal{A}}, \zeta)$ is a reachable realization of $\mathcal{D}_{\Omega,N}$. Then $\texttt{ComputeHybridRepresentation(}R, \widetilde{\mathcal{A}}, \zeta\texttt{)}$ always terminates. If R is a representation of Ψ_Ω and* rank $H_{\Omega,N,N} =$ rank H_Ω*, then $\texttt{ComputeHybridRepresentation(}R, \widetilde{\mathcal{A}}, \zeta\texttt{)}$ returns a hybrid representation isomorphic to the hybrid representation $HR_{R,\bar{\mathcal{A}},\zeta}$, where $\bar{\mathcal{A}}$ is obtained from $\widetilde{\mathcal{A}}$ as described in Lemma 56. That is, if $\widetilde{\mathcal{A}} = (Q, X_2, O \times \bar{O}_N, \delta, \widetilde{\lambda})$ then $(\bar{\mathcal{A}}, \zeta)$ is a realization of $\mathcal{D}_\Omega$, where $\bar{\mathcal{A}} = (Q, X_2, O \times \bar{O}, \delta, \bar{\lambda})$ and*

$$\bar{\lambda}(q) = (o, (\eta_N^{-1}(T_j))_{j\in J_2}) \iff \widetilde{\lambda}(q) = (o, (T_j)_{j\in J_2})$$

Notice that by Lemma 56 $(\bar{\mathcal{A}}, \zeta)$ is a reachable realization of $\mathcal{D}_\Omega$.

Proof. From Proposition 41 it follows that

$$\texttt{ComputeStateSpace(}R, \widetilde{\mathcal{A}}, \zeta\texttt{)}$$

terminates. Proposition 40 implies that

$$\texttt{ComputePath(}\mathcal{A}, \zeta, q\texttt{)}$$

always terminates too. Thus, we get that

$$\texttt{ComputeHybridRealization(}R, \widetilde{\mathcal{A}}, \zeta\texttt{)}$$

always terminates.

It follows from Proposition 41 that with the notation of

$$\texttt{ComputeHybridRepresentation}$$

$\mathrm{Im} U_q = \mathcal{X}_q$ and $U_q^T U_q = I$. Notice that if rank $H_{\Omega,N} =$ rank H_Ω, then by Lemma 56 $(\bar{\mathcal{A}}, \zeta)$ is a reachable realization of $\mathcal{D}_\Omega$, if $(\widetilde{\mathcal{A}}, \zeta)$ is a reachable realization of $\mathcal{D}_{\Omega,N}$. Thus, $HR_{R,\bar{\mathcal{A}},\zeta}$ is a well-defined realization of Ω. Use the notation of

$$\texttt{ComputeHybridRepresentation}.$$

Denote by HR the hybrid representation returned by the algorithm. Define the following map: $T_C : \bigoplus_{q\in Q} \widetilde{\mathcal{X}}_q \to \bigoplus_{q\in Q} \mathcal{X}_q$, $T_C(x) = U_q x$ for each $x \in \widetilde{\mathcal{X}}_q, q \in Q$. We claim that $(id, T_C) : HR \to HR_{R,\bar{\mathcal{A}},\zeta}$ is a hybrid representation morphism, where id is the identity map on Q. It is clear that id is a Moore-automata map. It is also clear that $T_C(\widetilde{\mathcal{X}}_q) = \mathrm{Im} U_q = \mathcal{X}_q$. It is easy to see that $U_q U_q^T y = y$ for each $y = U_q z$. Hence, for each $q \in Q$, $z \in X_1$, $T_C(\widetilde{A}_{q,z} x) = U_q U_q^T M_z U_q x = M_z U_q x = A_{q,z} T_C x$, since $M_z T_C x \in \mathcal{X}_q$ and therefore there exists $y \in \mathbb{R}^{n_q}$ such that $U_q y = M_z T_C x$.

Similarly, for each $q_1, q_2 \in Q$, $\gamma \in X_2$, $\delta(q_2, \gamma) = q_1$, $T_C \widetilde{M}_{q_1,\gamma,q_2} x = U_{q_1} U_{q_1}^T M_\gamma U_{q_2} x = M_\gamma U_{q_2} x = M_{q_1,\gamma,q_2} T_C(x)$, since $M_\gamma x \in \mathcal{X}_{q_1}$ and therefore $M_\gamma x = U_{q_1} y$ for some $y \in \mathbb{R}^{n_{q_1}}$. For each $q \in Q$, $z \in X_1, j \in J_2$ it holds that $T_C \widetilde{B}_{q,z,j} = U_q U_q^T M_z M_{w_q} B_{\zeta(f),j} = B_{q,z,j}$ and $\widetilde{C}_q x = C U_q x = C_q T_C x$. Finally, $T_C \widetilde{\mu}(f) = T_C U_q^T B_f = B_f = \mu(f)$, since $B_f = U_q z \in \mathcal{X}_q$ for some $z \in \mathbb{R}^{n_q}$, $q = \zeta(f)$. That is, (id, T_C) is indeed a linear hybrid system morphism. Moreover, id is a bijection and T_C is a linear isomorphism, since $\sum_{q \in Q} n_q = \sum_{q \in Q} \dim \mathcal{X}_q$. That is, (id, T_C) is a linear hybrid system isomorphism. But it means that HR is a realization of Ω. □

The algorithms above enable us to formulate algorithms for minimisation, observability and reachability reduction of hybrid representations. We will also be able to present an algorithm for constructing a hybrid representation from finite data. Recall from Section 10.2 the algorithm `ComputeAutomataRealization`. Recall from Section 10.1.3 the algorithm `ComputePartialRepresentation`. Consider the following algorithm

`ComputePartialHybRepr(`$H_{\Omega,N+1,N}, W_{\mathcal{D}_{\Omega,N},D,D}$`)`

1. $R = ComputePartialRepresentation(H_{\Omega,N+1,N})$
2. $(\widetilde{\mathcal{A}}, \zeta) = ComputeAutomataRealization(W_{\mathcal{D}_{\Omega,N},D,D})$
3. $HR = ComputeHybridRepresentation(R, \widetilde{\mathcal{A}}, \zeta)$
4. return HR

Proposition 43. *Assume that* rank $H_{\Omega,N,N}$ = rank $H_{\Omega,N+1,N}$ = rank $H_{\Omega,N,N+1}$ *and* $card(W_{\mathcal{D}_{\Omega,N},D,D}) = card(W_{\mathcal{D}_{\Omega,N},D+1,D}) = card(W_{\mathcal{D}_{\Omega,N},D,D+1})$. *The algorithm*

ComputePartialHybRepr($H_{\Omega,N+1,N}, W_{\mathcal{D}_{\Omega,N},D,D}$*)*

always terminates. If rank $H_{\Omega,N,N}$ = rank H_Ω *and* $card(W_{\mathcal{D}_{\Omega,N},D,D}) = card(W_{\mathcal{D}_{\Omega,N}})$ *then*

ComputePartialHybRepr($H_{\Omega,N+1,N}, W_{\mathcal{D}_{\Omega,N},D,D}$*)*

returns a minimal hybrid representation of Ω *which is isomorphic to* $HR_{N,D}$ *from Theorem 67.*

Proof. We will use the notation of the algorithm `ComputePartialHybRepr` and the proof of Theorem 67. If rank $H_{\Omega,N,N}$ = rank $H_{\Omega,N+1,N}$ = rank $H_{\Omega,N,N+1}$ it follows form Theorem 65 that

`ComputePartialRepresentation` terminates and it returns an N-partial representation of Ψ_{Ω}. Similarly, ff

$$card(W_{\mathcal{D}_{\Omega,N},D,D}) = card(W_{\mathcal{D}_{\Omega,N},D+1,D}) = card(W_{\mathcal{D}_{\Omega,N},D,D+1})$$

then it follows from Proposition 37 that `ComputeAutomataRealization` terminates and it returns a D partial realization of $\mathcal{D}_{\Omega,N}$.

Assume that rank $H_{\Omega,N} =$ rank H_{Ω} and $card(W_{\mathcal{D}_{\Omega,N},D,D}) = card(W_{\mathcal{D}_{\Omega,N}})$. It follows from Proposition 42 that HR is isomorphic to $HR_{R,\bar{\mathcal{A}},\zeta}$. From Theorem 65 it follows that there exists a representation isomorphism $T : R \to R_N$. From Proposition 37 it follows that there exists a Moore-automata isomorphism $\phi : (\widetilde{\mathcal{A}}, \zeta) \to (\mathcal{A}_D, \mu_D)$. It is easy to see that ϕ determines an automaton isomorphism $\phi : (\bar{\mathcal{A}}, \zeta) \to (\bar{\mathcal{A}}_D, \mu_D)$. Then by Lemma 16 $(\phi, T_C) : HR_{R\bar{\mathcal{A}},\zeta} \to HR_{N,D}$ is a surjective hybrid representation morphism $T_C(x) = Tx$ for all $x \in \mathcal{X}_q$. Thus, T_C is surjective. We will argue that T_C is injective too. Indeed, $T_C x = T_C y$ implies that $T_C x = T_C y \in \mathcal{X}_s$, where $\mathcal{X}_s$ is the continuous state-space belonging to the discrete state s of $HR_{N,D}$. Since ϕ is bijective, we get that $\phi^{-1}(s) = q$ for some $q \in Q$. Here Q is the discrete state-space of $\widetilde{\mathcal{A}}$. Thus, $x, y \in \mathcal{X}_q$, where $\mathcal{X}_q$ is the continuous state-space component belonging to to discrete-state q of $HR_{R\bar{\mathcal{A}},\zeta}$. Thus, $T_C x = T_C y = Tx = Ty$. But T is injective, therefore $x = y$. That is T_C is injective. It is also surjective hence T_C is a linear isomorphism. Consequently, (ϕ, T_C) is an isomorphism. Thus, we get that HR is isomorphic to $HR_{R\bar{\mathcal{A}},\zeta}$ and $HR_{R\bar{\mathcal{A}},\zeta}$ is isomorphic to $HR_{N,D}$. Hence HR is isomorphic to $HR_{N,D}$.

If rank $H_{\Omega,N} =$ rank H_{Ω} and $card(W_{\mathcal{D}_{\Omega,N},D,D}) = card(W_{\mathcal{D}_{\Omega,N}})$ by Theorem 67 $HR_{N,D}$ is a minimal realization of Ω. Since HR and $HR_{N,D}$ are isomorphic, it follows that HR is a minimal realization of Ω too. □

Using the algorithms above we can construct algorithms for minimality reduction of hybrid representations. It will also enable us to check reachability, observability of hybrid representations.

Assume that HR is a hybrid representation of Ω. The following algorithm constructs a minimal hybrid representation of Ω.

`ComputeMinimalHybRepresentation(`HR`)`

1. $R = ComputeRepresentation(HR)$

2. Assume that $\dim HR = (q, p)$. Let $N = qm + p$.

$$(\widetilde{\mathcal{A}}, \zeta) = ComputeMooreAutomaton(HR, N)$$

3. $R_{min} = ComputeMinimalRepresentation(R)$

4. $(\mathcal{A}_{min}, \zeta_{min}) = ComputeMinimalAutomata(\widetilde{\mathcal{A}}, \zeta)$

5. $HR_{min} = ComputeHybridRealization(R_{min}, \mathcal{A}_{min}, \zeta_{min})$

6. return HR_{min}

Proposition 44. *The algorithm* `ComputeMinimalHybRepresentation(`HR`)` *above computes a minimal realization of* Ω.

Proof. Indeed, by Proposition 41 R is a representation of Ω, therefore R_{min} is a minimal representation of Ψ_{Ω}. Similarly, by Proposition 40 $(\widetilde{\mathcal{A}}, \zeta)$ is a realization of $\mathcal{D}_{\Omega,N}$. It follows that $(\mathcal{A}_{min}, \zeta_{min})$ is a minimal realization of $\mathcal{D}_{\Omega,N}$. Thus, $(\mathcal{A}_{min}, \zeta_{min})$ is a reachable and observable realization. By Lemma 56 $(\bar{\mathcal{A}}_{min}, \zeta_{min})$ is a reachable and observable realization of $\mathcal{D}_{\Omega}$. Therefore it is a minimal realization of $\mathcal{D}_{\Omega}$. Thus, by Corollary 9 $HR_{R_{min},\bar{\mathcal{A}}_{min},\zeta_{min}}$ is a minimal realization of Ω. Since HR_{min} is isomorphic to $HR_{R_{min},\bar{\mathcal{A}}_{min},\zeta_{min}}$, we get that HR_{min} is a minimal realization of Ω. □

Reachability of HR can be checked by the following algorithm

`IsHybReprReachable(`HR`)`

1. $R = ComputeRepresentation(HR)$

2. $(\mathcal{A}, \zeta) = (\mathcal{A}_{HR}, \mu_D)$

3. if $IsReachable(R)$ and $IsReachableAutomata(\mathcal{A}, \zeta)$ then return true otherwise false

It follows easily from Lemma 13 that $IsReachable(HR)$ returns true if and only if HR is reachable and returns false otherwise. The following algorithm checks observability of HR.

`IsHybReprObservable(`HR`)`

1. $R = ComputeRepresentation(HR)$

2. Assume $\dim R = N$.

$$(\widetilde{\mathcal{A}}, \mu_D) = ComputeMooreAutomata(HR, N)$$

3. $O = ComputeObservabilityMatrix(R)$

4. Assume that $Q = \{q_1, \dots, q_d\}$ and $J_2 = \{j_1, \dots, j_m\}$. For each $i = 1, \dots, d$ define $I_{q_i} \in \mathbb{R}^{n \times n}$, $n = n_{q_1} + \cdots + n_{q_d}$, where n_{q_i} is the dimension of the continuous state-space associated with q, i.e. $\mathcal{X}_q = \mathbb{R}^{n_q}$, as follows

$$(I_{q_k})_{i,j} = \begin{cases} 1 & \text{if } i = j = \sum_{z=1}^{k-1} n_z + l \text{ for some } l = 1, \dots, n_{q_k} \\ 0 & \text{otherwise} \end{cases}$$

$$E_{q_k} = \begin{bmatrix} I_{q_k} & \mathbb{O}_{n,dm} \\ \mathbb{O}_{dm,n} & \mathbb{O}_{dm,dm} \end{bmatrix}$$

5. If $IsObservableAutomata(\widetilde{\mathcal{A}}, \mu_D)$ returns true and for each $i = 1, \dots, k$, rank $O \cdot E_{q_i} = n_{q_i}$ then return true else return false

Proposition 45. *The algorithm* `IsHybReprObservable(`HR`)` *always returns true if HR is observable and false otherwise*

Proof. We will use the notation of `IsHybReprObservable(`HR`)`. From Lemma 13 it follows that HR is observable if and only if R_{HR} is $\mathcal{X}_q$ observable for all $q \in Q$ and $(\bar{\mathcal{A}}, \mu_D)$ is observable. It follows from 64 that if $N = \dim R_{HR}$ then rank $H_\Omega \leq N$ and thus rank $H_{\Omega,N,N} =$ rank H_Ω. Hence by Lemma 58 it follows that $(\bar{\mathcal{A}}, \mu_D)$ is observable if and only if $(\widetilde{\mathcal{A}}, \mu_D)$ is observable, that is, if and only if `IsObservabelAutomata(`$\widetilde{\mathcal{A}}, \mu_D$`)` returns true. Notice `ComputeRepresentation(`HR`)` returns a representation R which is isomorphic to R_{HR}. We will use the notation of Subsection 3.3.2. Fix the isomorphism $T : R_{HR} \to R$. It is easy to see that for all $x \in \mathcal{X}_{q_i}$, $i = 1, \dots, d$, $(Tx)_j = x_l$ if $j = \sum_{k=1}^{i-1} n_{q_k} + l$ and $(Tx)_j = 0$ otherwise. For all $i = 1, \dots, d$, $z = 1, \dots, m$, $(Te_{q_i,j_z})_l = 1$ if $l = (i-1) * m + z$ and $(Te_{q_i,j_z})_l = 0$ otherwise. It is easy to see that for each $i = 1, \dots, d$, if $x = y + z$ such that $y \in T(\mathcal{X}_{q_i})$ and $z \in \bigoplus_{j=1,\dots,d, j \neq i} T(\mathcal{X}_{q_j} \oplus \mathbb{R}^{dm})$, then $E_{q_i} x = y$. Thus, $\ker O \cdot E_{q_i} = \ker O \cap T(\mathcal{X}_{q_i})$. Since `ComputeObservabilityMatrix(`$R$`)` returns a matrix O such that $\ker O = O_R$ we get that $\ker O \cdot E_{q_i} = O_R \cap T(\mathcal{X}_q)$. Notice that rank $O \cdot E_{q_i} = n_{q_i}$ if and only if $\ker O \cdot E_{q_i} = O_R \cap T(\mathcal{X}_q = \{0\})$. Thus, the condition that for each $i = 1, \dots, d$ rank $O \cdot E_{q_i} = n_{q_i}$ is equivalent to R being $T(\mathcal{X}_q)$ observable for each $q \in Q$. Since R_{HR} and R are isomorphic, R is $T(\mathcal{X}_q)$ observable if and only if R_{HR} is $\mathcal{X}_q$ observable. Thus, `IsHybReprObservable` returns true if and only if R_{HR} is $\mathcal{X}_q$ observable for each $q \in Q$ and $\bar{\mathcal{A}}_{HR}$ is observable, that is, if and only if HR is observable. $\square$

If $J_2 = \emptyset$ or $J_2 \neq \emptyset$ but we can decide whether $T_{q_1,j}(w) = T_{q_2,j}(w)$ for all $q_1, q_2 \in Q$, $w \in X^*, |w| < N$, $j \in J_2$, then the procedure `ComputeMinimalHybRep` and procedure `IsHybReprObservable` above can be implemented as a numerical computer algorithm. In particular, if the matrices $A_{q,z}, C_q, B_{q,z,j}, M_{q_1,y,q_2}$ are rational for all

$z \in X_1, y \in X_2$, $q, q_1, q_2 \in Q, j \in J_2$, or $J_2 = \emptyset$, then the procedure above yields a computer algorithm for computing a minimal hybrid representation of family of hybrid formal power series.

In fact, the procedures presented above imply the following. Assume that $\mathcal{X}_q = \mathbb{R}^{n_q}$, all matrices of $A_{q,z}, M_{q_1,y,q_q}, C_q, B_{q,z,j}$ are rational (have only rational elements) and for all $q \in Q$, $j \in J_2$, $z \in X_1$, $y \in X_2$ and $\mu(j)$ is a rational vector (has only rational entries) for all $j \in J_1$. Assume that J_1 is finite. Then the procedures `IsHybRepObservable`, `IsHybRepReachable` and `ComputeMinimalHybRepresentation` above are algorithms in the sense of classical Turing computability. That is, they can be implemented by a Turing machine. Thus, observability and reachability of hybrid representations is *algorithmically decidable* in this case. Similarly, minimal representation can be constructed by an algorithm.

10.4 Switched Systems

The section presents results on partial realization theory of switched systems. The following two subclasses of switched systems will be discussed: linear switched systems and bilinear switched systems. Switched systems with both arbitrary and constrained switchings will be investigated. The section also deals with the algorithmic aspects of realization theory for linear and bilinear switched systems. That is, algorithms for constructing minimal (bi)linear switched system realizations will be presented, along with algorithms for checking semi-reachability and observability of such systems or transforming (bi)linear switched systems to equivalent semi-reachable and observable (bi)linear switched systems.

An algorithm for computing a N partial realization of (bi)linear switched systems will be presented too, for arbitrary N. The realization computed by the algorithm will turn out to be a minimal realization, provided that the rank of the finite chunk of the Hankel-matrix equals the rank of the full Hankel-matrix. An lower bound on the size of the finite portion of the Hankel-matrix will be formulated, such that any finite portion of the Hankel-matrix of size greater than this lower bound will be of the same rank as the infinite Hankel-matrix. The algorithm uses matrix factorization, thus any matrix factorization algorithm, including SVD decomposition can be used for the implementation. The algorithm might serve as a basis for subspace identification-like methods for linear or bilinear switched systems.

A modified version of the algorithm above which computes a N partial (bi)linear switched system realization with constrained switching will be presented too. If the finite chunk of the Hankel-matrix if of the same rank as the Hankel-matrix, then the algorithm gives a realization by a (bi)linear switched system too. A lower bound on the size of the Hankel-matrix can be given, such that any finite portion of the Hankel-matrix, which is of greater size than the lower bound will have the same rank as the infinite Hankel-matrix. Unfortunately, the computed realization need not be a minimal one, but it will be semi-reachable and observable. As it was already described in [55, 51, 53], it is difficult to find a minimal realization for (bi)linear switched systems with constrained switching. Instead, we can find a semi-reachableand observable realization, which need not be of the smallest possible dimension. However, there exists a constant M, which depends on the set of admissible switching sequence, such that no other realization can have more than M times smaller dimension. The realization algorithm presented in this paper computes precisely such a "almost minimal" realization. The algorithm it might serve as a foundation for a method similar to subspace-identification.

The outline of the section is the following. Subsection 10.4.1 discusses partial realization theory for linear switched systems with arbitrary switchings. It also presents algorithms for computing a minimal realization and checking observability, reachability and minimality. Subsection 10.4.3 deals with bilinear switched systems with arbitrary switchings, it presents partial realization theory and related algorithms for this class of systems. Subsection 10.4.2 discusses partial realization theory for linear switched systems with constrained switching and Subsection 10.4.4 discusses partial realization theory for bilinear switched systems with constrained switching.

10.4.1 Partial Realization Theory for Linear Switched Systems: Arbitrary Switching

Recall from Section 4.1 the results on realization theory of linear switched systems. Recall that the realization problem for linear switched system can be reduced to finding a rational representation for a suitable family of formal power series. Recall that observability, reachability and minimality of linear switched systems is equivalent to observability, reachability and minimality respectively of suitable rational formal power series representations. The theory of rational formal power series allows us to formulate a partial realization theorem for linear switched systems. Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (Q \times T)^{+}, \mathcal{Y})$. Let $H_{\Phi,N,M} = H_{\Psi_{\Phi},N,M}$. Notice that

$$(H_{\Phi,N,M})_{(v,i),(u,j)} = \begin{cases} e_l^T D^{(0,\mathbb{I}_k,0}y_0^{\Phi}(e_z, quvq_h) & \text{if } i = p*(h-1)+l, k = |uv| \\ & \text{and } j = (q,z) \\ e_l^T D^{(\mathbb{I}_k,0)} f(0, uvq_h) & \text{if } i = p*(h-1)+l, k = |uv| \\ & \text{and } j = f \in \Phi \end{cases}$$

That is, $H_{\Phi,N,M}$ can be computed directly from the input-output maps belonging to Φ. Let (Σ, μ) be a linear switched system realization. We will say that (Σ, μ) is a

N-partial realization of Φ if $R_{\Sigma,\mu}$ is an N partial representation of Ψ_{Φ}. The intuitive interpretation of the concept is the following. If (Σ, μ) is an N-partial realization of Φ, then for all $f \in \Phi$ the Taylor series expansion of $y_{\Sigma}(\mu(f), ., .)$ and f coincide up to the elements of order N. That is, $y_{\Sigma}(\mu(f), ., .)$ can be thought as a some sort of approximation of f.

Theorem 11 and Theorem 12 imply the following.

Theorem 68 (Partial realization of linear switched systems). *Assume that* rank $H_{\Phi,N,N}$ = rank $H_{\Phi,N+1,N}$ = rank $H_{\Phi,N,N+1}$ *holds. Then there exists an N-partial realization* (Σ_N, μ_N) *of* Φ*. If* rank $H_{\Phi,N,N}$ = rank H_{Φ} *or there exists a linear switched system realization* (Σ, μ) *of* Φ *such that* $\dim \Sigma \leq N$*, then the realization* (Σ_N, μ_N) *is a minimal realization of* Φ*.*

Proof. The condition rank $H_{\Phi,N,N}$ = rank $H_{\Phi,N+1,N}$ = rank $H_{\Phi,N,N+1}$ can be rewritten as rank $H_{\Psi_{\Phi},N,N}$ = rank $H_{\Psi_{\Phi},N+1,N}$ = rank $H_{\Psi_{\Phi},N,N+1}$. Applying Theorem 64 we get that there exists a representation R_N such that the following holds. R_N is an N-partial representation of Ψ_{Φ} and if rank $H_{\Psi_{\Phi},N,N}$ = rank $H_{\Phi,N,N}$ = rank H_{Φ} = rank $H_{\Psi_{\Phi}}$ or there exists a representation R of Ψ_{Φ} such that $\dim R \leq N$, then R_N is a minimal representation of Ψ_{Φ}. Define the linear switched system realization $(\Sigma_N, \mu_N) = (\Sigma_{R_N}, \mu_{R_N})$. If R_N is a N-representation of Ψ_{Φ}, then (Σ_N, μ_N) is an N-realization of Φ, since $R_{\Sigma_N,\mu_N} = R_N$. Similarly, if R_N is a minimal representation of Ψ_{Φ}, then (Σ_N, μ_N) is a minimal realization of Φ. Notice that there exists a representation R of Ψ_{Φ} such that $\dim R \leq N$ if and only if there exists a linear switched system realization (Σ, μ) of Φ such that $\dim \Sigma \leq N$. Thus we get the second part of the theorem. ☐

The results of Section 10.1.3 allow us to formulate algorithms for computing a minimal realization of Ψ, deciding semi-reachable and observability and to transform a specified linear switched system realization to a minimal one. Below we will present these algorithms.

`ComputeLinSwitchRealization(`$H_{\Phi,N+1,N}$`)`

1. $R = ComputePartialRepresentation(H_{\Phi,N+1,N})$
2. If $R = NoRepresentation$ then retun $NoRealization$
3. Return (Σ_R, μ_R).

Proposition 46. *(i) The algorithm*

ComputeLinSwitchRealization

returns a linear switched system realization whenever

$$\texttt{ComputePartialRepresentation}$$

returns a formal power series representation. The realization (Σ, μ) *which is returned by*

$$\texttt{ComputeSwitchLinRealization}(H_{\Phi,N+1,N})$$

is an $N+1$*-partial realization of* Φ*.*

(ii) *If* $\mathrm{rank}\ H_{\Phi,N+1,N} = \mathrm{rank}\ H_{\Phi,N,N+1} = \mathrm{rank}\ H_{\Phi,N,N}$ *then*

$$\texttt{ComputeLinSwitchRealization}(H_{\Phi,N+1,N})$$

always returns a linear switched system realization (Σ, μ) *and this realization is isomorphic to the realization* (Σ_N, μ_N) *from Theorem 68.*

(iii) *If* $\mathrm{rank}\ H_{\Phi,N,N} = \mathrm{rank}\ H_{\Phi}$ *or* Φ *has a linear switched system realization* $(\widetilde{\Sigma}, \widetilde{\mu})$ *such that* $\dim \widetilde{\Sigma} \leq N$*, then the realization* (Σ, μ) *returned by* *ComputeLinSwitchRealization*$(H_{\Phi,N+1,N})$ *is a minimal realization of* Φ*.*

Proof. We will use the notation of the algorithm
`ComputeLinSwitchRealization` throughout the proof. `Part (i)`
It is clear that whenever `ComputePartialRepresentation` returns a valid representation, the algorithm `ComputeLinSwitchRealization` returns a valid linear switched system realization. By part (i) of Theorem 65 we get that R is a $N+1$ partial representation and thus (Σ_R, μ_R) is a $N+1$ partial realization.

`Part(ii)`
If $\mathrm{rank}\ H_{\Phi,N,N} = \mathrm{rank}\ H_{\Phi,N+1,N} = \mathrm{rank}\ H_{\Phi,N,N+1}$ holds, then we get that

$$\mathrm{rank}\ H_{\Psi_\Phi,N,N} = \mathrm{rank}\ H_{\Psi_\Phi,N,N+1} = \mathrm{rank}\ H_{\Psi,N+1,N}$$

Thus by part (ii) of Theorem 65 we get that that `ComputerPartialRepresentation` returns a valid representation and there exists an isomorphism $\xi : R \to R_N$. From [55] we know $\xi : (\Sigma_R, \mu_R) \to (\Sigma_{R_N}, \mu_{R_N}) = (\Sigma_N, \mu_N)$ is a linear switched system isomorphism.

`Part(iii)`
If $\dim \widetilde{\Sigma} \leq N$ then $\dim R_{\widetilde{\Sigma},\widetilde{\mu}} \leq N$ and $R_{\widetilde{\Sigma},\widetilde{\mu}}$ is a representation of Ψ_Φ. Then from Theorem 65 we get that R is a minimal representation of Ψ_Φ. If $\mathrm{rank}\ H_{\Phi,N,N} = \mathrm{rank}\ H_\Phi$ then by Theorem 65 it follows that R is a minimal representation of Ψ_Φ.

Thus, in both cases, (Σ_R, μ_R) is a minimal realization of Φ. □

Let (Σ, μ) be a linear switched system realization of Φ. It is clear from Proposition 11 that reachability of (Σ, μ) can be checked by checking if `IsReachable(`$R_{\Sigma,\mu}$ returns true. Similarly, observability of (Σ, μ) can be checked by checking if

`IsObservable(`$R_{\Sigma,\mu}$`)`

returns true or not. Consider the following algorithm.

`ComputeReachableRealization(`(Σ, μ)`)`

1. $R = ReachableTransform(R_{\Sigma,\mu})$
2. Return (Σ_R, μ_R).

Proposition 47. *The algorithm* `ComputeReachableRealization(`(Σ, μ) *returns a semi-reachable realization of* Φ.

Proof. Since $R_{\Sigma,\mu}$ is a representation of Ψ_Φ, we get that

$$R = ReachableTransform(R_{\Sigma,\mu})$$

is a reachable representation of Ψ_Φ. Then by Theorem 10 (Σ_R, μ_R) is a realization of Φ. Notice that $R_{\Sigma_R,\mu_R} = R$ is reachable, thus (Σ_R, μ_R) is semi-reachabletoo by Corollary 11. ☐

Similarly, (Σ, μ) can be transformed to an observable realization of Φ with the following algorithm.

`ComputeObservableRealization(`(Σ, μ)`)`

1. $R = ComputeObservableRepresentation(R_{\Sigma,\mu})$
2. Return $(\Sigma_o, \mu_o) = (\Sigma_R, \mu_R)$.

Proposition 48. *The algorithm* `ComputeObservableRealization(`(Σ, μ)`)` *returns an observable realization* (Σ_o, μ_o) *of* Φ. *If* (Σ, μ) *is semi-reachable, then* (Σ_o, μ_o) *is semi-reachable too.*

Proof. Since (Σ, μ) is a realization of Φ, $R_{\Sigma,\mu}$ is a representation of Ψ_Φ. Thus, $R = ComputeObservableRepresentation(R_{\Sigma,\mu})$ is an observable representation of Ψ_Φ. Moreover, if (Σ, μ) is semi-reachable, then $R_{\Sigma,\mu}$ is reachable, thus R is reachable. Thus, (Σ_R, μ_R) is a realization of Φ and it is observable. If (Σ, μ) is semi-reachable, then $R_{\Sigma_R,\mu_R} = R$ is reachable, and thus (Σ_R, μ_R) is reachable too. ☐

Finally, minimality realization of Φ can be computed as follows.

`ComputeMinimalRealization`$((\Sigma,\mu))$

1. $R_{min} = ComputeMinimalRepresentation(R_{\Sigma,\mu})$

2. return $(\Sigma_{R_{min}}, \mu_{R_{min}})$

It easy to deduce from Proposition 10 that `ComputeMinimalRealization`$((\Sigma,\mu))$ indeed returns a minimal realization of Φ.

In [69] reachability and observability of linear switched systems were studied. In particular, if $\mathrm{Im}\mu = \{0\}$, then Σ is reachable in sense of [69] whenever (Σ,μ) is semi-reachable. In [69] procedures were presented to compute the reachability and observability matrices. In fact, these matrices correspond spaces $W_{R_{\Sigma,\mu}}$ and $O_{R_{\Sigma,\mu}}$. The algorithms in [69] are quite similar to those presented here, but apply to a much more restricted class of problems.

10.4.2 Partial Realization Theory for Linear Switched Systems: Constrained Switching

Recall from Subsection 4.1.4 the results on realization theory of linear switched systems with constrained switching. The results from Subsection 4.1.4 allow us to develop partial realization theory for linear switched systems with constrained switching. We will use th notation of Subsection 4.1.4 in the sequel. Assume that L is regular. Then from Lemma 19 it follows that Ω is rational. Notice that Ω depends only on L. In fact, a representation of Ω can be computed from a finite automaton recognising L. Similarly to the case of arbitrary switching, we say that (Σ,μ) is an N-partial realization of Φ, if $R_{\Sigma,\mu}$ is an N-partial realization of Ψ_{Φ}. Let $H_{\Phi,N,M} = H_{\Psi_{\Phi},N,M}$.

Theorem 69. *Assume that* $\mathrm{rank}\ H_{\Phi,N,N} = \mathrm{rank}\ H_{\Phi,N+1,N} = \mathrm{rank}\ H_{\Phi,N,N+1}$. *Then there exists a N-partial realization (Σ_N,μ_N) of Φ. Assume that* $\mathrm{rank}\ H_{\Omega_{\Phi}} \le M$ *and there exists a realization (Σ,μ) of Φ with constraint L such that* $\dim\Sigma \le N$. *Then* $\mathrm{rank}\ H_{\Phi,NM,NM} = \mathrm{rank}\ H_{\Phi,NM+1,N} = \mathrm{rank}\ H_{\Phi,NM,NM+1}$ *and (Σ_{NM},μ_{NM}) is a realization of Φ with constraint L, it is semi-reachable, observable and it satisfies (4.15) and (4.16). Similarly, if* $\mathrm{rank}\ H_{\Phi,N,N} = \mathrm{rank}\ H_{\Phi}$ *then (Σ_N,μ_N) is realization of Φ with constraint L, its is semi-reachable, observable and satisfies (4.15) and (4.16).*

Proof. If $\mathrm{rank}\ H_{\Phi_{\Psi},N,N} = \mathrm{rank}\ H_{\Phi_{\Psi},N+1,N} = \mathrm{rank}\ H_{\Phi_{\Psi},N,N+1}$ then by Theorem 64 there exists a N representation R_N of Ψ_{Φ}. Then by Theorem 14 $(\Sigma_N,\mu_N) = (\Sigma_{R_N},\mu_{R_N})$ is an N realization of Φ. Assume that there exists a realization (Σ,μ)

of Φ such that $\dim \Sigma \leq N$. Then from Theorem 13 it follows that $\Psi_\Phi = \Omega_\Phi \odot K_{\Sigma,\mu}$. Since $R_{\Sigma,\mu \circ U(\mu)}$ is a representation of $K_{\Sigma,\mu}$ and $\dim R_{\Sigma,\mu\circ U(\mu)} = \dim \Sigma \leq N$, we get that $\text{rank } H_{K_{\Sigma,\mu}} \leq N$. Thus, by Lemma 6 $\text{rank } H_{\Psi_\Phi} \leq \text{rank } H_{\Omega_\Phi} \cdot \text{rank } H_{K_{\Sigma,\mu}} \leq NM$. Thus, by Theorem 64 R_{NM} is a minimal representation of Ψ_Φ. Similarly, if $\text{rank } H_{\Phi,N,N} = \text{rank } H_\Phi$, then by Theorem 64 R_N is a minimal representation of Ψ_Φ. Let $(\Sigma_p, \mu_p) = (\Sigma_{NM}, \mu_{NM}) = (\Sigma_{R_{NM}}, \mu_{R_{NM}})$ if there exists a realization Σ of Φ such that $\dim \Sigma \leq N$. Let $(\Sigma_p, \mu_p) = (\Sigma_{R_N}, \mu_{R_N})$ if $\text{rank } H_{\Phi,N,N} = \text{rank } H_\Phi$. Thus in both cases,by Theorem 14 (Σ_p, μ_p) is a realization of Φ with constrained L and (4.15) holds. Since R_{NM} (R_N) is reachable and observable by Corollary 11 we get that (Σ_p, μ_p) is semi-reachable and observable. Since R_{NM} (R_N) is a minimal realization, it holds that $\text{rank } H_{\Phi_\Psi} = \dim R_{NM} = \dim \Sigma_p$ (or $\text{rank } H_{\Phi_\Psi} = \dim R_N = \dim \Sigma_p$). Assume that $(\widetilde{\Sigma}, \widetilde{\mu})$ is a realization of Φ. Then $\Psi_\Phi = \Omega_\Phi \odot K_{\widetilde{\Sigma},\widetilde{\mu}}$, thus by Lemma 6 $\dim \Sigma_p = \text{rank } H_{\Psi_\Phi} \leq M \cdot \text{rank } H_{K_{\widetilde{\Sigma},\widetilde{\mu}}} \leq M \cdot \dim \widetilde{\Sigma}$. Thus, (Σ_p, μ_p) satisfies (4.16). $\square$

Consider the following algorithm.

`ComputeLinSwitchConstRealization(`$H_{\Phi,N+1,N}$`)`

1. $R = \mathit{ComputePartialRepresentation}(H_{\Phi,N+1,N})$
2. If $R = \mathit{NoRepresentation}$ then return $\mathit{NoLinConstRealization}$
3. Return (Σ_R, μ_R)

Recall the notion of linear switched isomorphism (algebraic similarity) from [55, 51].

Proposition 49. *(i) The algorithm*

`ComputeLinSwitchConstRealization`

returns a linear switched system realization whenever

`ComputePartialRepresentation`

returns a formal power series representation. The realization (Σ, μ) which is returned by

`ComputeLinSwitchConstRealization(`$H_{\Phi,N+1,N}$`)`

is an $N+1$-partial realization of Φ.

(ii) If $\text{rank } H_{\Phi,N+1,N} = \text{rank } H_{\Phi,N,N+1} = \text{rank } H_{\Phi,N,N+1}$ *then*

ComputeLinSwitchConstRealization($H_{\Phi,N+1,N}$)

always returns a linear switched system realization (Σ, μ) and this realization is isomorphic to the realization (Σ_N, μ_N) from Theorem 69.

(iii) If rank $H_{\Phi,N,N} =$ rank H_Φ *then the realization (Σ, μ) returned by*

ComputeLinSwitchConstRealization($H_{\Phi,N+1,N}$)

is semi-reachable and observable and it satisfies (4.15, 4.16).

(iv) If Φ has a linear switched system realization $(\widetilde{\Sigma}, \widetilde{\mu})$ such that $\dim \widetilde{\Sigma} \leq N$, and rank $H_{\Omega_\Phi} \leq M$ *then the realization (Σ, μ) returned by*

ComputeLinSwitchConstRealization($H_{\Phi,NM+1,NM}$)

is a realization of Φ, it is semi-reachable and observable and it satisfies (4.15,4.16)

Proof. We will use the notation of the algorithm `ComputeLinSwitchConstRealization` throughout the proof.

`Part (i)`

It is clear that whenever `ComputePartialRepresentation` returns a valid representation, the algorithm `ComputeLinSwitchConstRealization` returns a valid linear switched system realization. By part (i) of Theorem 65 we get that R is a $N+1$ partial representation and thus (Σ_R, μ_R) is a $N+1$ partial realization.

`Part(ii)`

If rank $H_{\Phi,N,N} =$ rank $H_{\Phi,N+1,N} =$ rank $H_{\Phi,N,N+1}$ holds, then we get that

$$\text{rank } H_{\Psi_\Phi,N,N} = \text{rank } H_{\Psi_\Phi,N,N+1} = \text{rank } H_{\Psi_\Phi,N+1,N}$$

Thus by part (ii) of Theorem 65 we get that that `ComputePartialRepresentation` returns a valid representation and there exists a isomorphism $\xi : R \to R_N$ from Theorem 64. From [55] we know $\xi : (\Sigma_R, \mu_R) \to (\Sigma_{R_N}, \mu_{R_N}) = (\Sigma_N, \mu_N)$ is a linear switched system isomorphism.

`Part(iii)` If rank $H_{\Phi,N,N} =$ rank H_Φ, then by Theorem 65 the representation R returned by `ComputePartialRepresentation` is a minimal representation thus, it is reachable and observable. Hence, by Corollary 11 (Σ_R, μ_R) is semi-reachable and observable and by Theorem 15 part (ii) it satisfies (4.15). Since (Σ_R, μ_R) is isomorphic to (Σ_N, μ_N) we get that $\dim \Sigma_R = \dim \Sigma_N$ and thus (Σ_R, μ_R) satisfies (4.16).

`Part(iv)`

If $\dim \widetilde{\Sigma} \leq N$ then by Theorem 69 (Σ_{NM}, μ_{NM}) is a semi-reachable and observable realization of Φ and it satisfies (4.15,4.16). Since R is isomorphic to R_{NM} and R_{NM} is a minimal, that is, it is reachable and observable, we get that R is reachable and observable too. Thus, (Σ_R, μ_R) is semi-reachable and observable. From Theorem 15 part (ii) it follows that (Σ_R, μ_R) satisfies (4.15). Since (Σ_R, μ_R) is isomorphic to (Σ_N, μ_N) we get that $\dim \Sigma_R = \dim \Sigma_N$ and thus (Σ_R, μ_R) satisfies (4.16). □

Below we will give an estimate on $M = \text{rank } H_{\Omega_\Phi}$. At the same time, the proof of the estimate will also demonstrate that M depends only on L. Recall from Subsection 4.1.4 the definition of the language $\widetilde{L}$.

Lemma 59. *Assume that L is regular. Denote by $n_{\widetilde{L}}$ the cardinality of the state-space of the minimal automaton recognising $\widetilde{L}$. Then*

$$\text{rank } H_{\Omega_\Psi} \leq n_{\widetilde{L}}$$

Proof. Notice [55] that $\widetilde{L}_q = \{w \in Q^* \mid wq \in \widetilde{L}\}$ and $\widetilde{L}_{q_1,q_2} = \{w \in Q^* \mid q_1 w q_2 \in \widetilde{L}\}$. Let $\mathcal{A} = (S, Q, \delta, s_0, F)$ be a minimal finite-state automaton recognising $\widetilde{L}$. Here S is the state-space, $\delta : S \times Q \to S$ is the state-transition function, s_0 is the initial state, F is set of accepting states. For more on automata and formal languages see [17]. For each $q \in Q$, let $s_q = \delta(s_0, q)$ and $H_q = \{s \in S \mid \delta(s,q) \in F\}$. Then it is easy to see that $\widetilde{L}_q$ is accepted by (S, Q, δ, s_0, H_q) and $\widetilde{L}_{q_1,q_2}$ is accepted by $(S, Q, \delta, s_{q_2}, H_{q_1})$. Assume that $S = \{s_1, \ldots, s_n\}$, $n = n_{\widetilde{L}}$. Consider the representation. $R = (\mathbb{R}^n, \{A_z\}_{z \in Q}, C, B)$ such that the following holds. Denote by e_i the ith unit vector of $\mathbb{R}^n$. For each $q \in Q$, $A_q(e_i) = e_k$ if $\delta(s_i, q) = s_k$, $i = 1, \ldots, n$. The map C is of the form $C = \begin{bmatrix} W_{q_1}^T & \ldots & W_{q_N}^T \end{bmatrix}$, where $W_{q_l} e_i = \begin{cases} (1, \ldots, 1)^T \in \mathbb{R}^p & \text{if } s_i \in H_{q_l} \\ 0 & \text{otherwise} \end{cases}$. For each $(q,j) \in Q \times \{1, \ldots, m\}$, $B_{(q,j)} = e_k$ such that $s_k = s_q$ and for each $f \in \Phi$, $B_f = e_k$ such that $s_k = s_0$. Then, it is easy to see that $W_{q_l} A_w B_f = (1, \ldots, 1)$ if and only if $w \in \widetilde{L}_{q_l}$ and $W_{q_l} A_w B_f = 0$ otherwise. Similarly, $W_{q_l} A_w B_{(q,j)} = (1, \ldots, 1)$ if $w \in \widetilde{L}_{q_l,q}$ and it is zero otherwise. That is, $C A_w B_{(q,j)} = \Gamma_q(w)$ and $C A_w B_f = \Gamma(w)$. Thus, R is a representation of Ω_Φ. Thus, $M = \text{rank } H_{\Omega_\Phi} \leq \dim R = n_{\widetilde{L}}$. □

Corollary 25. *With the notation and assumptions of Lemma 59*

$$\text{rank } H_{\Omega_\Phi} \leq 2^{n_L \cdot |Q| + 1}$$

where n_L is the cardinality of the state-space of a minimal automaton accepting L.

Proof. Let (S, Q, δ, s_0, F) be a minimal automaton accepting L. Consider the non-deterministic automaton $B = ((S \times Q) \cup \{s_0'\}, Q, \delta_B, s_0', F \times Q)$ defined in the proof of Lemma 8, [55]. Recall that $(s', x) \in \delta_B(s'_0, x)$ if and only if there exists $w \in Q^*$, such that $\delta(s_0, wx) = s'$ and $(s', u) \in \delta_B((s,x), u)$ if and only if either $u = x, s' = s$ or there exists $w \in Q^*$ such that $\delta(s, wu) = s'$. Then it follows from the proof of Lemma 8, [55] that B accepts $\widetilde{L}$. We can construct a deterministic automaton from B which accepts $\widetilde{L}$. This automaton will have at most $2^{|(S \times Q) \cup \{s'_0\}|} = 2^{n_L \cdot |Q|+1}$ states. Thus, $n_{\widetilde{L}} \leq 2^{n_L \cdot |Q|+1}$ and by Lemma 59 rank $H_{\Omega_\Phi} \leq 2^{n_L \cdot |Q|+1}$. $\square$

10.4.3 Partial Realization Theory for Bilinear Switched Systems: Arbitrary Switching

Recall from Section 4.2 the results on realization theory of bilinear switched systems and the role of formal power series in it. The results of Section 4.2 allow us to formulate partial realization theory for bilinear switched systems. In the sequel we will use notation and terminology of Section 4.2. Define $H_{\Phi,N,M} = H_{\Psi_\Phi,N,M}$. In fact, the following holds.

$$(H_{\Phi,N,M})_{(v,i),(u,f)} = e_l^T c_f(i(uv)(q_h, \epsilon))$$

for each $u, v \in \Gamma^*, |v| \leq N, |u| \leq M$, $i = p * (h-1) + l$, $f \in \Phi$. That is, the entries of $H_{\Phi,N,M}$ can be obtained from the generating convergent series which generate the elements of Φ. We will say that a bilinear switched system realization (Σ, μ) is an *N-partial realization of* Φ if $R_{\Sigma,\mu}$ is an N-partial representation of Ψ_Φ. Intuitively, (Σ, μ) is an N-partial realization of Φ, if the following holds. For any $f \in \Phi$ the values of the generating convergent series of f and $y_\Sigma(\mu(f), ., .)$ coincide for all words of length at most N. That is, $y_\Sigma(\mu(f), ., .)$ can be thought of as an approximation of f. With the notation of Theorem 64 the following holds.

Theorem 70 (Partial realization). *Let* $\Phi \subseteq F(PC(T, \mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$. *Assume that* rank $H_{\Phi,N,N}$ = rank $H_{\Phi,N+1,N}$ = rank $H_{\Phi,N,N+1}$. *Then there exists a N realization* (Σ_N, μ_N) *of* Φ. *If* rank $H_{\Phi,N,N}$ = rank H_Φ *or* Φ *has a realization* (Σ, μ) *such that* $N \geq \dim \Sigma$*, then the realization* (Σ_N, μ_N) *is a minimal realization of* Φ.

Proof. The condition rank $H_{\Phi,N,N}$ = rank $H_{\Phi,N+1,N}$ = rank $H_{\Phi,N,N+1}$ is equivalent to rank $H_{\Psi_\Phi,N,N}$ = rank $H_{\Psi_\Phi,N+1,N}$ = rank $H_{\Psi_\Phi,N,N+1}$. Thus, by Theorem 64, there exists a representation R_N, such that R_N is an N-representation of Ψ_Φ. Define $(\Sigma_N, \mu_N) = (\Sigma_R, \mu_R)$. Then $R_{\Sigma_N,\mu_N} = R_N$, and thus (Σ_N, μ_N) is an N-realization of Φ. Notice that Φ has a bilinear switched system realization (Σ, μ) such that $\dim \Sigma \leq N$ if and only if Ψ_Φ has a representation R, such that $\dim R \leq N$. Thus, if Φ has a bilinear switched system realization (Σ, μ), such that $\dim \Sigma \leq N$, then by Theorem 64 R_N is a minimal representation of Ψ_Φ. But it means that (Σ_N, μ_N) is a minimal realization of Φ. Similarly, if rank $H_{\Phi,N,N}$ = rank H_{Φ} then by Theorem 64 R_N is a minimal representation, therefore (Σ_N, μ_N) is a minimal realization. $\square$

The results of Section 10.1.3 allow us to compute a (partial) realization of Φ using SVD decomposition. It also enables us to formulate algorithms for deciding semi-reachableand observability of bilinear switched systems.

Consider the following algorithm

`ComputeBilinSwitchRealization(`$H_{\Psi,N+1,N}$`)`

1. $R = ComputePartialRepresentation(H_{\Phi,N+1,N})$
2. If $R = NoRepresentation$ then return $NoRealization$
3. Return (Σ_R, μ_R).

Proposition 50. *(i) The algorithm*

ComputeBilinSwitchRealization

returns a bilinear switched system realization whenever

ComputePartialRepresentation

returns a formal power series representation. The realization (Σ, μ) which is returned by

ComputeSwitchBilinRealization($H_{\Phi,N+1,N}$)

is an $N+1$-partial realization of Φ.

(ii) If

$$\text{rank } H_{\Phi,N+1,N} = \text{rank } H_{\Phi,N,N+1} = \text{rank } H_{\Phi,N,N+1}$$

then

ComputeBilinSwitchRealization($H_{\Phi,N+1,N}$)

always returns a linear switched system realization (Σ, μ) and this realization is isomorphic to the realization (Σ_N, μ_N) from Theorem 70.

(iii) If rank $H_{\Phi,N,N}$ = rank H_{Φ} *or Φ has a bilinear switched system realization $(\widetilde{\Sigma}, \widetilde{\mu})$ such that $\dim \widetilde{\Sigma} \le N$, then the realization (Σ, μ) returned by*

$$\texttt{ComputeLinSwitchRealization}(H_{\Phi,N+1,N})$$

is a minimal realization of Φ.

Proof. We will use the notation of the algorithm `ComputeBilinSwitchRealization` throughout the proof.

`Part (i)`

It is clear that whenever `ComputePartialRepresentation` returns a valid representation, the algorithm `ComputeBilinSwitchRealization` returns a valid bilinear switched system realization. By part (i) of Theorem 65 we get that R is a $N+1$ partial representation and thus (Σ_R, μ_R) is a $N+1$ partial realization.

`Part(ii)`

If rank $H_{\Phi,N,N} =$ rank $H_{\Phi,N+1,N} =$ rank $H_{\Phi,N,N+1}$ holds, then we get that

$$\text{rank } H_{\Psi_\Phi,N,N} = \text{rank } H_{\Psi_\Phi,N,N+1} = \text{rank } H_{\Psi,N+1,N}$$

Thus by part (ii) of Theorem 65 we get that that `ComputePartialRepresentation` returns a valid representation and there exists a isomorphism $\xi : R \to R_N$ from Theorem 64. From [55] we know $\xi : (\Sigma_R, \mu_R) \to (\Sigma_{R_N}, \mu_{R_N}) = (\Sigma_N, \mu_N)$ is a bilinear switched system isomorphism.

`Part(iii)`

If $\dim \widetilde{\Sigma} \leq N$ then $\dim R_{\widetilde{\Sigma},\widetilde{\mu}} \leq N$ and $R_{\widetilde{\Sigma},\widetilde{\mu}}$ is a representation of Ψ_Φ. Then from Theorem 65 we get that R is a minimal representation of Ψ_Φ. Thus, (Σ_R, μ_R) is a minimal realization of Φ. Similarly, if rank $H_{\Phi,N,N} =$ rank H_Φ, then by Theorem 65 R is a minimal representation and thus (Σ_R, μ_R) is a minimal realization. □

Let (Σ, μ) be a bilinear switched system realization of Φ. It is clear from Proposition 23 that reachability of (Σ, μ) can be checked by checking if `IsReachable(`$R_{\Sigma,\mu}$ returns true. Similarly, observability of (Σ, μ) can be checked by checking if `IsObservable(`$R_{\Sigma,\mu}$`)` returns true or not. Consider the following algorithm.

`ComputeBilinReachableRealization(`(Σ, μ)`)`

1. $R = ReachableTransform(R_{\Sigma,\mu})$
2. Return (Σ_R, μ_R).

Proposition 51. *The algorithm* `ComputeBilinReachableRealization(`(Σ, μ)`)` *returns a semi-reachable realization of* Φ.

Proof. Since $R_{\Sigma,\mu}$ is a representation of Ψ_Φ, we get that

$$R = ReachableTransform(R_{\Sigma,\mu})$$

is a reachable representation of Ψ_Φ. Then by Proposition 16 (Σ_R, μ_R) is a realization of Φ. Notice that $R_{\Sigma_R,\mu_R} = R$ is reachable, thus (Σ_R, μ_R) is semi-reachable too by Lemma 23. $\square$

Similarly, (Σ, μ) can be transformed to an observable realization of Φ with the following algorithm.

`ComputeBilinObservableRealization`$((\Sigma, \mu))$

1. $R = ComputeObservableRepresentation(R_{\Sigma,\mu})$
2. Return $(\Sigma_o, \mu_o) = (\Sigma_R, \mu_R)$.

Proposition 52. *The algorithm* `ComputeBilinObservableRealization`$((\Sigma, \mu))$ *returns an observable realization* (Σ_o, μ_o) *of* Φ. *If* (Σ, μ) *is semi-reachable, then* (Σ_o, μ_o) *is reachable too.*

Proof. Since (Σ, μ) is a realization of Φ, $R_{\Sigma,\mu}$ is a representation of Ψ_Φ. Thus, $R = ComputeBilinObservableRepresentation(R_{\Sigma,\mu})$ is an observable representation of Ψ_Φ. Moreover, if (Σ, μ) is semi-reachable, then $R_{\Sigma,\mu}$ is reachable, thus R is reachable. Thus, (Σ_R, μ_R) is a realization of Φ and it is observable. If (Σ, μ) is semi-reachable, then $R_{\Sigma_R,\mu_R} = R$ is reachable, and thus (Σ_R, μ_R) is reachable too. $\square$

Finally, minimality realization of Φ can be computed as follows.

`ComputeBilinMinimalRealization`$((\Sigma, \mu))$

1. $R_{min} = ComputeMinimalRepresentation(R_{\Sigma,\mu})$
2. return $(\Sigma_{R_{min}}, \mu_{R_{min}})$

It easy to deduce from Lemma 24 that `ComputeBilinMinimalRealization`$((\Sigma, \mu))$ indeed returns a minimal realization of Φ.

10.4.4 Partial Realization Theory for Bilinear Switched Systems: Constrained Switching

Recall from Subsection 4.2.4 the results on realization theory of bilinear switched systems with constrained switching and the role of formal power series in it. Below we will use those results to develop partial realization theory for bilinear switched systems with constrained switching. We will use the notation of Subsection 4.2.4 in the sequel. Assume that $\Phi \subseteq F(PC(T, \mathcal{U}) \times TL, \mathcal{Y})$ admits a generalized Fliess-series expansion. Assume that L is a *regular language*. Just as for linear switched systems, we will say that a bilinear switched system (Σ, μ) is an N *partial realization of* Φ, if $R_{\Sigma,\mu}$ is an N partial representation of Ψ_{Φ}. We will denote $H_{\Psi_{\Phi},N,M}$ by $H_{\Phi,N,M}$. Similarly to Subsection 10.4.3, by a realization of Φ we will always mean a bilinear switched system realization of Φ with constrained L. If (Σ, μ) is a (bilinear switched system) realization of Φ, then $R_{\Sigma,\mu}$ denotes the associated representation as defined in Subsection 10.4.3. Similarly, if R is a suitable representation, then (Σ_R, μ_R) denotes the *bilinear switched system* realization associated with R. With the notation above the results on partial realization theory literally coincide with those for linear switched system presented in Subsection 10.4.2. Thus the notation emphasises even further the similarity between the two theories.

Theorem 71. *Assume that* rank $H_{\Phi,N,N}$ = rank $H_{\Phi,N+1,N}$ = rank $H_{\Phi,N,N+1}$. *Then there exists a N-partial realization (Σ_N, μ_N) of Φ. Assume that* rank $H_{\Omega_{\Phi}} \leq M$ *and there exists a realization (Σ, μ) of Φ with constraint L such that* $\dim \Sigma \leq N$. *Then* rank $H_{\Phi,NM,NM}$ = rank $H_{\Phi.NM+1,N}$ = rank $H_{\Phi,NM,NM+1}$ *and (Σ_{NM}, μ_{NM}) is a realization of Φ with constraint L, it is semi-reachable, observable and it satisfies (4.25) and (4.26). Similarly, if* rank $H_{\Phi,N,N}$ = rank H_{Φ}, *then* rank $H_{\Phi,N,N}$ = rank $H_{\Phi,N+1,N}$ = rank $H_{\Phi,N,N+1}$ *and $(\Sigma_N, \mu_N) = (\Sigma_{R_N}, \mu_{R_N})$ is a semi-reachable and observable realization of Φ and it satisfies (4.25) and (4.26).*

Proof. If rank $H_{\Phi_{\Psi},N,N}$ = rank $H_{\Phi_{\Psi},N+1,N}$ = rank $H_{\Phi_{\Psi},N,N+1}$ then by Theorem 64 there exists a N representation R_N of Ψ_{Φ}. Then by Theorem 19 $(\Sigma_N, \mu_N) = (\Sigma_{R_N}, \mu_{R_N})$ is an N realization of Φ. Assume that there exists a realization (Σ, μ) of Φ such that $\dim \Sigma \leq N$. Then from Lemma 26 it follows that $\Psi_{\Phi} = \Omega_{\Phi} \odot \Theta_{\Sigma,\mu}$. Consider the map $\mu^{'} : y_{\Sigma}(\mu(f), ., .) \mapsto \mu(f)$. It can be shown (Subsection 4.2.4) that $\mu^{'}$ is well defined and $(\Sigma, \mu^{'})$ is a realization of $\Phi^{'}$ (without constraints). Thus $R_{\Sigma,\mu^{'}}$ is a representation of $\Theta_{\Sigma,\mu}$ and $\dim R_{\Sigma,\mu^{'}} = \dim \Sigma \leq N$, we get that rank $H_{\Theta_{\Sigma,\mu}} \leq N$. Thus, by Lemma 6 rank $H_{\Psi_{\Phi}} \leq$ rank $H_{\Omega} \cdot$ rank $H_{\Theta_{\Sigma,\mu}} \leq NM$. Thus, by Theorem 64 R_{NM} is a minimal representation of Ψ_{Φ}. Similarly, if rank $H_{\Phi,N,N}$ = rank H_{Φ} then by Theorem 64 R_N is a minimal representation of Ψ_{Φ}. Let $(\Sigma_p, \mu_p) = (\Sigma_{R_{NM}}, \mu_{R_{NM}})$ if there exists a realization Σ of Φ such that $\dim \Sigma \leq N$. Let $(\Sigma_p, \mu_p) = (\Sigma_{R_N}, \mu_{R_N})$ if rank $H_{\Phi,N,N}$ = rank H_{Φ}. Thus, by Theorem 19 (Σ_p, μ_p) is a realization of Φ with constrained L and (4.25) holds. Since R_{NM} (R_N) is reachable and observable by Lemma: 23 we get that (Σ_p, μ_p) is semi-reachable and observable. Since R_{NM} (R_N) is a minimal realization, it holds that rank $H_{\Phi_{\Psi}} = \dim R_{NM} = \dim \Sigma_p$ (rank $H_{\Phi_{\Psi}} = \dim R_N = \dim \Sigma_p$). Assume that $(\widetilde{\Sigma}, \widetilde{\mu})$ is a realization of Φ. Then $\Psi_{\Phi} = \Omega \odot \Theta_{\widetilde{\Sigma},\widetilde{\mu}}$, thus by Lemma 6 $\dim \Sigma_p$ = rank $H_{\Psi_{\Phi}} \leq M \cdot$ rank $H_{\Theta_{\widetilde{\Sigma},\widetilde{\mu}}} \leq M \cdot \dim \widetilde{\Sigma}$, where M = rank H_{Ω}. Thus, (Σ_p, μ_p) satisfies 4.26. $\square$

Consider the following algorithm.

`ComputeBilSwitchConstRealization(`$H_{\Phi,N+1,N}$`)`

1. $R = ComputePartialRepresentation(H_{\Phi,N+1,N})$
2. If $R = NoRepresentation$ then return $NoBilConstRealization$
3. Return (Σ_R, μ_R)

Recall the notion of bilinear switched isomorphism (algebraic similarity) from [55, 53].

Proposition 53. *(i) The algorithm*

`ComputeBilSwitchConstRealization`

returns a bilinear switched system realization whenever

`ComputePartialRepresentation`

returns a formal power series representation. The realization (Σ, μ) which is returned by

`ComputeBilSwitchConstRealization(`$H_{\Phi,N+1,N}$`)`

is an $N+1$-partial realization of Φ.

(ii) If rank $H_{\Phi,N+1,N} =$ rank $H_{\Phi,N,N+1} =$ rank $H_{\Phi,N,N+1}$ *then*

`ComputeBilSwitchConstRealization(`$H_{\Phi,N+1,N}$`)`

always returns a bilinear switched system realization (Σ, μ) and this realization is isomorphic to the realization (Σ_N, μ_N) from Theorem 69.

(iii) If rank $H_{\Psi,N,N} =$ rank H_{Ψ} *then the realization (Σ, μ) returned by*

`ComputeBilSwitchConstRealization(`$H_{\Phi,N+1,N}$`)`

is a realization of Φ, it is semi-reachableand observable and it satisfies (4.25,4.26)

(iv) If Φ has a bilinear switched system realization $(\widetilde{\Sigma}, \widetilde{\mu})$ such that $\dim \widetilde{\Sigma} \leq N$*, and* rank $H_{\Omega_\Phi} \leq M$ *then the realization (Σ, μ) returned by*

`ComputeBilSwitchConstRealization(`$H_{\Phi,NM+1,NM}$`)`

is a realization of Φ, it is semi-reachableand observable and it satisfies (4.25,4.26)

Proof. We will use the notation of the algorithm `ComputeBilSwitchConstRealization` throughout the proof.

`Part (i)`
It is clear that whenever `ComputePartialRepresentation` returns a valid representation, the algorithm `ComputeBilSwitchConstRealization` returns a valid bilinear switched system realization. By part (i) of Theorem 65 we get that R is a $N+1$ partial representation and thus (Σ_R, μ_R) is a $N+1$ partial realization.

`Part(ii)`
If rank $H_{\Phi,N,N} =$ rank $H_{\Phi,N+1,N} =$ rank $H_{\Phi,N,N+1}$ holds, then we get that

$$\text{rank } H_{\Psi_\Phi,N,N} = \text{rank } H_{\Psi_\Phi,N,N+1} = \text{rank } H_{\Psi,N+1,N}$$

Thus by part (ii) of Theorem 65 we get that that `ComputePartialRepresentation` returns a valid representation and there exists a isomorphism $\xi : R \to R_N$ from Theorem 64. From [55, 53] we know $\xi : (\Sigma_R, \mu_R) \to (\Sigma_{R_N}, \mu_{R_N}) = (\Sigma_N, \mu_N)$ is a bilinear switched system isomorphism.

`Part(iii)` If rank $H_{\Phi,N,N} =$ rank H_Φ, then by Theorem 65 the representation R returned by `ComputePartialRepresentation` is a minimal representation thus, it is reachable and observable. Hence, by Lemma 23 (Σ_R, μ_R) is semi-reachable and observable and by Theorem 20 part (ii) it satisfies (4.25). Since (Σ_R, μ_R) is isomorphic to (Σ_N, μ_N) we get that $\dim \Sigma_R = \dim \Sigma_N$ and thus (Σ_R, μ_R) satisfies (4.26).

`Part (iv)`
If $\dim \widetilde{\Sigma} \leq N$ then by Theorem 71 (Σ_{NM}, μ_{NM}) is a semi-reachable and observable realization of Φ and it satisfies (4.25,4.26). Since R is isomorphic to R_{NM} and R_{NM} is a minimal, that is, it is reachable and observable, we get that R is reachable and observable too. Thus, (Σ_R, μ_R) is semi-reachable and observable. From Theorem 19 it follows that (Σ_R, μ_R) satisfies (4.25). Since (Σ_R, μ_R) is isomorphic to (Σ_{NM}, μ_{NM}) we get that $\dim \Sigma_R = \dim \Sigma_{NM}$ and thus (Σ_R, μ_R) satisfies (4.26). $\square$

Below we will give an estimate on $M =$ rank H_Ω. At the same time, the proof of the estimate will also demonstrate that M depends only on L. Recall the definition of the language $\widetilde{L}$ from Subsection 10.4.2.

Lemma 60. *Assume that L is regular. Denote by $n_{\widetilde{L}}$ the cardinality of the state-space of the minimal automaton recognising $\widetilde{L}$. Then*

$$\text{rank } H_\Omega \leq n_{\widetilde{L}}$$

Proof. Recall from the proof of Lemma 17, [55] that $L_q = pr_Q^{-1}(\widetilde{L}_q)$. Thus, using the notation of the proof of Lemma 59, if $(S, Q, \delta, s_0, F,)$ is a minimal automaton accepting $\widetilde{L}$, then $(S, \Gamma, \widetilde{\delta}, s_0, F_q)$ is an automaton accepting L_q, where $\widetilde{(\delta)}(s,(r,j)) = \delta(s,r)$, $s \in S, (r,j) \in \Gamma$ and $F_q = \{s \in S \mid \delta(s,q) \in F\}$. Assume that $S = \{s_1, \ldots, s_n\}$. Define the following representation

$$R = (\mathbb{R}^n, \{A_z\}_{z \in \Gamma}, C, B)$$

where $A_{(q,j)} e_i = e_k$ if $\widetilde{\delta}(s_i,(q,j)) = s_k$, $(q,j) \in \Gamma$, $B_f = s_0$ and

$$C = \begin{bmatrix} W_{q_1}^T & \cdots & W_{q_N}^T \end{bmatrix}$$

such that $W_{q_l} e_i = \begin{cases} (1,1,\ldots,1)^T \in \mathbb{R}^p & \text{if } s_i \in F_{q_l} \\ 0 & \text{otherwise} \end{cases}$, $l = 1, \ldots, N$. Thus,

$$W_{q_l} A_w B_f = C_{q_l}(w)$$

and therefore R is a representation of Ω. Thus, $\operatorname{rank} H_\Omega \leq \dim R = n = n_{\widetilde{L}}$. □

Corollary 26. *With the notation and assumptions of Lemma 59*

$$\operatorname{rank} H_\Omega \leq 2^{n_L \cdot |Q| + 1}$$

where n_L is the cardinality of the state-space of a minimal automaton accepting L.

Proof. From the proof of Corollary 25 it follows that $n_{\widetilde{L}} \leq 2^{n_L \cdot |Q|+1}$ and thus by Lemma 60 we get the required inequality. □

10.5 Hybrid Systems Without Guards

The section presents partial realization theory for linear and bilinear systems. Algorithms for computing minimal realizations and checking observability, semi-reachability and minimality will be discussed too. Numerical examples will be presented too. The main tool for deriving these results is the partial realization theory of hybrid power series and the related algorithms presented in Section 10.3. The outline of the section is the following. Subsection 10.5.1 presents partial realization theory and the related algorithms for linear hybrid systems. Subsection 10.5.2 presents partial realization theory for bilinear hybrid systems.

10.5.1 Linear Hybrid Systems

The theory of hybrid formal power series developed in Section 3.3 allows us to formulate a partial realization theorem for linear hybrid systems. It also enables us to formulate algorithms for deciding observability and semi-reachability of linear hybrid systems and to give an algorithm for constructing a minimal linear hybrid system realization based on a specified linear hybrid system realization.

Let Φ be a set of input-output maps and assume that Φ has a hybrid kernel representation. Our first objective is to construct a linear hybrid system realization of Φ from finitely many data points. It is easy to see that all information needed for constructing the indexed set of hybrid formal power series $\Omega = \Psi_{\Phi}$ can be obtained (in theory) from the set of input-output maps Φ. In the remaining part of the section we will tacitly assume that Φ is finite, i.e., Φ consists of finitely many input-output maps.

Recall the results of Subsection 10.3. If Φ is a finite collection of input-output maps, then the index set $J = \Phi \cup (\Phi \times \{1, \ldots, m\})$ of Ψ_{Φ} is finite. It is easy to see that if Φ is finite then all the data for constructing $W_{\mathcal{D}_{\Psi_{\Phi},N},D,D}$ and $H_{\Psi_{\Phi},N,N}$ can be obtained from the input-output maps of Φ and the number of data points needed for constructing $W_{\mathcal{D}_{\Psi_{\Phi},N},D,D}$ and $H_{\Psi_{\Phi},N,N}$ is finite. Theorem 67 yields that the finite data from $W_{\mathcal{D}_{\Psi_{\Phi},N},D,D}$ and $H_{\Psi_{\Phi},N,N}$ can be used to compute a minimal hybrid representation of Ψ_{Φ}. But any minimal hybrid representation HR of Ψ_{Φ} yields a minimal linear hybrid realization (H_{HR}, μ_{HR}) of Φ. Thus, we get the following result. Let $H_{\Phi,N,M} = H_{\Psi_{\Phi},N,M}$, $\mathcal{D}_{\Phi,N} = \mathcal{D}_{\Psi_{\Phi},N}$ for all $N, M \in \mathbb{N}$, $N, M > 0$.

Theorem 72. *Assume that Φ is a finite collection of input-output maps and Φ has a hybrid kernel representation. Assume that* $\operatorname{rank} H_{\Phi,N,N} = \operatorname{rank} H_{\Phi}$ *and $card(W_{\mathcal{D}_{\Phi,N},D,D}) = card(W_{\mathcal{D}_{\Phi,N}})$. Let $HR_{N,D}$ be the hybrid representation from Theorem 67. Then $(H_{N,D}, \mu_{N,D}) = (H_{HR_{N,D}}, \mu_{HR_{N,D}})$ is a minimal linear hybrid system realization of Φ and it can be constructed from finite data which can be obtained directly from Φ. In particular, if Φ has a linear hybrid system realization (H, μ) such that $\dim H = (p, q)$ and $qm + p \leq N$, then $(H_{N,N}, \mu_{N,N})$ is a minimal linear hybrid system realization of Φ and it can be constructed from finitely many data which is directly obtainable from Φ.*

The results of Subsection 10.3 also allow us to check observability and semi-reachability of linear hybrid systems algorithmically. Indeed, consider a linear hybrid system realization (H, μ). It is easy to see that the construction of $HR_{H,\mu}$ can be carried out by a computer algorithm. It follows that $HR_{H,\mu}$ is reachable if and only if (H, μ) is semi-reachable and $HR_{H,\mu}$ is observable if and only if H is observable. Recall the procedures `IsHybRepObservable` and `IsHybRepReachable`. To check semi-reachability of (H, μ) we can use `IsHybRepReachable` on $HR_{H,\mu}$. To check observability of (H, μ) we can apply `IsHybRepObservable` to $HR_{H,\mu}$. Finally, we can apply `ComputeMinimalHybRepresentation` to $HR_{H,\mu}$ to obtain a minimal hybrid representation HR and then we can construct (H_{HR}, μ_{HR}) which will be a minimal linear hybrid system realization of Φ. Notice that the construction of (H_{HR}, μ_{HR}) can be carried out algorithmically. Thus, if all the entries of the system matrices of H are rational and all the values of μ are rational, then observability and semi-reachability of (H, μ) is *algorithmically decidable* and a minimal linear hybrid realization of Φ can be constructed from (H, μ) by an algorithm in sense of classical Turing computability.

As an illustration we will present below a numerical example.

Example

Consider the following linear hybrid system. Consider the Moore-automaton $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$, where $Q = \{q_1, q_2\}$, $\Gamma = \{a, b\}$ and $O = \{0\}$. Define the discrete state transition map by $\delta(q_1, a) = q_1, \delta(q_1, b) = q_2, \delta(q_2, b) = q_2, \delta(q_2, a) = q_2$. Define the readout map $\lambda(q_1) = \lambda(q_2) = o$. Consider the linear hybrid system

$$H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, B_q, C_q)_{q \in Q}, \{M_{q_1, \gamma, q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2, \gamma)\})$$

where $\mathcal{Y} = \mathcal{U} = \mathbb{R}$, $p = m = 1$, $\mathcal{X}_{q_1} = \mathbb{R}^3$ and $\mathcal{X}_{q_2} = \mathbb{R}^2$ and the matrices A_q, B_q, C_q, $q \in \{q_1, q_2\}$ are of the following form

$$A_{q_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad B_{q_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad C_{q_1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$A_{q_2} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad B_{q_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C_{q_2} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

The linear reset maps are the following

$$M_{q_1, a, q_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad M_{q_2, b, q_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_{q_1, a, q_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{q_2, b, q_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The form of the input/output map $\upsilon_H((q_2, x_0), .)$ induced by (q_2, x_0), $x_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ is quite complex, as a demonstration we will present below the output to the discrete input sequence $(b, t_1)(a, t_2)(a, t_3)(b, t_4)$.

$$\upsilon_H((q_2, x_0), u, (b, t_1)(a, t_2)(a, t_3)(b, t_4), t_5) = \\ (o, e^{2t_5} e^{3t_4} e^{3t_3} e^{2t_2} e^{2t_1} + \int_0^{t_1 + \cdots + t_5} e^{t_1 + \cdots t_5 - s} u(s) ds)$$

Consider a linear hybrid system H_m of the following form

$$(\mathcal{A}^m, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q^m, A_q^m, B_q^m, C_q^m)_{q \in Q^m}, \{M^m_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q^m, \gamma \in \Gamma, q_1 = \delta^m(q_2, \gamma)\})$$

where $Q^m = \{q\}$, $\mathcal{X}_q^m = \mathbb{R}^3$, the automaton $\mathcal{A}^m = (Q^m, \Gamma, O, \delta^m, \lambda^m)$ is given by

$$\delta^m(q, z) = q, z \in \{a, b\} \text{ and } \lambda^m(q) = o$$

The matrices $A_q^m, B_q^m, C_q^m, M^m_{q,z,q}, z \in \{a, b\}$ are specified below

$$A_q^m = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B_q^m = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad C_q^m = \begin{bmatrix} -1 & -1 & -1.414214 \end{bmatrix}$$

$$M^m_{q,b,q} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M^m_{q,a,q} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Define $\mu_m(\upsilon_H((q_2, x_0), .)) = (q, z_0)$ by $z_0 = \begin{bmatrix} -0 & -0 & -0.707107 \end{bmatrix}^T$. Then (H_m, μ_m) is a minimal linear hybrid system realization of $\upsilon_H((q_2, x_0), .)$. The realization H_m was computed using a Matlab implementation of the algorithm presented in the paper. ‘

10.5.2 Bilinear Hybrid Systems

The theory of hybrid formal power series developed in Section 3.3 allows us to formulate a partial realization theorem for bilinear hybrid systems. It also enables us to formulate algorithms for deciding observability and semi-reachability of bilinear hybrid systems and to give an algorithm for constructing a minimal bilinear hybrid system realization based on a specified hybrid system realization. In fact, the results presented below are more general than the ones described in [48]. Notice that the algorithmic aspects of realization theory are treated in this paper in a much more detailed manner than in [48].

Let Φ be a collection of input-output maps and assume that Φ admits a hybrid Fliess-series expansion. It is easy to see that all information needed for constructing the indexed set of hybrid formal power series $\Omega = \Psi_\Phi$ can be obtained (in theory) from the set of input-output maps Φ, more precisely, from the generating series c_f and discrete input-output maps f_D for all $f \in \Phi$. In fact, the values of c_f can be recovered from f by taking high-order derivatives with respect to time and continuous inputs.

Assume that Φ is finite collection of input-output maps. Notice that it also implies that the index set $J = \Phi$ of Ψ_{Φ} is finite. Unless stated otherwise, we will use this finiteness assumption in the rest of this section.

Our first goal is to construct a bilinear hybrid realization of Φ from finite number of data points. Recall the results of Subsection 10.3. It is easy to see that if Φ is finite then all the data for constructing $W_{\mathcal{D}_{\Omega,N},D,D}$ and $H_{\Omega,N,N}$ can be obtained from the input-output maps of Φ and the number of data points needed for constructing $W_{\mathcal{D}_{\Omega,N},D,D}$ and $H_{\Omega,N,N}$ is finite. Theorem 67 yields that the finite data from $W_{\mathcal{D}_{\Omega,N},D,D}$ and $H_{\Omega,N,N}$ can be used to compute a minimal hybrid representation of Ω. But any minimal hybrid representation HR of Ω yields a minimal bilinear hybrid realization (H_{HR}, μ_{HR}) of Φ. Thus, we get the following result. Denote $H_{\Phi,N,M} = H_{\Psi_{\Phi},N,M}$, $\mathcal{D}_{\Phi,N} = \mathcal{D}_{\Psi_{\Phi},N}$.

Theorem 73. *Assume that Φ is a finite collection of input-output maps and Φ admits a hybrid Fliess-series expansion. Assume that* $\operatorname{rank} H_{\Phi,N,N} = \operatorname{rank} H_{\Phi}$ *and* $card(W_{\mathcal{D}_{\Phi,N},D,D}) = card(W_{\mathcal{D}_{\Phi,N}})$. *Let $HR_{N,D}$ be the hybrid representation from Theorem 67. Then $(H_{N,D}, \mu_{N,D}) = (H_{HR_{N,D}}, \mu_{HR_{N,D}})$ is a minimal bilinear hybrid system realization of Φ and it can be constructed from finite data which can be obtained directly from Φ. In particular, if Φ has a linear hybrid system realization (H, μ) such that* $\dim H = (p, q)$ *and* $\max\{p, q\} \leq N$, *then $(H_{N,N}, \mu_{N,N})$ is a minimal bilinear hybrid system realization of Φ and it can be constructed from finitely many data which is directly obtainable from Φ.*

The results of Subsection 10.3 also allow us to check observability and semi-reachability of bilinear hybrid systems algorithmically. Indeed, consider a bilinear hybrid system realization (H, μ). It is easy to see that the construction of $HR_{H,\mu}$ can be carried out by a computer algorithm. It follows that $HR_{H,\mu}$ is reachable if and only if (H, μ) is semi-reachable and $HR_{H,\mu}$ is observable if and only if H is observable. Recall the procedures `IsHybRepObservable` and `IsHybRepReachable`. To check semi-reachability of (H, μ) we can apply `IsHybRepReachable` to $HR_{H,\mu}$. To check observability of (H, μ) we can apply `IsHybRepObservable` to $HR_{H,\mu}$. Finally, we can apply `ComputeMinimalHybRep` to $HR_{H,\mu}$ to obtain a minimal hybrid representation HR and then we can construct (H_{HR}, μ_{HR}) which will be a minimal bilinear hybrid system realization of Φ. Notice that the construction of (H_{HR}, μ_{HR}) can be carried out algorithmically. Thus, if all the entries of the system matrices of H are rational and all the values of μ are rational, then observability and semi-reachability of (H, μ) is *algorithmically decidable* and a minimal bilinear hybrid realization of Φ can be constructed from (H, μ) by an algorithm in sense of classical Turing computability.

Below we will present a numerical example

Example

Consider the following bilinear hybrid system. Consider the Moore-automaton $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$, where $Q = \{q_1, q_2\}$, $\Gamma = \{a, b\}$ and $O = \{0\}$. Define the discrete state transition map by $\delta(q_1, a) = q_1, \delta(q_1, b) = q_2, \delta(q_2, b) = q_2, \delta(q_2, a) = q_2$. Define the readout map $\lambda(q_1) = \lambda(q_2) = o$. Consider the linear hybrid system

$$H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, \{B_{q,j}\}_{j=1,\dots,m}, C_q)_{q\in Q}, \{M_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\})$$

where $\mathcal{Y} = \mathcal{U} = \mathbb{R}$, i.e. $p = m = 1$, $\mathcal{X}_{q_1} = \mathbb{R}^3$ and $\mathcal{X}_{q_2} = \mathbb{R}^2$ and the matrices $A_q, B_{q,1}, C_q$, $q \in \{q_1, q_2\}$ are of the following form

$$A_{q_1} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad B_{q_1,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad C_{q_1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$A_{q_2} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad B_{q_2,1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad C_{q_2} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

The linear reset maps are of the following form

$$M_{q_1,a,q_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad M_{q_2,b,q_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The input/output map $\upsilon_H((q_2, x_0), .)$ induced by (q_2, x_0), $x_0 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, is quite complex, as a demonstration we will present below the output to the discrete input sequence $(b, t_1)(a, t_2)(a, t_3)(b, t_4)$.

$$\begin{aligned} &\upsilon((q_2, x_0), u, (b, t_1)(a, t_2)(a, t_3)(b, t_4), t_5) = \\ &\quad (o, \sum_{w_1,\dots,w_5 \in \mathbb{Z}_m^*} 3^{nz(w_2)+nz(w_3)} V_{w_1,\dots,w_5}[u](t_1, \dots, t_5)) \end{aligned}$$

where $nz(w)$ is the number of occurrences of the symbol 0 in w, $V_{w_1,\dots,w_5}[u](t_1, \dots, t_5)$ – product of iterated integrals.

A minimal realization of $\upsilon_H((q_2, x_0), ., .)$ of the following form.

$$H_m = (\mathcal{A}^m, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q^m, A_q^m, \{B_{q,j}^m\}_{j=1,\dots,m}, C_q^m)_{q\in Q^m}, \{M_{\delta^m(q,\gamma),\gamma,q}^m \mid q \in Q^m, \gamma \in \Gamma\})$$

where $\mathcal{U} = \mathcal{Y} = \mathbb{R}$, $Q^m = \{q\}$, $\mathcal{X}_q^m = \mathbb{R}^2$, the automaton $\mathcal{A}^m = (Q^m, \Gamma, O, \delta^m, \lambda^m)$ is given by

$$\delta^m(q, z) = q, z \in \{a, b\} \text{ and } \lambda^m(q) = o$$

The matrices $A^m_q, B^m_{q,1}, C^m_q, M^m_{q,z,q}, z \in \{a, b\}$

$$A^m_q = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \quad B^m_{q,1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C^m_q = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Reset maps:

$$M^m_{q,b,q} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \quad M^m_{q,a,q} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Define $\mu_m(\upsilon_H((q_2, x_0), .)) = (q, z_0)$ by $z_0 = \begin{bmatrix} 0 & -1 & \end{bmatrix}^T$. Then (H_m, μ_m) is a minimal bilinear hybrid system realization of $\upsilon_H((q_2, x_0), .)$. The realization H_m was computed using a Matlab implementation of the algorithm presented in the paper.

Chapter 11

Conclusions

The aim of this chapter is to draw conclusions from and to outline further research direction based on the work presented in this thesis. We will start by recapitulating what was achieved in this thesis. Then we will proceed with formulating a number of claims concerning further research directions in the area of hybrid systems in general and in the field of realization theory of hybrid systems in particular. We will also suggest a number of mathematical theories, which, we believe, could be useful in study of hybrid systems.

11.1 Short Summary of the Thesis

In this thesis we presented realization theory for a number of classes of hybrid systems. With one exception only hybrid systems without guards, i.e. without autonomous switching were considered. The exception is the class of discrete-time piecewise-affine hybrid systems. This class is essentially the same as the class of PL systems introduced by Sontag in [15]. Unfortunately the results obtained for this class of hybrid systems are incomplete and inconclusive. Much more research is needed.

In contrast to PL systems, realization theory for continuous-time linear and bilinear switched and hybrid systems is rather complete. We were also able to present numerical algorithms for computing a minimal realization, checking semi-reachability, observability and minimality and for computing a realization from finite input-output data. These result have a great potential impact on control and identification methods for hybrid systems of the above class. The main tool in realization theory of these systems was the framework of rational formal power series. Precisely this classical theory allowed us to prove the presented results. We also managed to make the first inroad into realization theory of nonlinear hybrid systems without guards. The obtained results a promising but much further research needs to be done.

Unfortunately, we did not manage to obtain too many results concerning hybrid systems with guards. This stays entirely a topic of further research.

11.2 Conclusions

The present thesis offers a rather complete and coherent view of realization theory of hybrid systems without guards. In author's opinion, the main reason for the relative ease with which these results were obtained is the realization that hybrid systems in essence are *nonlinear systems*.

In fact, this statement is absolutely true for switched systems. Switched systems can be viewed as a collection of vector fields, which is exactly a point of view adopted by nonlinear systems theory. Perhaps the only difference between classical nonlinear systems and switched systems is the choice of admissible inputs. In case of classical nonlinear systems the admissible input were mostly either smooth or analytic or piecewise-continuous, or integrable or piecewise-constants. In case of switched systems the admissible input are such that one input component has to be piecewise-constant, the other can be smooth, continuous, analytic etc.

Switched systems with constrained switching stand already a bit further from classical nonlinear systems. In this case the set of admissible inputs is further restricted, in fact, it is not closed under composition. Indeed, if two finite switching sequences are admissible, then it does not follow that their composition will be admissible too. In some sense a switched system with constrained switching represents contains more information then its input-output behaviour. Indeed, if we now all the vector fields and readout maps, then we also know how the system would behave for non-admissible switching sequences, but of course we cannot measure such a behaviour. This basic fact is reflected in the problems we encountered while trying to obtain minimal switched system realizations under switching constrains.

General hybrid systems without guards are even further from nonlinear systems. There is no way to view them as classical nonlinear systems. In the author's opinion, developing realization theory for such systems was a much more challenging task than developing realization theory for switched systems. Nevertheless, nonlinear system theory, in particular, theory of rational formal power series did provide us with the necessary tools. The results derived for hybrid systems without guards present an interesting combination of automata theory and theory of formal power series. The latter has been a well-established tool of nonlinear control theory for a long time.

Thus, theory of hybrid systems without guards seems to be close to fulfilling the hope that hybrid systems theory can built by smart combination of automata theory and classical control systems theory.

Although we just argued that theory of hybrid systems without guards are a combination of the classical control theory and automata theory, we would like to note that we did not succeed in using too many off-the-shelf results from either of fields. That is, the developed theory is not so a much combination of classical theorem, rather, it is the result of rethinking and extending the classical results. This is also reflected in the fact that we use only very few classically known results. It is more the ideas, rather than the known results we used.

An important common feature of hybrid systems without guards and nonlinear systems is that in the analytic case the long-term behaviour of the system can be recovered from the local, small-time behaviour. That is, if two systems have the same behaviour locally, for small times, they will have the same behaviour globally too. This is a very important feature from the point of view of realization theory. Indeed, almost all results on realization theory of continuous time systems rely on collecting this local behaviour, represented in a way or another by high-order time derivatives at zero, in the Hankel-matrix and requiring different finiteness conditions to hold for the resulting infinite matrix. Together with the finite rank property it also allows designing numerical algorithms for computing realization from input-output data. The property that knowledge of local behaviour is enough for building a realization was heavily exploited in all the hybrid systems without guards considered in this thesis.

Unlike nonlinear systems, hybrid systems without guards have dynamics, which is not necessarily defined by action of a family of (local) diffeomorphism on a manifold. This is due to the presence of reset maps, which need not be invertible. In this sense hybrid systems are quite close to discrete-time nonlinear systems with non-invertible right-hand side. That property of hybrid systems without guards makes global analysis difficult. In particular, it makes it difficult to study accessibility and observability properties. For nonlinear systems the main tool for studying such properties is Sussmann's orbit theorem. But Sussmann's orbit theorem does not seem to work for systems, where the dynamics is not generated by diffeomorphism. In fact, the orbits of such systems need not be manifolds, they fall into the category of differentiable spaces.

The current thesis hardly scratches the surface of hybrid systems with guards. The only class of such hybrid systems which is dealt with in this thesis is the class of piecewise-affine hybrid systems in discrete-time, i.e. the rather Sontag's PL systems. The results which are presented in this thesis are quite elementary. They rely on techniques, which are in some sense similar to those for time-varying linear systems. As for other classes of hybrid systems with guards, the realization problem is still open for those systems.

11.3 Further Research

In this section we would like to give ideas and suggestions for future research in the topic. We would also like to draw attention to what we see as potential difficulties and suggest mathematical techniques, which, in our opinion, could help to solve those difficulties. We will divide this section into several subsections, each discussing a specific topic.

11.3.1 Coalgebraic Approach to Realization Theory of Nonlinear and Hybrid Systems

Although, as we have already pointed out, hybrid systems with guards are much more interesting object to investigate than hybrid systems without guards, we still think that it is worthwhile to clarify a number of issues for hybrid systems without guards too. Not the least because hybrid systems without guards can always occur as extremal versions of hybrid systems with guards. More precisely, if a hybrid system with guards is such that for any guard there are input values which immediately steer the system to the guard, then in fact we get a hybrid system without guards, where the role of external discrete events is taken over by those special input values. That is, by feeding in a suitable continuous input impulse, we can force an instantaneous discrete-state transition. Thus, certain continuous inputs will act as discrete inputs and will trigger discrete state transitions. That is, such hybrid systems with guards will function as hybrid systems without guards.

Another motivation for studying hybrid systems without guards is that while hybrid systems without autonomous switching do not seem to occurs too often, hybrid systems with both autonomous and external switching seem to be quite frequent. Investigating hybrid systems with external switching only seems to be a logical step towards exploring hybrid systems with both autonomous and external switching.

As the reader has already seen, theory of Hopf-algebras and bi algebras played a prominent role in realization theory of hybrid systems without guards. In fact, relevance of Hopf-algebra in realization theory was noticed much earlier, see [27, 29]. So far, Sweedlet-type coalgebra theory was used only for local realization. In fact, as the reader could see, even local realization is not yet fully solved for hybrid systems.

Nevertheless, we are quite confident that it can be solved using the approach and results of the current paper as a starting point.

In fact, we believe that the theory of Sweedler-like coalgebras could be successfully used for solving the global realization problem too. Let us outline how, in our opinion, it could be done.

Consider the duality between manifolds and algebra of smooth functions on manifolds. It is known that the algebra of smooth functions over a manifold has a natural topology which makes it a Frechet-algebra. The underlying manifold corresponds to the set of those algebraic homomorphisms from the algebra to reals, which are continuous in the topology of the algebra. The topology of the manifold itself coincides with the Zariski topology of it as a real spectrum of the algebra of smooth functions. We suggest to make a step further, and view a manifold together with the covariant tensors spaces at each point as a coalgebra. This coalgebra will be a direct sum of pointed cocommutative irreducible cofree coalgebras. To each point of the manifold there corresponds a pointed irreducible component, which is the cofree cocommutative irreducible pointed coalgebra generated by the tangent space at that point. The unique group-like element of this coalgebra is the point of the manifold. The topology of the manifold induces a topology on this coalgebra. Then the algebra of smooth functions over the manifold is in fact the topological dual of this coalgebra. That is, we have duality between topological coalgebras and topological algebras. This leads as to studying objects which are very similar to those of formal groups.

The ideas above lead us to the following approach to control systems on manifolds. Let us view the state-space manifold as a coalgebra. Let us view the input-space as a coalgebra. Notice that regardless of whether the input-space is a manifold or discrete set, we can always adopt such a point of view. By input-space we mean the index set of the semi-group of transformations, which define the dynamics of the system. For instance, if this transformation semi-group is generated by flows of a family of vector fields, then our input space will be the set of timed sequences of indices of the vector fields which make up the family. Notice that the input-space has a natural algebra, moreover a bialgebra structure. If the input space forms a group with respect to concatenation, then it can be assigned a pointed cocommutative Hopf-algebra structure.

By adopting the point of view described above, the dynamics is just a coalgebra map from tensor product of the input- and the state-spaces to state-space, the initial states just group-like elements of the state-space and the readout maps just elements of the (topological) dual of the state-space. If we consider the (topological) dual of the state-space, then the coalgebra map describing the dynamics yields a *measuring.* A measuring is a map from a coalgebra and algebra to an algebra, such that the map has certain special properties. Thus, we naturally arrive to the framework of coalgebra and algebra systems, after possibly defining a suitable topology on the state-space.

The author does not see it inconceivable that realization theory could be carried out in a way, similar to what was done for local realization of nonlinear and hybrid systems. Of course, topological arguments should be taken into account too, the constructions would not be purely algebraic. But one can still hope that the approach described above could serve as a unifying framework for a number of constructions.

Let us remark that the point of view described above is quite similar to the concept of k systems described in [64]. Indeed, the spectrum of any algebra can be identified with a coalgebra formed by the space spanned by algebraic maps from the algebra to the ground field (we tacitly assumed that the algebra is reduced). The duality between spectrum and algebra is analogous to the duality between algebra and coalgebra described above.

It would be interesting to explore further the possible use of Hopf-algebra theory in solving the realization problem. It would be also interesting to see how the already known results could be proven with this approach. However, the suggested coalgebraic approach might not be suitable for tackling all types of systems. In particular, in case of polynomial systems it is questionable how useful the coalgebraic approach is, as the author is not aware of nice characterisation of finitely generated algebra or algebra with finite transcendence degree as duals of suitable coalgebras. Hence, what the author suggests is not to use the coalgebraic approach as a universal tool. It seems more appropriate to use duality and look at algebras and coalgebras, depending on which point of view seems to be more fruitful.

In fact, Sontag's "algebraic approach" ([64]), which is more or less explicitly present in many of his papers, is dual to the coalgebraic approach. We believe that our approach is more useful for general nonlinear systems, while his approach is more useful for systems, algebra of which is finitely generated or has finite transcendence degree, such as rational or polynomial systems.

We conjecture that even global results on realization theory of nonlinear and hybrid systems could be successfully dealt with in our framework. In particular we think that Jakubczyk's approach to realization theory could be recast in our framework. As in his case the input-space forms a group, we would get that the input bialgebra is in fact a Hopf-algebra. Hopf-algebras have nice properties and can be easily related to enveloping algebras of Lie-algebras. This algebraic relationship could perhaps be combined with topological arguments to yield the construction of a state-space manifold.

11.3.2 Realization Theory of Hybrid Systems with Guards

As we mentioned earlier, this thesis contains very few results on hybrid systems with guards. The results on discrete-time hybrid systems which were presented in this thesis are rather elementary. The approach which we used to obtain them is not likely to be extendable to hybrid systems with inputs. That is because we assumed the the initial state uniquely determines the switching sequence. This is true for the autonomous case, but false for non-autonomous case, as the inputs can influence the switching sequence. In fact, it is easy to construct such an example where any switching sequence can be obtained by feeding in suitable inputs. On the other hand, the approach presented in this thesis may still work if the inputs do not influence the switching sequence. Even if the switching sequence depends only on the initial state, there are a number of questions which remain unanswered. In particular, it remains to be explored how the presented conditions can be checked algorithmically and how the construction of a realization can be carried out by an algorithm. The issue of minimality remains unexplored too.

Continuous-time hybrid systems with guards, in particular piecewise-affine hybrid systems with guards were not mentioned at all in this thesis. Realization theory for this class of hybrid systems remains completely a topic of further research.

A possible line of attack would be extension of the approach presented in this thesis. That is, we assume that the switching depends only on the initial state and we translate the realization problem for hybrid systems with guards to realization problem for time-varying systems of a particular structure. This approach might also give results and insights to realization theory and identification of PV and LPV systems.

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Samenvatting

De nederlandse vertaling van de titel van dit proefschrift is ‘ Realisatietheorie van Hybride Systemen ’ .

Het onderwerp van wiskundige regeltheorie is het regelen van systemen die voorkomen in natuur of techniek. Voorbeelden van zulke systemen zijn vliegtuigen, auto's, transportbanden en zelfs apparaten voor de automatische dosering van geneesmiddelen. Zulke systemen hebben de volgende eigenschapen gemeen. Ten eerste, hun gedrag verandert in de loop van de tijd. Ten tweede, het doel van het regelen is het systeem een bepaald gedrag af te dwingen. Zoals de naam suggereert, bestudeert men in wiskundige regel- en systeemtheorie de wiskundige modellen van systemen die in het praktijk voorkomen. Door naar de wiskundige modellen van systemen te kijken wordt het probleem van de regeling van systemen vertaald naar een goed gedefinieerd wiskundig probleem. Juist het oplossen van wiskundige problemen die op zulke manier zijn ontstaan is de taak van wiskundige systeem- en regeltheorie.

Om een voorbeeld te geven, differentiaalvergelijkingen worden vaak gebruikt voor het modelleren van in de praktijk voorkomende systemen. In dit geval worden de mogelijke regelacties gemodelleerd als een ingangsfunctie in de rechterkant van de differentiaalvergelijkingen. Het regelen van het systeem correspondeert met het kiezen van een functie die bij elk tijdstip een ingangswaarde kiest. Op deze manier verkrijgt men een differentiaalvergelijking die tijdsafhankelijk is. De wiskundige vertaling van het regelprobleem wordt het kiezen van zo'n functie, zodanig dat de oplossing van de resulterende tijd-variabele differentiaalvergelijking aan de vereiste voorwaarden voldoet.

Wiskundige systeem- en regeltheorie is een multidisciplinair vakgebied, dat wiskunde, techniek en informatica combineert.

Een van de kernproblemen van systeem- en regeltheorie is het vinden van realistische wiskundige modellen van in de praktijk voorkomende systemen. In de meeste gevallen zijn de wiskundige modellen van in de praktijk voorkomende systemen slechts gedeeltelijk bekend. Om een volledig wiskundig model te vinden, moet men gebruik maken van experimentele gegevens. Op deze manier ontstaat er het volgende wiskundige probleem. Welke wiskundige modellen van een bepaald type kunnen het waargenomen gedrag van het systeem beschrijven ? Als het waargenomen gedrag gewoon uit een eindig aantal experimentele gegevens bestaat, dan spreekt men over een *identificatieprobleem.* Als het waargenomen gedrag een abstrakte wiskundige relatie is, die de samenhang tussen de waarnemingen (de uitgang) en de regelacties (de ingang) beschrijft, dan spreekt men over een *realisatieprobleem.* De abstrakte wiskundige relatie tussen ingang en uitgang wordt vaak als *het ingangs-uitgangsgedrag van het systeem* genoemd. Het vakgebied van *systeemidentificatie* bestudeert het oplossen van het identificatieprobleem voor verschillende klassen van systemen. Het vakgebied van *realisatietheorie* bestudeert het oplossen van het realisatieprobleem voor verschillende klassen van systemen.

Het is duidelijk dat systeemidentificatie enorm belangrijk is voor de praktijk. Echter men kan zich afvragen waarom realisatietheorie van belang is. De reden voor het bestuderen van realisatietheorie is de volgende. Realisatietheorie beantwoordt een heel fundamentele vraag over regelsystemen. Hij legt verband tussen het waargenomen gedrag van het systeem en zijn interne structuur. Deze kennis, die op zichzelf heel waardevol is, kan heel goed toegepast worden bij het oplossen van een aantal meer praktische problemen, onder andere, voor het systeemidentificatieprobleem. Realisatietheorie kan beschouwd worden als een geïdealiseerd systeemidentificatieprobleem, waarbij heel veel praktische problemen buiten beschouwing zijn gelaten. Dus, als het realisatieprobleem onvolledig is begrepen, dan is er weinig kans voor het vinden van een bevredigende oplossing van het identificatieprobleem. In feite, vormt realisatietheorie de grondslag voor een groot aantal identificatiemethoden.

Realisatietheorie speelt ook een belangrijke rol bij een reeks andere problemen van de regeltheorie. Een van de belangrijke bijdragen van realisatietheorie is het bestuderen van de structuur van minimale systemen en het leggen van een verband tussen de minimaliteit van systemen en zulke belangrijke systeemeigenschapen als *regelbaarheid* en als *waarneembaarheid.* Op hun beurt, zijn de eigenschappen als waarneembaarheid en regelbaarheid meestal noodzakelijke voorwaarden voor het bestaan van oplossingen van regelproblemen.

Dit proefschrift behandelt het onderwerp van realisatietheorie van hybride systemen. Hybride systemen zijn systemen die zijn opgebouwd uit discrete deelsystemen en continue deelsystemen. Continue systemen zijn systemen die in een oneindig aantal toestanden kunnen verkeren. Discrete systemen zijn systemen waarvan de toestand slechts een eindig of aftelbaar aantal waarden kan aannemen. Een auto met een versnellingsbak levert een goede analogie op. Als wij de rijdende auto als een systeem beschouwen, dan zijn de positie en de snelheid van de auto en de versnelling waarin hij rijdt componenten van het systeem. Omdat de snelheid en de positie een oneindig aantal verschillende waarden kunnen aannemen, behoren zij tot de continue componenten van het systeem. De versnellingsbak kan maar een eindig aantal verschillende toestanden aannemen (in de meeste auto's zijn er maar vier versnellingen), dus behoort de versnellingsbak tot de discrete deelsystemen van het systeem. Om in wiskundige termen te spreken: de continue deelsystemen worden meestal door differentiaalvergelijkingen beschreven, terwijl de discrete deelsystemen worden beschreven door een eindig aantal regels, die de toestand van de discrete componenten bepalen. Deze regels hebben de vorm: ‘ als de voorwaarde A geldt dan moet de discrete component X in de toestand Y verkeren ’ . Vaak worden deze regels met behulp van een automaat met een eindig aantal toestanden beschreven. De motivatie voor het bestuderen van zulke systemen is de volgende. Ten eerste, een reeks van verschijnselen in de natuur vertoont een hybrid karakter en kunnen deze verschijnselen op natuurlijke wijze met hybride systemen gemodelleerd worden. Ten tweede, worden heel veel technische systemen met behulp van computers bestuurd. Vaak is het nuttig om het onderliggende technische systeem en de besturende computer als één systeem te beschouwen. Terwijl technische systemen meestal goed beschreven kunnen worden door differentiaalvergelijkingen, moet de besturende computer als een automaat met een eindig aantal toestanden gemodelleerd worden. Op deze manier krijgen wij systemen waarvan sommige deelsystemen een continu en andere deelsystemen een discreet gedrag tonen. Door de aanwezigheid van zowel discrete als continue onderdelen ligt het vakgebied van hybride systemen op het kruispunt van wiskundige regel- en systeemtheorie en informatica.

Dit proefschrift behandelt bijna uitsluitend hybride systemen, waarvan het gedrag van de discrete deelsystemen onafhankelijk is van het gedrag van de continue deelsystemen. Zulke hybride systemen zijn makkelijker te bestuderen dan hybride systemen van meer algemene vorm. Toch hoopt men dat het bestuderen van deze, meer beperkte, klasse van hybride systemen zal helpen in het bestuderen en begrijpen van hybride systemen van meer algemene aard.

De hoofdstuk 1 bevat een informele inleiding tot de inhoud van dit proefschrift. Hoofdstuk 2 bevat de belangrijkste wiskundige begrippen en notaties, die verder gebruikt zullen worden in dit proefschrift. Hoofdstuk 3 is een van de belangrijkste hoofdstukken van dit proefschrift. Dit hoofdstuk formuleert de abstrakte wiskundige theorie van hybride en klassieke formele machtreeksen. Deze theorie vormt de theoretische grondslag van realisatietheorie van een reeks klassen van hybride systemen. Hoofdstukken 4 en 7 behandelen de realisatiethorie voor de volgende klassen van hybride systemen: *linear switched systems*, *bilinear switched systems*, *linear hybrid systems* en *bilinear hybrid systems*. Hoofdstuk 8 bevat enkele resultaten over realisatietheorie van niet-lineaire hybride systemen. Dit hoofdstuk maakt gebruik van de theorie van zogenaamde *coalgebras*, in de zin van het beroemde boek van Sweedler. Hoofdstuk 9 bevat enkele voorlopige resultaten over realisatietheorie van de zogenaamde *discrete-time piecewise-affine hybrid systems*. Deze klasse van hybride systemen is de enige klasse van hybride systemen, die in dit proefschrift bestudeerd wordt en die hybride systemen toelaat, waarvan het gedrag van de discrete componenten wel afhankelijk is van het gedrag van de continue componenten. Hoofdstuk 10 behandelt de algorithmische aspecten van realisatietheorie van hybride systemen. Hoofdstuk 5 wijkt een beetje af van het hoofdkader van dit proefschrift. Het onderwerp van dit hoofdstuk is geen realisatietheorie, maar de structuur van de verzameling van bereikbare toestanden van *linear switched systems*. Hoofdstuk 6 beschrijft een alternatieve methode voor het ontwikkelen van realisatietheorie voor linear switched systems, waarbij er geen gebruik wordt gemaakt van de theorie van formele machtreeksen. Tenslotte, worden er in Hoofdstuk 11 enkele conclusies getrokken en enkele suggesties gedaan over toekomstige onderzoekthema's.

Summary

Mathematical control theory is concerned with control of natural and engineering systems. The range of such systems includes aeroplanes, conveyor belts, cars and even systems for automated injection of medicines. A common property of such systems is that their behaviour changes with time and the goal of the control is to achieve a particular behaviour of the system as time advances. As its name suggests, mathematical control theory studies the mathematical models of such systems. By looking at the mathematical models the problem of controlling the system translates into a well-defined mathematical problem. Solving mathematical problems which arise in this way is the primary task of mathematical control theory.

For example, differential or difference equations are widely used to model real-life systems. In this case the possible control actions correspond to input functions in the right-hand side of the equations. The process of controlling the system is modelled as a function of time, taking values in the input space. Substitution of such a function into the right hand side of the equation results in a time-varying differential/difference equation. The mathematical reformulation of the control problem in this case is to choose this function in such a way that the solution of the resulting time-varying differential/difference equation meets the specified control objectives.

Mathematical control theory is inherently a multidisciplinary subject, which combines (applied) mathematics, engineering, and computer science.

One of the core problems of control theory is to find proper models of real-life systems. Usually the mathematical models of real-life phenomena are only partially known. In order to obtain a full mathematical model, experimental data of the real-life system is required. In this way the following mathematical problem arises. Which mathematical models of a certain type can generate the observed behaviour of the system ? If the observed behaviour is a finite collection of experimental data, then the question above is usually referred to as the *system identification problem.* If the observed behaviour is an abstract mathematical relation describing the relationship between controls (inputs) and observable features of the system (outputs), then we

speak of the *realization problem*. This abstract relationship between controls and observed behaviour is often referred to as the *input-output behaviour of the system*. The field of *system identification* studies the solution of the system identification problem for various classes of systems. The field of *realization theory* studies the realization problem for various classes of systems.

Clearly, the field of system identification is of huge practical importance. But one may wonder why realization theory is important at all. The reason for studying realization theory is the following. Realization theory answers a very fundamental question about systems, by establishing a relationship between the observed behaviour of the system and its inner structure. This knowledge, which is valuable on its own, can also be used for solving a number of more practical problems. One of those problems is system identification. The realization problem can be thought of as system identification problem under idealised circumstances. Thus, if realization theory is poorly understood for a class of systems, then there is little hope for finding a satisfactory solution for the identification problem. In fact, a great deal of system identification techniques are based on realization theory.

Realization theory plays an important role in other branches of control theory too. One of the important contributions of realization theory is the study of the structure of minimal systems and the investigation of the relationship between minimality and such important properties of systems as *controllability* and *observability*. In turn, these properties and the structure of the minimal system play an important role in developing control methods.

This thesis deals with realization theory of hybrid systems. Hybrid systems are control systems which contain both discrete and continuous components. Roughly speaking, continuous components are components which can take infinitely many different states. Discrete components are components which can have only finitely many different states. Perhaps a car with a gear box offers a good analogy. If we consider the position, speed and the position of the gear of the car as components of the system describing the motion of the car, then the speed and position can have infinitely many values, while the gear can only be in four or five different positions. Thus, the position and the speed of the car are the continuous components and the position of the gear is the discrete component. In mathematical terms, the continuous components are described by differential/difference equations, and the discrete components are described by a finite set of rules, which prescribe the state of the discrete components. These rules are of the form ïf condition A holds, then discrete component X has to be in state Y". Very often these rules are specified by a finite state automaton. The motivation for the study of hybrid systems is the following. First, a number of natural phenomena can be naturally viewed as hybrid

systems. Second, many engineering systems are controlled by computers. Very often it makes sense to model the engineering system and the computers controlling it as one system. While the underlying engineering system is usually best modelled by differential/difference equation, the controlling computers have to be modelled by a finite state automaton. Hence, we get systems some components of which exhibit continuous behavior and some components exhibit discrete behavior. Due to the presence of discrete components, hybrid systems lie on the junction of control theory and computer science.

This thesis deals mostly with hybrid systems *without guards*. Hybrid systems without guards are such hybrid systems in which the time evolution of the discrete components is independent from the time evolution of continuous components. Hybrid systems without guards are easier to study than more general hybrid systems. One can hope that the results obtained for hybrid systems without guards will help studying more general hybrid systems.

Chapter 1 contains an informal introduction to the thesis, Chapter 2 presents the main mathematical notions used in the thesis. Chapter 3 describes the abstract mathematical framework of hybrid power series. This framework forms the theoretical basis for realization theory of hybrid systems. Chapters 4 and 7 present realization theory of linear and bilinear switched systems and linear and bilinear hybrid systems. Chapter 8 presents some preliminary results on realization theory of nonlinear hybrid systems. This chapter uses the machinery of Sweedler-style coalgebras. Chapter 9 presents preliminary results on realization theory of piecewise-affine discrete-time hybrid systems. This is the only class of hybrid systems dealt with in this thesis, which contains hybrid systems with guards. Chapter 10 addresses the algorithmic aspects of realization theory. Chapter 5 is a bit different from the other parts of the thesis. It does not address realization theory, rather it deals with a related topic, the structure of reachable sets of hybrid systems. Chapter 6 discusses an alternative approach to realization theory of linear switched systems, such that no use of formal power series theory is required. Finally, Chapter 11 presents some conclusions and sketches some possible future research directions.