Exogenous and Endogenous Spatial Growth Models

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Abstract

In this paper, we investigate the impact on aggregate regional utility as a result of both exogenous growth and endogenous growth in a spatial system. We will first analyze the case of two closed regions, followed by the case of two open regions. The main instrument used in our approach to study the changes in collective regional welfare is Dynamic Programming. The traditional exogenous Solow growth model forms the basis of our paper. The analysis of this model will be extended to a comparison of two closed regions with exogenous growth. By introducing a case of a common labour market, we are able to investigate exogenous growth between two open regions. For the analysis of endogenous growth, we adopt the same structure as the one used for the investigation of exogenous growth models. In this framework, an investment in knowledge is considered as the endogenous driving force. Finally, we take a closer look at the timing of cost-reducing investments. In total, seven related but distinct cases are identified and studied in more detail.

1. Introduction

In the 1950s and 1960s various theories of economic growth have been developed and intensively studied. Especially Harrod (1948), Solow (1956), Verdoorn (1956), Domar (1957), Inada (1963) and Kaldor (1961) offered eye-opening insights into the theory and praxis of economic growth. In the fashion of that time, Bos and Tinbergen (1962) connected also economic growth with planning models in order to utilize economic progress in a low developed country or region as a strategy towards a higher level of development. Interesting remarks about different types of economic growth and multi-regional decision-making are made by Armstrong and Taylor (1993).

Since the late 1980's growth models have again become a popular topic in the economic literature. Especially the role of technological progress received intensive attention, in particular in the recent endogenous growth theory. A nice overview about this subject is presented by Stoneman (1983). The present paper studies exogenous as well as endogenous growth models, with a particular emphasis on the effect different spatial background conditions may have on the welfare position of one or more regions. In our analysis, economically closed as well as open regions are considered. We will adopt an optimization approach but, in contrast to the analysis performed by Bos and Tinbergen (1962) who presuppose the existence of a planner or policymaker in order to ensure a certain rate of growth, we will place more emphasis on decision-making by rational actors in a competitive space. The main analysis instrument used in our approach to the study of changes in collective welfare is Dynamic Programming. This tool from optimal control analysis (Kirk, 1978) can be applied in order to investigate the way in which welfare is affected as a result of different parameter values in the model under consideration. Instead of a completely dynamic approach, we will mainly apply comparative statics in order to be able to make a comparison between various interesting cases of closed and open, exogenous and endogenous models. In total, seven such cases will be investigated in the paper.

In the first part of the paper the traditional Solow growth model is presented. This exogenous growth model will form the basis of our analysis. We follow the study of Wan (1971) who investigated the differences between two closed regions under conventional conditions. Our paper will extend this analysis by considering a situation where the input factor labour is free to move from one region to the other. In the second part of the paper endogenous growth is studied more closely. Thereto, we use a model developed by Nijkamp and Poot (1997) who introduced investment in knowledge as a mechanism to create a system with endogenous growth. Finally, the paper considers the question at which point in time an optimal investment in human capital can take place and how competition among regions to implement the obtained new knowledge affects the (sum of) the welfare position of regions. In this context, game theory may be useful to study this in more detail.

2. The Solow Growth Model

This section is devoted to the standard growth model in economics, as developed by Solow (1956). This model is a neoclassical growth model which considers solely the real side of the economy. In particular, it investigates the relationship between the growth of the labour force, capital investment and total production within a "closed" economic system. As argued later on, the specified model is an exogenous growth model (which to some extent may be seen as a degenerated case of an endogenous growth model). For the sake of convenience, we will first offer a brief introduction to this general class of growth models.

In the one sector Solow growth model, firms produce a homogeneous good. All firms in the model operate in a perfectly competitive input and output market. These firms, J in total, are identical and act rationally (i.e. they maximize profit subject to the available input factors). Their objective function is assumed to have the following form:

$$\Pi_j = \max \int_0^{+\infty} \pi_j(t) e^{-\rho t} \, dt \tag{2.1}$$

where ρ is the discount factor which is equal for all firms, π the profit at time t and where Π_j denotes the aggregate profit of firm j over time. Flexible prices, "market clearing" and "perfect foresight" are (implicitly) assumed. It is taken for granted that capacity can fulfil demand. If K(t) represents the homogeneous input factor capital at time t and L(t) the homogeneous input factor labour, the total production by each firm j, $j \in J$, can over time be described by a production function $Y_j(t) = F(K_j(t), L_j(t))$, which is defined on the non-negative orthant of $\mathbb{R}^2$, further denoted as $\mathbb{R}_+^2$. Solow assumes a production function that generates constant returns to scale. One of the consequences of constant returns to scale is that the production function is homogeneous of degree 1. This makes it possible to apply the well-known Euler condition.

For the sake of completeness we will first define the relevant symbols in this part of the paper.

Y(t):	the optimal aggregate production function
C(t):	total consumption, with $C(t) > 0$
I(t):	total investment, with $I(t) \geq 0$
K(t):	the homogeneous production factor capital
L(t):	the homogeneous production factor labour
A(t):	the exogenous factor of technical progress
t:	time
Ω_t:	the information set at time t
w(t)	the wage rate at time t
r(t)	the interest rate at time t
μ:	rate of depreciation
s:	average propensity to save, with $0 < s < 1$
n:	growth rate of labour, with $n > 0$
L_0:	amount of labour available at $t = 0$, with $L_0 > 0$
$c(t) \equiv C(t)/L(t)$:	per capita consumption
$k(t) \equiv K(t)/L(t)$:	capital-labour ratio, with $0 \leq k(t) < +\infty$
$f(k(t)) = F(K(t)/L(t),1)$:	the production function in the per capita form
$y(t) \equiv Y(t)/L(t)$:	production per capita.

Where possible, we will use a prime (′) to indicate a first-order derivative and two primes (′′) for a second-order derivative.

With the use of the Euler condition the optimal aggregate production function can be derived as follows.

$$\sum_{j=1}^{J} [F_{K_j}(K_j(t), L_j(t)) \cdot K_j(t) + F_{L_j}(K_j(t), L_j(t)) \cdot L_j(t))] \\ \sum_{j=1}^{J} F(K_j(t), L_j(t)) = \sum_{j=1}^{J} Y_j(t) = Y(t) \tag{2.2}$$

with

$$F_{K_j} \equiv \frac{\partial F}{\partial K_j} \;\; ; \;\; F_{L_j} \equiv \frac{\partial F}{\partial L_j} \tag{2.3}$$

The aggregate production function in the economic system considered is thus $Y(t) = F(K(t), L(t))$. Constant returns to scale make it also possible to write the production function in a per capita form, i.e. $f(k(t)) \equiv F(K(t)/L(t),1)$. This function is well-defined over k(t). The ratio k(t) is called the capital-labour ratio.

We have the following standard equations for the Solow model.

$$Y(t) = F(K(t), L(t)) \tag{2.4}$$

$$Y(t) = C(t) + I(t) \tag{2.5}$$

$$I(t) = K(t)' + \mu K(t) \tag{2.6}$$

$$L(t) = L_0 e^{nt} \tag{2.7}$$

Important properties of F(K(t),L(t)) are:

□ $F(K(t),L(t))$ is at least C^2 on the interval (0,), (2.8)

which means that the production function is twice continuously differentiable.

□ $$\frac{\partial F(K(t),L(t))}{\partial L(t)} > 0 \ , \ \frac{\partial F(K(t),L(t))}{\partial K(t)} > 0 \ ,$$
$$\frac{\partial^2 F(K(t),L(t))}{\partial L^2(t)} < 0 \ , \ \frac{\partial^2 F(K(t),L(t))}{\partial K^2(t)} < 0 \tag{2.9}$$

□ $$F(\lambda K(t),\lambda L(t)) = \lambda F(K(t),L(t)) \tag{2.10}$$

where λ is a scalar. The properties (2.8) and (2.10) are by assumption.

Under some (additional) assumptions given in (2.11.a)-(2.11.f) Solow's basic exogenous growth model can be obtained. These assumptions are:

$$F(0) = 0 \tag{2.11.a}$$

$$f(k(t))' > 0, \ f(k(t))'' < 0, \ \forall k(t) \geq 0 \tag{2.11.b}$$

$$\lim_{k \to 0} f(k(t))' = +\infty \tag{2.11.c}$$

$$\lim_{k \to +\infty} f(k(t))' = 0 \tag{2.11.d}$$

$$\mu = 0 \tag{2.11.e}$$

$$L(t) > 0 \tag{2.11.f}$$

Assumptions (2.11.a)-(2.11.d) are called the Inada-conditions. These conditions guarantee the existence of an optimum for (2.1), which is a unique asymptotically stable global maximum. Assumption (2.11.e) is imposed by Solow to create a situation in which net investment equals gross investment; from (2.5) we find that $K(t)' = sY(t)$. In this situation, capital stock can only increase. The last assumption, (2.11.f), avoids that k(t) is improperly defined. Considering the assumptions (2.11.a), (2.11.c) and (2.11.d) simultaneously, it is clear that these eliminate the possibility that the economy vanishes regardless of the optimality condition.

It should be noticed that the value of K(t) in equation (2.4) is the amount of capital used, whereas in (2.6) the available capital stock is considered. Thus, it is usually assumed that capital is fully "employed". Similarly, the model assumes full employment of the production factor labour: L(t) in equation (2.4) is the amount of employed labour, while L(t) in (2.7) is the amount of available labour. Since $k(t) \equiv K(t)/L(t)$, with $0 \leq k(t) < +\infty$, is a non-fixed ratio due to the change in value of K(t) and/or L(t), it follows that factor substitution is allowed in the Solow growth model. Combining (2.3), (2.4) and (2.5) we obtain

$$F(K(t),L(t)) = C(t) + K(t)' + \mu K(t) \tag{2.12}$$

Given (2.11.e) and dividing (2.12) by L(t) results in:

$$k(t)' = f(k(t)) - nk(t) - c(t) \tag{2.13}$$

Since $c(t) = (1-s)[f(k(t)) - \mu k(t)]$ and given (2.11.e), it follows that:

$$sf(k(t)) = nk(t) + k(t)' \tag{2.14}$$

The optimum can be found by setting k(t)′ equal to zero. This yields an optimal solution, which we will denote by k^*, such that $k^* \in \mathbb{R}_+^2$. The balanced growth path is the path described by the value of k^* over time, or in symbols: $k^*\ \forall t,\ t \in [0, +\infty)$. Without solving the non-linear differential equation $sf(k(t)) = nk(t) +$
$k(t)'$, it can be shown that the Solow growth model generates a unique stable global maximum. If k^* is this maximum, then by using the Lyapunov function this property can easily be proven. The basis of the proof is that there should be two converging sequences over time. Following Brock and Malliaris (1989) let us assume $v(x) = 0.5x^2$, where $x = k(t) - k^*$, and where v(x) represents the distance between the two sequences. Then:

$$\text{if } \frac{\partial v(x(t))}{\partial t} = v(x(t))' \tag{2.15}$$

we may state that

$$\begin{aligned} &v(x(t))' = xx' = xk(t)' \Leftrightarrow \\ &x[sf(x+k^*) - n(x+k^*)] \leq x[sf(k^*) + xsf(k\ )\quad n(x\quad k\ )] \\ &x[sf(x+k^*) - n(x+k^*)] \leq x^2[sf(k^*)' - n] \Leftarrow \frac{x^2 n}{f(k\ )}[k\quad f(k\ )] < 0 \end{aligned} \tag{2.16}$$

In equation (2.14) the variable x is the difference in value between the capital-labour ratio at time t and the optimum k^*, while v(.)′ represents the behaviour of the difference $k(t)-k^*$ with respect to time. The Lyapunov function shows that for all initial values on the interval $(0, +\infty)$ the total investment, i.e. the value of k(t)′, converges over time to the real value k^*. Thus, k^* is an unique asymptotic stable global maximum. **Q.E.D.**

ANNEX 2 will consider the Lyapunov function more closely. Note that the inequality follows from the strict concavity of f(k(t)). In other words; the Solow growth model has a quasi-stable global equilibrium. This maximum, k^*, is an asymptotic attractor with the interval $(0, +\infty)$ as the immediate-basin-of-attraction (Nusse and Yorke, 1993).

It is noteworthy that Takayama (1991) has more recently shown that an infinite amount of time is needed to reach the value k^* in the Solow growth model. In fact, the value k^* is never exactly reached in this model. He also demonstrated that the decision process to find the optimal consumption for all t takes place over an infinite time horizon.

To conclude this part, the Solow growth model is a continuous time neoclassical growth model which has a unique quasi-stable global equilibrium under the given assumptions over the interval $[0, +\infty)$. This optimum, k^*, is an attractor with the interval $(0, +\infty)$ as the immediate-basin-of-attraction. A consequence of this type of equilibrium is that there exists a path (k(t),c(t)) that, *independent of the initial values of all variables, monotonically converges to a balanced growth path, i.e. k^*. On the balanced growth path, all variables of the tuple (L(t),K(t),Y(t),C(t),I(t)) grow at the same rate over time.* It is also possible to include in the standard growth model technical progress in explicit form. This does not alter the main conclusions (see ANNEX 1).

3. Exogenous Growth in a Multi-regional System

This section aims to derive the properties of our exogenous growth model in a multi-regional system. The analysis will be conducted in two steps: first a closed multi-regional system, followed by an open multi-regional system. The result will be based on optimization theory and we will use here a Dynamic Programming (DP) framework as specified in more detail in ANNEX 2. As ANNEX 2 makes clear, under plausible conditions DP can be applied to find the optimum in a growth model.

The multi-regional system we will consider is one of a simple kind. We neglect, for example, that regions are embodied in a larger (economic) structure. This structure may be a national economy or even the global economy at the final stage. Interesting in this case is the study of the "seamless world" from Krugman and Venebles (1995). Fujita and Krugman (1995) also take into account the structure of the economic geography. Their paper focuses on the relationship between manufacturing and agglomeration, which is an element we will leave out of consideration too.

The decision process over time in a multi-regional growth model with I regions ($i = 1,..,I$) and exogenous growth can be formalized within a general DP-form. The control variable will be the per capita consumption $c_i(t)$, while the capital-labour ratio $k_i(t)$ is treated as a state variable. To model properly the decision process of the actors over more than one subsequent time period, a discount factor ρ_i is required to weight the value of the different time periods. The actors' utility is represented by the variable $u_i(.)$. We will, without loss of generality, throughout the paper assume two regions, i.e. $I = 2$. Note that we are here considering the demand side of the market instead of the production side, as analyzed in the previous section. This is permissible, since the model generates a Pareto optimum (Maddala and Miller, 1989, p. 247), as a result of the rational behaviour of all actors. Assuming that the optimization takes place over a finite period of time and knowing the initial values (k_0) as well as the terminal values (denoted by k_T), the system to be optimized for two regions is the following one:

$$\begin{cases} W_1 = \max \int_0^T u_1(c_1(t))e^{-\rho_1 t}\, dt \\ W_2 = \max \int_0^T u_2(c_2(t))e^{-\rho_2 t}\, dt, \end{cases} \tag{3.1}$$

subject to

$$\begin{cases} f_1(k_1(t) - n_1 k_1(t) - c_1(t) = 0 \\ f_2(k_2(t)) - n_2 k_2(t) - c_2(t) = 0 \end{cases} \tag{3.2}$$

where the utility function is concave with respect to its domain. It should be noted that this model -in an open spatial system- allocates flows of inputs on the basis of the multitemporal utility expressions in the normative DP model, as this models considers only the real side of the multi-regional economy.

Considering the system (3.1)-(3.2), a region i will maximize its welfare under the constraint that

$$f_i(k_i(t)) - n_i k_i(t) - c_i(t) = 0 \tag{3.3}$$

The specific form of the constraint follows from the fact that $k_i(t)' = 0$.

Proof:

As Section 2 made clear, the following condition is valid at the optimum:

$$k_i(t)' = 0 \Rightarrow 0 = s_i f_i(k_i(t)) - n_i k_i(t)) \tag{3.4}$$

which can be rewritten as:

$$(1-s_i)f_i(k_i(t)) - f_i(k_i(t)) - n_i k_i(t) = 0 \Leftrightarrow f_i(k_i(t)) \quad n_i k_i(t) \quad c_i(t) \quad 0 \tag{3.5}$$

Q.E.D.

In words, the constraint for region i follows from the optimal value of $k_i(t)$, i.e. k_i^*. Since actors of region i optimize their welfare, we need to identify the optimal level of per capita consumption, i.e. $c_i(t)$. Note that $k_i(t)' = 0$ is a necessary but not a sufficient condition for obtaining the optimal value for $c_i(t)$. The optimal per capita consumption in time will be denoted by c_i^*. Since we are no longer interested in the optimal value of the capital-labour ratio solely, but rather in the k_i^* and c_i^* at the same point in time, we have to look for the tuple (k_i^*,c_i^*) instead of a tuple on the balanced growth path, i.e. $(k_i^*,c_i(t))$. Thereto, we need the following first-order condition:

$$\frac{\partial[c_i(t)]}{\partial[k_i(t)]} = \frac{\partial[f_i(k_i(t)) - n_i k_i(t)]}{\partial[k_i(t)]} = 0 \tag{3.6}$$

from which follows that

$$f_i(k_i(t)^{\prime} = n_i \tag{3.7}$$

This implies that the marginal physical product of capital, $f_i(k_i(t))'$, equals the growth rate of labour, n_i. If (2.11.b) holds for all i, and given (3.7), it is clear that there exists a unique global value for $k_i(t)$ which maximizes per capita consumption. Thus, a unique global value of the optimal consumption, c_i^* does exist. The tuple obtained, (k_i^*,c_i^*), is a point on a path known as the *golden rule* path. It is the path of all balanced growth paths which maximizes per capita consumption.

Rationally behaving actors maximize their welfare over time. The actors have an incentive to choose a level of consumption equal to c_i^* for each point in time. If they choose another level of per capita consumption, the loss of utility will be equal to

$$\int_0^T [u_i(c_i^*(t)) - u_i(c_i(t))]e^{-\rho t}dt \geq 0 \quad if \quad c_i^* \geq c_i(t) \quad t \tag{3.8}$$

As a consequence, for each level of per capita consumption different from c_i^* actors can always do better, so that such a deviation from c_i^* would mean a contradiction with (assumed) rational behaviour. It is thus clear, that actors have an incentive to reach a value on the golden rule path.

In conclusion, *when our growth model is in a Pareto optimum at all points in time, the values of the tuple are such that $(k_i(t),c_i(t)) = (k^*,c^*)$. As a consequence, the spatial economy is on the so-called golden rule path.*

Several observations can now be made about the consequences of actions that affect a single region or both regions due to spill-over effects. Throughout the paper we will assume that at $t=0$ the two regions considered in our analysis are completely identical and have the same absolute variable values; in symbols: $K_1(0) = K_2(0)$, $Y_1(K(t),L(t)) = Y_2(K(t),L(t))$, $L_1(0) = L_2(0)$, $n_1/s_1 = n_2/s_2$, $n_1 = n_2 \geq 0$, $u_1(.) = u_2(.)$, such that the first-order derivative of the utility function with respect to consumption is positive and the second-order derivative of the utility function with respect to the same variable is negative, i.e. the utility function is concave over c(t). Actors will optimize the system of equations of (3.1)-(3.2) under the restriction that $K_1(0) = K_2(0) \geq 0$ and $L_1(0) = L_2(0) > 0$ which implies that $k_i(0) \geq 0$ $\forall i$, with $i = \{1,2\}$. Now, several interesting cases, in total seven, will be studied. In the original Solow growth model the case of exports and imports are left out. Nevertheless, some remarks can be made about differences in growth between open economic systems. We will first assume the existence of two closed economic systems, which are closed in the sense that there does not exist an economically interrelated market.

Case 1: comparison of two closed regions

In Case 1 two closed economic regions will be compared with each other. Assume that the labour force growth as well as the discount factors are identical, in symbols:

$$0 < \rho_1 = \rho_2 \; , \; 0 < n_1 = n_2 \quad \forall \; t > 0 \qquad (3.9)$$

Assume also that the production function in both economic systems are identical. Comparing these systems it can first be shown that *a difference in the marginal propensity of saving leads to a difference in per capita consumption among both economic systems*.

Proof:

Let $c_i(t) = (1\text{-}s_i)f(k(t))$ with $i = \{1,2\}$ and $s_1 \geq s_2$; then the following equation is valid:

$$c_1(t) = (1-s_1)f(k(t)) \leq (1-s_2)f(k(t)) = c_2(t) \qquad \textbf{Q.E.D.} \quad (3.10)$$

Next, we will analyze the two economic systems, 1 and 2, in a slightly different setting. Assume that both regions are on the balanced growth path. Let these two regions have an identical production function, but different absolute values of their variables. Assume also that the ratio $n_1/s_1 = n_2/s_2$ is valid. Assume further that the regions differ in population growth rate and have also a different marginal propensity of saving. Compared to Case 1 we drop the assumption that $n_1 = n_2$ and that $s_1 = s_2$. If (the monetary variable) w(t) is the wage rate and (the monetary variable) r(t) the interest rate at each moment of time, as a consequence of the above mentioned assumptions both systems have the same ratios of:

$$\frac{K(t)}{L(t)} \; , \; \frac{Y(t)}{L(t)} \; , \; \frac{Y(t)}{K(t)} \; , \; w(t) \; , \; r(t) \text{ and } \frac{w(t)}{r(t)} \qquad (3.11)$$

Proof:

(a) Given the specification above, the difference between the absolute values of the two regions will be a scalar, for example, α. Since the production functions of both regions are assumed to be equal and have the property of constant returns to scale, it follows that: $k_1(t) = K_1(t)/L_1(t) = \alpha K_1(t)/\alpha L_1(t) = K_2(t)/L_2(t) = k_2(t)$. The occurrence of a situation in which the optimum value of region 1 is equal to the value of the optimum of region 2 depends on the exogenous growth rate (n) as well as the marginal propensity of saving (s) among the two regions at a certain point in time.

(b) For $i = \{1,2\}$, the equation $n_i k_i(t) = s_i f(k_i(t))$ can be rewritten as:

$$\frac{s_i}{n_i} \frac{L_i(t) f(k_i(t))}{L_i(t)} = k_i(t) \quad \text{for} \quad i = \{1,2\} \qquad (3.12)$$

By using $L_i(t)/L_i(t) = 1$, knowing that $n_1/s_1 = n_2/s_2$ as well as $Y_1(t) = Y_2(t)$ and reminding that $k_1(t) = k_2(t)$ from the proof under (a), it is clear that $Y_1(t)/L_1(t) = Y_2(t)/L_2(t)$.

(c) Given (a) and (b) it follows that:

$$\frac{Y_1(t)}{K_1(t)} \cdot \frac{L_1(t)}{L_1(t)} = \frac{Y_2(t)}{K(t)} \cdot \frac{L_2(t)}{L_2(t)} \Leftrightarrow \frac{Y_1(t)}{K_1(t)} = \frac{Y_2(t)}{K_2(t)} \qquad (3.13)$$

(d) Since the ratio $\frac{Y_1(t)}{K_1(t)} = \frac{Y_2(t)}{K_2(t)}$ is fixed, it follows now that $f_1(k(t))' = f(k_2(t))'$. As a consequence, the following result holds:

$$r_1(t) = r_2(t) \qquad (3.14)$$

(e) Following the same line of reasoning for the labour market results in:

$$f_1(k(t)) - f_1(k(t))' \, k_1(t) = f_2(k(t)) - f_2(k(t))' \, k_2(t) \qquad (3.15)$$

(f) Since $r_1(t) = r_2(t)$ and $w_1(t) = w_2(t)$, we can derive

$$\frac{w_1(t)}{r_1(t)} = \frac{w_2(t)}{r_2(t)} \qquad \textbf{Q.E.D.} \quad (3.16)$$

In Case 1, the per capita consumption is equal among the two regions, i.e. $c_1(t) = C_1(t)/L_1(t) = C_2(t)/L_2(t) = c_2(t)$. If $c_1(t) = c_2(t) \Rightarrow u_1(c_1(t)) = u_2(c_2(t))\ \forall t > 0$. In other words, if the per capita consumption is equal for the two regions and both regions have the same utility function and discount rate, the aggregate utility is equal for both regions concerned. This can be expressed in symbols as follows:

$$\lim_{t \to +\infty} u_1(c_1(t)) e^{-\rho_1 t} = \frac{u_1(c_1(t))}{\rho_1} = \frac{u_2(c_2(t))}{\rho_2} = \lim_{t \to +\infty} u_2(c_2(t)) e^{\;_2 t} \qquad (3.17)$$

So, at each point in time the utility for both regions is equal as a result of the equality of the per capita consumption in both regions for all t.

It can thus be concluded that *when two identically acting closed regions, which only differ in absolute variable values, are on the balanced growth path, both regions will generate the same (aggregate) utility over time. They also have an equal ratio of production with respect to each input, capital-labour ratio, wage as well as interest rate*.

Case 2: comparison of two closed regions with a different discount factor

In Case 2, the situation where two regions have a different discount factor will be examined. The conditions for Case 2 are:

$$0 < \rho_1 < \rho_2 \,,\ 0 < n_1 = n_2 \qquad \forall t > 0 \qquad (3.18)$$

The effect of differences in the discount rate over time is rather straightforward. It is clear that the following inequality is valid:

$$\lim_{t \to +\infty} u_1(c_1(t)) = \frac{u_1(c_1(t))}{\rho_1} < \frac{u_2(c_2(t))}{\rho_2} = \lim_{t \to +} u_2(c_2(t)) \qquad t > 0 \quad (3.19)$$

In words, when over the entire time horizon a region whose actors prefer future consumption more than actors in another region do, the aggregate utility of that region is negatively affected.

This case shows that *the accumulated utility over time of the region with the lowest discount rate is higher compared to an identically acting region with a higher discount rate*.

Case 3; a comparison of two closed regions with unequal population growth

In this case a situation is assumed in which the population of region 1 grows faster than that population of region 2. It is still assumed that both regions are closed. For our model this means, in symbols, that:

$$0 < \rho_1 = \rho_2 \; , \; 0 < n_2 < n_1 \quad \forall t > 0 \tag{3.20}$$

Since two identical, economically closed, regions with a different population growth are considered, this implies that

$$c_1(t) = f_1(k_1(t)) - n_1 k_1(t) < f_2(k_2(t)) - n_2 k_2(t) \quad c_2(t) \quad u_1(c_1(t)) < u_2(c_2(t)) \tag{3.21}$$

with the consequence that

$$0 < u_1(c_1(t)) < u_2(c_2(t)) \tag{3.22}$$

Thus, *a region with a population growth rate which is higher compared to another region will receive a lower utility at time t*.

In case of $n_2 < n_1$ for all values of $t > 0$, the aggregate utility of region 1 is lower than the aggregate utility of region 2. When t tends to $+\infty$, the aggregate utilities are:

$$\lim_{t \to +\infty} u_1(c_1(t)) e^{-\rho_1 t} = \frac{u_1(c_1(t))}{\rho_1} < \frac{u_2(c_2(t))}{\rho_2} = \lim_{t} u_2(c_2(t)) e^{\quad_2 t} \tag{3.23}$$

A large population results -given the amount of available consumption C(t)- in a lower per capita consumption. Since the utility is based on the per capita consumption, in our analysis the utility of the region with the smallest population group is higher than the utility of the region with the larger population.

The model specified at the beginning of this section makes it impossible to analyze the behaviour of actors of two *open* economic regions. To investigate a situation with open regions a mechanism is required that connects each region with the other region(s). A slight modification of the model (3.1)-(3.2) will make it possible to overcome this difficulty. Several options can be chosen to model such a mechanism. Here we will focus on the difference in the consumption (per capita) among regions on the one hand and the mobility of labour on the other. For a two-region model we will now redefine n_1 and n_2. Part of the redefinition is the introduction of the function which takes into account the difference in (total) consumption between the two regions; let this function be $\Xi(C_1(t)-C_2(t))$. By assumption, the properties of this function are:

$$\frac{\partial[\Xi(C_1(t)-C_2(t))]}{\partial[C_1(t)-C_2(t)]} > 0 \; , \; \lambda\Xi(C_1(t)-C_2(t)) = \Xi(\; C_1(t) \quad C_2(t)) \tag{3.24}$$

Since both consumption functions are continuous, $\Xi(.)$ is a continuous function. In a per capita form denoted by ξ, the function will be based on the difference d(t) in the per capita consumption among the two regions. In symbols:

$$d(t) = \xi(c_1(t)-c_2(t)) = [f_1(k_1)t))-n_1k_1(t)] \quad [f_2(k_2(t)) \; n_2k_2(t)] \tag{3.25}$$

For reasons to be explained later, we may place a scalar θ, $0 \leq \theta \leq 1$, in front of the function $\xi(.)$. This scalar reflects the degree of openness between the two regions concerned. Finally, the growth of the population will be based on exogenous population growth in the region, represented by ν_i. Given the exogenous population growth and the common labour market, we (re-)define the growth of the labour force in (3.2) as follows:

$$e^{n_1(t)t} = e^{[\nu_1 + \theta \cdot \xi(c_1(t) - c_2(t))]t} \; , \; e^{n_2(t)t} = e^{[\nu_2 + \theta \cdot \; (c_2(t) \quad c_1(t))]t} \tag{3.26}$$

In words, population growth within a region i, n_i, depends on the exogenous population growth, ν_i, and on the movement of labour into the region at hand or towards another region. The function $\xi(.)$ makes clear that *if the per capita consumption is lower in a region compared to another region, an amount of labour will move away* *from the region with the lower per capita consumption towards the region(s) with the higher per capita consumption*. It is easy to see that the movement of labour is bounded. Note that the value of $n_i(t)$ must be strictly positive. Given the Inada-conditions, a value of $n_i(t) > 0 \ \forall i \in I$ guarantees a solution in the non-negative orthant of $\mathbb{R}^I$ for $y_i(t) = f_i(k_i(t)) > 0$. Thus, when there is no movement of labour from one region towards other regions, the following condition must be satisfied:

$$\nu_i > 0 \quad \forall i \in I \tag{3.27}$$

It should be noted that the values for ν_i are exogenous in the model. If two regions are considered, the question is what can be said about the following three inequalities?

$$\nu_2 < \nu_1 \ , \ 0 < \nu_1 \ , \ 0 < \nu_2 \tag{3.28}$$

First, we notice that:

$$0 < \nu_1 - \nu_2 < \nu_1 \tag{3.29}$$

from which we can derive

$$0 < \theta \cdot \xi(c_1(t) - c_2(t)) \leq \xi(c_1(t) - c_2(t)) < \ _1 \quad n_1(t) \ \text{ for } 0< \ 1 \tag{3.30}$$

Thus, a movement of labour is limited to the size of the difference in the (exogenous) growth rate between the two regions and is, as a consequence, bounded. Looking at the definition of the function $\xi(c_1(t)\text{-}c_2(t))$, the incentive for actors to move towards another region is based on the difference in the per capita consumption among regions. Keeping in mind the formalized decision process of actors in (3.1)-(3.2), a lower per capita consumption leads to a lower utility at the same point in time. Since actors are assumed to act rational, i.e. they are assumed to maximize their utility, a move to a region with a higher utility is part of rational behaviour. A consequence of the movement of labour, in a two-region situation, is that labour will decline in the region with the larger population and rise in the region with the smaller population. As a consequence, the difference in per capita consumption between the two regions will be completely levelled off. At the time that the per capita consumption is equal among the two regions, the movement of labour stops, because the utilities of the regions have become equal in each region. Due to full information, the movement of labour from one region towards the other will not lead to overshooting in the model. Fluctuations of ν_i over time will neither lead to an instability of the system due to full information. A more explicit analysis of the two-region case, with an unequal population growth and allowing for labour mobility, will be presented in Case 4.

Case 4: comparison of two open regions with unequal population growth

In Case 4, the system (3.1)-(3.2) has to be optimized, in which n_1 and n_2 are replaced by $n_1(t)$ respectively $n_2(t)$. In the analysis of our multi-region model with two regions, several possible settings can now be considered.

(a) First, if $\theta = 0$, no labour movement is possible among the regions. As a consequence, the results of Case 2 and 3, which considered closed regions, are again obtained.

(b) The above mentioned scalar θ can have the range $0 < \theta \leq 1$. When $\theta = 1$, there is a free interregional movement of labour. If $0 < \theta < 1$, the flow of labour is constrained. As a consequence, the different positions in welfare between the two regions will last more than one point in time. In fact, a lag is introduced into the model. This implies that the desire to adjust to the needed level of inputs cannot be entirely fulfilled at the point in time the difference in utility emerges, since $0 < \theta < 1$. Thus, with $0 < \theta < 1$ more than one point in time is required to level off the difference in utility among the regions.

Let us now assume that at a point in time $\tau \in t$ the population of region 1 grows faster than that of region 2, i.e. $\nu_1 > \nu_2$. This implies that the per capita consumption of region 1 is smaller than that of region 2, $c_1(t) < c_2(t)$. As a consequence of the definition of the function $\xi(.)$, the value of $n_1(t)$ will decline due to the outflow of labour. Since

$$n_1(t) = \nu_1 + \theta \cdot \xi(c_1(t) - c_2(t)),\ n_2(t) = \nu_2(t) \qquad (c_2(t) \quad c_1(t)) \qquad (3.31)$$

the outflow of labour from region 1 equals the inflow of labour in region 2. Due to this inflow of labour, the value of $n_2(t)$ will rise. This rise will last until $n_1(t)$ and $n_2(t)$ are equal in value. It depends on the value of θ whether the movement of labour between the two regions can take place right away, i.e. at the point in time the deviation in the exogenous growth rate occurs. If θ has a value between 0 and 1, a levelling off of the difference in population growth is not possible at a single point in time. More than one discrete time period is required to reach an equal population size among regions. Since the inequality in population size remains for more than one point in time, the per capita consumption remains unequal among these regions for this time. As a consequence, the utility in the smallest region is higher compared to the larger region. Since the aggregate utility is the sum of the utilities at each point in time, this affects the aggregate utility of both regions. The smaller region will benefit from the delay in the movement of labour. It prefers a value of θ close to 0. Since we consider an open multi-region model, a parameter value $\theta = 0$ cannot be chosen; as a consequence, the region with the smaller population growth will instead prefer the lowest possible value for θ. The region with a larger population growth prefers an immediate movement of labour to a delayed movement, but will choose under a situation of open regions as a second best solution a value of θ close to 1.

It can now be concluded, that *a mechanism that allows at all points in time an immediate movement of labour between two open regions with an unequal population growth leads to an aggregate utility which is equal among the two regions. A restriction on the movement of labour will cause a difference in per capita consumption among the two regions for more than one point in time; the aggregate utility of a region, which is the sum of the utility at each point in time, is affected by the deviation in the per capita consumption among the regions. Thus, when the two regions have a open labour market, the region with a small population growth prefers a long lasting delay in the movement of labour. The region with a larger population growth prefers a very fast movement, i.e. a free movement of labour.*

4. Endogenous Growth in a Multi-regional System

We will now turn to the case of endogenous growth in a spatial system. A very interesting study on long-term endogenous growth from a neoclassical perspective with convergence implications was recently offered by Barro and Sala-I-Martin (1995). We refer in this context also to Carlino and Mills (1996) and Kohno and Ide (1993).

Before we present the optimization process for a multi-regional endogenous growth model, the optimization process for a single-region situation will be presented. Thereto, we will proceed with an extensive presentation of an endogenous growth model based on the recent study of Nijkamp and Poot (1997). In their study, the technical progress is endogenized. As the mentioned at the end of Section 2, technical progress does not necessarily alter the main conclusions of the basic Solow model. Thus, without any problem we may use such a form of technical progress here. Interesting in this context is the study of Romer (1986), who analyzed long-run economic growth in a competitive equilibrium model with endogenous technical progress, based on increasing returns to scale.

Since technical progress is labour-augmenting, the homogeneous production factor labour will be of the following form:

$$L(t) = A(t)e^{nt} \quad \text{with} \quad A(t) \equiv e^{gt} \qquad (4.1)$$

where the rate of technical growth g depends on the stock of knowledge. Let the variable N(t) measure the effective labour input of region i such that

$$N(t) \equiv L(t)B(t) \tag{4.2}$$

where L(t) is the amount of employed workers and B(t) an index representing the average quality of labour input. The index B(t) depends on the total stock of knowledge and training. In this model the accumulation of knowledge is assumed to take place in the following way:

$$B(t)' = H\left(\frac{R(t)}{L(t)}, B(t)\right) \tag{4.3}$$

where the rate of change in human capital is dependent on the ratio R(t)/L(t) (i.e., the total expenditure per worker for improving the capabilities of the working force), as well as on the existing stock of knowledge. The function H(.) is supposed to be identical for each region. Now we assume that the total expenditure R(t) per unit of time is a (constant) fraction of national income Y(t), or in symbols:

$$R(t) = mY(t) \qquad m,\ R(t),\ Y(t) \in \mathbb{R}_+ \tag{4.4}$$

The function H(.) is assumed to be at least twice continuously differentiable (i.e. C^2), homogeneous of degree one and concave. The relationship with the constituent variables is rather straightforward, as is illustrated in (4.5):

$$\frac{\partial H(.)}{\partial R(t)} > 0\ ,\ \frac{\partial H(.)}{\partial B(t)} > 0\ ,\ \frac{\partial H(.)}{\partial m} > 0\ ,\ \frac{\partial H(.)}{\partial L(t)} < 0 \tag{4.5}$$

Given the properties of H(.), it is possible to rewrite and substitute the function into the Solow growth model:

$$\frac{B(t)'}{B(t)} = H\left(\frac{R(t)}{L(t)B(t)}, 1\right) = H(my^{@}(t), 1) \equiv h(mf(k^{@}(t))), \tag{4.6}$$

with $y^{@}(t) \equiv Y(t)/N(t)$ and $k^{@}(t) \equiv K(t)/N(t)$, where the scalar m can be interpreted as the marginal propensity of saving of the region to finance the investment in human capital. Clearly, $0 \leq m \leq 1$. Actors of a closed economic region will maximize their utility over time. The optimization process of a closed endogenous single-region model can be formalized, by assuming an objective function of the form analogous to (3.1):

$$W = \max \int_0^T u(c(t))e^{-\rho t}\, dt \tag{4.7}$$

subject to

$$f(k^{@}(t)) - nk^{@}(t) - h(mf(k^{@}(t))) - c(t) = 0 \tag{4.8}$$

As a result, in the model two decision variables are found, viz. the marginal propensity of saving (like in the exogenous growth model) and additionally the value of m. For given values of s and m, the effective capital intensity, k(t), is represented by the following equation (provided $\mu = 0$):

$$k_i^{@}(t)' = s_i f_i(k_i^{@}(t)) - [n_i + h(m_i f_i(k_i^{@}(t)))]k_i^{@}(t) \qquad i\ I \tag{4.9}$$

Like in the exogenous variant of the Solow growth model, for a single-region endogenous growth model the long run equilibrium can be found without solving the non-linear differential equation. Since h(.) is concave and given the conditions specified for the exogenous growth model, a unique asymptotic stable global equilibrium exists. The immediate-basin-of-attraction is equal to that of the exogenous growth model.

In conclusion, *a single-region endogenous growth model has a unique asymptotic stable global equilibrium*.

The immediate-basin-of-attraction of the single-region endogenous growth model is identical to the one of the exogenous growth model.

Now we start discussing a multi-regional growth model, based on the Nijkamp and Poot (1997). All variables introduced above will receive a subscript i, indicating the region under consideration, i.e. region i. The optimization process of a closed endogenous multi-regional model consisting of I regions can be formalized by adding a subscript i to equations (4.7)-(4.9). The long run equilibrium can then be found for each region i in an analogous way. For the same reason, a unique asymptotic stable global equilibrium exists also for the multi-regional case.

To analyze next a two-region endogenous growth model, we start from the same point where the analysis of the exogenous model started. Like in the previous section, we assume that at $t = 0$ there exist two identical regions under the following conditions: $Y_1(K(t),L(t)) = Y_2(K(t),L(t))$, $K_1(0) = K_2(0)$, $L_1(0)T_1(0) = L_2(0)T_2(0)$, $n_1/s_1 = n_2/s_2$, $n_1 = n_2 \geq 0$, $u_1(.) = u_2(.)$ such that the utility function is concave over its domain, $c_i(t)$.

Compared to the system (3.1)-(3.2), defined for multi-regional exogenous growth in Section 3, the only difference is that the scalars m_1 and m_2 are introduced here. Like we have done in the previous section, a set of several simple but systematic cases on a multi-regional system will now be dealt with.

Case 5: investment in human capital in one of the two closed regions

Case 5 will use the same conditions as specified in Case 1, i.e.,

$$0 < \rho_1 = \rho_2 \;,\; 0 < n_1 = n_2 \quad \forall t > 0 \tag{4.10}$$

To show the effects of an investment in human capital, we assume that at a certain point in time τ, $\tau \in t > 0$, region 1 chooses a value of m_1 such that $0 = m_2 < m_1 < 1$. At this point in time it is obvious that:

$$c_1(t) = f_1(k_1(\tau)) - n_1 k_1(\tau) - h(m_1 f_1(k_1(\tau))) < f_2(k_2(\;)) \quad n_2 k_2(\;) \quad h(m_2 f_2(k_2 \qquad \because \; 0 = h(m_2 f_2(k_2(\tau))) < h(m_1 f_1(k_1(\;)) \tag{4.11}$$

If utility is solely defined over the variable consumption, then at τ the following inequality is valid:

$$0 < u_1(c_1(\tau))e^{-\rho_1 \tau} < u_2(c_2(\tau))e^{-\rho_2 \tau} \tag{4.12}$$

Let the scalar delta, δ, be a non-negative small number. This scalar represents formally the length of a time period. Due to the increase of B(t), the production function for region 1 yields a higher production compared to the initial production function. More of the homogeneous good can be produced with a certain mix of K(t) and L(t). Let an investment made in time τ become productive in $\tau + \delta$; then

$$f_2(k_2(\tau + \delta)) = f_1(k_1(\tau + \delta)) < f_1^{\$}(K_1(\tau + \delta)) \tag{4.13}$$

where $f_1^{\$}(.)$ describes the production function after the investment in human capital by region 1. A rise in labour productivity means that with a lower amount of input factors the same level of production can be reached. So, it seems plausible to rewrite (4.13) as follows:

$$f_2(k_2(\tau{+}\delta)) = f_1(k_1^{@}(\tau{+}\delta)) \quad \text{with} \quad k_1^{@}(\tau{+}\delta) < k_2(\quad) \tag{4.14}$$

The expression (4.13) shows that an investment in human capital leads to an upward shift of the production function, while (4.14) indicates that for the same level of production less of the input factor labour is required. We prefer to use (4.13) for our analysis here. We assume now that after the investment in time τ no further investment in human capital takes place, i.e. $0 = m_1 = m_2$ for $\forall t > \tau$. At $\tau + \delta$, the consumption of region 1 exceeds that of region 2 at the same moment of time, since:

$$c_1(\tau+\delta) = f_1^{\$}(k_1(\tau+\delta)) - n_1k_1(\tau+\delta) > f_2(k_2(\tau+\ )) \quad n_2k_2(\quad) \quad c_2(\quad) \tag{4.15}$$

In general, a region i will only invest in human capital if

$$\int_{\tau}^{T} u_i^{\$}(k_i(t))e^{-\rho t}\,dt - \int_{\tau}^{T} u_i(k_i(t))e^{-\rho t}\,dt \geq 0 \tag{4.16}$$

where $u_i^{\$}(.)$ denotes the utility obtained after the investment in human capital. For our two-region model this means that region 1 will invest in human capital, if

$$\int_{\tau}^{T} u_1^{\$}(k_1(t))e^{-\rho t}\,dt \geq \int_{\tau}^{T} u_1(k_1(t))e^{-\rho t}\,dt = \int_{\tau}^{T} u_2(k_2(t))e^{\quad t}\,dt \tag{4.17}$$

Equation (4.17) states that the aggregate utility of region 1 will at least be equal to the aggregate utility obtained in the initial situation (without an investment in human capital). In this case, the investment generates a utility gain which can solely offset the total cost of investment, i.e. the break-even point is reached at the final period of time T. Since we only consider the real side of the economy, the cost of investment will be the loss of utility. When (4.16) is (strictly) positive, an investment in human capital will generate a rise in utility that exceeds the loss caused by the investment. As a consequence, the aggregate utility of region 1 as well as the global utility will rise. This loss is the difference between the utility obtained after the investment and the utility that could have been obtained without an investment over the same period of time. If (4.16) has a value equal to 0 at a point in time $\tau+\chi$, with $\tau < \tau+\chi < T$, the aggregate utility of region 1 will rise compared to the situation without an investment in human capital. The aggregate utility of region 1 will be

$$W_1 = \int_{0}^{\tau} u_1(k_1(t))e^{-\rho t}\,dt + \int_{\tau}^{\tau+\chi} u_1^{\$}(k_1(t))e^{-\rho t}\,dt \ - \ \int^{T} u_1^{\$}(k_1(t))e^{\quad t}\,dt \tag{4.18}$$

The last term of (4.18) measures the gain of utility generated by the investment in human capital. In view of (4.16), this term must have a value equal to or greater than 0. It is obvious that the second term at the right-side of (4.18) is equal to 0 at time $\tau+\chi$. Region 2 has an aggregate utility that is equal to

$$W_2 = \int_{0}^{\tau} u_2(k_1(t))e^{-\rho t}\,dt + \int_{\tau}^{\tau+\chi} u_2(k_1(t))e^{-\rho t}\,dt + \int^{T} u_2(k_2(t))e^{\quad t}\,dt \quad \int_{0}^{T} u_2(k_2(t))e \tag{4.19}$$

Thus we may conclude that *a region will only invest if the utility obtained after the investment equals or exceeds the utility obtained from a situation without an investment. When the gain exceeds the cost of investment, it follows that an investment in human capital leads to an increase of the aggregate utility in the region in which the investment takes place. A rise in the aggregate utility of the investing region means at the same time a rise in global welfare.*

Given the set-up for the single-region endogenous growth model in the beginning of this section, an *open* multi-regional system with endogenous growth consisting of two identical regions can be modelled in the following way:

$$\begin{cases} W_1 = \max \int_0^T u_1(c_1(t))e^{-\rho t}\, dt \\ W_2 = \max \int_0^T u_2(c_2(t))e^{-\rho t}\, dt \end{cases} \tag{4.20}$$

subject to

$$\begin{cases} f_1(k_1^@(t) - n_1(t)k_1^@(t) - h(m_1 f_1(k_1^@(t))) - c_1(t) \quad 0 \\ f_2(k_2^@(t)) - n_2(t)k_2^@(t) - h(m_2 f_2(k_2^@(t))) - c_2(t) \quad 0 \end{cases} \tag{4.21}$$

with $k_i^@(t) \equiv K_i(t)/N_i(t)$, $N_i(t) \equiv L_i(t)T_i(t)$ and under the condition that $n_1(t)$, $n_2(t)$, ν_1 and ν_2 have a value greater than 0. The function $\xi(.)$ is as defined before. For our analysis we use the same starting point as in the previous cases, so that $K_1(0) = K_2(0) \geq 0$, $L_1(0) = L_2(0) > 0$, $N_1(0) = N_2(0) > 0 \Rightarrow k_i^@(0) \geq 0\ \forall i$, with $i = \{1,2\}$. Assume that at $t = 0$ two regions exist under the following conditions: $K_1(0) = K_2(0)$, $L_1(0) = L_2(0)$, $n_1/s_1 = n_2/s_2$, $n_1(t) = n_2(t) \geq 0$, $Y_1(K(t),L(t)) = Y_2(K(t),L(t))$, $u_1(.) = u_2(.)$ such that the utility function is concave over $c_i(t)$. If then $m_1 = m_2 = 0$ $\forall t$, the analysis of Case 1 of Section 3 is valid, since this value of m implies for the entire time period exogenous growth. The situation in which m_1 is unequal to m_2 will be treated in Case 6.

Case 6: investment in human capital in one of the two identical open regions

In Section 3 we have seen that a difference in growth rates among two regions will be levelled off, when a movement of labour is possible. The effects of a mechanism that allows for a movement of labour among two regions will now be studied.

Assume that ν_1 and ν_2 have a very low value and that δ is a very small, non-negative number. Let an investment yield a gain that exceeds the total cost of investment. Given the size of the population at time τ, an investment in human capital by region 1 at τ leads to a reduction of the available consumption and thus to a lower per capita consumption. Due to the function $\xi(.)$, an outflow of labour from region 1 towards region 2 will take place until the utility is equal among both regions. Given the definition of the utility function, an investment in human capital at time τ will cause a reduction in global welfare, since

$$[(u_1(\tau) - \Delta u_1(\tau)) - u_2(\tau)]e^{-\rho\tau} < [u_1(\tau) - u_2(\tau)]e \tag{4.22}$$

where $\Delta u_1(.)$ denotes the volume of loss of utility due to the investment in human capital at τ. We will denote the sum of utilities by $u^g(.)$ and the utility of region 1 after levelling off by $u_1^-(.)$, where $u_1^-(.)$ has a value on the open interval $u_1^- \in ((u_1(.) - \Delta u_1(.)), u_1(.))$. Looking at the left-hand side of (4.22), it is clear that the global utility, which we will denote by $u^g(.)$, will have a value $(u_1(.) - \Delta u_1(.)) + u_2(.) = u^g(.) < u_1(.) + u_2(.)$. The right-hand side of (4.22) represents the global utility, if no investment would have been undertaken by region 1. If at $\tau + \delta$ the investment becomes productive, it will lead to an increase in global utility, i.e.,

$$[(u_1(\tau + \delta) + \Delta u_1(\tau + \delta)) - u_2(\tau + \delta)] > [u_1(\quad) \quad u_2(\quad)] \tag{4.23}$$

We will now write $u_1^+(.)$ to denote the utility of region 1 after levelling off. It is clear that the utility of region 1 has a value such that $u_1^+(.) \in (u_1(.), u_1(.) + \Delta u_1(.))$. The rise of utility in region 1 will cause a movement of labour from region 2 towards region 1. The size of the labour flow from region 2 towards region 1 at $\tau + \delta$ exceeds the movement from the reverse movement at τ, due to the increase in global welfare. As (4.24) shows, the size of the labour flow is in the first place caused by the rise of utility from $u_1^-(.)$ to its original level, $u_1(.)$, and in the second place by the gain in (global) utility as a result of the increased productivity in region 1, i.e.

for the initial level of production less labour input is required.

$$\int_{\tau}^{\tau+\delta}[u_1^+(t)-u_1^-]e^{-\rho t}dt = \int_{\tau}^{\tau+\delta}[u_1(t)-u_1^-]e^{-\rho t}dt + \int_{\tau}^{\tau+\xi}[u_1^+(t)\ u_1(t)]e^{\ t} > 0 \quad u_1(t) > \tag{4.24}$$

Equation (4.24) means that the utility of region 1 exceeds the utility of region 2 from $\tau+\delta$ on. As a consequence, region 1's gain of the investment will partly decline due to the increased population size at $\tau+\delta$ compared to τ, and before. Region 2 will definitely gain from the investment made by region 1, since the gain in utility in region 1 leads to a movement of labour from region 2 towards region 1 at $\tau+\delta$ which is larger than the reverse movement at τ. Finally, notice that only when an investment can take place without cost, the point in time where the rise in productivity takes place, $(\tau+\delta)$, equals the point in time where the break-even point is reached, i.e. $(\tau+\chi)$ in our model.

Thus, it can so far be concluded, that *when one of the two identical open regions -open in the sense of a common labour market- invests in human capital, it will have to share the gain with the non-investing region due to the spatial mobility of labour. Since an investment will only be initialized, if it at least offsets the loss caused by the investment, it follows that an investment in human capital leads to a rise in global welfare.*

If the population is limited in its movement to the region with the higher utility, i.e. $0 < \theta < 1$, this will have some implications for the results derived above. Considering an extreme case in which $\theta = 0$, it simply follows that no levelling off can take place in utility among the two regions in time τ. In our specified model with two identical regions, the lack of levelling off has no affect on the reduction in global welfare at τ. Since the investing region, region 1, loses more utility when $\theta = 0$ compared to a value of $\theta = 1$, region 1 prefers an immediate movement of labour at τ. The opposite is true for the non-investing region. At $\tau+\delta$ the preferences are reversed. The investing region prefers a situation without an inflow of labour to keep the entire gain of utility from the investment in human capital. Region 2 benefits only from the investment made by region 1, if its population size is lower compared to the time τ where the investment by region 1 took place. Region 2 prefers a immediate movement of labour towards region 1. Given the decision problem as defined below, it follows that region 1 prefers a value of θ close to 1, i.e. a high level of spatial mobility, if

$$\int_{\tau}^{\tau+\delta} u_1(t)e^{-\rho t}dt + \int_{\tau+\delta}^{T} u_1(t)e^{-\rho t}dt < 0 \tag{4.25}$$

and a very low spatial mobility of labour, if

$$\int_{\tau}^{\tau+\delta} u_1(t)e^{-\rho t}dt + \int_{\tau+\delta}^{T} u_1(t)e^{-\rho t}dt > 0 \tag{4.26}$$

The preferences of the non-investing region are exactly the opposite of the preferences of the investing region.

In conclusion, *if the gain in utility due to the investment in human capital exceeds the related loss, an investment in human capital by a region is initiated. In a two-region model with a common labour market where only one of the two regions invest, the investing region prefers a very slow spatial movement of labour among the regions. The non-investing region prefers in this situation a high speed of outflow of labour in order to benefit as much as possible from the labour-augmenting investment undertaken by the other region.*

It should be recognized that the optimal point in time to invest in human capital remained undetermined so far. Game theory may offer here a quick insight into the solution to this problem. The final case of our paper will show that some form of competition among the regions will determine the optimal point in time an investment will take place (Tirole, 1988). Investigating the timing of adoption of a new technology in a broader sense than we will has been studied by Koski and Nijkamp (1996).

Case 7: competing open regions and the timing of investments

If there exists an economic link between two regions, a form of competition may show up in the model. An interactive mechanism that connects two economic regions may then lead to the introduction of a game-theoretic setting. This may, for example, be used to consider the effects of the diffusion of a technology within a region or among regions. In our analysis thus far, we have considered the possibility of a movement of labour from one region to the other. Now, firms of both regions produce the homogeneous good at a constant unit cost. Assume that at $t=0$ the unit cost, mc, is equal among the regions, i.e. $mc_1=mc_2$. Neither firm makes a profit due to the market form of perfect competition which is present in both regions. The relationship between both regions, based on the physical flow of goods, is thus one of a Bertrand duopoly. In this context the technical (labour-augmenting) innovation reduces the unit cost to MC such that $MC<mc_i$, with $i=\{1,2\}$. Let the expected profit of the adoption of the innovation be $P=(mc_i\text{-}MC)e^{-rt}$.

Over a bounded time horizon, the value of the profits tends towards $(mc_i\text{-}MC)/r$, where r denotes the interest rate, when $T\in t$ is sufficiently large. Assume that, in an extreme case, only one region can adopt the new technology. It is obvious that the region which adopts the new technology first becomes the "leader". The profit that each firm of the innovating region i makes per unit of product is $mc_i\text{-}MC_i$. If both regions can adopt the innovation, this does not benefit one region more compared to the other, if the implementation of the new technology takes place at the same time t. Due to the cost of adoption of a new technology, an introduction may turn out to be non-profitable. Let the break-even point where the profit from the new technology, P, equals the cost of implementation, IC(t), be at time t^c. Adopting the new technology by a region after t^c will turn out to be non-optimal, while the other region can do better by adopting the new technology slightly earlier. The other region becomes in that case the leader. Clearly, an adoption before t^c will not come into consideration. As a consequence, both regions will adopt the new technology at the same time t^c, leaving no gain for the consumers or to both regions. The pay-offs of the firm remain zero after the adoption of the new technology from t^c on.

The game-theoretic analysis mentioned above considers prices, while our growth models are based on the real-side of the economy. Nevertheless, the result of the game-theoretic analysis has already been outlined in (4.18) and (4.19). In the first place, the cost of investment in Case 5 and 6 is the loss of welfare by the investing region. Secondly, the other region in a two-region model will lose welfare too, if there is a flexible common labour market (negative externality). If the welfare loss is a sunk cost, it is clear that an early investment needs a longer time before the investment generates a positive return, as is shown by the last term of (4.18). It is clear that each region has the desire to invest in human capital. Such an investment will take place at a point in time τ equal to $t=0$. If both regions will invest in human capital, it turns out that the investment for region 1 as well as for region 2 will take place at $t=0$. Let there be a situation in which a region is willing to invest, i.e. (4.16) is strictly positive. Each region wants to invest at $t=0$, since

$$\int_0^{\tau}[u_i^{\$}(t) - u_i(t)]e^{-\rho t}\,dt > 0 \tag{4.27}$$

In words, a region can do better by investing as early as possible, because the utility derived from the investment is strictly higher than that obtained without an investment. If we choose to investigate endogenous growth from a situation in which less inputs are required to produce the initial quantity of output, the same desire to invest from $t=0$ on exists, since

$$\int_0^{\tau}[u(c_i^{@}(t)) - u(c_i(t))]e^{-\rho t}dt > 0 \tag{4.28}$$

where $c_i^{@}(t)$ is per capita consumption after the investment. This follows from (4.14) which results in the inequality

$$c_i(t) = f_i(k_i(t)) - n_i k_i(t) < f_i(k_i^@(t)) - n_i k_i^@(t) \quad c_i^@(t) \tag{4.29}$$

Equations (4.7)-(4.8) together with the analysis of Case 7 show that rationally behaving actors have an incentive to maximize their consumption in the first place, and that they will invest in human capital from $t = 0$ on. Note that the level of investment in human capital in bounded. This is clear when an extreme case is studied in which there is no growth of the labour force and no consumption at a certain period in time, i.e.,

$$f_i(k_i^@(t)) - h(m_i f_i(k_i^@(t))) = 0 \tag{4.30}$$

Thus, the investment per unit of time can be at most equal to the volume of production at this point in time. Since consumption will be positive (otherwise the utility is equal to 0) the investment in human capital will be lower than the volume of production per unit of time. This result is valid given the assumption of rational behaviour of actors. As Case 6 has shown, if region 2 does not invest, it will gain from the investment in human capital by region 1. Nevertheless, region 2 can have a higher aggregate utility when it initiates an identical investment. Like region 1, region 2 will invest at $t = 0$, due to (4.27). Both regions will then benefit from the common investment in human capital. Since both regions invest at $t = 0$, no movement of labour will take place among the regions. As a consequence, no levelling off, neither of the loss nor of the gain, among the regions will then occur. Since we have assumed that (4.16) is positive, aggregate utility will rise. Compared to the situation in which only one region invests, this rise in collective utility exceeds the rise of utility obtained in Case 6. The rise in total utility, e(t), for our two-region model will in case of economic similarity of regions with the same to be equal to:

$$e(t) = 2 \cdot \int_{\tau+\chi}^{T} u^{\$}(t) e^{-\rho t} \tag{4.31}$$

where each region contributes equally to the gain in global utility. Region 1 receives the full gain of utility and is clearly better off. It is also clear, that in Case 7 under equal regional behaviour the value of θ has no effect on the results of the analysis. Since identical regions act simultaneously, no movement of labour will take place for any value of t.

From Case 7 it can be concluded, that *when two regions compete on a cost reducing innovation, the implementation of the new technology takes place as soon as possible. Two competing identical regions will invest at the same point in time, i.e. at $t = 0$. Due to their identical behaviour, no movement of labour among the regions will take place. As a consequence, the loss of global welfare is higher, since, firstly, the loss cannot be levelled off among the regions, and secondly, both investing regions contribute to the loss in global welfare instead of one region. Each region obtains an aggregate utility that exceeds the aggregate utility from a situation in which no investment would be initialized. The rise of global welfare is twice the utility gain one single-region can generate. When only one region invests, for example due a patent, it faces a levelling off of the losses as well as a utility gain resulting from the investment. Given the results of Case 6, this means that the non-investing region will obtain a part of this gain, i.e. its utility rises even when it does not invest at all.*

5. Concluding Remarks

In this paper the decision process of competitive actors and regions with perfect information has been considered. The analysis was based on dynamic optimization strategies. Several interesting cases have been investigated, which generated a fascinating series of intriguing theoretical results. They might to be tested empirically in subsequent research. The main conclusion of our paper is that the assumption of rational behaviour on the consumer-side as well as on the supply-side implies that the optimum will be on the golden rule path. On this path the capital-labour ratio and the per capita consumption are maximized. Further, we have explicitly demonstrated that there is an incentive to invest in endogenous growth at the very first period in time, i.e. $t = 0$. We were also able to show that in case of a Cobb-Douglas production function the model ensures convergence towards a unique quasi-stable global equilibrium.

Any form of uncertainty will turn the deterministic model into a stochastic model. Interesting will be a situation in which two open regions face a fluctuation in the exogenous population growth on the one hand and a free mobility of labour on the other hand in a situation of choice uncertainty. As long as the actors use adaptive expectations, the DP approach can be used to find the optimal adjustment process in an uncertain environment. When actors form their expectations rationally, the DP model can no longer be used to derive the optimal adjustment path, since it is based on expectations formation in an adaptive way. Kydland and Prescott (1977) have proven this for an economic system based on difference equations. A proof for a continuous time (growth) model has not yet been given thus far, nor have the consequences for the actor's decision process in the (growth) models been investigated.

Clearly, there is also a need to analyze the dynamics of decision-making within a multi-regional growth model. Another issue of interest is the study of different types of production functions in relation with multi-sectoral growth models. Finally, we have only considered a two-region growth model. A multi-sectoral growth model with more than two regions is a subject that also requires more study in the future, especially in the case of structurally different regional economies.

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ANNEX 1

The Standard Solow Growth Model with Technological Change

To model the impact of a productivity rise caused by technological progress a similar line of reasoning can be followed as the one used for modelling the decision process of the actors in the exogenous growth model. The impact of a productivity rise can be modelled as follows: $Y(t) = F(A(t),K(t),L(t))$ with $K(t) \geq 0$, $L(t) > 0$ and a derivative of A(t) with respect to t of

$$\frac{dA(t)}{dt} > 0 \qquad \text{(A1.1)}$$

where A(t) is the rate of productivity change as a result of technical progress. Variables used in the traditional Solow growth model which are affected by the labour-augmenting technical progress will be marked by the symbol #. The utility of the actors in the one sector growth model is represented by the variable u(.). Assuming that the optimization takes place over a finite time and knowing the initial values (k_0) as well as the terminal values (denoted by k_T), the optimization problem can be written as follows:

$$W = \max \int_0^T u(c^{\#}(t))e^{-\rho t}dt \tag{A1.2}$$

subject to

$$k^{\#}(t)^{/} = f(k^{\#}(t)) - nk^{\#}(t) - mk^{\#}(t) - c^{\#}(t) \quad 0 \tag{A1.3}$$

with $k^{\#}(0) = k_0^{\#}$, $k^{\#}(T) = k_T^{\#}$, $k_0^{\#} > 0$, $k_T^{\#} > 0$, $k^{\#}(t) > 0$, $c^{\#}(t) > 0$ and $\rho \geq 0$, where the utility function has the same properties as formulated by the exogenous growth part. It is obvious that for a technical progress the scalar m must have a value greater than 0. This can be rewritten in a Hamiltonian function. If $p_i(t)$ are the Hamiltonian multipliers, which are piecewise continuous derivatives, such that $p_0(T) = 1$, $p_0(t) = 1 \forall t$ and $p_i(T) = 0$ for $i = 1,2,..,I$, then this results in:

$$H(k^{\#}(t),c^{\#}(t),p_i(t),t) \equiv u(c^{\#}(t))e^{-\rho t} + p_i(t)[f(k^{\#}(t)) \quad (n \quad m)k^{\#}(t) \quad c^{\#} \tag{A1.4}$$

since

$$H(c^{\#}(t),u^{\#}(t)^{*},p(t),t) \geq H(c^{\#}(t),u^{\#}(t),p(t),t) \quad u^{\#}(t) \; U^{\#} \tag{A1.5}$$

where U is the set of possible utility functions. It is clear that

$$u(c^{\#}(t))e^{-\rho_i t}+p(t)[f(k^{\#*})-(n+m)k^{\#*}-c^{\#}(t)] \leq u(c^{\#}(t))e^{\quad_i t} \; p(t)[f(k^{\#}) \; (n$$
$$\forall\, c^{\#}(t) \geq 0,\; 0 \; t \; T,\; i \quad 1,2 \tag{A1.6}$$

Since labour is the driving force in the exogenous Solow model, a labour-augmenting technical progress is chosen in our analysis. This means that we have chosen for a production function of the generalized form $Y(t) = F(K(t),A(t)L(t))$. Formalizing the production function in this way the technical progress affects only the production factor labour. For example, labour becomes more productive due to the application of a new technology. For the traditional Solow growth model we have chosen to model the effect of a change in technology by assuming that $A(t) = e^{mt}$. Each type of technical progress will cause an upward shift of the production function, i.e. given a certain amount of inputs more can be produced. Several grounds can be found that causes that $d[A(t)L(t)]/dt > dL(t)/dt$. For example, education or a learning-process on-the-job. Another possibility is a market force, e.g. competitive actors may take measures to increase the productivity of labour. Since such actors do not appear in Solows exogenous growth model, we will introduce them here. Assume that market actors aim to increase the labour productivity on the one hand and to have a balanced growth economy on the other hand. These actors are assumed to follow the market rules or to act in a "discretionary" way. First of all, the assumption of perfect information remains valid, though this assumption may be dropped. How will the adjustment process takes place and is it different compared to the one found in the exogenous model with the same k_0? Does there still exist a stable equilibrium? These questions will be answered in the remaining part of this paper. First, we (still) assume that the production function with labour-augmenting technical progress has constant returns to scale. This makes it possible to keep on using the per capita notation, i.e.

$$y^{\#}(t) = f(k^{\#}(t)) \equiv F(k^{\#}(t),1) \tag{A1.7}$$

Compared to the previous exogenous growth model, an adjusted model then emerges:

$$Y^{\#}(t) = F(K(t),L^{\#}(t)) \qquad (A1.8)$$

$$Y(t) = C(t) + I(t) \qquad (A1.9)$$

$$I(t) = K(t)^{/} + \mu K(t) \qquad (A1.10)$$

$$L^{\#}(t) = L_0 e^{(n+m)t} \qquad (A1.11)$$

$$k^{\#}(t) \equiv \frac{K(t)}{L^{\#}(t)} \qquad (A1.12)$$

The properties of $F(K(t),L^{\#}(t))$ are analogous to this given in Section 2, and are assumed to be the following:

- □ $F(K(t),L^{\#}(t))$ is at least C^2 on the interval$(0,\ \ )$ (A1.13)

- □ $$\frac{\partial F(K(t),L^{\#}(t))}{\partial L^{\#}(t)} > 0\ ,\ \frac{\partial F(K(t),L^{\#}(t))}{\partial K(t)} > 0\ ,$$ $$\frac{\partial^2 F(K(t),L^{\#}(t))}{\partial L^2(t)} < 0\ ,\ \frac{\partial^2 F(K(t),L^{\#}(t))}{\partial K^2(t)} < 0 \qquad (A1.14)$$

- □ $F(\lambda K(t),\lambda L^{\#}(t)) = \lambda F(K(t),L^{\#}(t))$ (A1.15)

Like in Solow's exogenous growth model, the production function is twice continuous differentiable and λ is a scalar.

$$F(0) = 0 \qquad (A1.16.a)$$

$$f(k^{\#}(t))^{/} > 0,\ f(k^{\#}(t))^{//} < 0,\ \ \forall k^{\#}(t) \geq 0 \qquad (A1.16.b)$$

$$\lim_{k^{\#}\to 0} f(k^{\#}(t))^{/} = +\infty \qquad (A1.16.c)$$

$$\lim_{k\to +\infty} f(k^{\#}(t))^{/} = 0 \qquad (A1.16.d)$$

$$\mu = 0 \qquad (A1.16.e)$$

$$L^{\#}(t) > 0 \text{ and } f(+\infty) < +\infty \qquad (A1.16.f)$$

Following the same procedure as applied to the first exogenous model and knowing that $sY(t) = K(t)' = I(t)$, it follows that $sf(k(t)) = (n+m)k(t) + k(t)'$. The optimum can be found by setting $k(t)'$ equal to zero. This yields the optimum $k^{\#*} \in \mathbb{R}_+^2$. The balanced growth path for the endogenous growth model is the path described by the value of $k^{\#*}$ over time, or in symbols: $\bar{k}^{\#*}\ \forall t,\ t \in [0, +\infty)$. When $A(t) = e^0$, the optimal capital-labour ratio of the exogenous growth model of Section 2 is equal to the optimal capital-labour ratio derived in this section, in symbols: $k^{\#*} = k^*$. This is rather obvious since for a value of $A(t) = e^0 = 1$ a situation with no technical progress is obtained. Given a certain amount of capital and labour at a point in time t, only for values of $m > 0$, i.e. $A(t) > 1$, a rise in productivity will be found. Important is that *a capital-augmenting or a neutral-augmenting technical progress, in which both inputs are affected equally by A(t), generate the same results as the labour-augmenting technical progress does*.

It can be concluded that the impact of *a productivity rise leads to the same results as the standard exogenous Solow growth model*. Like in the model of Section 2, the model with a labour-augmenting technical progress has a unique quasi-stable global equilibrium somewhere on the interval $(0, +\infty)$ under the given assumptions. This optimum, $k^{\#*}$, is an attractor with the interval $(0, +\infty)$ as the immediate-basin-of-attraction. A consequence of this type of equilibrium is that there exists a path $(k^{\#}(t), c^{\#}(t))$ that, independent of the initial values of all variables, monotonically converges to a balanced growth path, i.e. $k^{\#*}$. On the balanced growth path, all variables of the tuple $(L^{\#}(t), K(t), Y(t), C(t), I(t))$ grow at the same rate over time. If and only if $A(t) = e^0 = 1$, which is a situation without technical progress, both long run equilibria are equal in value, i.e. $k^{\#*} = k^*$. *For a situation with capital-augmenting or a neutral-augmenting technical progress the same conclusions can be drawn as for labour-augmenting technical progress*.

ANNEX 2

A Dynamic Programming Formulation of the Solow Growth Model

This annex will interpret the Solow growth model in an appropriate dynamic programming (DP) context. In the Solow growth model the adjustment process towards the balanced growth path is not derived in an explicit form, since the non-linear differential equation is not solved. Nevertheless, something more can be said about this adjustment process. The chosen route towards the equilibrium depends on the decision made by the actors of a single region in the model. In this part of the paper, the decision process and the adjustment process that it initiates, will be analyzed. Let, at $t = 0$, K_0 be the initial capital stock, L_0 the initial size of the labour force and $k_0 \equiv K_0/L_0$ the capital-labour ratio that follows from these initial values. Two types of initial situations can theoretically appear in the model. First of all, at $t = 0$ the initial value for k(t) on $(0. +\infty)$ is equal to k^*. If k_0 can be set equal to k^* at $t = 0$, the optimum is already reached from the very beginning. Following the balanced growth path from $t = 0$ on means that the optimal path in the Solow growth model is followed along the entire time-span. Limitations of any kind on the one hand and/or initial (starting) values of $k_0 \neq k^*$ on the other hand may make it impossible to attain this optimal growth path for all t. Then an adjustment process needs to be initiated to reach this optimal path. Note that when every element from the open interval $(0, +\infty)$ has k^* as limit point and when the limit of k^* equals the limit point, it follows that the function f(k(t)) is continuous over this interval.

To simplify the analysis, and leaving the structure of the problem definition of the traditional Solow growth model unaffected, we assume that k^* is a unique stable global attractor. We assume also that the balanced growth path can be reached by an adjustment process over a finite time interval T which starts at $t = 0$, with $T \in t$. As a consequence, the interval $(0, +\infty)$ can be split-up into two intervals; $U = (0, k^*]$ and $V = [k^*, +\infty)$. This can be done, since no "overshooting" can take place in the new setting due to the two previously mentioned additional assumptions about the attractor and the time-path. It is clear that $U \subset \mathbb{R}^+$ and that $V \subset \mathbb{R}^+$. For decision-making an unsolvable problem appears on the scene due to the fact that $\mathbb{R}$ is an uncountable set. When $\mathbb{R}$ is an uncountable set, then this also holds for $\mathbb{R}_2^+$, since $\mathbb{R}_2^+ \subset \mathbb{R}_2$. From this follows that the sub-intervals $U \subset \mathbb{R}^+$ and that $V \subset \mathbb{R}^+$ are uncountable subsets. This means that actors face an uncountable infinite set of decision possibilities (adjustment routes) for $\forall k_0 \in (0, k^*)$ as well as for $\forall k_0 \in (k^*, +\infty)$.

To overcome this problem, we will make the Solow growth model less theoretical and more realistic. It is obvious that not all values of k_0 can appear in the real world, for example, due to the measuring systems used.

It can be defended that the amount of each homogeneous input factor in the production process must be an element of the set of natural numbers, i.e. $L(t),K(t)\in\mathbb{N}$. The set of natural numbers is a "countable" infinite set. Note that since $L(t),K(t)\in\mathbb{N}$, the ratio $K(t)/L(t)=k(t)\in Q^{+}$ (where Q^{+} is the set of non-negative rational numbers). As a result the values of k(t) are elements of a countable infinite set too! Thus Q^{+} is a countable set. As a consequence actors have a countable infinite amount of possible adjustment paths to choose from. Since a part of the original domain $\mathbb{R}^{+}$ of the Solow growth model is used, due to $L(t),K(t)\in\mathbb{N}$, only a subset of the original range remains. Consequently, the plotted continuous curves in the space f(k(t))xk(t) will become discontinuous. This causes no problem, since the set which contains the values of the discontinuous curve is a subset of the set which contains the entire curve. Thus, the range after the first iteration is feasible in the Solow growth model. If the trajectory of the adjustment process consist of points that are feasible in the real world on the one hand and the non-linear general solution of the equation $sf(k(t))=nk(t)+k(t)'$ on the other, this will lead to a set of points obtained after each necessary iteration of which each element is a subset of Q^{+}. This set is, in fact, a Cantor-set. Let W denote this Cantor-set. Then the global attractor is not necessarily an element of this set, since the limit of k^{*} may not exist due to the non-continuous function f(k(t)). If $k^{*}\notin W$, an alternative point can be the best possible (second-best) result that can be reached. Let S be a countable set containing all feasible initial points with the possible adjustment paths to the optimum k^{*}. It is obvious that the set S is not necessarily non-empty, i.e. it is possible that there exists no adjustment path towards the unique asymptotic stable global maximum.

Let us make a step aside and consider a Cobb-Douglas function. This example will give more detailed insight into the adjustment paths within the Solow exogenous growth model. By choosing this form of the Solow growth model we can find the optimum by solving the differential equation. The production function $Y(t)=K(t)^{\alpha}L(t)^{1-\alpha}$ can be written in a per capita form, since the sum of the exponents equals one. In symbols:

$$y(t) = k(t)^{\alpha} \tag{A2.1}$$

so that

$$k(t)' = sk(t)^{\alpha} - nk(t) \tag{A2.2}$$

Now let $m(t)\equiv k(t)^{1-\alpha}$. This results in a differential equation of a linear form, i.e. $m(t)'+n(1-\alpha)m(t)=s(1-\alpha)$. When D is the integration constant, the solution of this differential equation is:

$$k(t) = \left[De^{-n(1-\alpha)t} + \frac{s}{n}\right]^{\frac{1}{1-\alpha}} \quad \text{with} \quad D = \left[k_0 - \frac{s}{n}\right] \tag{A2.3}$$

With this type of function more can be said about the adjustment paths. Since the sequence of a given k_0 tends towards a real number, the sequence is convergent. The real number is equal to

$$\lim_{t\to+\infty} D = k_0\left[\frac{s}{n}\right]^{\frac{1}{1-\alpha}} < +\infty \tag{A2.4}$$

As this expression indicates, every initial value which is an element of the interval $(0,+\infty)\subset\mathbb{R}$ will generate a convergent sequence. Let k_0 and k_1 be starting values of k(t) at $t=0$ such that $k_1-k_0>0$ and $k_0,k_1\in U$ or $k_0,k_1\in V$. Using the adjustment path described by k(t), it follows that:

$$\lim_{t\to+\infty} (k_1 - k_0)\left[e^{-n(1-\alpha)t} + \frac{s}{n}\right]^{\frac{1}{1-\alpha}} = 0 \tag{A2.5}$$

The difference between two convergent sequences is a convergent sequence too! Looking at (A2.5) the term $[e^{-n(1-\alpha)}+(s/n)]^{1/(1-\alpha)}$ is a monotonically decreasing function over time in the non-negative orthant of $\mathbb{R}$. It is clear that the initial ordering of k_1 and k_0 does not change over time. In words, the ordering of the adjustment paths remains equal over time within the Solow growth model when the production is modelled by a Cobb-Douglas function. Since k_0-$k_1 > 0$ can be written as -(k_1-$k_0 < 0$), it is not necessary to prove the situation in which $k_1 < k_0$. The result of (A2.5) is comparable with the result of (2.13) in Section 2. Looking at (2.12), the variable x is the difference in value between the capital-labour ratio at t and the optimum k^* while $v(.)'$ represents the behaviour of the difference $k(t)$-k^* with respect to time.

The adjustment path can be such that the resulting values for K(t) and L(t) are not in $\mathbb{N}$. Removing these points from the set of initial starting points after each iteration will result in a Cantor set. As a consequence, the value of k^* may not be a member of the set with feasible solutions. It is also possible that no adjustment path exists under these conditions. Given the ordering of, for example, the Cobb-Douglas function, a second-best solution can be found. Let the set S^f contain all feasible adjustment paths within the Solow growth model. In the (countable) set S^f there exists at least one $s \in S^f$ which represents the optimal adjustment path. This element of S^f will be denoted by s^*. This path minimizes the "cost" of the adjustment process. Assume that only the costs necessary for the adjustment process other than the difference in the optimal k^* and the actual k(t) is skipped in our analysis. Denote the elements which describes the adjustment path s by s_i, $s_i \in W$. The Solow model based on the Cobb-Douglas function has an asymptotic attractor too! An infinite period of time is thus needed for each initial value of k(t) unequal to the optimum to "reach" the equilibrium. Assume that the adjustment process requires a time period $T \in t$, where T is sufficiently large. Initially, the path of adjustment is described by a continuous non-linear differential equation which consists of an infinite number of elements. Since we remove all non-feasible results after each iteration, we end up with a discontinuous range of which each element belongs to Q^+. Again, Q^+ is a countable infinite set. The finite time period follows from the elimination of the asymptotic feature of the attractor. This is necessary, as otherwise the optimum k^* can never exactly been reached. What can be said about the optimal adjustment path towards k^*? Given any value of k_0, $k_0 \in (0, +\infty)$, there is at least one path $s \in S^f$ that describes the optimal adjustment path to the optimum, i.e. the balanced growth path. Denote this optimal adjustment path by s^*, $s \subset W$. This optimal path is described by the solution of the differential function $sf(k(t)) = nk(t) + k(t)'$.

The following remark can be made concerning the adjustment process. Values of k_0 can be found on the interval $0 \le k_0 < +\infty$. Due to the assumption that no depreciation can take place in the growth model, the adjustment on $k^* < k(t) < +\infty$ can solely be generated by a rising value of L(T). Since the growth of L(t) is exogenous to the actors, the adjustment can be seen as "passive". Initial values of k(t) left from k^* can be changed by investments of actors as well as by the growth of the labour force. Thus this may be seen as an "active" adjustment process. Given the assumption of perfect foresight in the first place and the market structure in the second place, the optimal adjustment path will be chosen until k^* is reached. When this is the case, the balanced growth path will be followed from there on. It should be noted that the structure of the above defined traditional growth model with perfect competition among the actors makes a game-theoretic situation not possible.

For the adjustment process the initial value k_0 plays a crucial role. The path that describes the adjustment over time depends on k_0. If k_0 and k^* are seen as state variables, it is clear that the optimal path s^* can be obtained using dynamic programming (DP). Applying DP is possible due to the assumptions of perfect foresight and full information.

Solow's growth model is a time-invariant model with perfect foresight which makes it possible to apply DP to derive the optimal adjustment path towards the optimum k^*, given the initial value k_0. In such a situation the continuous time Bellman equation must be used. In this way actors can derive the optimal sequences of states, given the initial value, k_0, and the dynamic system. What happen if $k^* \notin W$, given the Cobb-Douglas function and exogenous growth? In the case k^* can never be reached, since it is not a member of the domain/range of the Cantor set. Which value is now the second-best optimum? We have proven that the ordering of the adjustment paths depends on the initial values and this will remain so after subsequent iterations. Nevertheless, the distance between the adjustment paths decreases and tends to 0, if t goes to $+\infty$. If $k_1 < k_0$ and if $[k^*$-$k_i^N]$ is the distance between the optimum and the N-th iteration of the initial value k_i, $i \in \{0,1\}$, then it is clear that, if time $T < +\infty$ and if $k_i^N \in W$, the value k_i^N that is nearest to k^* is the second best optimum.

Given the ordering of the Cobb-Douglas function, it is clear that $[k^*-k_0^N] < [k^*-k_1]$. Thus, the closer the initial value to k^*, the better the optimum k^* can be approached over a finite time as $k^* \notin W$. This holds also under overshooting.

DP, thus models the decision process over subsequent stages to obtain a sequence of decisions which define the optimal policies for each point in time. Thereto, a performance measure is needed. For our problem this measure will be minimizing the cost generated by the adjustment process. In general notation, the performance of the system is represented by a measure of the form:

$$W = \min \ [h(k(T),T) + \int_0^T g(k(t),K(t))dt] \tag{A2.6}$$

where h(.) describes current state of the system and g(.) the decision chosen to transform this state to the next state. L(t) is given (i.e. exogenous), while K(t) is a controllable variable. k(t) is the state variable in the optimization process. Note that f(k(t)) is a monotonic transformation of k(t). $k(t)' = h(c(t),k(t))$ is the general notation of the adjustment process. From this follows that K^* is the optimal control variable and as result k^* is the optimal trajectory. K(t) causes that

$$W^* \doteq h(k(T)^*,T) + \int_0^T g(k(T)^*,K(t)^*)dt \leq h(k(T),T) + \int_0^T g(k(t),K(t))dt \tag{A2.7}$$

Thus, there exists at least one optimal value for K(t) that minimizes the cost of the adjustment process. We are interested in a "global minimum" or, in different words, in a unique adjustment path, given the initial value k_0.

Under the assumption of perfect information, it is evident that an "open-loop" DP-procedure yields the same results compared to those obtained by a "close-loop" DP-procedure; see Buiter (1981). The information available at time t is $\Omega(t)$. Let the function $\mathcal{K}(.)$ describe the relation between the information on one hand and the control variable on the other hand. This means that

$$K(t) = \mathcal{K}_t(\Omega(t)) \ \text{ with } \mathcal{K}_t \equiv \mathcal{K}(t), \ \ t \in [0,T] \tag{A2.8}$$

Due to the assumption of perfect foresight it follows that $\Omega(0)$ is equal to $\Omega(t)$ $\forall t$. All information is known a priori. Feedback of information does not lead to new insights into the decision process after $t=0$. Thus, the optimal adjustment path chosen at $t=0$ is optimal $\forall t$. This knowledge is needed when a government will become an actor in the model or when the assumption of perfect information is dropped.

The decision process over time in a one region growth model with exogenous growth can be formalized within a DP-form. Thereto, several changes will be made compared to the general form of the Bellman equation presented above. The control variable will be the consumption c(t), while the k(t) is treated as a state variable. For modelling the decision process of the actors over more than one subsequent time period properly, a discount factor is required to weight the value of the different time periods. Let ρ denote the value of time. Actor's utility is represented by the variable u(.). Assuming that the optimization takes place over a finite time and knowing the initial values (k_0) as well as the terminal values (denoted by k_T), then the optimization problem can be written as follows:

$$W = \max \int_0^T u(c(t))e^{-\rho t} \, dt \tag{A2.9}$$

subject to

$$k(t)^{/} = f(k(t)) - nk(t) - c(t) \qquad \text{(A2.10)}$$

with $k(0) = k_0$, $k(T) = k_T$, $k_0 > 0$, $k_T > 0$, $k(t) > 0$, $c(t) > 0$, $\rho \geq 0$ and such that the utility function is concave over its domain, c(t). This can be rewritten in a Hamiltonian function. If $p_i(t)$ are the Hamiltonian multipliers, which are piecewise continuous derivatives, such that $p_0(T) = 1$, $p_0(t) = 1$ $\forall t$ and ${}_i p(T) = 0$ for $i = 1,2,..,I$, then this results in:

$$H(k(t),c(t),p_i(t),t) \equiv u(c(t))e^{-\rho t} + p_i(t)[f(k(t)) \quad nk(t) \quad c(t)], \qquad \text{(A2.11)}$$

since

$$H(c(t)^{*},u(t)^{*},p_i(t),t) \geq H(c(t),u(t),p_i(t),t) \quad u(t) \; U \qquad \text{(A2.12)}$$

where U is the set of possible utility functions. It is clear that

$$u(c(t))e^{-\rho t} + p(t)[f(k(t)^{*}) - nk(t)^{*} - c(t)] \quad u(c(t))e^{\;\; t} \quad p(t)[f(k(t) \qquad \text{(A2.13)}$$
$$\forall \;\; c(t) \; 0, \; 0 \; t \; T$$

It can now formally be concluded that *when the decision process of actors of the Solow growth model is explicitly modelled, the final result depends on the initial value. Dynamic Programming (DP) to derive the optimal strategy for the actors requires a finite set of adjustment paths. Introducing the set of decisions possible in the real world guarantees a finite set of adjustment paths. As a consequence of a finite set of possible adjustment paths, this can create a situation in which a non-optimal, or second-best, adjustment path exists towards the balanced growth path (i.e. the global attractor) over a finite time horizon. This result is opposite to the traditional Solow growth model which has an infinite amount of adjustment paths that approaches the attractor over an infinite time interval.*