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On the Inefficiency of Portfolio Insurance and Caveats to the Mean/Downside-Risk Framework

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Abstract

Portfolio insurance strategies based on options typically treat the investment in the risky asset, e.g., stock, as fixed. We show in a mean/downside-risk framework that such a strategy is inefficient. Using at the money put options, expected returns can be increased by more than 250 basis points without taking on more risk. Gains can become arbitrarily large when one uses options with extremely high strike prices. This is due to a serious caveat to the mean/downside-risk framework that is typically adopted in the literature by substituting downside-risk measures for standard risk measures such as the variance of returns. These pathologic results vanish when one maximizes an appropriately chosen HARA utility function. In this framework, fixing the holding of the risky asset in advance leads to efficiency losses that vary between 250 and 650 basis points depending on the degree of risk aversion.

Key words: mean/downside-risk efficiency; option strategies; optimal portfolio choice; portfolio insurance; downside-risk.

JEL Codes: G11

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1 Introduction

Risk management and optimal asset allocation are key issues in modern financial economics. Fostered by an increased integration and liberalization of financial markets and a spectacular growth in the number of financial products, managers have felt an increased need to efficiently allocate the available resources given an acceptable risk profile. Though the basic allocation problem is easy to formulate, formalizing it and implementing its solution are a far more difficult task. Most of the results one obtains hinge on the efficiency framework and the risk measure one adopts.

Since Markowitz (1952), the dominating efficiency paradigm in financial economics has been that of mean/variance efficiency. In this framework, expected return is taken as a measure of profitability, while the variance captures the risk. Efficient strategies in this paradigm attain a given expected return at minimum risk, i.e., at minimum variance. The use of the variance as a measure of risk has a well-known drawback: it penalizes positive and negative deviations from the expected return in a perfectly symmetric way. This does not match the notion of risk of practitioners, who typically link risk to adverse price movements rather than favorable ones, see Sortino and van der Meer (1991). Moreover, the use of the variance can lead to perverse efficiency results if non-linear instruments like options are available, see Leland (1996) and Lhabitant (1997).

An alternative for the variance is a downside-risk measure. A prototypical example of such a measure is value-at-risk (VaR). VaR measures the maximum amount one can loose over a given horizon given a certain confidence level, see Jorion (1997). It is a popular risk measure that is used by practitioners as well as supervisory institutions. If one substitutes VaR (or some other downside-risk measure) for the variance, one obtains an alternative to the mean/variance efficiency framework. Such alternatives were. proposed and studied by, e.g., Roy (1952), Telser (1955), and Kataoka (1963).

In the present paper we adopt the mean/downside-risk framework to assess the efficiency of portfolio insurance strategies if nonlinear instruments, in particular options, are available. The liquidity in derivative markets has improved dramatically over the last decade. This enables investors to include derivative instruments in their investment strategies. Strategies using derivatives allow investors to efficiently attain return distributions that suit their preferences better than strategies based on traditional investment categories only. Studies such as Merton, Scholes, and Gladstein (1978, 1982), Figlewski, Chidambaran, and Kaplan (1992), and Bodie and Crane (1998) clearly illustrate that the use of options or equivalent dynamic trading strategies results in superior return distributions. They do so mainly by comparing simulation

results for investment strategies without options and strategies that employ options according to prespecified rules of thumb. In contrast to these studies, we adopt an analytic approach and formally derive optimal portfolios. Analytical results on optimally optioned portfolios in relation to the adopted risk/return framework are scarce, see, e.g., Ahn, Boudoukh, Richardson, and Whitelaw (1998).

Our findings extend the results of Ahn, Boudoukh, Richardson, and Whitelaw (1998), who analytically consider optimal portfolio insurance based on options given a constraint on the cost of the insurance. In their framework, optimality is evaluated in terms of the VaR profile of the insured portfolio. A typical portfolio insurance approach as in Ahn et al. (1998), however, treats the investment in the basic risky asset, e.g., stock, as fixed. We show that such a strategy leads to large efficiency losses in a mean/downside-risk framework. Given a constraint on the allowable VaR, substantially higher expected returns can be achieved if the amount invested in the basic risky asset is allowed to vary and if options on this asset are available. We present realistic numerical examples in which the efficiency gains range from 250 to 650 basis points.

We analytically derive the optimal asset allocation in the framework of perfect and complete markets, see Black and Scholes (1973). If at the money, in the money, or slightly out of the money put options are available, the typical pay-off funtion of the mean/downside-risk optimal portfolio mimics that of a call option combined with a riskfree investment. The riskfree investment effectively ensures a tolerable (VaR) risk profile, while the call option creates the upward potential of a stock investment without the associated downside-risk of a naked stock strategy. We also show that for sufficiently low strike prices it is optimal to short unlimited amounts of put options if VaR is used as a downside-risk measure. This illustrates a first pathology in the mean/VaR efficiency framework: as VaR only accounts for the event of adverse price movements, and not for the extent of their impact, the mean/VaR framework may lead to unrealistic asset allocation policies.

As a second result in our paper, we establish some serious pathologies in the more general mean/downside-risk framework, see also Dert and Oldenkamp (1997). It turns out that if the optimal asset allocation for a fixed strike price of the option is optimized over the strike price, the expected return diverges to infinity for a given VaR profile. Given the typical structure of the optimal asset allocation, the manager effectively ends up taking a bet on an extreme stock price realization by going long in an extremely far out of the money call option, given that the downside-risk constraint is met by a riskfree investment. This signals that the mean as a measure of profitability combined with a measure of downside-risk gives an incomplete

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characterization of the preference ordering of typical investors over different pay-off profiles. Both measures fail to take account of the variability of the pay-offs above the VaR critical level, whereas investors are not completely indifferent in this respect.

We argue that general utility specifications for optimal asset allocation are more appropriate than either the mean/variance or the mean/downside-risk efficiency frameworks. Such utility specifications should incorporate the tradeoff between downside-risk, upward potential or expected return, and variability of returns with an acceptable downside-risk profile. We therefore propose an n-attribute efficiency framework rather than the usual two-attribute one, with n at least equal to three. Even in such a framework, however, the portfolio insurance strategy is dominated in terms of efficiency by a strategy that allows the investment in the basic risky asset to vary. To substantiate these claims, we consider an example using a threshold power utility function. For this utility specification, effeciency losses of portfolio insurance with respect to an unrestricted optimal strategy vary between 250 and 650 basis points for different degrees of risk aversion.

The article is set up as follows. In Section 2, we present the model and give the analytical derivations. Section 3 presents the efficiency comparison between the stock-only investment strategy, portfolio insurance based on options, and an unrestricted optimal investment strategy. This leads to the pathology in the mean/downside-risk framework, which is discussed in Section 4. Section 5 concludes by summarizing the main results and suggesting possible directions for future research.

2 Analytical results

In Subsection 2.1, we set up our basic model. We focus on portfolios that may consist of holdings in a non dividend paying stock, a put option, and a riskfree asset, see also, e.g., Merton et al. (1982) and Ahn, Boudoukh, Richardson, and Whitelaw (1998). In Subsection 2.2 we present the exact analytic characterization of the optimal portfolio.

2.1 Set-up of the model

Consider an investor who has \$1 to invest. The manager's planning period starts at time $t = 0$ and ends at time $t = 1$. At time $t = 0$, there are three investment categories: (i) a riskfree asset, earning a (continuously compounded) rate of return r^c, (ii) a non dividend paying stock, giving a risky (continuously compounded) rate of return r^s, and (iii) a European put option

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on the stock with exercise price K. For simplicity, we assume that the option matures at the planning horizon $t = 1$. We also assume that the 'perfect market conditions' of Black and Scholes (1973) are satisfied, such that the usual option pricing formula can be applied. Note that in the framework of Black and Scholes (1973) options are in fact redundant assets as they can be perfectly replicated by a dynamic investment strategy. Although this may be a valid statement for the market as a whole, the large marjority of investors is unable to replicate an option efficiently by implementing a dynamic trading strategy. Proper use of options may dramatically improve their ability to create pay-off patterns that suit their investment preferences. Alternatively, one can interpret our results as pertaining to dynamic investment strategies that are equivalent to static option investments.

We focus on the case where only a single option series can be included in the portfolio. This suffices to obain insight in the structure of optimal portfolios in the stylized framework presented below, see also Dert and Oldenkamp (1997) and Ahn et al. (1998).

One of the implications of the framework of Black and Scholes (1973) is that stock returns r^s are normally distributed with mean μ and variance σ^2. To save on notation, we define $\bar{\mu} = \mu + \sigma^2/2$, such that the expected $t = 1$ price of the stock is $\exp(\bar{\mu})$. Without loss of generality, we assume the initial stock price and the intial asset level to be equal to $A_0 = \$1$ and $P_0^s = \$1$, respectively. Notice that the intial stock price of $1 allows us to interpret the strike price K as the money-ness of the option. The final asset value A_1 is given by

$$A_1 = e^{r^c} \cdot (1 - y - P_0^o \cdot x) + e^{r^s} \cdot y + \max[K - e^{r^s}, 0] \cdot x, \qquad (1)$$

with P_0^o denoting the initial price of the option, x denoting the number of options in the portfolio, and y denoting the number of stocks in the portfolio. The amount of initial funds left after buying stock and options is invested (lent or borrowed) in cash at the riskfree rate. Note that by allowing for negative values of x and y, we also allow the investor to select combinations of put options, stocks and cash that are equivalent to holding long or short positions in call options. We formulate the problem in terms of put options, however, following the usual convention in the literature on portfolio insurance, see, e.g., Ahn et al. (1998).

We assume that the investor maximizes the return on his portfolio, given a constraint on the portfolio's risk. If the variance of the returns is used as a measure of risk, we obtain the familiar mean/variance framework of standard portfolio theory, Markowitz (1952). As argued in the introduction, however, it is more useful to consider downside-risk measures in our present

context with options. We concentrate on the following class of downside-risk measures:

$$E\left((A^* - A_1)^\kappa \cdot 1_{A^*}(A_1)\right),\qquad(2)$$

where A^* denotes a benchmark asset level, $1_{A^*}(A_1)$ is a step function, $1_{A^*}(A_1) = 1$ for $A_1 < A^*$, and $1_{A^*}(A_1) = 0$ otherwise, and $\kappa \geq 0$ defines the specific downside-risk measure. Well-known downside-risk measures are obtained for $\kappa = 0$ (shortfall probability), $\kappa = 1$ (expected shortfall), and $\kappa = 2$ (semi-variance). In the remainder of this paper, we concentrate primarily on the shortfall probability ($\kappa = 0$). Shortfall probabilities form the basis of value-at-risk (VaR) analyses, which are now common practice in risk management, see e.g., Jorion (1997). VaR measures the maximum loss over a certain period given a certain desired confidence level. Further below in this section, we also discuss how our results generalize to settings where the expected loss or the semivariance is used to measure risk.

Note that for $\kappa = 0$, (2) can be rewritten as

$$P(A_1 \leq A^*),\qquad(3)$$

where $P(\cdot)$ denotes the probability measure of the stock price $\exp(r^s)$. An efficient investment strategy maximizes expected return given a bound on the risk profile as given in (3). Formally, the. optimal strategy for a given risk profile, i.e., a maximum VaR level, is the solution to

$$\max_{x,y}\quad E(A_1)\qquad(4)$$
$$\text{s.t.}\quad P(A_1 \leq A^*) \leq \psi,$$

where ψ is a prespecified constant relating to the confidence level of the VaR. Specifically, following Ahn et al. (1998) and Artzner et al. (1997), we define $A_0 \cdot \exp(r^c) - A^*$ as the $(1 - \psi)$ confidence level VaR over the planning period. So the VaR is defined with respect to the asset level that can be attained by investing all initial funds in the riskfree asset, i.e., $\exp(r^c)A_0$. The constraint in (4) is also labeled a (probabilistic) shortfall constraint in the literature, see Leibowitz and Kogelman (1991) and Leibowitz, Kogelman, and Bader (1992). Given the above investment problem, we are interested in two questions.

1. Does the optimal portfolio contain options?

2. If so, what is the magnitude of the improvement that can be achieved over (i) a stock-only strategy (i.e., x = 0) and (ii) an optimal portfolio insurance strategy (i.e., y is fixed)?

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These questions are addressed in the remainder of the paper.

2.2 An analytic characterization of the global optimum

The following theorem is proved in the Appendix.

Theorem 1 *The optimal solution to (4), if it exists and is bounded,* takes
the form:

$$
\begin{cases}
x = y = \dfrac{e^{r^c} - A^*}{(1 + P_0^o) \cdot e^{r^c} - \max(K, P^{-1}(\psi))} & \textit{for } c_1 \geq c_0 \geq 0, \\[4mm]
y = 0, \; x = \dfrac{e^{r^c} - A^*}{P_0^o \cdot e^{r^c} - \max(0, K - P^{-1}(\psi))} & \textit{for } c_0 \geq c_1 \geq 0,
\end{cases}
\tag{5}
$$

where

$$
c_0 = \frac{P_0^o \cdot e^{r^c} - \alpha}{e^{\bar{\mu}} - e^{r^c}},
\tag{6}
$$

and

$$
c_1 = \frac{P_0^o \cdot e^{r^c} - \max(0, K - P^{-1}(\psi))}{P^{-1}(\psi) - e^{r^c}},
\tag{7}
$$

$\alpha = E[\max(0, K - \exp(r^s))]$, *and* $P(\exp(r^s) \leq P^{-1}(\psi)) = \psi$. *For* $c_1 < 0$, (4)
is unbounded.

Proof: see Appendix.

So far, the strike price of the option has not been treated as a decision
variable in the optimization problem. We relax this restriction in Section 3.
As can be verified in (5), the holdings in the optimal portfolio strongly depend
on the strike price K of the available option series. If the strike price is far
out of the money or deep in the money, peculiar results emerge which are
discussed in Subsections 2.2.2 and 2.2.3, see also Pelsser and Vorst (1995).
We first interpret the portfolios presented in (5). It is perhaps illustrative
to mention right away that for the vast majority of reasonable strike prices,
the first solution in (5) is optimal, i.e., $y = x$. This portfolio contains an
equal number of stocks and options. Via put-call parity, this boils down to
a riskfree investment in cash combined with a long position in call options.
The riskfree investment guarantees that the shortfall constraint is met with
certainty, while the call option serves to maximize expected return. More
details follow below.

To facilitate the interpretation of Theorem 1, first note that the numerator
in (5) always equals the investor's maximum allowable VaR, see below (4). If

the investor wants maximum certainty, $A*$ equals $\exp(r^c)$ and, consequently, both stock and option investments are zero.

The constant c_0 in (6) gives the negative ratio of the risk premium of the option to that of the stock. Similarly, c_1 gives the negative ratio of pay-offs of an option-only investment to a stock-only investment, both in excess of an equivalent riskfree investment, and both pay-offs evaluated at the VaR critical realization of the stock price, $P^{-1}(\psi)$. The constants c_0 and c_1 also have a geometric interpretation. The constant c_0 is the slope coefficient of the contour lines of the objective function in $(x,$ y$)$-space. Similarly, c_1 is the slope of the VaR constraint in $(x,$ y$)$-space, i.e., the slope of the boundary of the feasible region, see also the Appendix. As such, c_0 and c_1 reflect the relative steepness of the boundary of the feasible region in $(x,$ y$)$-space with respect to the iso-objective lines.

Figure 6 plots the constants c_0 and c_1 as functions of K for a specific set of parameter values. The plot remains very similar as long as the risk premium of stock is positive $(\bar{\mu} > r^c > 0)$ and the VaR critical return on stock is below the riskfree rate $(P^{-1}(\psi) < \exp(r^c))$. It is clear that c_0 lies between 0 and 1 and is monotonically increasing in the strike price K. The constant c_1 is negative and monotonically decreasing for $K \leq P^{-1}(\psi)]$, and monotonically increasing for $K > P^{-1}(\psi)$.

Given the definitions of c_0 and c_1, we now interpret the optimal portfolios in (5) in more detail.

2.2.1 Optimality of cash and call

If $c_1 \geq c_0$, the top asset allocation presented in (5) is optimal. As remarked earlier, a portfolio containing an equal number of stocks and put options is equivalent to a riskfree cash investment combined with a long position in call options. Such investment strategies have often been investigated in the literature without a formal proof of their optimality properties in the mean/VaR framework, see, e.g., Merton, Scholes, and Gladstein (1978) or Zimmerman (1996). The range of strike prices for which $c_1 \geq c_0$ is the halfline starting from the most rightward vertical dashed line in Figure 1.

The economic intuition for the condition $c_1 \geq c_0$ can be seen by solving (4) under an additional constraint that fixes either the number of stocks, $y = y^*$, or the number of options, x $= x^*$, with y^* and x^* fixed constants. It can then be proved using standard Lagrangean optimization that the condition $c_1 \geq c_0$ boils down to the requirement that the shadow price of either of the above two constraints is positive. Consequently, if the number of stocks or options is fixed in advance and if a feasible solution exists under this constraint, it pays to relax the constraint and increase the number of stocks
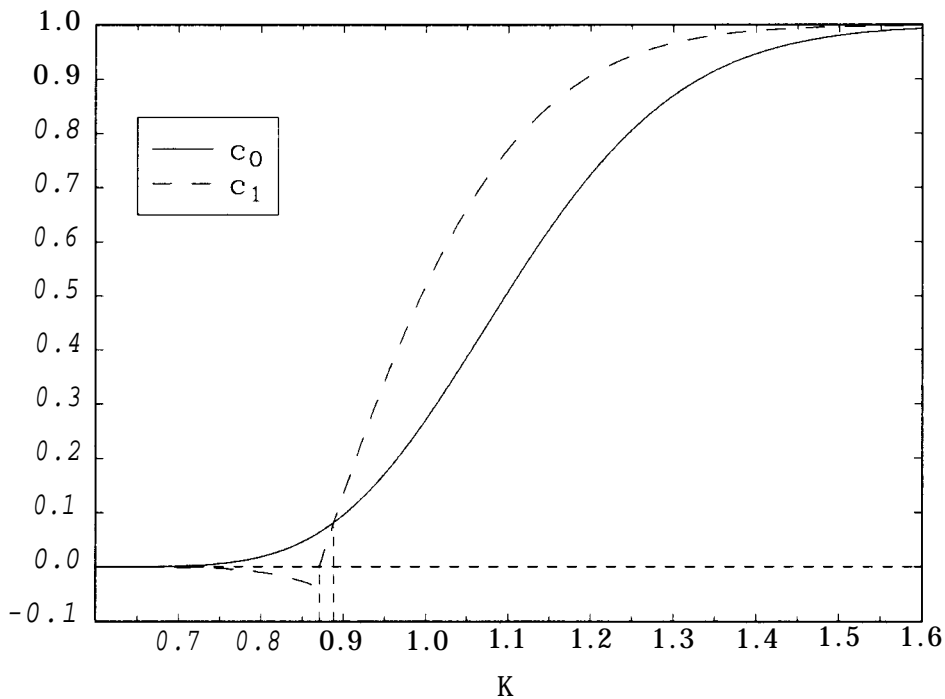
Figure 1: Slope coefficients c_0 and c_1 from (6) and (7) as a function of the strike price K of the put option; $r^c = 5\%$, $\mu = 10\%$, $\sigma = 15\%$, $\psi = 0.05$.

and/or options. It follows from the Appendix that this line of reasoning only holds as long as the final asset level A_1 is a monotonically non-decreasing function of the stock price, see also Dybvig (1988). This is the case if $y \geq 0$ and $x \leq y$. In this region, $c_1 \geq c_0$ thus implies that it is profitable from an expected return perspective for a given VaR profile to increase the number of options and the number of stocks up to point where $x = y$. After this point the final asset level A_1 will no longer be non-decreasing in the stock price $\exp(r^s)$, see Figure 6 in the Appendix.

The cash and call investment scheme provides the investor with the upward potential of stock, without the downside-risk associated with a naked stock strategy. It is easy to see that excess cash is needed in order to implement a long call strategy, i.e., $A^\star < \exp(r^c)$. $A_0 = \exp(r^c)$. This boils down to the requirement that the maximum allowable VaR of the investor must be positive, implying that he is willing to take some risk. The 'spare cash' left after guaranteeing the critical asset level $A^\star$ with certainty should, from a maximum expected return perspective, be used to buy call options.

2.2.2 Optimality of short put options

There is a small region of strike prices for which it is optimal to invest nothing in stocks while simultaneously shorting put options and investing the proceeds at the riskfree rate. This region corresponds to the small interval characterized between the vertical dashed lines in Figure 1. In this region, $c_0 \geq c_1 > 0$. Using similar constrained optimization programs as in Subsection 2.2.1, we obtain that $c_0 \geq c_1 > 0$ implies that the shadow prices on fixed stock and option investments are negative. It thus pays to decrease option and stock holdings up to the point where the VaR constraint is binding and the final asset level A_1 becomes non-monotonic in the stock price $\exp(r^s)$. Just as in the case of an unbounded program (4), see Subsection 2.2.3, the optimality of short put options is due to the choice of VaR as the downside risk measure. It gives rise to aggressive pay-off functions and may, therefore, have limited practical relevance. The optimality of the short put strategy vanishes if the extent of shortfall is also taken into account, e.g., by using expected shortfall or semivariance as a downside-risk measure. Finally note that the short put strategy is only feasible if the investor is willing to take some risk, i.e., $A^* < \exp(r^c)$.

2.2.3 Unbounded

The problem (4) is unbounded if $c_1 < 0$, which occurs for a large interval of (far out of the money) strike prices, see Figure 1. For typical parameter values, we have $P^{-1}(\psi) < \exp(r^c)$, such that $c_1 < 0$ implies that the VaR critical pay-off of the option is smaller than that of an equivalent riskfree investment. As a result, it is optimal to short an infinite amount of put options. The proceeds of this transaction can be invested in cash and in stock in such a way that the VaR constraint in (4) is just met, Similar results were established by Pelsser and Vorst (1995).

The above investment strategy gives rise to extremely aggressive pay-off patterns and is generally not practically implementable. In fact, one can argue that the solution is driven by an inadequate specification of risk constraints: huge losses with a small probability are acceptable. Note that even in cases where no feasible solution exists if the number of options is restricted to zero, the unbounded solutions can be feasible and optimal. For example, if $r^c = 0.05$, $A^* = 1.1$, and $A_0 = 1$, it is clear that the investor's maximum VaR is negative, such that no riskfree investment has a tolerable risk profile. This implies that no feasible investment strategy exists if one can invest in stock and cash only. Depending on the set of parameters' there may

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[1]It suffices that $\bar{\mu} > r^c$, $P^{-1}(\psi) < \exp(r^c)$, and $A^* < \exp(r^c)$.

exist a "bust or boom" strategy: for $c_1 < 0$ one can (theoretically) short huge amounts of options and invest the proceeds partly in stock and partly in cash. In this way, one obtains a feasible strategy if the strike price of the option is sufficiently low, i.e., if the probability of the put option being exercised is sufficiently small. It is clear that the unbounded solution is an artifact which is due to the specific choice of the downside-risk measure: shorting sufficiently far out of the money puts does not increase the VaR, i.e., the probability of shortfall. It does, however, (i) marginally increase the amount of funds available, and (ii) substantially increase the extent of shortfall. If we incorporate the latter in our (downside) risk measure, aggressive pay-off patterns as described above are no longer optimal.

2.2.4 Infeasible

The program (4) has no feasible solution if c_0, $c_1 > 0$ and $A^* - \exp(r^c) \cdot A_0 > 0$. If $A^* > \exp(r^c) \cdot A_0)$, the investor's maximum VaR is negative, implying he wants to earn more than the riskfree rate with a probability of at least $1 - \psi$. If c_0, $c_1 > 0$ however, this cannot be achieved.

3 Efficiency evaluation

We now turn to a comparison of the efficient portfolio derived in Section 2 with a stock-only strategy and an optimal portfolio insurance strategy. We concentrate on VaR as the relevant downside-risk measure, i.e., $\kappa = 0$. Section 4 contains some remarks as to how our results generalize to alternative downside-risk measures such as expected shortfall and semivariance. The optimal stock-only investment strategy $(x^{so} = 0)$ is given by

$$y^{so} = \frac{A^* - e^{r^c}}{P^{-1}(\psi) - e^{r^c}},\tag{8}$$

if the pair (x^{xo}, y^{so}) is a feasible strategy, with the superscript ^{so} denoting stock-only. The optimal portfolio insurance strategy under a VaR constraint is derived in Ahn et al. (1998). These authors minimize the cost of the option investment instead of maximizing the expected pay-off as in (4). This is equivalent to our approach for a fixed value of the strike price K. If we also want to optimize over the strike price as in Ahn et al. (1998), the two approaches differ slightly (for fixed stock investments). In our present setting with the expected pay-off as the objective, we obtain for fixed K the optimal
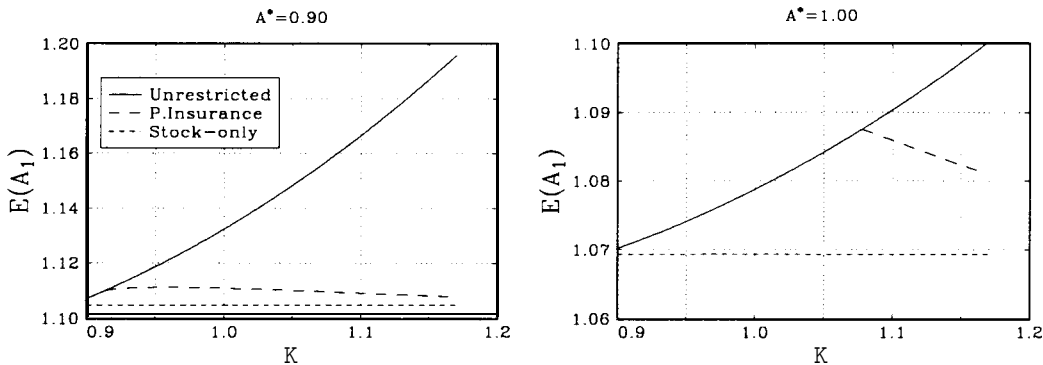
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Figure 2: Expected final asset value $E(A_1)$ as a function of the strike price K of the option for the unrestricted investment strategy, a portfolio insurance strategy with fixed stock exposure $y = y^* = 1$, and a stock-only investment strategy; $r^c = 5\%$, $\mu = 10\%$, $\sigma = 15\%$, $\psi = 0.05$, $A_0 = 1$.

strategy

$$x^{pi} = x^{pi}(y^*) = \frac{(A^* - e^{r^c}) - y^* \cdot (P^{-1}(\psi) - e^{r^c})}{\max(0, K - P^{-1}(\psi)) - P_o^0 \cdot e^{r^c}}, \tag{9}$$

where y^* denotes the fixed stock investment which has to be insured. Again we assume that the optimal portfolio insurance pair (x^{pi}, y^*), is feasible, with the superscript ^{pi} denoting portfolio insurance.

Figure 2 presents the expected return of the three investment strategies as a function of the strike price of the option. The result is computed for a VaR probability of $\psi = 5\%$ and two critical asset levels, $A^* = 0.9, 1.0$. At the riskfree rate of 5% used in the plots, this corresponds to VaR values of 15 cents and 5 cents per dollar of the invested notional. The results remain qualitatively similar as long as $\bar{\mu} > r^c$, $A^* < \exp(r^c)$, and $P^{-1}(\psi) < \exp(r^c)$, which are relevant restrictions from an empirical point of view.

Note that for both values of A^* the allowable VaR is positive, given the initial asset level $A_0 = 1$ and the riskfree rate $r^c = 5\%$. Therefore, the efficient portfolio derived in Subsection 2.2 contains a positive number of stocks and options for sufficiently high strike prices K. The precise portfolio compositions for the different investment strategies are presented in Figure 3. For the portfolio insurance strategy, we follow Ahn et al. (1998) by assuming a fixed number of stocks equal to $y^{pi} = y^* = 1$. Note that this implies that the initial asset level consists of stock only. To limit the VaR of this portfolio, cash can be borrowed at the riskfree rate to buy put options, compare Ahn et al. (1998).

Obviously from Figures 2 and 3, the stock-only investment strategy is invariant to changes in the strike price of the option. This investment strategy
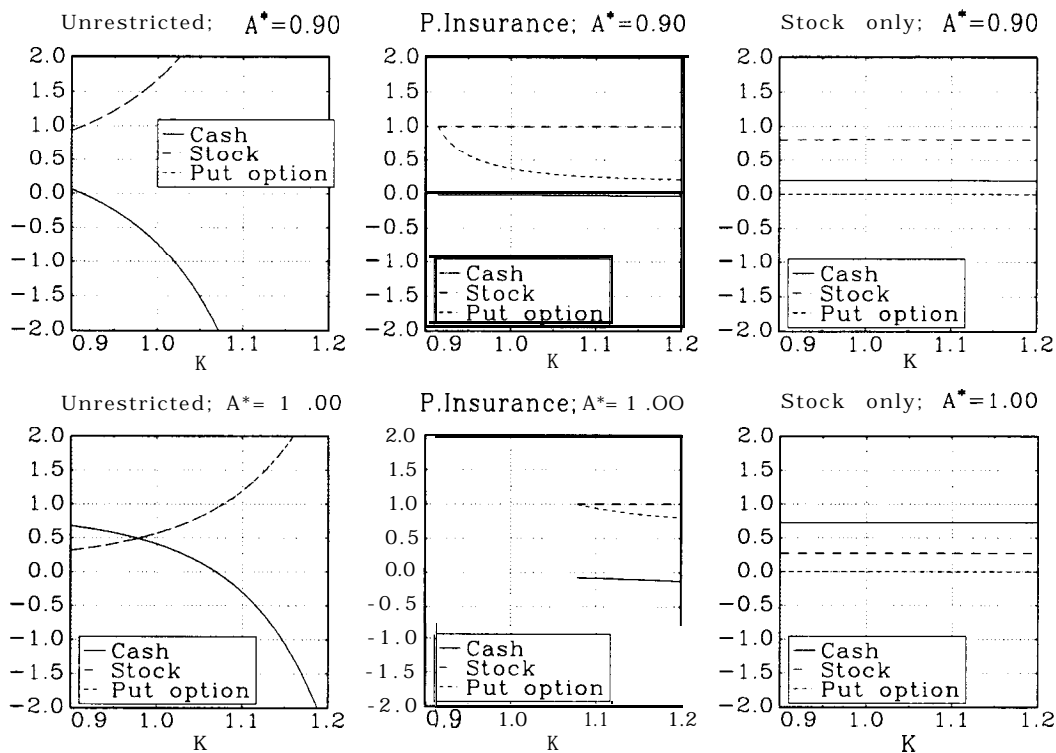
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Figure 3: Composition of optimal portfolios using unrestricted optimization, a portfolio insurance approach (fixed one unit stock investment $y = y^* = 1$), and a stock-only investment strategy (i.e., no options x = 0). The figure presents the number of stocks and options and the amount invested in the riskfree asset, all as a function of the strike price K of the option; $r^c = 5\%$, $\mu = 10\%$, $\sigma = 15\%$, $\psi = 0.05$, $A_0 = 1$.

has the lowest expected return of the three strategies considered, illustrating the usefulness of incorporating options in the portfolio if options are available.

The expected return on the optimal portfolio insurance strategy, if feasible, is always higher than that of the stock-only strategy. For feasibility, the available option needs a sufficiently high strike price. This result hinges on the level of the fixed exposure to stock and on the VaR level. If the risk tolerance is higher, i.e., if the VaR $\exp(r^c) - A^*$ is higher, the portfolio insurance strategy for $y^* = 1$ becomes feasible for a wider range of strike prices K. As in Ahn et al. (1998), we can optimize the expected pay-off of the portfolio insurance strategy over the strike price of the option. For $A^* = 1$, for example, this implies that the same strategy is chosen as in the unrestricted optimum. The optimal strike price in this case corresponds to a 7.8% in the money put option. For $A^* = 0.9$, the optimal portfolio insurance strategy uses a 4%

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out of the money put option. Optimal portfolio insurance now leads to a different asset allocation strategy than unrestricted optimal asset allocation. This also appears from the fact that the expected pay-off for $K = 0.96$ of the unrestricted strategy exceeds that of the portfolio insurance strategy by about 100 basis points. The portfolio composition for the portfolio insurance strategy as presented in Figure 3 shows that cash is borrowed at the riskfree rate in order to buy protective put options. The number of options bought is decreasing in the strike price K for K sufficiently large. This is intuitively clear, as options with a higher K provide more protection and are also more expensive.

As seen in Figure 2, the unrestricted strategy has by far the highest expected return of the three strategies considered. This is especially clear for sufficiently high strike prices K. In accordance with Theorem 1, Figure 3 shows that the optimal portfolio contains a long riskfree cash investment to guarantee compliance with the VaR constraint. The remainder is invested in an equal number of stocks and options. The expected return of the unrestricted strategy over the portfolio insurance approach (for fixed K) is monotonically increasing in the strike price, see Figure 2. Moreover, if we optimize the expected return for a given VaR over the strike price K, the difference in expected pay-offs even tends to infinity. These results can easily be explained in the mean/downside-risk framework we consider. Nevertheless, such solutions are undesirable from a practitioner's point of view. We elaborate on these pathologic findings in the next section.

4 Caveats to the mean/downside-risk framework

In order to obtain insight into the efficiency results of the previous section, we plot the pay-off patterns of the optimal portfolio insurance and the unrestricted strategy at the investment horizon as a function of the stock price. The result is presented in Figure 4. We consider the same critical asset levels as in Figure 2. We assume that the available put options have strike prices of $K = 0.96$ and $K = 1.078$ for $A^* = 0.9$ and $A^* = 1.0$, respectively. These strike prices are optimal for the portfolio insurance strategy.

We note that the portfolio insurance strategy has a less non-linear pay-off pattern if it does not coincide with the unrestricted strategy. As explained earlier, this situation arises for a sufficiently high risk tolerance, i.e., allowable VaR level. The optimal pay-off of the unrestricted strategy, however, always has the same structure: a call option combined with a riskfree investment.
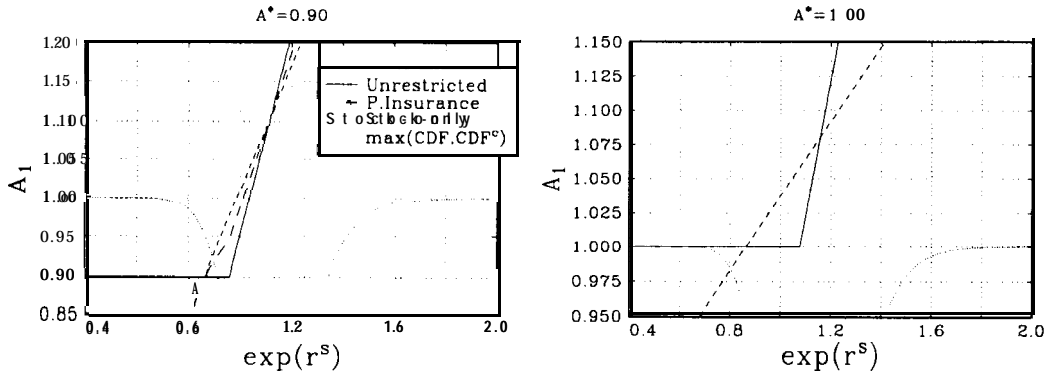
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Figure 4: Final asset value A_1 as a function of the stock price realization at $t = 1$. The strike price K of the option is 0.96 and 1.078 for $A^* = 0.9$ and $A^* = 1.0$, respectively; $y^* = 1$, $r^c = 5\%$, $\mu = 10\%$, $\sigma = 15\%$, $\psi = 0.05$, $A_0 = 1$. $\max(CDF, CDF^c)$ denotes the maximum of the probability of the stock price falling above and below $\exp(r^s)$.

Optimizing the unrestricted pay-off over the strike price, we obtain $K \rightarrow \infty$. The corresponding expected pay-off also diverges to infinity, implying an infinite efficiency gain over the optimal portfolio insurance strategy. The pay-off pattern of the optimal (over K) unrestricted strategy, however, is highly unrealistic. As the call option for $K \rightarrow \infty$ is extremely far out of the money, the pay-off of the strategy will be equal to the riskfree cash investment with very high probability. With a very small probability, the option expires in the money, creating a large leverage effect and an extremely large pay-off. This amounts to the investor taking a bet on an extreme realization of the stock price, something appropriately labeled the casino *effect* by Dert and Oldenkamp (1997). The casino effect is not an artifact due to the choice of the specific downside-risk measure. Dert and Oldenkamp (1997) prove that in complete markets for general values of $\kappa > 0$ in (2), extreme bets similar to the one described above are optimal.

Preventing the casino effect requires a reformulation of the investment problem as laid out in (4). The main problem with (4) is that no realistic distinction is made between different pay-off functions that have identical downside-risk values, see also Figure 4. As argued in the literature, downside-risk is the most relevant notion of risk from a practitioners point of view, see, e.g., Sortino and van der Meer (1991). Combined with the expected return as a measure of profitability, however, downside-risk does not provide a sufficiently rich characterization of the (utility) ordering of different pay-off profiles. An extreme example of this was given above for the optimal portfolio insurance strategy and the optimal unrestricted strategy. Though

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investors may perceive risk in terms of the event and extent of falling below a certain benchmark return or asset level, it is not true that they are indifferent with respect to the variability of the pay-off above the benchmark. Both the expected return and the downside-risk measure are insensitive to this type of variability. The combination of these two measures in an efficiency framework, therefore, does not provide an adequate description of the investor's preference ordering over different pay-off profiles.

By contrast, the mean/variance paradigm does not suffer from this deficiency, as the variance takes the variability of the pay-off into account both above and below the shortfall asset level. The main drawback of the variance, however, is that it assigns equal weight to deviations above and below the threshold asset value, which is less realistic, see Sortino and van der Meer (1991). Moreover, with the asymmetric return distributions that come with optioned portfolios, the use of the variance can lead to perverse efficiency results, see Leland (1996) and Lhabitant (1997).

There are several ways to cope with the above deficiency in the mean/downside-risk framework. The first approach controls the casino effect by introducing additional shortfall constraints in (4). For example, in addition to (4), the manager might require that the assets A_1 exceed a second threshold value $A** > A*$ with a sufficiently high probability. This effectively reduces the size of the feasible region in Figure 6. Such an approach is adopted by Dert and Oldenkamp (1997). Their findings reveal that the introduction of additional shortfall constraints indeed yields more realistic portfolios, but mitigates or masks rather than solves the casino effect. As an alternative to the solution of Dert and Oldenkamp (1997), one can fix either the allowable range of strike prices K or the maximum amount invested in stock. The latter approach closely links to the portfolio insurance strategy, where stock holdings are fixed in advance. It explains why the casino effect has failed to emerge clearly from the literature on portfolio insurance using options.

A second approach to solve the casino effect uses alternative specifications of the objective function. For example, one can replace the expected asset level by a nonlinear transformation of the final asset value. This reflects the tradeoff between expected return and variability of VaR acceptable pay-offs. Alternatively, one can replace the expectations operator by a different measure of profitability, e.g., the median pay-off or some other quantile. Both the first and second approach are quite ad-hoc and may to some extent prevent the casino effect.

A third possible approach that encompasses the previous two is to use general utility functions to derive optimal portfolios. Such utility functions should incorporate the appropriate trade-off between downside-risk, upward potential or expected return, and variability of risk-acceptable pay-offs. De-

signing an empirically relevant version of a utility function that incorporates all these issues, however, is not trivial, and more research has to be directed to this area. A simple utility function that goes some way in incorporating the above three key characteristics of pay-off distributions is the well-known threshold power utility function:

$$
U(A_1) = \begin{cases} (A_1 \quad A^*)^{1-\gamma}/(1-\gamma) & \text{for} \quad A_1 > A^*, \\ \\ -\infty & \text{for} \quad A_1 < A^*, \end{cases} \tag{10}
$$

where $\gamma > 0$ denotes the risk aversion parameter.[2] The threshold level A^* ensures a restriction on downside-risk: portfolios with a positive probability of a pay-off smaller than A^* cannot be optimal, because $U(A_1) = -\infty$ for $A_1 < 0$, and thus $E(U(A_1)) = -\infty$. Variability of the pay-offs above the threshold level are taken into account by the concave shape of the utility function. Finally, profitability is accounted for by the fact that the utility function is monotonically increasing.

The expected utility based on (10) is maximized with respect to (x, y), both with and without the restriction $y = 1$. The threshold A^* is set to 0.9 as in the left-hand panel of Figure 4. The results are presented in Figure 5.

To compare the portfolio insurance strategy with the optimal unrestricted strategy, we compute the additional amount of initial funds (A_0) needed to equate the expected utility of the portfolio insurance strategy to that of the unrestricted strategy. To facilitate the comparison, we optimize over the strike price of the option in the range $[0.9, 1.2]$,[3] i.e., we compare the K-optimal portfolio insurance strategy with the K-optimal unrestricted strategy for a given degree of risk aversion γ. The top-left panel of the figure reveals that the efficiency loss varies from 250 basis points for $\gamma = 2$ to 650 basis points for $\gamma = 6$. This is intuitively clear. The increase in. flexibility for the investor caused by dropping the constraint on the fixed investment in stock becomes more worthwhile if risk is penalized more heavily in the objective function. The top-right and bottom-left panels of the figure demonstrate that increased risk aversion leads to lower stock investments and lower option investments. In particular, for $\gamma \geq 3$, the manager starts short selling instead of buying put options, though the amounts remain small for near and at the money options. It is useful to re-emphasize that even if one agrees on solving or mitigating the casino effect by resorting to threshold power utility optimization, the unrestricted strategy still provides significant efficiency

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[2]For $\gamma = 1$, (10) has to be replaced by $\ln(A_1 - A^*)$.

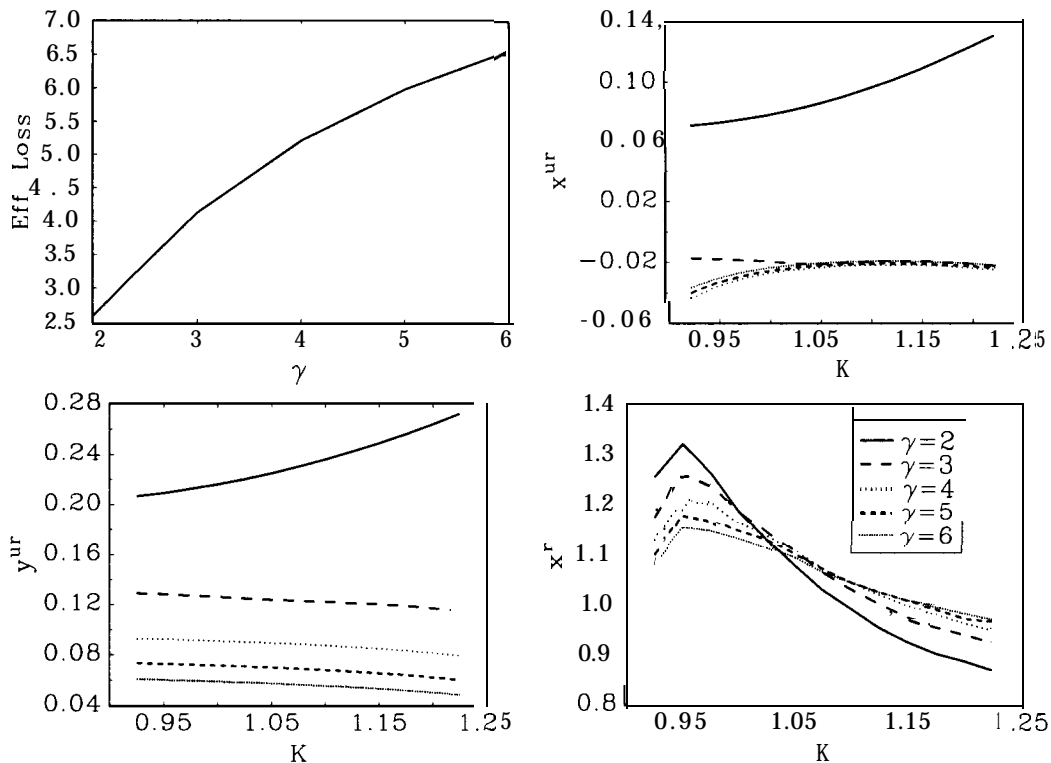[3]This range of strike prices was used to facilitate the numerical computations, which are quite involved.

Figure 5: Efficiency loss for various degrees of risk aversion γ of optimal portfolio insurance versus an optimal unrestricted strategy computed over a grid of strike prices (top-left). The efficiency loss is computed as the percentage of additional funds needed to equate the expected utility of the portfolio insurance strategy to that of the unrestricted strategy. The fractions invested in options and stock for the unrestricted strategy are in the top-right and bottom-left panels, respectively. The bottom-right panel contains the option investment for the portfolio insurance strategy. The plots are based on (10) and $A^* = 0.9$, $y^* = 1$, $r^c = 5\%$, $\mu = 10\%$, $\sigma = 15\%$, $A_0 = 1$.

gains over the portfolio insurance strategy.

To conclude this section, note that the reported caveats in principle also apply to optimal portfolio insurance as in Ahn et al. (1998). The extent to which these caveats can be exploited, however, is hindered significantly by the constraint that holdings in stock may not be changed. As mentioned before, fixing the number of stocks in advance is one possible way to overcome the casino effect. Fundamentally, however, fixing the number of stocks masks rather than solves the pathologies in the mean/downside-risk efficiency framework.

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5 Conclusions

In this paper we analytically derived the optimal portfolio in a $\mathrm{mean/downside}$-risk efficiency framework if options are available. Using this optimal solution, we showed that portfolio insurance and stock-only investment strategies are inefficient from a mean/downside-risk efficiency perspective. Given a downside-risk limit, the expected return can be increased substantially by relaxing constraints on the amount invested in the risky asset and/or the amount invested in options on this asset.

For large ranges of strike prices of the option, the optimal unrestricted portfolio consists of a riskfree investment that suffices to meet the downside-risk constraint. The remaining funds, if any, are invested in a call option. By optimizing over the strike price of the call option, we were able to point out a serious caveat to the mean/downside-risk efficiency framework. This framework implies optimality of portfolios that essentially bet on extreme realizations of the stock price. With high probability, the pay-off equals the shortfall level, while with small probability, the pay-off is extremely high. Dert and Oldenkamp (1997) appropriately label this the casino effect. Such portfolios seem very unlike the pay-off function desired by a typical investor. This peculiar result is robust to the specification of the downside-risk measure, see Dert and Oldenkamp (1997). Moreover, if a portfolio insurance strategy is augmented with the selection of the exposure to the basic risk factor, identical caveats apply to this strategy.

A second caveat was reported if value-at-risk (VaR) is used as a downside-risk measure. In that case, it is optimal to sell unlimited amounts of far out of the money put options and to invest the proceeds in stock, which gives rise to extremely aggressive pay-off functions. This is caused by the fact that VaR as a downside-risk measure only accounts for the event of adverse price movements, and not for the extent of their impact. In that sense it is an inadequate measure for downside-risk.

We mentioned several alternatives to overcome the pathologies in the mean/downside-risk framework. The most relevant of these uses general utility functions and expected utility maximization to derive optimal portfolios. Such utility functions should include an appropriate tradeoff between downside-risk, upward potential or expected return, and variability of the pay-offs that have an acceptable downside-risk profile. Such a tradeoff effectively requires a preference ordering of an investor over pay-off distributions that are characterized by multiple (> 2) attributes. We discussed the effect of the familiar threshold power utility specification, which goes some way in dealing with these issues. Much research effort, however, has still got to be put into designing empirically relevant versions of utility functions for asset
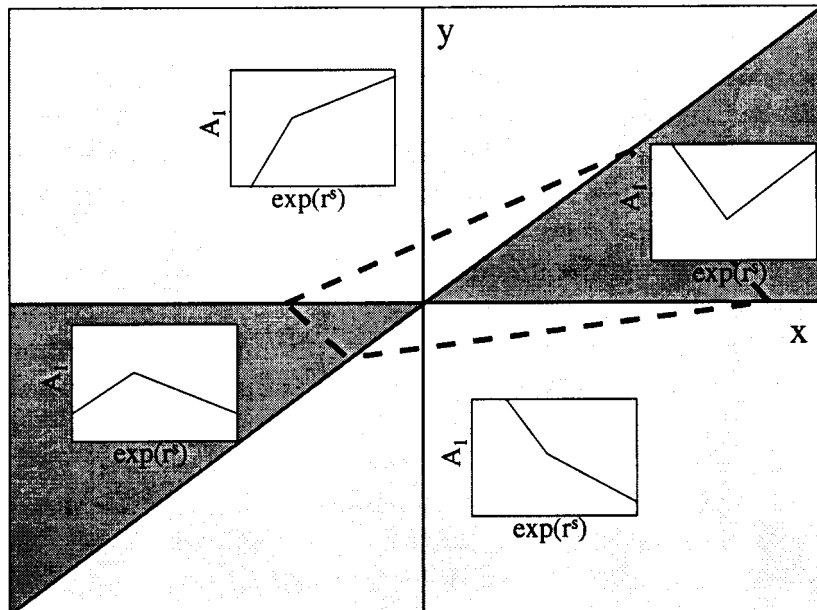
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Figure 6: Typical shape of the pay-off function for different values of the fractions invested in stock (y) and options (x); the dashed polygon reveals the typical shape of the feasible region.

allocation problems.

Irrespective of the approach adopted to overcome the caveats to the mean/downside-risk framework, unrestricted optimal asset allocations provide significant efficiency improvements over optimal portfolio insurance type allocations. This stresses the need for simultaneous rather than partial asset allocation procedures.

A Appendix: Proof

Proof of Theorem 1: Figure 6 contains 4 subdiagrams giving the shape of the final asset value A_1 as a function of the stock price. The (x, y)-plane is divided into 4 sections, dictated by the lines y = x and y = 0. The figure also shows a typical shape of the feasible region of problem (4).

In the region $\{(x, y) \quad y \geq 0, x \leq y\}$, A_1 is monotonically increasing in the stock price $\exp(r^s)$. With a slight abuse of notation, define $A_1(\exp(r^s))$ as the asset level as a function of the stock price. We establish that

$$P(A_1 \leq A^*) \leq \psi \Leftrightarrow$$

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$$P(A_1(\exp(r^s)) \leq A^*) \leq \psi \Leftrightarrow$$
$$P(\exp(r^s) \leq A_1^{-1}(A^*)) \leq \psi \Leftrightarrow$$
$$A_1^{-1}(A^*) \leq P^{-1}(\psi) \Leftrightarrow$$
$$A_1(P^{-1}(\psi)) \geq A^*, \tag{A1}$$

where A_1^{-1} denotes the inverse function of $A_1(\cdot)$, and where $P^{-1}(\cdot)$ denotes the inverse c.d.f. As $A_1(\cdot)$ is linear, (Al) presents the border of the feasible region in $(x,\ y)$-space. Note that this part of the border is only valid in the region $\{(x,\ y)|\ y \geq 0,\ \mathrm{x} \leq \mathrm{y}\}$. In the other regions, similar derivations as (Al) can be used to establish the remaining parts for the border.

Clearly, the border in the region $\{(\mathrm{x},\ y)|\ \mathrm{y} \geq 0,\ \mathrm{x} \leq \mathrm{y}\}$ has slope coefficient c_1 as in (7). It is also clear that the iso-objective curves have slope c_0, which exceeds zero for usual parameter values, see Section 2. Note that the iso-objective curves are increasing in the upper-right direction. We can now distinguish three situatios, assuming the feasible region is non-empty.

Case (i): $c_1 > c_0 > 0$, now the slope of the border of the feasible region is steeper than that of the iso-objective curve, resulting in the intersection point of the x = y-line and the border of the feasible region being optimal. Solving for this intersection point results in the first portfolio given in (5).

Case (ii): $c_0 > c_1 > 0$, we now have the reverse situation of case (i). The slope of the iso-objective curve is steeper than that of the border of the feasible region. Given that the iso-objective curves indicate increasing expected returns in the upper-right direction in the $(x,\ y)$-plane, the left intersection point of the border of the feasible region with the line y = 0 gives the optimal portfolio. Solving for this intersection point gives second portfolio presented in (5).

Case (iii): $c_1 < 0$, now the feasible region is unbounded.
This proves the theorem.

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