AN ADJUSTMENT PROCESS FOR NONCONVEX PRODUCTION ECONOMIES

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Abstract: We consider a slightly adapted version of the general equilibrium model with possibly nonconvex production technologies presented by Villar (1994). Typical for such models is that the behaviour of a producer is modelled by a pricing rule that relates market prices and production vectors - a combination to which we refer as the market condition - with a set of acceptable prices for this producer. We prove the existence of a path of market conditions that links any arbitrarily chosen market condition with an equilibrium market condition. At an equilibrium market condition all markets are cleared and all producers accept the market prices. The adjustment of the market prices and production quantities along the path can be given some economic interpretation as a tatonnement process. Along this process the market prices are adjusted according to the sign of the excess demands on the underlying markets, and the production quantities according to the difference between market prices and acceptable prices. The existence theorem holds for any semi-algebraic version of the model, i.e. all sets and mappings in the model can be described by polynomial (in-)equalities. Any path connecting the initial market condition with an equilibrium market condition can be approximated arbitrarily close by applying a simplicial algorithm. By restarting this algorithm in a different market condition, we may find more than one equilibrium.

Key words General equilibrium, nonconvex production, semi-algebraic economy, globally convergent adjustment process, simplicial algorithm.

1 Introduction

We consider a standard model of an exchange economy with nonconvex production technologies that is based on Villar (1994) where, given a combination of market prices and production vectors, - a combination to which we refer as a market condition -, each consumer determines his utility maximizing bundle of commodities and each producer determines whether the market price vector is acceptable to him or not. The behaviour of the producer is therefore modelled by a pricing rule that relates a set of acceptable price vectors to every market condition. In an equilibrium market condition the market prices clear the commodity markets and they are acceptable to all the producers.

In order to justify the use of any equilibrium concept in an economic model, it is relevant to provide an underlying dynamic motivation, for example by proving the existence of a tatonnement process that results in this equilibrium. Our main theorem states that, starting from any market condition, there exists a path of market conditions converging to an equilibrium, under the assumptions that the model is semi-algebraic. The latter condition roughly means that all sets and functions in the model can be described by poiynomial (in-)equalities. We remark that the analysis remains valid if all sets and functions are finitely sub-analytic (see Blume and Zame (1992)).

The path of market conditions can be interpreted as being generated by a tatonnement process. The adjustment of the market prices resembles the price adjustment process introduced in van der Laan and Talman (1987). Along the path of market conditions generated by the price-quantity adjustment process, the market prices of the commodities in excess demand (excess supply) are maximal (minimal) relative to their initial value, while the market prices of the commodities in equilibrium are allowed to vary between this lower and upper bound such as to keep these markets in equilibrium. The adjustment of the production quantities is similar. Each producer compares the market price vector with his nearest acceptable price vector. We refer to the latter price vector as the producer's reference price. If the market price of a commodity is higher (lower) than the corresponding reference price for a producer, then the production quantity of this commodity in his production technology is kept relatively maximal (minimal). If the market price of a commodity equals the corresponding reference price, then its production quantity in this producer's technology is allowed to vary between this lower and upper bound such as to keep the market price acceptable.

In this way we have explicitly formulated a price-quantity adjustment process, contrary to Brown (1991) and Villar (1994) who only informally mention such a process. Another adjustment process for this type of economy is given in Kamiya (1988). This process initially adjusts the production quantities until all producers face acceptable prices. Subsequently, the markets are brought into equilibrium while keeping the prices acceptable for all producers. The path generated by this adjustment process can be approximated by a simplicial algorithm, as explained in Kamiya (1991).

The path of market conditions generated by our price-quantity adjustment process can be approximated by applying the simplicial algorithm introduced in Doup and Talman (1987). This algorithm and the fact that the process can be started anywhere allows us to compute possibly more than one equilibrium. In this respect we improve upon Kamiya (1988) where the set of initial market conditions is limited. The ability to compute more than one equilibrium obtains its relevance from policy analysis and the occurrence of inefficient equilibria.

We state the model in Section 2. Section 3 then describes the price-quantity adjustment process and provides our main convergence theorem, while we also address the problem of practically following the path. Finally, in Section 4 we illustrate the performance of the algorithm in two simple examples where the producer's behaviour is characterized by respectively marginal cost pricing and average cost pricing. In these examples, we show that our algorithm can find all equilibria.

2 The model

Let us first introduce some notation. For any integer $k > 0$, I_k denotes the set $\{1, \ldots, k\}$, $\mathbb{R}^k_+$ ($\mathbb{R}^k_-$) denotes the nonnegative (nonpositive) orthant of the k-dimensional Euclidean space, whereas $\mathbb{R}^k_{++}$ denotes its strictly positive orthant. For any subset $A \subset \mathbb{R}^k$, $\text{int}(A)$, bd(A), and *co(A)* denote the interior, the boundary, and the convex hull of A respectively, and they are defined with respect to the affine hull of A. The vector with all components equal to zero is denoted by $\underline{0}$, and e denotes the vector with all components equal to one. The dimension of these vectors will be clear from the context. The n-dimensional unit simplex $\{x \in \mathbb{R}^{n+1}_+ \mid e^\top x = 1\}$ is denoted by $\mathcal{S}^n$.

In this paper we use the concept of semi-algebraicness. A semi-algebraic set in $\mathbb{R}^k$ is a finite union of sets of the form

$$\{x \in \mathbb{R}^k \quad f_1(x) = 0, \ldots, f_a(x) = 0; v_1(x) < 0, \ldots, v_d(x) < 0\},$$

where f_h, $h \in I_a \cup \{0\}$, and v_h, $h \in I_d \cup \{0\}$, are polynomials with real coefficients. Let A and B be semi-algebraic sets. The correspondence $F : A \to B$ is semi-algebraic if its graph is a semi-algebraic set. For more details on the concept of semi-algebraicness, we refer to Blume and Zame (1992).

We consider an exchange economy E with $n + 1$ commodities, c consumers, and m producers, indexed by ℓ, i, and j respectively. The commodity prices are represented by a price vector $p \in \mathbb{R}^{n+1}_+ \setminus \{\underline{0}\}$.

Each producer j is characterized by a tuple $(Y^j, \bar{\phi}^j)$, where the set $Y^j \subset \mathbb{R}^{n+1}$ denotes his production set and the mapping $\bar{\phi}^j$ denotes the pricing rule that describes his economic behaviour. A production plan of producer j is represented by a vector $y^j \in Y^j$. We assume that Y^j is closed, $\underline{0} \in Y^j$, and Y^j satisfies free disposal, i.e. $Y^j - \mathbb{R}^{n+1}_+ \subset Y^j$. Under these assumptions, the set of weakly efficient production plans for producer j equals $\mathrm{bd}(Y^j)$. The whole economy's production plan is given by the $m(n+1)$-tuple $y = (y^1, \ldots, y^m) \in Y$, where Y denotes $\prod_j Y^j$. Hence, the set of economically relevant production plans in the economy equals $\prod_j \mathrm{bd}(Y^j)$.

In the sequel, a market condition consists of a price vector $p \in \mathbb{R}^{n+1}_+ \setminus \{\underline{0}\}$ and a production vector $y \in \prod_j \mathrm{bd}(Y^j)$. Producer j's pricing rule $\bar{\phi}^j : (\mathbb{R}^{n+1}_+ \setminus \{\underline{0}\}) \times \prod_j \mathrm{bd}(Y^j) \to \mathbb{R}^{n+1}_+ \setminus \{\underline{0}\}$ relates the market condition (p, y) to the set $\bar{\phi}^j(p, y)$ of price vectors being acceptable to producer j. We assume that, for each producer j, the pricing rule $\bar{\phi}^j$ is an upper hemicontinuous correspondence with nonempty, closed, and convex values.

The current definition of a pricing rule is a generalization of well-known pricing rules such as marginal (cost) pricing and average cost pricing. Under a marginal (cost) pricing rule, a price vector is acceptable to producer j at production vector $y^j \in \mathrm{bd}(Y^j)$ if y^j fulfills the first-order conditions for profit maximization at this price vector p. The marginal (cost) pricing rule can then be denoted by a mapping $\overline{\mathrm{MC}}^j : \mathrm{bd}(Y^j) \to \mathbb{R}^{n+1}_+ \setminus \{\underline{0}\}$, defined for producer j as

$$\overline{\mathrm{MC}}^j(y^j) = \{p \in \mathbb{R}^{n+1}_+ \setminus \{\underline{0}\} \mid p \in \mathcal{N}_{Y^j}(y^j)\},$$

where $\mathcal{N}_{Y^j}(y^j)$ denotes the Clarke normal cone to $\mathrm{bd}(Y^j)$ at y^j (see Quinzii (1992)).

The average cost pricing rule states that all prices where producer j makes zero profits, i.e. breaks even, at production bundle $y^j \in \mathrm{bd}(Y^j)$ are acceptable to producer j. This pricing rule can be denoted by a mapping $\overline{\mathrm{AC}}^j : \mathrm{bd}(Y^j) \to \mathbb{R}^{n+1}_+ \setminus \{\underline{0}\}$, for producer j given by

$$\overline{\mathrm{AC}}^j(y^j) = \{p \in \mathbb{R}^{n+1}_+ \setminus \{\underline{0}\} \mid p^\top y = 0\}.$$

Each consumer i is characterized by a tuple (X^i, u^i, ω^i), with $X^i \subset \mathbb{R}^{n+1}$ his consumption set, $u^i : X^i \to \mathbb{R}_+$ his utility function, and $\omega^i \in \mathbb{R}^{n+1}_+$ his initial endowments. More specifically, consumer i is assumed to have a consumption set $X^i = \mathbb{R}^{n+1}_+$, a continuous, strictly quasi-concave, and strictly monotone utility function u^i, and a vector of initial endowments $\omega^i >> \underline{0}$. The distribution of the initial endowments over the consumers is denoted by $w = (\omega^1, \ldots, \omega^c)$. The total endowments in the economy are denoted by $W = \sum_{i=1}^{c} \omega^i$.

Let the set of attainable allocations in the economy be given by

$$\mathcal{A} = \left\{(x, y) \in \mathbb{R}^{c(n+1)}_+ \times Y \mid \sum_{i=1}^{c} x^i - W \leq \sum_{j=1}^{m} y^j \right\}.$$

Then we assume that, for every $x = (x^1, \ldots, x^c) \in \mathbb{R}^{c(n+1)}_+$, the set of attainable productions of producer j is compact. This implies that it is not possible for producer j nor for the whole economy to obtain an unlimited amount of production out of a finite amount of inputs.

Consumer i acts as a price taker and maximizes his utility given his wealth, which is described by the continuous function $r^i : (\mathbb{R}^{n+1}_+ \setminus \{\underline{0}\}) \times \mathbb{R}^{m(n+1)} \to \mathbb{R}$, defined by $r^i(p, y) = p^\top(\omega^i + \sum_{j=1}^{m} \theta^i_j y^j)$, where θ^i_j denotes the share of consumer i in the profits of producer j. Of course it holds that $\sum_i \theta^i_j = 1$ for all $j \in I_m$. Notice that for certain market conditions some producers make negative profits leading to an empty budget set $\{x^i \in \mathbb{R}^{n+1}_+ \mid p^\top x^i \leq r^i(p, y)\}$. We then follow Villar (1994) by defining the mapping $\bar{g} : (\mathbb{R}^{n+1}_+ \setminus \{\underline{0}\}) \times \mathrm{bd}(Y) \to \mathbb{R}^{m(n+1)}$ which associates with any market condition (p, y) the set of vectors $t = (t^1, \ldots, t^m) \in \mathbb{R}^{m(n+1)}$ solving

$$\begin{array}{ll} \min_t & \|t - y\|_2 \\ \text{s.t.} & r^i(p, t) \geq 0, \ \forall i \in I_c. \end{array} \tag{2.1}$$

The optimization problem in (2.1) contains a convex and closed constraint set since its constraints are linear in t, while its objective function is strictly convex in t. Hence, there exists a uniquely determined and continuous solution $\bar{g}(p, y) = (\bar{g}^1(p, y), \ldots, \bar{g}^m(p, y))$. Let $\bar{r}^i : (\mathbb{R}^{n+1}_+ \setminus \{\underline{0}\}) \times \mathrm{bd}(Y) \to \mathbb{R}$ be such that $\bar{r}^i(p, y) = r^i(p, \bar{g}(p, y))$.

Next, Villar (1994) considers the set of attainable allocations

$$\hat{\mathcal{A}} = \{(x, y) \in \mathbb{R}^{c(n+1)}_+ \times \mathrm{bd}(Y) \mid \sum_{i=1}^{c} x^i - W \leq \sum_{j=1}^{m} \bar{g}^j(p, y),\ p \in \mathbb{R}^{n+1}_+ \setminus \{\underline{0}\}\},$$

which is, under the assumptions made, a nonempty and compact subset of $\mathbb{R}^{(c+m)(n+1)}$. Let $\mathcal{K}$ be a closed cube in $\mathbb{R}^{n+1}$ which contains the projections of $\hat{\mathcal{A}}$ on the production sets and consumption sets in its interior. Following Villar (1994) we can then construct the compactified set of relevant production vectors $\mathcal{F}^j$, $j \in I_p$, and consumption sets $\hat{X}^i$, $i \in I_c$, by taking the intersection of $\text{bd}(Y^j)$ with $\mathcal{K}$ respectively of $\mathbf{R}_+^{n+1}$ with $\mathcal{K}$. The compactified set $\mathcal{F}$ denotes the intersection of bd(Y) with $\mathcal{K}$.

We now obtain that, given any market condition (p, y) $\in (\mathbf{R}_+^{n+1} \setminus \{\underline{0}\}) \times \mathcal{F}$, consumer i solves

$$\begin{array}{ll} \max_{x^i} & u^i(x^i) \\ \text{s.t} & p^\top x^i \leq \bar{r}^i(p, y) \\ & x^i \in \hat{X}^i. \end{array} \tag{2.2}$$

This optimization problem results in a continuous demand function $\bar{d}^i : (\mathbf{R}_+^{n+1} \setminus \{\underline{0}\}) \times \prod_j \mathcal{F}^j \rightarrow \mathbf{R}^{n+1}$ for each consumer i. Aggregation over all consumers results in the market excess demand function $\bar{z} : (\mathbf{R}_+^{n+1} \setminus \{\underline{0}\}) \times \prod_j \mathcal{F}^j \rightarrow \mathbf{R}^{n+1}$ defined by

$$\bar{z}(p, y) = \sum_{i=1}^{c} \bar{d}^i(p, y) - \sum_{j=1}^{m} \bar{g}^j(p, y) - W.$$

Observe that $\bar{z}$ is continuous in the market conditions and homogeneous of degree zero in the market prices ($\bar{z}(\lambda p, y) = \bar{z}(p, y)$ for all $\lambda > 0$), it satisfies Walras' Law ($p^\top \bar{z}(p, y) = 0$ for all market conditions (p, y)) and the desirability assumption (if $p_\ell = 0$ then $\bar{z}_\ell(p, y) > 0$ for each commodity ℓ).

The economy can be characterized by $E = \{ (u^i, \omega^i)_{i=1}^c, \left(Y^j, \bar{\phi}^j, (\theta_j^i)_{i=1}^c\right)_{j=1}^m \}$. Due to the homogeneity of $\bar{z}$ only relative prices matter, so we may restrict the commodity prices to S^n. The market condition $(p^*, y^*) \in S^n \times \prod_j \mathcal{F}^j$ constitutes an equilibrium in E if the market prices p^* clear the commodity markets and all producers approve of the market prices p^*, i.e.

$$\begin{array}{lll} \text{i)} & \bar{z}(p^*, y^*) \leq \underline{0} & \text{(Market Equilibrium)} \\ \text{ii)} & p^* \in \cap_{j=1}^m \bar{\phi}^j(p^*, y^*) & \text{(Production Equilibrium),} \end{array} \tag{2.3}$$

with $\bar{g}(p^*, y^*) = y^*$. Villar (1994) proves that there exists an equilibrium (p^*, y^*) in the economy E under

Condition V: Every consumer in E obtains a positive wealth under any market condition constituting a production equilibrium.

Let $\bar{k}$ equal the radius of $\mathcal{K}$, and take $K > \bar{k}$. Following Kamiya (1988), we then derive that there exists a homeomorphism b_K^j from $\mathcal{S}^n$ to a subset of $\mathrm{bd}(Y^j)$ containing $\mathcal{F}^j$, whose inverse is given by

$$(b_K^j)^{-1}(y^j) = \frac{y^j + Ke}{e^\top(y^j + Ke)}.$$

Observe that the vector $\underline{0} \in \mathcal{F}^j$ is related to the barycentre of $\mathcal{S}^n$ and that $\mathcal{F}^j$ is projected into the interior $\bar{\mathcal{S}}^n$ of S". Notice also that, for any $q_j \in \mathcal{S}^n$, a relatively high value of $q_{j\ell}$ indicates that commodity ℓ is an output of the production process of producer j, whereas a relatively low value of $q_{j\ell}$ indicates that commodity ℓ is an input of the production process of producer j.

Take $b_K = (b_K^1, \ldots, b_K^m)$, $S = \Pi_{j=1}^m \mathcal{S}^n$, and A = $\mathcal{S}^n$ x S. Now, any relevant market condition is related to a tuple $(p, q) \in$ A. Define, for each producer j, the mapping ϕ_K^j : A $\rightarrow \mathcal{S}^n$ with $\phi_K^j(p, q) = \bar{\phi}^j(p, b_K(q))$, and similarly the mapping g_K : A $\rightarrow \mathbb{R}^{m(n+1)}$ with $g_K(p, q) = \bar{g}(p, b_K(q))$. Since the set of attainable allocations is compact, the bounded losses assumption, which is explicitly made in Villar (1994), appears more or less endogeneously. Furthermore, the compactification leads to a continuous demand function $d_K^i : \Delta \rightarrow \mathbb{R}^{n+1}$ for each consumer i. Next we obtain the market excess demand function z_K : A $\rightarrow \mathbb{R}^{n+1}$ defined by

$$z_K(p, q) = \sum_{i=1}^{c} d_K^i(p, q) - \sum_{j=1}^{m} g_K^j(p, q) - W,$$

that is continuous, satisfies Walras' Law and the desirability assumption. For ease of notation we will suppress the use of K in our notation.

3 The price-quantity adjustment process

From now on we assume a compactified economy. In order to define our price-quantity adjustment process let, for producer j, the mapping π^j : A $\rightarrow \mathcal{S}^n$ be defined by $\pi^j(p, q) = \arg\min_{\bar{p} \in \phi^j(p,q)} \|p - \bar{p}\|_2$. For any market condition $(p, q) \in$ A, $\pi^j(p, q)$ denotes producer j's reference price in $\phi^j(p, q)$. It is uniquely determined due to the convexity of $\phi^j(p, q)$. Furthermore, let the mapping Γ : A $\rightarrow \mathbb{R}^{(m+1)(n+1)}$ be defined by $\Gamma(p, q) = \left(z(p, q), (p - \pi^j(p, q))_{j=1}^m\right)$.

Then, given any initial market condition $(p^0, q^0) \in \text{int}(\Delta)$, we define the set $\mathcal{P}(p^0, q^0; \Gamma)$ as the set of market conditions (p, $q) \in$ A satisfying, for each commodity ℓ,

$$\begin{aligned}
&\text{if } z_\ell(p, q) < 0 \text{ then } \min_k {p_k}/{p_k^0} = {p_\ell}/{p_\ell^0} \\
&\text{if } z_\ell(p, q) = 0 \text{ then } \min_k {p_k}/{p_k^0} \leq {p_\ell}/{p_\ell^0} \leq \max_k {p_k}/{p_k^0} \\
&\text{if } z_\ell(p, q) > 0 \text{ then } \quad {p_\ell}/{p_\ell^0} = \max_k {p_k}/{p_k^0}
\end{aligned}$$

and, for each producer j, (3.1)

$$\begin{aligned}
&\text{if } p_\ell - \pi_\ell^j(p, q) < 0 \text{ then } \min_k {q_{jk}}/{q_{jk}^0} = {q_{j\ell}}/{q_{j\ell}^0} \\
&\text{if } p_\ell - \pi_\ell^j(p, q) = 0 \text{ then } \min_k {q_{jk}}/{q_{jk}^0} \leq {q_{j\ell}}/{q_{j\ell}^0} \leq \max_k {q_{jk}}/{q_{jk}^0} \\
&\text{if } p_\ell - \pi_\ell^j(p, q) > 0 \text{ then } \quad {q_{j\ell}}/{q_{j\ell}^0} = \max_k {q_{jk}}/{q_{jk}^0},
\end{aligned}$$

with $\min_k {p_k}/{p_k^0} = \min_k {q_{jk}}/{q_{jk}^0}$.

Theorem 3.1. Let the economy $E = \left\{ (u^i, \omega^i)_{i=1}^c, \left(Y^j, \bar{\phi}^j, \left(\theta_j^i\right)_{i=1}^c\right)_{j=1}^m \right\}$ satisfy Condition V. Furthermore, the utility functions u^i, i ∈ I,, the production sets Y^j, $j \in$ I,, and the pricing rules $\bar{\phi}^j$, $j \in I_m$, are semi-algebraic, and satisfy all conditions stated in Section 2. Then, for any market condition $(p^0, q^0) \in \text{int}(\Delta)$, there exists a path $\bar{P} \in \mathcal{P}(p^0, q^0; \Gamma)$ connecting (p^0, q^0) with an equilibrium market condition $(p^*, q^*) \in \text{int}(\Delta)$.

Sketch of the proof: The proof in fact mimics a similar proof in van den Elzen (1997), so we confine ourselves to stipulating some details which are specific to the theorem here.

First we observe that the set $\mathcal{P}(p^0, q^0; \Gamma)$ is a finitely analytic set. The semi-algebraicness of Y^j results in the semi-algebraicness of b^j. The construction of g entails the minimization of a semi-algebraic expression over a semi-algebraic set, which makes g semi-algebraic. Then also the wealth functions $\bar{r}^i$ are semi-algebraic. The consumer problem is thus described by semi-algebraic functions and sets. From this we derive that the individual demand functions d^i, i ∈ I,, and the market excess demand function z are semi-algebraic. Similarly, π^j is semi-algebraic due to the semi-algebraicness of ϕ^j. Obviously, the restrictions on the relative prices and production quantities are given by linear inequalities and therefore lead to semi-algebraicness of (3.1).

Next we take a sequence $\{G^h\}_{h=1}^\infty$ of simplicial subdivisions of A with a grid size that converges to zero. We replace I' by its piecewise linear approximation Γ_{G^h} on each simplicial subdivision G^h in this sequence. Although Γ may not be a continuous mapping due to π^j possibly not being continuous, Γ_{G^h} is continuous and well-defined for each simplicial subdivision G^h in the sequence. Consecutively applying the simplicial algorithm introduced in Doup and Talman (1987) using lexicographic pivoting to compute a stationary point of Γ_{G^h} on A results into a sequence of paths $\{\bar{P}^h\}_{h=1}^{\infty}$ with $\bar{P}^h \subset \mathcal{P}(p^0, q^0; \Gamma_{G^h})$ for all $h \geq 1$ connecting the starting point (p^0, q^0) with a stationary point $(\hat{p}^h, \hat{q}^h)$ of Γ_{G^h} on A. While $z_{G^h} \rightarrow z$ $(h \rightarrow \infty)$, there exists an upper hemicontinuous mapping $\hat{\pi}^j = \limsup_{h\rightarrow\infty} \pi^j_{G_h}$. This mapping $\hat{\pi}^j$ depends on $\{\bar{P}^h\}_{h=1}^{\infty}$. Furthermore, due to the upper hemicontinuity of ϕ^j, it follows that $\hat{\pi}^j(p,q) \subseteq \phi^j(p,q)$ for each $(p, q) \in$ A. Then, as argued in van den Elzen (1997), the limes superior of the sequence of paths $\{\bar{P}^h\}_{h=1}^{\infty}$, $\hat{P}$, is a connected subset of $\mathcal{P}(p^0, q^0; \Gamma)$ and contains the starting point (p^0, q^0) and a stationary point (p^*, q^*) of Γ on A. For more details on this part of the proof, see Herings (1997). Due to the semi-algebraicness of $\mathcal{P}(p^0, q^0; \Gamma)$, $\hat{P}$ is also path connected, i.e. $\hat{P}$ contains a path $\bar{P}$ connecting (p^0, q^0) and (p^*, q^*).

Notice that the stationary point (p^*, q^*) lies in int(A). Suppose not. Then for some commodity ℓ, ${}^{p^*_\ell}/_{p^0_\ell}$ = mink ${}^{p^*_k}/_{p^0_k}$ = 0 or ${}^{q^*_{j\ell}}/_{q^0_{j\ell}}$ = mink ${}^{q^*_{jk}}/_{q^0_{jk}}$ = 0 for some producer j. Due to the 'desirability'-property of the excess demand function, $z_\ell(p^*, q^*) > 0$ if $p^*_\ell = 0$, which contradicts the conditions in (3.1). Hence, $z(p^*, q^*) = \underline{0}$ and $p^* \gg \underline{0}$, so $(p^*, q^*) \in$ bd(A) constitutes a market equilibrium with $q^*_{j\ell} = 0$ for some commodity ℓ and some producer j. But the construction of the inverse homeomorphism b^j prohibits this since $\mathcal{F}^j$ is projected into the interior of $\mathcal{S}^n$. Therefore $(p^*, q^*) \in \text{int}(\Delta)$ and contains an equilibrium. □

The path $\bar{P}$ can be interpreted as a tâtonnement process in the following way. Along $\bar{P}$ relative market prices ${}^{p_\ell}/_{p^0_\ell}$ are kept minimal (maximal) for those commodities ℓ which are in excess supply (excess demand). The relative market prices ${}^{p_\ell}/_{p^0_\ell}$ of the commodities ℓ whose markets are in equilibrium are allowed to vary between their lower bound mink ${}^{p_k}/_{p^0_k}$ and their upper bound $\max_k {}^{p_k}/_{p^0_k}$ in order to keep these markets in equilibrium for as long as possible. As soon as the relative market price on a market in equilibrium reaches its upper (lower) bound, the equilibrium is disturbed and the market is brought into excess demand (excess supply).

Given market conditions $(p, q) \in$ A, each producer j compares the market price vector p with his reference price vector $\pi^j(p, q)$. Suppose $p_\ell - \pi^j_\ell(p, q) < 0$ $(p_\ell - \pi^j_\ell(p, q) > 0)$. This means that producer j obtains less (more) than he considers acceptable if commodity ℓ is an output of his production technology while, if commodity ℓ is an input in his production technology then this means that he pays less (more) for this input than he considers acceptable. Producer j then prefers to have a relatively low (high) quantity of output commodity ℓ in his production technology, while he prefers to have a relatively high (low) quantity of input commodity ℓ in his production technology. Following the interpretation of b^j_K mentioned in the previous section, the auctioneer therefore decides to keep $q_{j\ell}/q^0_{j\ell}$, which corresponds to the relative production quantity of commodity ℓ in the production bundle of producer j, minimal (maximal) compared to the other commodities. Suppose $p_\ell - \pi^j_\ell(p, q) = 0$. Then producer j considers the market price p_ℓ acceptable and the auctioneer decides to keep this price acceptable for as long as possible by allowing $q_{j\ell}/q^0_{j\ell}$ to vary between its lower and upper bound.

In order to compute an equilibrium in E, one should find a way to approximately follow the path $\bar{P}$. One way to do this has already been suggested in the proof of Theorem 3.1. By consecutively applying the algorithm of Doup and Talman (1987) for a sequence of simplicial subdivisions $\{G^h\}_{h=1}^H$, $1 \leq H < \infty$, with smaller and smaller simplices, we obtain an arbitrarily accurate approximation $\bar{P}_{G^H}$ in $P(p^0, q^0; \Gamma_{G^H})$ of $\bar{P}$. This approximating path $\bar{P}_{G^H}$ connects the initially announced market condition $(p^0, q^0) \in \text{int}(\Delta)$ with an approximating equilibrium $(p^{*0}, q^{*0}) \in \text{int}(\Delta)$. As Doup and Talman (1987) show, the path $\bar{P}^h \subset \mathcal{P}(p^0, q^0; \Gamma_{G^h})$ can be generated by performing a sequence of lexicographic linear programming pivoting steps. In a similar way we can find alternative equilibria. We illustrate this in the next section.

4 Two examples

We consider an economy $E = \{(u, \omega), (Y, \bar{\phi})\}$ consisting of two commodities, one consumer, and one producer. The consumer has a utility function $u(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$ and initial endowments $w = (6, 3)^\top$. The production technology of the producer is described by the set $Y = (Y_1 \cap Y_2) \cup Y_3$, with

$$\begin{aligned} Y_1 &= \{(y_1, y_2) \in \mathbb{R}_- \times \mathbb{R}_+ \mid y_2 \leq \sqrt{-y_1}\}, \\ Y_2 &= \{(y_1, y_2) \in \mathbb{R}_- \times \mathbb{R}_+ \mid y_2 \leq \tfrac{1}{4}(-y_1)^2\}, \\ Y_3 &= \{(y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}_- \mid y_1 \leq -y_2\}. \end{aligned}$$

Y_1 exhibits decreasing returns to scale, Y_2 increasing returns to scale, and Y_3 constant returns to scale. Figure 4.1 provides an illustration of the production set Y. Observe that Y satisfies all'assumptions made in Section 2.

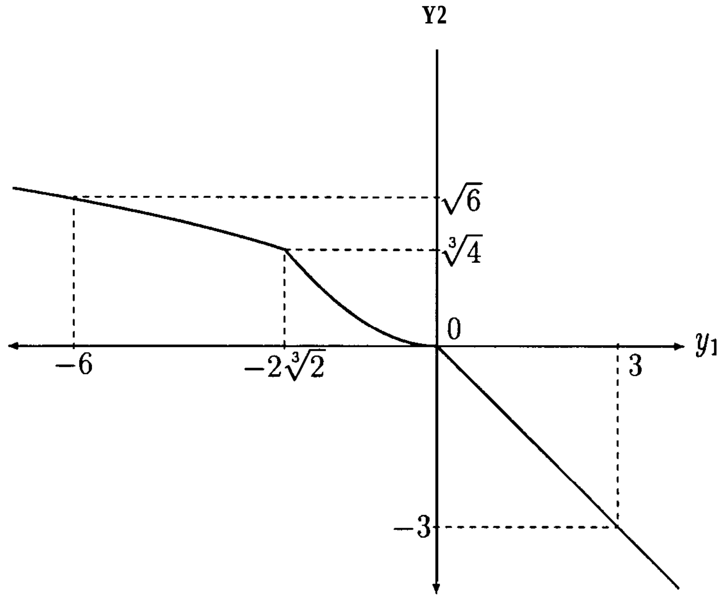

FIGURE 4.1: The production set Y consists of all production vectors on and to the left of the curve.

In this example, we can take $\mathcal{F} = \{y \in \text{bd}(Y) \mid -6 \le y_1 \le 3\}$ as the set of relevant production vectors, since for production vectors outside this set there does not exist a market equilibrium. The economy does not contain enough resources to sustain such production vectors. The excess demand function $\bar{z}$ of the consumer is given by

$$\bar{z}(p, y) = \left(\frac{-2p_1(6+y_1)+p_2(3+y_2)}{3p_1}, \frac{2p_1(6+y_1)-p_2(3+y_2)}{3p_2}\right)^{\mathsf{T}}.$$

The set $\mathcal{F}$ is the intersection of bd(Y) and a cube with radius 6 around the origin. Let $K = 8$ to construct a mapping b such that

$$b^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{y_1+8}{y_1+y_2+16} \\ \frac{y_2+8}{y_1+y_2+16} \end{pmatrix}$$

projects the set of relevant production vectors 3 on the convex and compact subset $\bar{\mathcal{S}}^1 := \{q \in \mathcal{S}^1 \mid \frac{2}{10+\sqrt{6}} \le q_1 \le \frac{11}{16}\}$ of $\text{int}(\mathcal{S}^1)$. Using b we can transform the excess demand function $\bar{z}$ into the mapping z.

Concerning the producer's behaviour we assume that $\bar{\phi}$ is given by the marginal cost pricing rule MC : bd(Y) $\rightarrow \mathcal{S}^1$. In this example we obtain

$$\text{if } y_1 < -2\sqrt[3]{2} \text{ then } \overline{\text{MC}}(y) = \left\{ \left(\tfrac{1}{1+2\sqrt{-y_1}}, \tfrac{2\sqrt{-y_1}}{1+2\sqrt{-y_1}} \right)^\top \right\},$$

$$\text{if } y_1 = -2\sqrt[3]{2} \text{ then MC(y) = cone} \left\{ \left(\tfrac{1}{1+2\sqrt[3]{4}}, \tfrac{2\sqrt[3]{4}}{1+2\sqrt[3]{4}} \right)^\top, \left(\tfrac{\sqrt[3]{2}}{1+\sqrt[3]{2}}, \tfrac{1}{1+\sqrt[3]{2}} \right)^\top \right\},$$

$$\text{if } -2\sqrt[3]{2} < y_1 < 0 \text{ then } \overline{\text{MC}}(y) = \left\{ \left(\tfrac{-y_1}{2-y_1}, \tfrac{2}{2-y_1} \right)^\top \right\},$$

$$\text{if } y_1 = 0 \text{ then MC(y) = cone} \left\{ (0,1)^\top, \left(\tfrac{1}{2}, \tfrac{1}{2} \right)^\top \right\},$$

$$\text{if } y_1 > 0 \text{ then } \overline{\text{MC}}(y) = \left\{ \left(\tfrac{1}{2}, \tfrac{1}{2} \right)^\top \right\}.$$

Using b we can transform the mapping MC into a mapping MC: $\mathcal{S}^1 \rightarrow$ S'. In combination with z we obtain Γ_{MC} as $\Gamma_{\text{MC}} = (z, p - \pi)$, with $\pi(p, q)$ the reference price in MC(q). In Figure *4.2,* we have drawn the market equilibrium curve ME := $\{(p, q) \in A \mid z_1(p, q) = z_2(p, q) = 0\}$ and the production equilibrium curve PE := $\{(p, q) \in A \quad p \in \text{MC}(q)\}$.

There exists an excess demand for commodity 1 (2) in all market conditions in $\text{int}(\Delta)$ lying above (below) the market equilibrium curve ME. The sign pattern of $p - \pi(p, q)$ in any market condition $(p, q) \in \text{int}(\Delta)$ is straightforward. The intersections of the two curves ME and PE, A = $((0.4, 0.6)^\top, (0.36, 0.64)^\top)$, B = $((0.22, 0.78)^\top, (0.48, 0.52)^\top)$, and C = $((0.2, 0.8)^\top, (0.5, 0.5)^\top)$, provide the equilibria in the economy E under marginal cost pricing. The market condition C constitutes the pure exchange equilibrium, since $q^C = (0.5, 0.5)^\top$ corresponds to a production vector $y^C = 0$. Furthermore, $q^A = (0.36, 0.64)^\top$ corresponds to a production vector $y^A = (-2\sqrt[3]{2}, \sqrt[3]{4})^\top$, whereas $q^B = (0.48, 0.52)^\top$ is related to a production vector $y^B = (-0.546, 0.075)^\top$.

For $(p^0, q^0) = ((0.6, 0.4)^\top, (0.2, 0.8)^\top)$, the set $\mathcal{P}(p^0, q^0;$ I',,) consists of the paths $\bar{P}^0_{\text{MC}} = (a_0, a_1, a_2, a_3, \text{A})$ connecting $a_0 = (p^0, q^0) \in \text{int}(\Delta)$ with the equilibrium A, and $P^0_{\text{MC}} = (\text{C}, c_1, c_2, \text{B})$ connecting the pure exchange equilibrium C with equilibrium B. The price-quantity adjustment process starting in a_0 generates $\bar{P}^0_{\text{MC}}$. Under this initial market condition, the consumer's excess demand is $z(p^0, q^0) = (0.77, -1.155)^\top$ urging an increase in the market price of commodity 1 and a decrease in the market price of commodity 2. The producer is initially requested to produce $y^0 = b(q^0) = (-5.4, 2.3)^\top$ at prices p^0. According to the marginal cost pricing rule however, only the reference price vector $\pi(p^0, q^0) = (0.177, 0.823)^\top$ is considered as acceptable to the producer when producing y^0. Since $p^0_1 = 0.6 > 0.177 = \pi_1(p^0, q^0)$

for input commodity 1 and $p_2^0 = 0.4 < 0.823 = \pi_2(p^0, q^0)$ for output commodity 2, the quantity of input commodity 1 in the production process is kept relatively low by increasing q_1 from q_1^0 and the quantity of output commodity 2 in the production process is kept relatively low by decreasing q_2 from q_2^0. Thus, $\bar{P}_{\mathrm{MC}}^0$ leaves a_0 towards the vertex $\left((1,0)^\top, (1,0)^\top\right)$ of A.

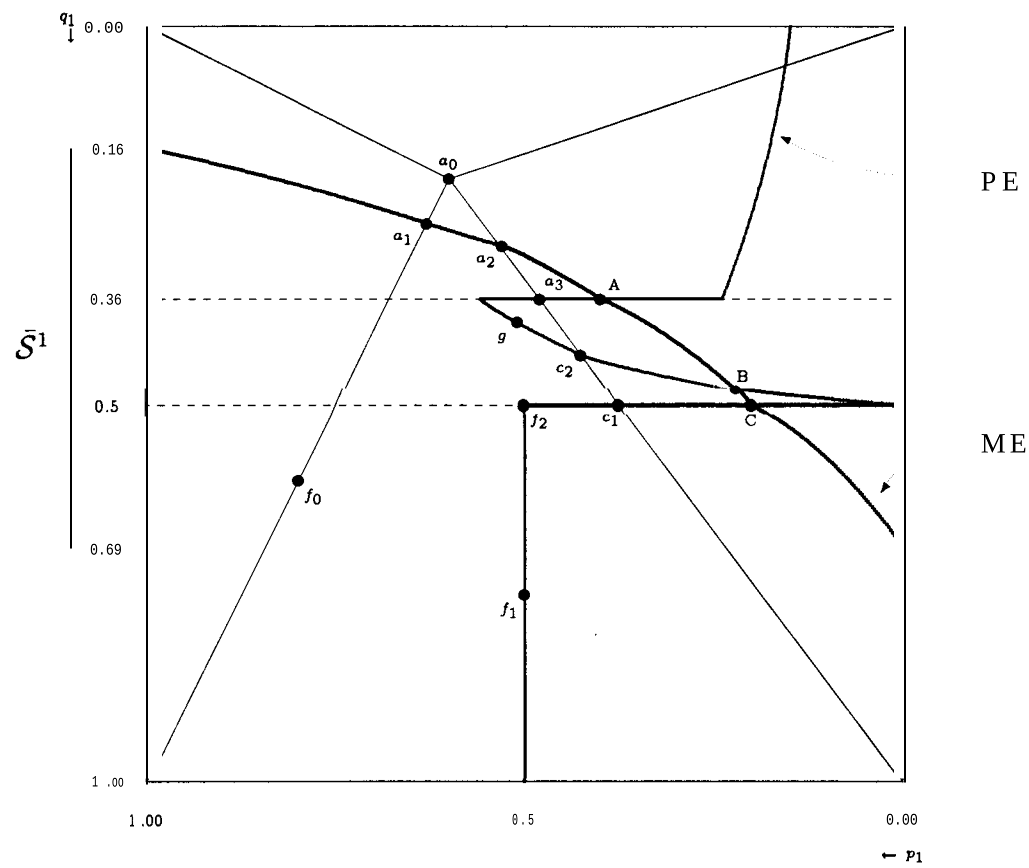

FIGURE *4.2:* $\mathcal{P}(p^0, q^0; I',,)$, where $a_0 = (p^0, q^0) = ((0.6, 0.4)^\top, (0.2, 0.8)^\top)$, consists of the paths $\bar{P}_{\mathrm{MC}}^0 = (a_0, a_1, a_2, a_3, \mathrm{A})$ and $P_{\mathrm{MC}}^0 = (\mathrm{C}, c_1, c_2, \mathrm{B})$. $\mathcal{P}(p^1, q^1; I',,)$, where $f_0 = (p^1, q^1) = ((0.8, 0.2)^\top, (0.6, 0.4)^\top)$, consists of the paths $\bar{P}_{\mathrm{MC}}^1 = (f_0, f_1, f_2, \mathrm{C})$ and $P_{\mathrm{MC}}^1 = (\mathrm{A}, a_3, g, \mathrm{B})$.

The market condition $a_1 = ((0.63, 0.37)^\top, (0.26, 0.74)^\top)$ contains a market equilibrium as $a_1 \in$ ME. Subsequently the market conditions are adjusted in such a way that the commodity markets stay in equilibrium for as long as possible. To this end p_1 is lowered from its upper bound (p_2 is raised from its lower bound) until market condition $a_2 =$

$\left((0.53, 0.47)^\top,\ (0.29, 0.71)^\top\right)$ is generated. The conditions on q haven't changed, hence q is adjusted as described in the preceeding paragraph.

At a_2 the relative market price of commodity 1 (2) hits its lower (upper) bound, i.e. it becomes equal to the relative minimum (maximum) of the production quantities. In order to fulfil the conditions in (3.1) they are kept equal to this lower (upper) bound and to this end the commodity markets are disequilibrated. The process moves towards $a_3 = \left((0.48, 0.52)^\top, (0.36, 0.64)^\top\right)$, where commodity 1 is in excess supply and commodity 2 is in excess demand. Again, the conditions on q haven't changed, hence q is adjusted as described before.

In us, $p \in$ MC($(0.36, 0.64)^\top$), i.e. the producer accepts the market prices p. Subsequently, in Figure 4.2, q is kept constant in order to keep the market prices acceptable for the producer. The conditions on the market prices p haven't changed in a_3, hence p is adjusted as before. The path generated by the adjustment process moves towards A. In A, $(p^A, q^A) \in$ ME and $(p^A,\ q^A) \in$ PE, hence $\bar{P}^0_{\mathrm{MC}}$ stops in the equilibrium A, where the agents engage in actual trade. Notice that $\hat{\pi}(p,\ q) = \phi(p,\ q)$ for all $(p,\ q) \in \bar{P}^0_{\mathrm{MC}}$.

If one chooses the initial market conditions $(p^1, q^1) = ((0.8, 0.2)^\top,\ (0.6, 0.4)^\top)$, then there exists a path in $\mathcal{P}(p^1,\ q^1;$ I',,) connecting $(p',\ q^1) \in \mathrm{int}(\Delta)$ with the pure exchange equilibrium C, namely the path $\bar{P}^1_{\mathrm{MC}} = (f_0, f_1, f_2,$ C) where $f_0 = (p^1, q^1)$, $f_1 = ((0.5, 0.5)^\top, (0.75, 0.25)^\top)$, and $f_2 = ((0.5,\ 0.5)^\top, (0.5,\ 0.5)^\top)$. The end point C of $\bar{P}^1_{\mathrm{MC}}$ is also an end point to the path $P^0_{\mathrm{MC}} \subset \mathcal{P}(p^0,\ q^0;$ I',,). By following this path P^0_{MC}, one also finds equilibrium B. Hence, in this way we are able to find all equilibria in E.

Observe that the price-quantity adjustment process introduced in Kamiya (1988) cannot be started in p^0, since there does not exist a $\bar{q}$ such that $p^0 \in \phi(p^0,\ \bar{q})$. For such a reason, Kamiya (1988) imposes the conditions $p^0 \in \{t \in \mathcal{S}^1$ 0.386 < t_1 < 0.5) and $q^0 = (0.5, 0.5)^\top$ on the initialization of his price-quantity adjustment process. Under all these conditions on the initialization, Kamiya (1988) finds the pure exchange equilibrium C. Hence, this example illustrates that Kamiya (1988) can only start in a limited set of market conditions, and therefore only finds equilibrium C in Figure 4.2.

As a second example we assume that the producer now behaves according to the average cost pricing rule AC : bd(Y) → S'. For the production set Y, this definition reduces to the following conditions, if $y_1 \leq -2\sqrt[3]{2}$ then $\overline{\mathrm{AC}}(y) = \left\{ \left(\frac{\sqrt{-y_1}}{\sqrt{-y_1} - y_1}, \frac{-y_1}{\sqrt{-y_1} - y_1} \right)^\top \right\}$,

if $-2\sqrt[3]{2} \leq y_1 < 0$ then $\overline{\mathrm{AC}}(y) = \left\{ \left(\frac{-y_1}{4 - y_1}, \frac{4}{4 - y_1} \right)^\top \right\}$,

if $y_1 = 0$ then AC(y) $= \mathcal{S}^1$,

if $y_1 > 0$ then $\overline{\mathrm{AC}}(y) = \left\{ \left(\frac{1}{2}, \frac{1}{2} \right)^\top \right\}$,

which we transform into a mapping AC: $\mathcal{S}^1 \to \mathcal{S}^1$ using b. In Figure 4.3 we have again drawn the market equilibrium curve ME and the production equilibrium curve PE := $\{(p, q) \in \Delta \mid p \in \mathrm{AC}(q)\}$ in A. The intersections of these curves, A = $((0.4, 0.6)^\top, (0.36, 0.64)^\top)$, B = $((0.33, 0.67)^\top, (0.40, 0.60)^\top)$, and C = $((0.2, 0.8)^\top, (0.5, 0.5)^\top)$, provide an equilibrium in E under average cost pricing. The market conditions C again constitute the pure exchange economy equilibrium as these market conditions contain no production.

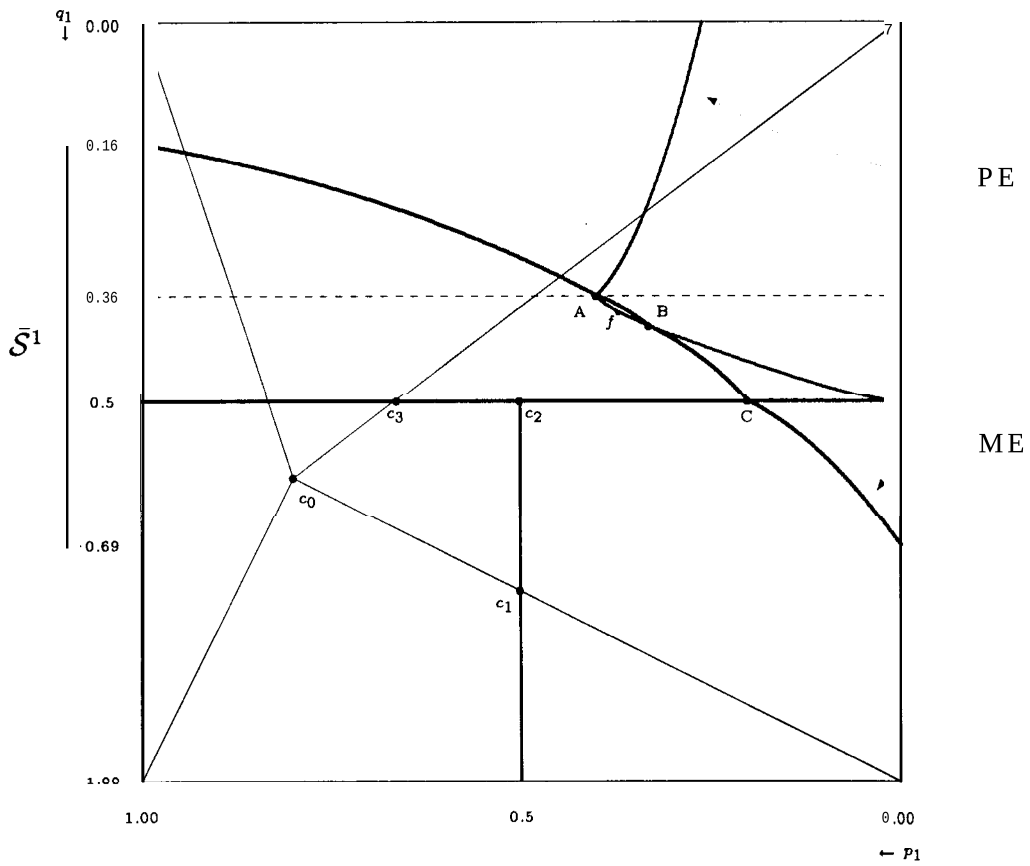

FIGURE 4.3: $\mathcal{P}(p^1, q^1; \Gamma_{\mathrm{AC}})$, where $c_0 = (p^1, q^1) = ((0.8, 0.2)^\top, (0.6, 0.4)^\top)$, consists of the sets $\bar{P}^1_{\mathrm{AC}}$ = (co, c_1, c_2, c_3, C) and P^1_{AC} = (A, f, B).

Now, $\mathcal{P}(p^1, q^1; \Gamma_{AC})$ consists of the sets $\bar{P}^1_{AC} = (c_0, c_1, c_2, c_3, C)$ and $P^1_{AC} = (A, f, B)$. $\bar{P}^1_{AC}$ has three end points, the starting point c_0, the pure exchange equilibrium C, and a market condition c_3. Kamiya (1988) imposes an extra condition on the definition of average cost pricing in order to force his adjustment process to end up in C. Namely, he takes AC(Q) equal to MC(Q) which, in this example, is given by the set $\{p \in \mathcal{S}^1 \mid p_1 \leq f\}$. Notice that, if we follow the proof of Theorem 3.1 by computing an approximating stationary point of Γ_{G^h} on A for a sequence $\{G^h\}_{h=1}^{\infty}$ of simplicial subdivisions of A, with mesh converging towards zero, we obtain a sequence of piecewise linear paths starting in c_0 and converging to the path $\bar{P}_{AC} = (c_0, c_1, c_2, C)$ which is a subset of $\bar{P}^1_{AC}$.

Notice that $\hat{\pi}$ is given by $\hat{\pi}(p, q) = \{(0.5, 0.5)^\top\}$ for all $(p, q) \in \bar{P}^1_{AC}$ such that $q_1 > 0.5$, and $\hat{\pi}(p, q) = \text{co}(\{(0.5, 0.5)^\top, (0, 1)^\top\}) \subset \phi(p, q)$ for all $(p, q) \in \bar{P}^1_{AC}$ such that $q_1 = 0.5$. It's this feature which makes $\bar{P}^1_{AC}$ end up in the pure exchange equilibrium C, and not in c_3.

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