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Spatial Autocorrelation
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February 16, 199'7

Abstract

In this paper we analyze the supply of network infrastructure at the regional level. Special attention is paid to spatial autocorrelation aspects. A method is developed to test which impact international borders have on the sup ply of network infrastructure in border regions. An empirical illustration is given for the regions in the EU. We find that in the case of railways spatial spill-overs are indeed significant. However, the hypothesis that border effects play a role in railway densities has to be rejected.

Keywords: **Network infrastructure; Spatial autocorrelation; Borders; Rail ways**

JEL classifaction: R10

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1 Introduction

Network infrastructure (highways, railways, canals, etc.) is mainly supplied by the public sector operating at various spatial levels: local, regional, national, and supranational. In most parts of the world the supranational level of the public sector is weakly developed, however, although a tendency can be observed that this level is gaining importance. An example is the European Union that is becoming more active in the supply of infrastructure, as witnessed by the planning of Trans European Networks.

The lack of importance of the supranational level in the planning of infrastructure leads to the conjecture that international transport links are weakly developed, if compared with intranational transport links. An example can be found for rail and road links through the Alps connecting Italy, Austria, Switzerland, Germany and France. In this part of Europe there indeed seems to be a lack of supply of international links. However, it is not clear whether this is due to a neglect of international links in the planning of infrastructure networks, or to the high costs involved in construction in mountainous areas. Therefore, we need a more refined (i.e., multivariate) analysis of infrastructure supply.

In this contribution we shortly discuss a simple model of infrastructure supply at the regional level. Since infrastructure in a region serves both intraregional and interregional demand, we may expect interregional spill-overs in the supply of infrastructure. These spatial spill-overs can be analyzed by means of spatial autocorrelation. Border effects in the provision of spatial infrastructure would imply that the spatial spill-overs will be different between regions in the same country compared with regions in different countries. We develop a method to allow for different spill-overs in both cases. Finally we test the method for a data set with 91 European regions.

2 Regional infrastructure supply

In the context of the present article we do not develop a complete politico economic model of interregional supply of transport infrastructure by the public sector. Instead we list some factors that probably influence the distribution of infrastructure investments among regions.

Demand for transport.
A possible proxy is the total regional population. Thus in regions

with a high level of population we expect a large infrastructure stock to accommodate the demand.

There is reason to expect economies of scale: infrastructure density increases at a decreasing rate with population density because in densely populated areas the average distance of trips may be expected to be smaller than in other areas. In addition indivisibilities in transport infrastructure may easily lead to excess supply of infrastructure capacities in sparsely populated areas. This can be tested by adding a quadratic term in the model to be estimated.

Possibilities to finance the investments.

A possible proxy for the capacity of the public sector to spend money is the income per capita. Two possibilities can be mentioned in this respect- First, when the regional government is the major actor, the investment is paid by regional taxes. In this case, the regional income per capita is the relevant determinant. The other possibility is that the investment projects are financed by the national government on the basis of a national investment fund. In this case, national income per capita is the relevant determinant.

The costs of construction of infrastructure.

These costs vary in many respects between regions and countries. Since a regional price index of infrastructure investment costs does not exist, we are forced to use a proxy by defining an altitude variable that captures the high construction costs for bridges and tunnels in mountainous regions-

A regional policy variable.

Some governments use infrastructure investments to stimulate regional growth. Thus it may happen that -ceteris paribus- lagging regions have relatively large infrastructure stocks.

Interregional spill-overs.

Infrastructure serves both intra- and interregional demand. Therefore, one may expect that a region where the surrounding regions have high infrastructure densities, will also have high densities themselves. Usually, roads do not suddenly stop when they cross a border, and this leads one to expect that infrastructure densities in contiguous regions are positively related. However, in the case of a lack of international

cooperation, or when demand for border crossing trips is small, this spill-over effect may be smaller or even absent.

In the next two sections we will discuss econometric aspects of estimating these differentiated spill-over effects.

3   Modelling spatial autocorrelation

An important implication of the spatial spill-over effects mentioned in section 2 is that observations on infrastructure density are interdependent. This dependency is known as spatial autocorrelation, as it is due to the spatial structure within the data. Ignoring this spatial dependency can introduce specification error and therefore cause bias in estimators. The generally accepted spatial model to cope with this problem is specified as follows, see ANSELIN (1988a):

$$Ay = X\beta + B^{-1}\epsilon \qquad (1)$$

where y is a (N x 1) vector, $\epsilon \sim N(0,\Omega)$ and β a (k x 1) vector of parameters corresponding to the exogenous variables X, Ω is a diagonal matrix', and

$$
\begin{aligned}
A &= I - \phi W \\
B &= I - \theta V.
\end{aligned}
\qquad (2)
$$

The binary weight matrices **W** and **V** (both (N x N)) are assumed to be known and define the spatial structure within the data. The weight or connectivity matrices are filled with ones and zeros according to whether the corresponding locations are connected or not. In many applications **W** and **V** are identical. Connected means 'have a common border' in this specific context. By definition, a location cannot be connected to itself, thus the connectivity matrix has a zero diagonal. I is the unit matrix and ϕ and θ are spatial interdependence parameters.

Special cases of (1) are obtained by setting ϕ or θ equal to zero. Thus we arrive at:

'In ANSELIN (1988a) heteroskedasticy is implemented by letting the diagonal elements of Ω be a function of exogenous variables

3

the spatial AR model[2](θ = 0) :

$$Ay = X\beta + \epsilon \qquad (3)$$

and the spatial ERROR model (ϕ = 0) :

$$y = X\beta + B^{-1}\epsilon. \qquad (4)$$

Note that the spatial ERROR model can be rewritten as

$$By = BX\beta + \epsilon. \qquad (5)$$

If $W = V$ these two models generate the same variance matrix[3] for y. This does not mean that the two models are equivalent; as we can see in (5) the exogenous variables are treated differently.

A more general formulation of the spatial autocorrelation model would be, see e.g. FLORAX and FOLMER (1992):

$$Ay = CX\beta + B^{-1}\epsilon \qquad (6)$$

where $C = I - \lambda U$, with U a binary weight matrix and λ a spatial dependence parameter.

Alternative extensions of models (1) and (6) are

$$Ay = X\beta + B\epsilon \qquad (1')$$

and

$$Ay = CX\beta + B\epsilon. \qquad (6')$$

The models are the spatial equivalents of the temporal ARMAX models, where both local and global effects are incorporated via the correlation structure of y, see HUANG (1984). This specification generates estimation problems, but opens a richer class of covariance structures. Details can be found in WINTERSHOVEN (1996). In the present paper we will confine ourselves however to models of the AR and ERROR type.

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[2]AR comes from autoregressive. This is to keep the analogy to the temporal models. In spatial terms it is mostly reffered to as the spatial *lag* model

[3]In practice this variance matrix is entirely filled with non-zero elements, which indicates that observations from all locations depend on each other. Therefore both the AR and the ERROR model can be interpreted to represent "global " spatial interrelationships.

4 Border effects in spatial aut ocorrelation

We consider the possibility to differentiate between two types of contiguity, i.e., contiguity of regions being part of the same country, and contiguity of regions separated by a national border. Given the above discussion on the importance of nation states in infrastructure supply, it is relevant to investigate whether borders play a role in the supply of network infrastructure. To analyze this problem we start with the model

$$Ay = X\beta + B^{-1}\epsilon$$

discussed in the section above. We restrict ourselves to $W = V$ and let W fall apart in W_1 and W_2.

The entries of W_1 equal 1 if two regions are *contiguous* and part of the *same* country and 0 otherwise. The entries of W_2 equal 1 if two regions are *contiguous* and part of *different* countries and 0 otherwise. This distinction implies

$$
\begin{aligned}
A &= I - \phi_1 W_1 - \phi_2 W_2 \\
B &= I - \theta_1 W_1 - \theta_2 W_2.
\end{aligned}
\tag{7}
$$

where $W_1 + W_2$ is 'rowsum = 1' standardized[4]. Therefore one retains for $\phi_1 = \phi_2$ and $\theta_1 = \theta_2$ the analysis for the 'rowsum = 1' standardized W case, thus making no difference between the types of contiguity[5].

Thus the introduction of the border effect leads to an additional W matrix, an idea already proposed by HORDIJK (1979). We expect that the national border coefficients, ϕ_2 and θ_2, take on values between 0 and their corresponding regional border coefficients, ϕ_1 and θ_1. When a national border coefficient is 0, the national border completely "destroys" the contiguity effect: there is no difference between two contiguous regions separated by a national border, and two regions in a country that are not contiguous. On the other hand, when the national border coefficient equals the regional border coefficient, the contiguity effect is not disturbed at all by the national

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[4]See ANSELIN (1988a) for other standardization options

[5]We have chosen for the sequence of first standardizing for all contiguous regions and then separating the two groups of regions (domestic versus foreign neighbours). An alternative would have been to first separate the two groups and standardize the two weight matrices afterwards. Since the number of foreign neighbours is smaller than the number of domestic neighbours this would lead to bigger values in the standardized W_2 compared with W_1. This would make ϕ_1 difficult to compare with ϕ_2.

border effect: it does not make any difference for the spatial interdependence whether the border separating two regions is a national one or not.

By the maximum likelihood (ML) estimation procedure we search for those parameters, which make the empirically observed data most likely. The likelihoodfunction has the same functional form as the probability function of y, but has to be interpreted as a function of the parameters instead of the observations. If we take the logarithm of the likelihood, which does not affect the estimate, we obtain the log-likelihood.

Now, from the first order conditions (see the Appendix) we see that $\hat{\beta}$ and $\hat{\sigma}^2$ can be solved analytically, namely:

$$\hat{\beta} = [(\hat{B}X)'(\hat{B}X)]^{-1} (\hat{B}X)'(\hat{B}\hat{A}y) \qquad (8)$$

$$\hat{\sigma}^2 = \frac{[\hat{B}(\hat{A}y - X\hat{\beta})]'[\hat{B}(\hat{A}y - X\hat{\beta})]}{N} \qquad (9)$$

The estimates of the spatial parameters have to be found numerically. Substitution of analytical solutions (8) and (9) in the log-likelihood gives:

$$L_C = C - \left(\frac{N}{2}\right)\ln(e'e) + \ln|A| + \ln|B| \qquad (10)$$

where C is a constant, and e = $B(Ay - X\beta)$.
This is the so called concentrated log-likelihood where we concentrate only on spatial parameters. If $\psi = [\phi_1, \phi_2, \ldots, \phi_p, \theta_1, \theta_2, \ldots, \theta_r]'$, with $p \geq 0$ and $r \geq 0$, then estimation can proceed according to the following steps:

[1] maximize L_C based on the starting value $\psi_0 = 0$; result $\hat{\psi}$.

[2] given $\hat{\psi}$ compute $\hat{\beta}$ and $\hat{\sigma}^2$.

5 Empirical results for railways

The model is estimated using observations for 91 regions in Europe at the NUTS-II level for the year 1986. The analysis is carried out for railroads; a similar analysis could be carried out for highways. The dependent variable y is formulated in density terms: the length of railroads in km per square km in the region. The dependent variables are represented as:

POP: population density (persons per square km)

GDP: Gross domestic product per capita (in ECU per head)

COST: dummy variable which attains the value 1 when the highest altitude in a region is at least 2000 m, and which is 0 in all other cases.

POL: dummy variable which attains the value 1 when a region is the subject of regional development policies, and which is 0 in all other cases.

When we ignore the spatial spill-over effects we arrive at the following estimation results (see Table 1):

OLS		
Variable	Coefficient	SE
$CONST$	-20.846	9.479
POP	115.786	21.567
POP^2	2217	11.952
GDP	0.729	0.089
$COST$	-15.039	5.853
L	-418.022	
R^2	0.813	

Table 1. OLS regression results of regional infrastructure supply.

The estimation results mainly support the hypotheses of section 2. Population density has a positive and significant impact on railway density. The result for the quadratic term, however, indicates that there are no economies of scale in the provision of railway infrastructure[6]. The income per capita variable has the expected positive sign and is significant. The result presented here relates to income levels measured at the national level. Estimation results with regional income per capita are less satisfactory. This result confirms our expectation that financial resources for investments in railway infrastructure in a region are mobilized at the national rather than the regional level. The cost dummy is significant: in mountainous areas the supply of railway infrastructure is smaller than in other areas, other features being equal. Estimations with the regional policy variable do not yield any significant results. Therefore this variable has been dropped.

As a first step to take spatial spill-overs into account, we estimate the *spatial AR(I) model* **and the** *spatial ERROR(l) model* **ignoring** the difference

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[6]When we judge the plot of POP towards infrastructure, a non-linear monotone increasing relation between the infrastructure variables and the population appears. To find this relation a Box-Cox transformation of **POP** seems to be the solution. However to incorporate this non-linearity we include the quadratic term POP^2.

between interregional borders and national borders. The estimation results read as follows (see Table 2):

Variable	AR(1) ML	AR(1) SE	ERROR(1) ML	ERROR(1) SE	ARERROR(1,1) ML	ARERROR(1,1) SE
$\hat{\phi}$	0375	0.083			0.038	0.149
$\hat{\theta}$			0.699	0.080	0.664	0.114
$CONST$	-15.031	8456	4.707	15219	0.970	15299
POP	89.534	20225	106.919	19.709	107.652	19.899
POP^2	16.831	10.939	10.777	10.565	10.904	10.721
GDP	0.418	0.103	0.438	0.140	0.450	0.151
$COST$	-8.086	5285	-6355	5.659	-6.661	5.7012
L	-410.402		-405.552		-405545	
$\hat{\sigma}^2_{ML}$	467.621	69.744	376.703	58.095	383.170	60272

Tabel 2. Maximum likelihood estimates of regional infrastructure supply taking into account spatial autocorrelation.

In both the spatial AR(1) and the spatial ERROR(1) model a significant spatial autocorrelation coefficient is found with a value that is substantial. In the spatial ARERROR(1,1) specification, only $\hat{\theta}$ is significant. Clearly, interregional spill-overs play an important role in the supply of railway infrastructure. In a qualitative sense the introduction of the spatial autocorrelation terms does not lead to changes for most of the explanatory variables, except for the cost dummy which becomes insignificant in all cases.

Finally we use the method described in section 3, to estimate to what extent there is a difference between intra-national and international spill-overs (see Table 3) ,

	AR(2)		ERROR(2)	
Variable	**ML**	**SE**	**ML**	**SE**
ϕ_1	**0276**	**0.083**		
ϕ_2	**0.523**	**0.106**		
θ_1			0.668	0.097
θ_2			0.852	0.191
CONST	-19.995	8.290	5.203	21.213
POP	100.650	19.663	108265	28.062
POP^2	5.504	10.653	10.680	15.045
GDP	0.652	0.114	0.392	0.192
COST	-14.065	5.223	-5.465	8.097
L	-407.619		-405.337	
$\hat{\sigma}^2_{ML}$	442.767	65.936	372.015	57.413

Table 3. Maximum likelihood estimates of regional infrastructure supply taking into account spatial autocorrelation and border effects.

	ARERROR(2,1)		ARERROR(1,2)		ARERROR(2,2)	
Variable	**ML**	**SE**	**ML**	**SE**	**ML**	**SE**
ϕ_1	0.113	0.127	-0.116	0.150	0.159	0.117
ϕ_2	0.311	0.158			0.465	0.132
θ_1	0.481	0.143	0.740	0.098	0.496	0.156
θ_2			0.935	0.163	-0.247	0.321
CONST	-12.135	11.981	13.703	17.387	-16.631	10.438
POP	108.772	19.891	105.760	19.365	98.792	19.012
POP^2	12.173	10.888	10.198	10.283	17.913	10.492
GDP	*0.495*	0.140	0.369	0.148	0.513	0.141
COST	*-7.729*	5.644	-4.610	5.565	-10.056	5.211
L	-404.234		-405.211		-402.2 12	
$\hat{\sigma}^2_{ML}$	395.493	60.201	352.848	56.690	377.833	57.173

Table 3 continued.

The introduction of the bordereffect coefficients leads to an unexpected result: the estimated coefficients $\hat{\phi}_2$ and $\hat{\theta}_2$ are bigger than respectively $\hat{\phi}_1$ and $\hat{\theta}_1$, except for $\hat{\theta}_2$ in the spatial ARERROR(2,2) model. Only in the AR(2) specification $\hat{\phi}_2$ is significantly higher than $\hat{\phi}_1$, which is bold printed. This is a surprising result; it means that railway supply in border regions is more strongly affected by the network density in the neighbour country than

by the density in neighbouring regions in the own country. Thus we have to reject the hypothesis that spatial autocorrelation in railway densities is smaller for "cross-border" compared with "domestic" pairs of regions-

In our search for the 'right 'model according to the highest likelihood, we present a summary with the likelihoods of our estimated models (Table 4).

$ERROR \backslash^{AR}$	0	1	2
0	-418.022	-410.402	-407.619
1	-405.552	-405545	-404234
2	-405.337	-405211	-402212

Table 4. Values of log-likelihoods for $ARERROR(i,j)$ models.

The ERROR(1) model deserves our preference. This again confirms that the introduction of the distinction between cross-border versus domestic contiguity does not lead to a significant improvement of the loglikelihood.

6 Concluding remarks

In this short paper we have shown that spatial spill-overs play a significant role in infrastructure supply. Ignoring these spatial autocorrelation issues can introduce specification error and therefore cause bias in estimators. A rather surprisii result of our analysis is that there are no signs of barrier effects related to national borders in the provision of railway infrastructure. In most specifications the national border coefficient is not significantly different from the regional one, which implies that the impact from contiguous foreign regions is not different from that of contiguous regions in the own country.

We conclude that as far as railway infrastructure density is concerned there are no signs of lack of international cooperation leading to a neglect of opportunities to connect networks at different sides of a national border. In another analysis with similar data (RIETVELD 1993), it was found that indeed border regions at the NUTS-II level do not suffer from relatively low infrastructure densities. For railway infrastructure a possible explanation of this lack of border effect is that the construction of the network mainly took place before World War II, a period where national borders probably did not have such a large impact compared with the period afterwards- One should be aware, however, that in the present paper we only discuss network density, and not the level of service. An examination of railway services reveals a clear gap in cross border services in terms of frequency

10

(RIETVELD, 1993). Our present research implies that the lack of service in cross border connections is not caused by a lack of infrastructure in border regions. Thus, other factors, such as a lack of demand for cross border transport play a role.

Further research on this topic of barrier effects on infrastructure supply is possible by addressing other modalities (especially highways), by changing the spatial scale of analysis (using smaller regions than the rather large NUTS-II regions) and by making the analysis dynamic since the evolution of transport infrastructure networks is a matter of many decades.

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A Appendix

We define the Spatial ARERROR(p,r) [7] model as follows:

$$Ay = X\beta + B^{-1}\epsilon$$

where y is a (N x 1) vector, $\epsilon \sim N(0, \Omega)$ and β a $(k \times 1)$ vectorofparameters associated with the exogenous variables X, Ω is a diagonal matrix, and

$$A = I - \phi_1 W_1 - \phi_2 W_2 - \ldots - \phi_p W_p$$
$$B = I - \theta_1 V_1 - \theta_2 V_2 - \ldots - \theta_q V_q.$$

The matrices W_i and V_j are possibly different connectivity matrices to allow for greater flexibility. The ϕ's and θ's are spatial AR and ERROR parameters.

We obtain the likelihood as:

$$L = -\frac{N}{2} ln(2\pi) - \frac{1}{2} ln|\Omega| + ln|B| + ln|A| - \frac{1}{2}\nu'\nu$$

where,

$$\nu'\nu = (Ay - X\beta)'B'\Omega^{-1}B(Ay - X\beta)$$

see also ANSELIN (1988a).

The first order partial derivatives are, with $\Omega_\sigma = \frac{\partial \Omega}{\partial \sigma}$:

$$\frac{\partial}{\partial \beta}L = (BX)'\Omega^{-\frac{1}{2}}\nu$$

$$\frac{\partial}{\partial \phi_i}L = -tr[A^{-1}W_i] + \nu'\Omega^{-\frac{1}{2}}BW_iy$$

$$\frac{\partial}{\partial \theta_k}L = -tr[B^{-1}V_k] + \nu'\Omega^{-\frac{1}{2}}V_k(Ay - X\beta)$$

$$\frac{\partial}{\partial \sigma^2}L = -\frac{1}{2}tr[\Omega^{-1}\Omega_\sigma] + \frac{1}{2}\nu'\Omega^{-\frac{1}{2}}\Omega_\sigma\Omega^{-\frac{1}{2}}\nu$$

The second order partial derivatives are:

$$\frac{\partial^2}{\partial\beta\partial\beta'}L = -(BX)'\Omega^{-1}(BX)$$

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[7] In ANSELIN (1988) the Spatial AR(1) and Spatial Error(1) model is presented

$$\frac{\partial^2}{\partial\beta\partial\phi_i}L = \frac{\partial^2}{\partial\phi_i\partial\beta}L$$
$$= -(BX)'\Omega^{-1}BW_iy$$

$$\frac{\partial^2}{\partial\beta\partial\theta_k}L = \frac{\partial^2}{\partial\theta_k\partial\beta}L$$
$$= -(V_kX)'\Omega^{-\frac{1}{2}}\nu - (BX)'\Omega^{-1}V_k(Ay - X\beta)$$

$$\frac{\partial^2}{\partial\beta\partial\sigma^2}L = \frac{\partial^2}{\partial\sigma^2\partial\beta'}L$$
$$= -(BX)'\Omega^{-1}\Omega_\sigma\Omega^{-\frac{1}{2}}\nu$$

$$\frac{\partial^2}{\partial\phi_i\partial\phi_j}L = \frac{\partial^2}{\partial\phi_j\partial\phi_i}L$$
$$= -tr[A^{-1}W_jA^{-1}W_i] - (BW_jy)'\Omega^{-1}(BW_iy)$$

$$\frac{\partial^2}{\partial\phi_i\partial\theta_k}L = \frac{\partial^2}{\partial\theta_k\partial\phi_i}L$$
$$= -(V_k(Ay - X\beta))'\Omega^{-1}BW_iy - \nu'\Omega^{-\frac{1}{2}}V_kW_iy$$

$$\frac{\partial^2}{\partial\phi_i\partial\sigma^2}L = \frac{\partial^2}{\partial\sigma^2\partial\phi_i}L$$
$$= -\nu'\Omega^{-\frac{1}{2}}\Omega_\sigma\Omega^{-1}BW_iy$$

$$\frac{\partial^2}{\partial\theta_k\partial\theta_l}L = \frac{\partial^2}{\partial\theta_l\partial\theta_k}L$$
$$= -tr[B^{-1}V_lB^{-1}V_k] - (V_l(Ay - X\beta))'\Omega^{-1}(V_k(Ay - X\beta))$$

$$\frac{\partial^2}{\partial\theta_k\partial\sigma^2}L = \frac{\partial^2}{\partial\sigma^2\partial\theta_k}L$$
$$= -\nu'\Omega^{-\frac{1}{2}}\Omega_\sigma\Omega^{-1}V_k(Ay - X\beta)$$

$$\frac{\partial^2}{\partial\sigma^2\partial\sigma^2}L = +\frac{1}{2}tr[\Omega^{-1}\Omega_\sigma\Omega^{-1}\Omega_\sigma] - \frac{1}{2}tr[\Omega^{-1}\Omega_{\sigma\sigma}] + \frac{1}{2}\nu'\Omega^{-\frac{1}{2}}\Omega_{\sigma\sigma}\Omega^{-\frac{1}{2}}\nu$$
$$-\nu'\Omega^{-\frac{1}{2}}\Omega_\sigma\Omega^{-1}\Omega_\sigma\Omega^{-\frac{1}{2}}\nu$$

We assume that the regularity conditions hold such that the ML estimates that are found as solutions to the system of first order partial derivatives are consistent, asymptotically normally distributed and asymptotically efficient, see HUANG (1984) for a special case. This implies that asymptotically the

covariance matrix of the estimates is given by:

$$[I\,(\psi)]^{-1} = -E\left[\frac{\partial^2 L}{\partial\psi\partial\psi'}\right]^{-1} = E\left[\frac{\partial L}{\partial\psi}\frac{\partial L}{\partial\psi'}\right]^{-1}.$$

To obtain the expected values, the following definitions and relations between the error terms are used:

$$\mu = B^{-1}\epsilon = Ay - X\beta$$
$$\epsilon = B(Ay - X\beta) = B\mu$$
$$\nu = \Omega^{-\frac{1}{2}}B(Ay - X\beta) = \Omega^{-\frac{1}{2}}\epsilon = \Omega^{-\frac{1}{2}}B\mu$$

It follows that, in terms of expected values:

$$E[\epsilon] = E[\mu] = E[\nu] = \mathbf{0}$$
$$E[\mu\mu'] = B^{-1}\Omega(B^{-1})'$$
$$E[\epsilon\epsilon'] = \Omega$$
$$E[\nu\nu'] = I$$

and, for y:

$$y = A^{-1}X\beta + A^{-1}B^{-1}\Omega^{\frac{1}{2}}\nu = A^{-1}X\beta + A^{-1}B^{-1}\epsilon = A^{-1}X\beta + A^{-1}\mu$$
$$E[y] = A^{-1}X\beta$$
$$E[yy'] = (A^{-1}B^{-1})\Omega(A^{-1}B^{-1})' + (A^{-1}X\beta)(A^{-1}X\beta)'$$

An application of these to the above partial derivatives, in combination with judicious use of the trace operator, yields the elements of the information matrix given below. For the various combinations of parameters, the following results are obtained:

$$I\beta\beta' = (BX)'\Omega^{-1}(BX)$$
$$I\beta\phi_i = (BX)'\Omega^{-1}BW_iA^{-1}X\beta$$
$$I\beta\theta_j = 0$$
$$I\beta\sigma^2 = 0$$
$$I\phi_i\phi_j = tr A^{-1}W_jA^{-1}W_i + tr(BW_j)'\Omega^{-1}(BW_i)(A^{-1}B^{-1})\Omega(A^{-1}B^{-1})'$$
$$\qquad\quad + tr(BW_j)'\Omega^{-1}(BW_i)(A^{-1}X\beta)(A^{-1}X\beta)'$$
$$I\phi_i\theta_k = tr V_k\Omega^{-1}BW_iA^{-1}B^{-1}\Omega(B^{-1})' + tr V_kW_iA^{-1}B^{-1}$$
$$I\phi_i\sigma^2 = tr\Omega_\sigma\Omega^{-1}BW_iA^{-1}B^{-1}$$

14

$$I\theta_k\theta_l = trB^{-1}V_lB^{-1}V_k + trV_l'\Omega^{-1}V_kB^{-1}\Omega(B^{-1})'$$
$$I\theta_k\sigma^2 = \Omega_\sigma\Omega^{-1}V_kB^{-1}$$
$$I\sigma^2\sigma^2 = \frac{1}{2}tr[\Omega^{-1}\Omega_\sigma\Omega^{-1}\Omega_\sigma]$$