Faculteit der Economische Wetenschappen en Econometrie

Serie Research Memoranda

A State-Space Calculus for Rational Probality Density Functions and Applications to Non-Gaussian Filtering

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Research Memorandum 1997-2 1

May 1997

vrije Universiteit amsterdam

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Submitted for publication

Abstract

We propose what be believe to be a novel approach to perform calculations for rational density functions using state space representations of the densities. By standard results from realisation theory, a rational probability density function is considered to be the transfer function of a linear system with generally complex entries. The stable part of this system is positive-real, which we call the density summand. The existence of moments is investigated using the Markov parameters of the density summand. Moreover, explicit formulae are given for the existing moments in terms of these Markov parameters. One of the main contributions of the paper are explicit state space descriptions for products and convolutions of rational densities.

As an application which is of interest in its own right, the filtering problem is investigated for a linear time-varying system whose noise inputs have rational probability density functions. In particular state space formulations are derived for the calculation of the prediction and update equations. The case of Cauchy noise is treated as an illustrative example.

1 Introduction

We are going to consider the filtering problem for the first order system

$$x_{t+1} = f_t x_t + \eta_t,$$

$$y_t = h_t x_t + \epsilon_t,$$

$t = 0, 1, 2, \ldots$, where f_t, h_t, are assumed to be known real numbers, and for ease of exposition are assumed to be such that $f_t \neq 0$ and $h_t > 0$, $t \geq 0$. The noise sequences $\{\eta_t\}_{t\geq 0}$ and $\{\epsilon_t\}_{t\geq 0}$ are assumed to be mutually independent sequences of independent random variables whose probability density functions are rational. The initial state x_0 is also assumed to be a random variable which is independent of the noise sequences and also has a rational density. No assumption is made that any of the random variables are identically distributed.

This filtering problems with non-Gaussian noise has applications in econometrics, for example in the analysis of financial time series. Studies have shown that the quantities that

*This research was supported by a NATO collaborative research grant CRG 940733. The work by the second author was also supported in part by NSF grant DMS 9501223

are encountered there often do not admit a Gaussian distribution ([5], [3], see also [9]), since these distributions have 'heavy tails'. As one of the consequences, higher order moments may not exist. It has therefore been proposed (see e.g. [8]) that these distributions be modelled by rational densities, both because they do have 'heavy tails' and because of the richness of the class of distributions. Examples of rational probability densities which have been used in the literature are Cauchy densities and Student densities with odd number of degrees of freedom.

The state filtering problem is defined as the problem of finding the best estimate $\hat{x}_t$ of x_t given knowledge of the distribution of x_0 and the values of $y_0, y_1, \ldots, y_t$. Since

$$\hat{x}_t = \int_{-\infty}^{\infty} x p(x)_{x_t|y_t,y_{t-1},\ldots,y_0} dx,$$

this estimate can be found if the conditional density $p_{x_t|y_t,y_{t-1},\ldots,y_0}$ of x_t is known and the first moment exists, given the measured values of $y_t, y_{t-1}, \ldots, y_0$ and knowledge of the distribution of x_0.

In principle the calculation of the conditional densities is not difficult. The unnormalized conditional densities, denoted by ρ instead of p, are given by

Update step:

for $\mathbf{t} = 0$:

$$\rho_{x_0|\mathcal{Y}_0}(x) = \rho_{x_0|y_0}(x) = \rho_{y_0|x}(y_0)\rho_{x_0}(x) = \rho_{\epsilon_0}(y_0 - h_0 x)\rho_{x_0}(x),$$

for $t \geq 1$:

$$\rho_{x_t|\mathcal{Y}_t}(x) = \rho_{y_t|x}(y_t)\rho_{x_t|\mathcal{Y}_{t-1}}(x) = \rho_{\epsilon_t}(y_t - h_t x)\rho_{x_t|\mathcal{Y}_{t-1}}(x)$$

$x \in \Re$.

Prediction *step:*

for $\mathbf{t} \geq 0$:

$$\rho_{x_{t+1}|\mathcal{Y}_t}(x) = \left(\rho_{f_t x_t|\mathcal{Y}_t} * \rho_{\eta_t}\right)(x) = \int_{-\infty}^{\infty} \rho_{x_t|\mathcal{Y}_t}\left(\frac{\xi}{f_t}\right)\rho_{\eta_t}(x - \xi)d\xi,$$

$x \in \Re$. Here we have set $\mathcal{Y}_t$ to be the collection of observations $y_t, y_{t-1}, \ldots, y_0$ . .

In [8] is was noted that the various probability densities occurring in the filtering problem are all rational functions if the noise variables and the initial state have rational probability densities and explicit formulas are given. The practical problem in doing these calculations for large numbers of observations is that the conditional densities are fairly complicated to calculate. To alliviate this problem we propose to use state space techniques for these calculations. Since by assumption the initial state and the noise sequences have rational densities this is indeed possible. For this purpose we are going to develop a 'state-space calculus' for rational probability density functions. We believe that this point of view is novel and may be of relevance beyond the application to non-Gaussian filtering as discussed here.

Let ρ be a not necessarily normalized rational probability density, i.e. p(z) is a rational function in the independent variable x, such that $p(x) \geq 0$, $x \in \Re$, and $0 < \int_{-\infty}^{\infty} \rho(x)dx < \infty$. This implies that ρ is *strictly proper,* i.e. $\lim_{|x|\to\infty} \rho(x) = 0$. By standard realization there exists a minimal state space realization such that

$$\rho(x) = c(ixI - A)^{-1}b, \qquad x \in \Re.$$

In particular we will present here state-space formulae for the translation, scaling, product and convolution of rational probability density functions. Most of our results will be

formulated in terms of state space realizations for the density summand, which is defined to be the 'stable' part of the probability density function. One reason for doing this is that in this way the dimensions of the realizations are typically half of what they would be otherwise. For actual implementations of our results this could lead to significant computational advantages, in particular when repeated applications are necessary such as can be expected for the filtering case. Moreover, we will investigate the existence of moments from the state-space point of view and give state-space formulae for the existing moments in terms of the Markov parameters of the density summand. A major part of the investigation will be built on a careful analysis of the connections between impulse responses, transfer functions and characteristic functions of the various objects. In a result that may be of independent interest a state-space formula is given for the system whose impulse response is the product of impulse responses of two systems.

2 Notation and Preliminaries

The symbol C stands for the complex field and the symbol $\Re$ stands for the real field. If *(A, b,* c, d) is a linear state space system we also often use the notation $\left(\begin{array}{c|c} A & b \\ \hline c & dd \end{array}\right)$ If M is a complex matrix M^* denotes the adjoint matrix. If G is a rational function, G* is defined by G*(s) = $(G(-\bar{s}))^*$, $s \in$ C. If G has the realization *(A, b,* c, d) then G^* has the realization *(-A*,* $c^*, -b^*$, d*). We call a system *(A, b,* c, d) *stable* if all eigenvalues of A are in the open left half plane. Note that such systems are often also called asymptotically stable. A rational function G is called strictly proper if $\lim_{|s|\to\infty}$ G(s) = 0. An unnormalized probability density function ρ is a nonnegative integrable function on $\Re$ such that $\int_{-\infty}^{\infty} \rho(x)dx > 0$, but not necessarily 1. Then $p = \frac{\rho}{\int_{-\infty}^{\infty} \rho(x)dx}$ is a normalized density function. The set of functions $\mathcal{P}$ is defined in Section 3.

3 State space representations of rational densities

If ρ is a not necessarily normalized rational probability density function, then ρ is strictly proper, i.e. $\lim_{|x|\to\infty} \rho(x) = 0$. Therefore by standard realization theory (see e.g. [2]) there exists a minimal linear state space system *(A, b,* c) such that

$$p(x) = c(ixI - A)^{-1}b, \qquad x \in \Re.$$

If n denotes the McMillan degree of ρ, the system matrix A will in general be a square complex matrix of size n x n, b will be a complex n x 1 matrix and c will be a complex 1 x n matrix. A rational probability density function which is symmetric with respect to 0 could however be realized with real system matrices.

Note also that we have set up the realization in such a way that we consider the rational function to be defined on the imaginary axis. While in principle the choice of axis is arbitrary it is convenient to choose the imaginary axis since then standard realization theoretic methods can be adopted without having to change axis. In particular we will be using the formal analogy of methods developed for spectral densities which are most naturally considered to be defined on the imaginary axis. To make this convention clear set

$$\Phi(ix) := p(x), \qquad x \in \Re.$$

Since Φ is a rational function defined on the imaginary axis it can be extended as a rational function to the whole complex plane. This rational function has the following properties:

1. $\Phi(s) = \Phi^*(s)$, $s \in \mathcal{C}$.

2. Φ has no poles on the imaginary axis.

3. @(ix) $\geq$ 0, x $\in \Re$.

4. $\lim_{|s|\to\infty} \Phi(s)$ = 0.

The set of rational functions that satisfy properties 1.,2.,3. and 4. is denoted by $\mathcal{P}$. Many of our calculations are going to be based on the following well-known additive decomposition of Φ:

$$\Phi(s) = Z(s) + Z^*(s), \qquad s \text{ EC},$$

where Z is a stable rational function, i.e. all poles of Z are in the open left half plane. This decomposition is unique if we assume that $Z(\infty)$ = 0 which can be done since $\Phi(\infty)$ = 0. The function Z is called the *spectral summand* of Φ. We *will also* call Z the *density summand* of p.

In the following Lemma some elementary and standard state space properties are collected concerning this additive decomposition of Φ.

Lemma 3.1 *Let (A, b,* c) be a minimal *realization of* Φ, *i.e.* $\Phi(s) = c(sI - A)^{-1}b$, *and (A, b, c) is minimal. There exists an equivalent realization*

$$\left(\begin{array}{cc|c} A_1 & 0 & b_1 \\ 0 & A_2 & b_2 \\ \hline c_1 & c_2 & 0 \end{array}\right)$$

of (A, b, c) such *that all eigenvalues* of A_1 *are* in *the open* left *half plane and all* eigenvalues *of* A_2 *are in the open right half* plane. *The state space* system (A_1, b_1, c_1) is a minimal realization of Z and (A_2, b_2, c_2) is a minimal realization *of* Z*.

Moreover, (A_2, b_2, c_2) is equivalent *to* $(-A_1^*, -c_1^*, b_1^*)$. *In particular there exists a minimal* realization *of* Φ *such that*

$$\left(\begin{array}{c|c} 0 & b_1 \\ -A_1^* & c_1^* \\ \hline -b_1^* & 0 \end{array}\right). \quad \_\_$$

Example: As a special case we are going to consider the Cauchy density, which was suggested for example in [5] as a suitable density to study financial time series. The normalized Cauchy density is defined as

$$p(x) = \frac{1}{\pi} \frac{k}{(x - x_0)^2 + k^2},$$

$x \in \Re$, where $x_0 \in \Re$ and $k > 0$. A state space realization of $\Phi(ix) := p(x)$, $x \in \Re$, is given by

$$\left[\begin{array}{c|c} A_\Phi & b_\Phi \\ \hline c_\Phi & 0 \end{array}\right] := \left[\begin{array}{cc|c} -k + ix_0 & 0 & \frac{1}{2\pi} \\ 0 & k + ix_0 & -\frac{1}{2\pi} \\ \hline 1 & 1 & 0 \end{array}\right].$$

The density summand of p is

$$Z(s) = \frac{1}{2\pi} \frac{1}{s - (-k + ix_0)},$$

which has one pole at $-k + ix_0$. A state space realization of Z is given by

$$\left[\begin{array}{c|c} A & b \\ \hline c & 0 \end{array}\right] := \left[\begin{array}{c|c} -k+ix_0 & \frac{1}{2\pi} \\ \hline 1 & 0 \end{array}\right].$$

□

4 Fourier transforms, moments and Markov parameters

In order to obtain state space formulae for the moments of probability density functions and for the convolution of such densities we need to employ the Fourier transform. The main tool will be to interpret the density summand as the Fourier transform of the impulse response of a stable linear state space system. Actually we introduce the Fourier transform as the Laplace transform evaluated on the imaginary axis. For a general reference on Fourier transforms see e.g. [7], [4]. This way of proceeding is of course closely related to the use of the characteristic function in statistics, but there are a few more minor technical differences.

For an integrable function f on $\Re$ define the Fourier transform as usual by

$$(\mathcal{F}(f))(iw) = \int_{-\infty}^{\infty} f(t)e^{-iwt}dt, \qquad iw \in i\Re.$$

If *(A, b,* c) is a stable minimal system, let m+(t) := $ce^{tA}b$ for $t \geq 0$, and m+(t) := 0 for $t < 0$. Then the Fourier transform of m^+ is given by

$$(\mathcal{F}m^+)(iw) = \int_0^{\infty} ce^{tA}be^{-itw}dt = c(-iwI + A)^{-1}e^{(-iwI+A)t}|_0^{\infty}b = c(iwI - A)^{-1}b$$

$$=: G(iw), \qquad iw \in i\Re.$$

If we set m-(t) := $b^*e^{-tA^*}c^*$ for $t < 0$, and $m^-(t) := 0$ for $t \geq 0$, then the Fourier transform of m^- is given by

$$(\mathcal{F}m^-)(iw) = \int_{-\infty}^{0} b^*e^{-tA^*}c^*e^{-itw}dt = b^*(-iwI - A^*)^{-1}e^{(-iwI-A^*)t}|_{-\infty}^{0}c^*$$

$$= -b^*(iwI - (-A)^*)^{-1}c^* = G^*(iw), \qquad iw \in i\Re.$$

The l^{th}-derivative of m^+ at $t > 0$ is given by $(m^+)^{(l)}(t) = cA^le^{tA}b$. Hence the right-hand side limit of the l^{th} derivative at 0 is given by $(m^+)^{(l)}(0+) = cA^lb$. The l^{th}-derivative of m^- at $t < 0$ is given by $(m^-)^{(l)}(t) = b^*(-A^*)^le^{-tA^*}c^*$. Hence the left-hand side limit of the l^{th} derivative at 0 is given by $(m^-)^{(l)}(0-) := b^*(-A^*)^lc^* = (-1)^l(cA^lb)^* = (-1)^l\left((m^+)^{(l)}(0+)\right)^*, l \geq 0.$

Assume now that *(A, b,* c) is the minimal realization of the spectral summand Z of the function $\Phi \in \mathcal{P}$. Then $(\mathcal{F}m^+)(iw) = Z(iw)$, $(\mathcal{F}m^-)(iw) = Z^*(iw)$, and for m := $m^+ + m^-$ we have that $(\mathcal{F}m)(iw) = \Phi(iw)$, $iw \in i\Re$. Hence m is the inverse Fourier transform of Φ. Note that m is l times continuously differentiable at $t = 0$, $l \geq 0$, if and only if $cA^kb = (-1)^k(cA^kb)^*$, $k = 0, 1, \ldots, l$.

If G is a strictly proper rational function on C, then G admits a Laurent expansion around ∞ such that

$$G(s) = \sum_{j=1}^{\infty} M(j)\frac{1}{s^j}, \qquad s \text{ EC.}$$

The parameters M(j), $j = 1, 2, \ldots$, are the Markov parameters of G (see e.g. [2]). If (A, b, c) is a realization of G then

$$G(s) = c(sI - A)^{-1}b = \frac{1}{s}c(I - \frac{A}{s})^{-1}b = \frac{1}{s}c\sum_{k=0}^{\infty}\left(\frac{1}{s}A\right)^k b = \sum_{j=1}^{\infty}\frac{1}{s^j}cA^{j-1}b.$$

Hence the Markov parameters of G are given by

$$M(j) = cA^{j-1}b, \qquad j = 1, 2, 3, \ldots .$$

The Markov parameters of a rational strictly proper function in $\mathcal{P}$ and its spectral summand are easily determined.

Lemma 4.1 *Let Φ be a strictly proper rational* function *in $\mathcal{P}$ with spectral summand Z. If (A, b,* c) is a minimal *realization of Z then*

1. the Markov parameters of Z are given by

$$cA^{j-1}b, \qquad j = 1, 2, 3, \ldots .$$

2. the Markov parameters of Z^* are given ***by***

$$(-1)^j b^*(A^*)^{j-1}c^* = (-1)^j(cA^{j-1}b)^*, \qquad j = 1, 2, 3, \ldots .$$

3. the Markov ***parameters*** *of* Φ are ***given by***

$$cA^{j-1}b - (-1)^{j-1}(cA^{j-1}b)^*, \qquad j = 1, 2, 3, \ldots .$$

In the following Lemma a basic result on the integrability of rational functions is summarized.

Lemma 4.2 *Let* $G = \frac{n_G}{d_G}$ ***with*** n_G and d_G *coprime* polynomials. ***Then***

$$\int_{-\infty}^{\infty} |G(x)|dx < \infty$$

if ***and*** *only if* $degree(n_G) \leq degree(d_G) - 2$ ***and*** $d_G(x) \neq 0$ *for* ***all*** $x \in \Re$.

If G is as defined in the Lemma then $degree(d_G) - degree(n_G)$ is called the co-degree of the rational function G. Therefore G is integrable if and only if the co-degree of G is greater or equal to 2. This Lemma also implies that if the random variable X has the rational probability density function $p = \frac{n_p}{d_p}$, then the moments EX^k exist for $k = 0, 1, 2, \ldots$, co-degree(p) $-$ 2.

Let k be such that M(j) = 0, for $j = 1, 2, \ldots, k - 1$ and $M(k) \neq 0$. Then the co-degree of G is k ([2]).

Summarizing the previous remarks we obtain the following Proposition.

Proposition 4.1 ***Let*** Φ ***be*** a ***strictly proper*** rational ***function in*** $\mathcal{P}$ ***with spectral summand*** Z. ***Let (A, b,*** c) ***be a*** minimal realization of Z. ***Let*** $m(t) := ce^{tA}b$, ***for*** $t \geq 0$ ***and*** $m(t) := b^*e^{-tA^*}c^*$ *for* $t < 0$. ***Then***

1. ***the co-degree of*** Φ ***is k if and only if*** $M(j) = 0$ *for all* $j \in (1,\ldots,$ k-l} and $M(k) \neq 0$, ***where M(j) is the*** j^{th} ***Markov parameter*** *of* Φ.

2. ***the*** co-degree *of* Φ *is* ***k*** *if and only if*

$$cA^{j-1}b = (-1)^{j-1}(cA^{j-1}b)^*$$

for all $j \in \{1,\ldots,k-1\}$ ***and***

$$cA^{k-1}b \neq (-1)^{k-1}(cA^{k-1}b)^*.$$

3. m is $k - 1$ times continuously ***differentiable at 0*** *if* and ***only if the first k Markov parameters*** *of* Φ *are zero.*

4. Φ ***has*** co-degree ***k*** *if* and ***only*** *if* ***m is*** $k-2$ ***times continuously*** *differentiable* ***but not*** $k - 1$ *times* continuously *differentiable* ***at 0.***

The following theorem provides important results concerning moments of a random variable with rational probability density.

Theorem 4.1 ***Let X be a random variable with unnormalized rational probability density function*** p. ***Let (A, b,*** c) ***be*** a realization *of* ***the density*** summand Z *of* ***p. Then***

1. ***the*** co-degree *of* ρ *is* ***k*** *if and only if*

$$cA^{j-1}b = (-1)^{j-1}(cA^{j-1}b)^*$$

for *all* $j \in \{1,\ldots,$ $k-1\}$ and $cA^{k-1}b \neq (-1)^{k-1}(cA^{k-1}b)^*$.

2. ***the*** l^{th} ***moment*** EX^l *of* X, ***with*** 1 a non-negative *integer, exists if and only if* $l \in \{0, 1, \ldots,$ $k-2)$.

3. $EX^l = (-i)^l \frac{cA^lb}{cb}$, *for* ***all*** $1 \in \{0, 1, \ldots,$ $k-2)$.

Proof: 1.) The follows immediately from Proposition 4.1.
2.) Recall that the l^{th} moment of X is given by

$$EX^l = \frac{1}{R}\int_{-\infty}^{\infty} x^l \rho(x)dx,$$

where $R := \int_{-\infty}^{\infty} \rho(x)dx$. The co-degree of the integrand is $k - l$. By Lemma 4.2 the integrand is integrable if and only if its co-degree is greater of equal to 2. Hence the claim.
3.) Let $0 \leq 1 \leq k - 2$. Set $\Phi(ix)$:= p(z), x $\in \Re$, and use the notation of Proposition 4.1. Then m is $k - 2$ times continuously differentiable at 0 and therefore on $\Re$. Since the co-degree of ρ is greater or equal to 2, m is continuous on $\Re$. Since ρ and m are continuous and integrable we have by the inversion theorem for Fourier transforms (see e.g. [4], Theorem 60.1, p.296) that

$$m(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(iw)e^{iwt}dw, \qquad t \in \Re.$$

Note that differentiation up to order $k - 2$ under this integral is justified by the usual argument (see e.g [4], Theorem 53.5, p. 268) as $|\omega^l \Phi(i\omega) e^{i\omega t}| = |\omega^l \Phi(i\omega)|$ is integrable for each $t \in \Re$ and $0 \leq l \leq k - 2$. Hence for $t \in \Re$,

$$\frac{d^l}{dt^l} m(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(iw) \frac{d^l}{dt^l} e^{iwt} dw = (i)^l \frac{1}{2\pi} \int_{\infty}^{\infty} w^l \Phi(iw) e^{iwt} dw.$$

Evaluating at $t = 0$, we have

$$\frac{d^l}{dt^l} m(t)|_{t=0} = \frac{1}{2\pi} (i)^l \int_{\infty}^{\infty} w^l \Phi(iw) e^{iwt} dw|_{t=0} = \mathbf{R}\ (i)^l \frac{1}{2\pi} EX^l.$$

Since $\frac{d^l}{dt^l} m(t)|_{t=0} = cA^l b$, $l = 0, \ldots, k - 2$, we have that

$$EX^l = \frac{2\pi}{R} (-i)^l cA^l b.$$

The constant R is determined by considering this equation for $l_0 = 0$. Since $EX^0 = 1$ we have that $R = 27rcb$. Hence $EX^l = (-i)^l \frac{cA^l b}{cb}$. □

In most of this paper we will be dealing with unnormalized rational probability densities ρ. If (A, b, c) is a state space realization of the density summand of ρ, the normalized probability density function is given by $p := \frac{\rho}{\int_{\infty}^{\infty} \rho(x) dx}$. By the above proposition $\int_{\infty}^{\infty} \rho(x) dx = 2\pi cb$, which provides a state space formula for the normalization constant.

If X is a random variable with rational probability density function ρ whose density summand has the state space realization (A, b, c), then the first moment exists if the codegree of ρ is at least three. This is the case if and only if

$$cb = (cb)^*$$

and

$$cAb = -(cAb)^*.$$

If the first moment, i.e the mean, exists then by the Theorem it is given by

$$EX = -i \frac{cAb}{cb}.$$

In the above discussion we gave a state space construction for the inverse Fourier transform m of a not necessarily normalized rational probability density function ρ, i.e.

$$m(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\omega) e^{i\omega t} d\omega, \qquad w \in \Re.$$

In the statistical literature an important object is the characteristic function of a random variable X which is defined by $E(e^{itX})$, $t \in \Re$. If X has the unnormalized probability function ρ, then

$$E(e^{itX}) = \frac{1}{\int_{-\infty}^{\infty} \rho(x) dx} \int_{-\infty}^{\infty} e^{itx} \rho(x) dx = \frac{2\pi}{\int_{-\infty}^{\infty} \rho(x) dx} m(t), \qquad t \in \Re.$$

Hence up to a (known) scaling factor the function m is identical to the characteristic function.

Example continued: We continue the discussion of the Cauchy density from Section 3. Note that for all $x_0 \in \Re$ and $k > 0$

$$cAb = \frac{1}{2\pi}(-k + ix_0) \neq -f(-k - ix_0) = -(cAb)^*.$$

Hence by the Theorem the mean EX does not exist. This is of course also directly evident by consideration of the integral $\int_{-\infty}^{\infty} xp(x)dx$.

If $m^+(\tau) := \frac{1}{2\pi}e^{\tau(-k+ix_0)}$ for $t \geq 0$ and $m^+(\tau) := 0$ for $t < 0$, then $\mathcal{F}(m^+)(iw) = \frac{1}{2\pi}\frac{1}{iw-(-k+ix_0)}$, $iw \in i\Re$. If $m^-(\tau) := \frac{1}{2\pi}e^{-\tau(-k-ix_0)}$ for $t < 0$ and $m^-(\tau) := 0$ for $t \geq 0$, then $\mathcal{F}(m^-)(iw) = \frac{1}{2\pi}\frac{1}{iw+(k-ix_0)}$, $iw \in i\Re$. With $m := m^+ + m^-$, we have that m is continuous at 0. The derivative is given

$$\frac{d}{dt}m(t) = \frac{1}{2\pi}(-k + ix_0)e^{\tau(-k+ix_0)}, \qquad \tau > 0,$$

$$\frac{d}{dt}m(t) = \frac{1}{2\pi}(k + ix_0)e^{-\tau(-k-ix_0)}, \qquad \tau < 0.$$

Note that the left-hand side limit and the right-hand side limit do not agree at 0. Hence m is not differentiable at 0. As the co-degree of p is 2 this is in agreement with Proposition 4.1. The Markov parameters of Φ are

$$c_\Phi b_\Phi = 0, \qquad c_\Phi A_\Phi b_\Phi = \frac{-k}{\pi}.$$

Hence the second Markov parameter is nonzero which is also in agreement with Proposition 4.1.

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5 Operations on probability densities

In this section we are going to discuss state-space formulations of operations on rational probability densities. Given state space realizations for the density summands of two probability densities we will give state space realizations for the density summand of the translation, scaling, the product and convolution of the densities.

5.1 Translation and scaling of a probability density

In the next Lemma the effect of translation and scaling of a random variable on the state space realization of the density is considered.

Lemma 5.1 *Let* X *be* a random *variable with unnormalized rational density p. Let (A, b, c) be a realization such that* $p(x) = c(ixI - A)^{-1}b$, $x \in \Re$.

Let $x_0 \in \Re$. *Then the* random variable $X + x_0$ *has an unnormalized probability density function* $q(x) = p(x - x_0)$ *which has a realization* $(A + ix_0I, b, c)$, *so*

$$q(x) = c(ixI - (A + ix_0I))^{-1}b, \qquad x \in \Re.$$

Let $a \in \mathbb{R}$, $a \neq 0$, ***then the random variable*** aX ***has the unnormalized probability density*** function $q(x) = \frac{1}{|a|}\rho(\frac{x}{a})$ ***which has*** a realization $(aA, b, \frac{a}{|a|}c)$, ***so***

$$q(x) = \frac{a}{|a|}c(ixI - aA)^{-1}b, \qquad x \in \mathbb{R}.$$

In the following Lemma we are going to write down the analogous results for the case when a state space realization is given for the density summand of the probability density.

Lemma ***5.2 Let X be a random variable with unnormalized rational density p. Let (A, b, c) be a realization of the density*** summand Z of ***p.***

Let $x_0 \in \mathbb{R}$, ***then the random variable X*** + x_0 ***has the*** unnormalized probability ***density function q(x)*** = $\rho(x -$ xc), x $\in \mathbb{R}$, ***whose*** density summand ***has*** a *realization*

$$\left(\begin{array}{c|c} A + ix_0 I & b \\ \hline c & 0 \end{array}\right).$$

Let $a \in \mathbb{R}$, $a \neq 0$, ***then the random variable*** aX ***has the unnormalized probability density function q(x)*** = $\frac{a}{|a|}\rho(\frac{x}{a})$ ***whose density*** summand ***has*** a ***realization***

$$\left(\begin{array}{c|c} aA & b \\ \hline c & 0 \end{array}\right),$$

if $a > 0$ *and*

$$\left(\begin{array}{c|c} -aA^* & c^* \\ \hline b^* & 0 \end{array}\right),$$

if $a < 0$.

5.2 Product of two rational probability densities

In the update step of the filtering problem it is necessary to calculate the product of two density functions. We are going to do this also by state space techniques using the decomposition into density summands. The following Lemmas will be useful.

Lemma ***5.3 Let*** G_1 ***and*** G_2 ***be two stable strictly proper rational functions with state space realizations*** (A_1, b_1, c_1) ***and*** (A_2, b_2, c_2). ***Then the product*** $G_1^*G_2$ ***can be decomposed as***

$$G_1^*G_2 = F + H^*,$$

where ***F, H*** are ***stable strictly proper*** rational *functions such that F has the realizations given by*

$$\left(\begin{array}{c|c} A_2 & b_2 \\ \hline b_1^*T_1 & 0 \end{array}\right), \qquad \left(\begin{array}{c|c} A_2 & T_2c_1^* \\ \hline c_2 & 0 \end{array}\right)$$

and H* has ***the realizations given by***

$$\left(\begin{array}{c|c} -A_1^* & T_1b_2 \\ \hline -b_1^* & 0 \end{array}\right), \qquad \left(\begin{array}{c|c} -A_1^* & -c_1^* \\ \hline c_2T_2 & 0 \end{array}\right),$$

where T_1 ***is the unique solution to the*** Sylvester ***equation***

$$A_1^*T_1 + T_1A_2 + c_1^*c_2 = 0$$

and T_2 ***is the unique solution to the Sylvester equation***

$$A_2T_2 + T_2A_1^* + b_2b_1^* = 0.$$

Proof: Note that a realization of G_1^* is given by

$$(-A_1^*, c_1^*, -b_1^*)$$

and a realization of $G_1^* G_2$ is given by

$$\left(\begin{array}{cc|c} -A_1^* & c_1^* c_2 & 0 \\ 0 & A_2 & b_2 \\ \hline -b_1^* & 0 & 0 \end{array}\right).$$

Performing a state space basis transformation with transformation matrix $\begin{pmatrix} I & T_1 \\ 0 & I \end{pmatrix}$ we obtain the equivalent realization

$$\left(\begin{array}{cc|c} -A_1^* & A_1^* T_1 + T_1 A_2 + c_1^* c_2 & T_1 b_2 \\ 0 & A_2 & b_2 \\ \hline -b_1^* & b_1^* T_1 & 0 \end{array}\right) = \left(\begin{array}{cc|c} -A_1^* & 0 & T_1 b_2 \\ 0 & A_2 & b_2 \\ \hline -b_1^* & b_1^* T_1 & 0 \end{array}\right)$$

since T_1 is such that $A_1^* T_1 + T_1 A_2 + c_1^* c_2 = 0$. Note that such a T_1 exists and is unique since both A_1^* and A_2 have all their eigenvalues in the open left half plane (see e.g [6]). This representation implies the first set of realizations. The other set of realizations follows analogously by considering the state space formula which corresponds to $G_2 G_1^*$. □

Remark. A method to generate explicit formulas for the solutions of Sylvester equations is presented in [1].

We can now derive the desired representation for the density summand of the product of two rational probability density functions.

Proposition 5.1 *Let* ρ_1 and ρ_2 *be two* ***unnormalized*** rational *probability density functions with density* summands Z_1 and Z_2. *Let* (A_i, b_i, c_i) *be* a realization *of* Z_i, $i = 1, 2$. *Then the density summand* Z *of the* ***un-normalized*** *rational probability density* ***function*** $p = \rho_1 \rho_2$ *has a realization given by*

$$\left(\begin{array}{cc|c} A_1 & b_1 c_2 & T_2^* c_2^* \\ 0 & A_2 & b_2 \\ \hline c_1 & b_1^* T_1 & 0 \end{array}\right),$$

where T_1, T_2 *are the* unique *solutions to the Sylvester equations*

$$A_1^* T_1 + T_1 A_2 + c_1^* c_2 = 0,$$

$$A_2 T_2 + T_2 A; + b_2 b_1^* = 0.$$

Proof: We have that

$$\rho = \rho_1 \rho_2 = (Z_1 + Z_1^*)(Z_2 + Z_2^*) = Z_1 Z_2 + Z_1 Z_2^* + (Z_1 Z_2^*)^* + (Z_1 Z_2)^*.$$

By Lemma 5.3 a state space realization for the stable part of this expression is given by

$$\left(\begin{array}{cccc|c} A_1 & b_1 c_2 & 0 & 0 & 0 \\ 0 & A2 & 0 & 0 & b_2 \\ 0 & 0 & A_1 & 0 & T_2^* c_2^* \\ 0 & 0 & 0 & A_2 & b_2 \\ \hline c_1 & 0 & c_1 & b_1^* T_1 & 0 \end{array}\right),$$

where T_1 is the unique solution of the equation

$$A_1^* T_1 + T_1 A_2 + c_1^* c_2 = 0$$

and T_2 is the unique solution of the equation

$$A_2 T_2 + T_2 A_1^* + b_2 b_1^* = 0.$$

Performing a state space basis transformation with transformation matrix

$$T = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & -I & 0 & I \end{pmatrix}$$

we obtain the equivalent realization

$$\left(\begin{array}{cccc|c} A_1 & b_1 c_2 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & b_2 \\ 0 & 0 & A_1 & 0 & T_2^* c_2^* \\ 0 & 0 & 0 & A_2 & 0 \\ \hline c_1 & b_1^* T_1 & c_1 & b_1^* T_1 & 0 \end{array}\right),$$

which is equivalent to

$$\left(\begin{array}{ccc|c} A_1 & b_1 c_2 & 0 & 0 \\ 0 & A_2 & 0 & b_2 \\ 0 & 0 & A_1 & T_2^* c_2^* \\ \hline c_1 & b_1^* T_1 & c_1 & 0 \end{array}\right).$$

On this realization perform another state space basis transformation with transformation matrix

$$T = \begin{pmatrix} I & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

to obtain

$$\left(\begin{array}{ccc|c} A_1 & b_1 c_2 & 0 & T_2^* c_2^* \\ 0 & A_2 & 0 & b_2 \\ 0 & 0 & A_1 & T_2^* c_2^* \\ \hline c_1 & b_1^* T_1 & 0 & 0 \end{array}\right),$$

which is equivalent to

$$\left(\begin{array}{cc|c} A_1 & b_1 c_2 & T_2^* c_2^* \\ 0 & A_2 & b_2 \\ \hline c_1 & b_1^* T_1 & 0 \end{array}\right).$$

□

It was noted before that the co-degree of a rational probability density function is at least two. Therefore the product of two such probability density functions has co-degree at least four. Hence for a random variable whose density is given by such a product at least the first and second moments exist. This will be used in the next section to show the existence of a conditional mean and variance.

5.3 Convolution of probability densities

We now come to determine a state space formulation for the convolution of two probability densities. Recall that if X and Y are two random variables with rational probability densities ρ_X and ρ_Y, then the probability density of X + Y is given by the convolution $\rho_X * \rho_Y$.

Let ρ_1 and ρ_2 be two unnormalized rational probability functions with corresponding spectral summands Z_1 and Z_2. Let (A_j, b_j, c_j) be a realization of Z_j, $j = 1, 2$. Let for j = 1, 2,

$$m_j^+(\tau) := \begin{cases} c_j e^{\tau A_j} b_j & \tau \geq 0, \\ 0 & \tau < 0, \end{cases}$$

$$m_j^-(\tau) := \begin{cases} b_j^* e^{-\tau A_j^*} c_j^* & \tau < 0, \\ 0 & \tau \geq 0. \end{cases}$$

Then $(\mathcal{F}m_j^+)(iw) = Z_j(iw)$, $(\mathcal{F}m_j^-)(iw) = Z_j^*(iw)$, $iw \in i\Re$ and

$$(\rho_1 * \rho_2)(iw) = \int_{-\infty}^{\infty} \rho_1(iw - i\nu)\rho_2(i\nu)\, d\nu = \mathcal{F}\left(\mathcal{F}^{-1}\int_{-\infty}^{\infty} \rho_1(iw - i\nu)\rho_2(i\nu)d\nu\right)$$

$$= \mathcal{F}\left((\mathcal{F}^{-1}\rho_1)(\mathcal{F}^{-1}\rho_2)\right)(iw) = \mathcal{F}\left((\mathcal{F}^{-1}(Z_1 + Z_1^*)(\mathcal{F}^{-1}(Z_2 + Z_2^*)\right)(iw)$$

$$= \mathcal{F}\left(m_1^+ + m_1^-)(m_2^+ + m_2^-)\right)(iw) = \mathcal{F}\left(m_1^+ m_2^+ + m_1^- m_2^-\right)(iw)$$

$$= \mathcal{F}(m_1^+ m_2^+)(iw) + \mathcal{F}(m_1^- m_2^-)(iw).$$

It follows that the spectral summand Z of $\rho_1 * \rho_2$ is given by $Z(iw) = \mathcal{F}(m_1^+ m_2^+)(iw)$.

In the following Proposition we are going to give the state space formulae for the product of the impulse responses of two single-input single-output state space systems. This will be the key step to determine a state-space realization for the convolution of two rational probability density functions.

Proposition 5.2 *Let $m_j^+(\tau) := c_j e^{\tau A_j} b_j$ for $\tau \geq 0$, and $m_j^+(\tau) := 0$ for $\tau < 0$, where (A_j, b_j, c_j) is an n_j-dimensional* single-input single-output *system, $j = 1, 2$. Then*

$$m^+(\tau) := m_1^+(\tau) m_2^+(\tau), \qquad \tau \geq 0$$

has a realization $m^+(\tau) = ce^{\tau A}b$ for $\tau \geq 0$ and $m^+(\tau) = 0$ for $\tau < 0$, where

$$A = A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2,$$

$$b = b_1 \otimes b_2,$$

$$c = c_1 \otimes c_2.$$

(Here $\otimes$ denotes the Kronecker product.)

Proof: This follows immediately from basic rules on the Kronecker product (see e.g. [6]), since for $\tau \geq 0$

$$m^+(\tau) = ce^{\tau A}b = (c_1 \otimes c_2)e^{\tau(A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2)}(b_1 \otimes b_2)$$

$$= (c_1 \otimes c_2)(e^{\tau A_1} \otimes e^{\tau A_2})(b_1 \otimes b_2) = c_1 e^{\tau A_1} b_1 \otimes c_2 e^{\tau A_2} b_2 = c_1 e^{\tau A_1} b_1 c_2 e^{\tau A_2} b_2$$

$$= m_1^+(\tau) m_2^+(\tau).$$

□

The Proposition is of interest in its own right, as it allows one to find state space formulas for products of impulse response functions.
Summarizing we have the following result.

Proposition 5.3 ***Let*** ρ_1 ***and*** ρ_2 ***be unnormalized rational probability densities whose spectral*** summands Z_1 ***and*** Z_2 ***have the*** n_1 ***dimensional and*** n_2 ***dimensional state space realizations*** (A_1, b_1, c_1) ***and*** (A_2, b_2, c_2). ***Then the density summand*** Z ***of the convolution*** $p = \rho_1 * \rho_2$ ***has the state space realization***

$$\left(\begin{array}{c|c} A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2 & b_1 \otimes b_2 \\ \hline c_1 \otimes c_2 & 0 \end{array} \right). \tag{1}$$

Proof: Suppose Z has realisation 1. Then the inverse Fourier transform of Z is $m_1^+ m_2^+$, **showing** that Z is the spectral summand of ***p.*** □

Note that the state space dimension of this realisation is $n_1 n_2$, which implies that the McMillan degree of Z is at most $n_1 n_2$.

6 State space expressions for the filtering equations

We are now in a position to derive state space expressions for the unnormalized conditional densities in the filter equations which were discussed in the introduction.

Theorem 6.1 ***Assume the notation and assumptions for the filtering problem as presented*** in ***the*** introduction.

Let $t \geq 0$ *and let* $(A_{x_{t|t-1}}, b_{x_{t|t-1}}, c_{x_{t|t-1}})$ ***be a*** n_{x_t} ***-dimensional state space realization of the density summand of the unnormalized conditional density*** $\rho_{x_t|\mathcal{Y}_{t-1}}$. For $t = 0$, ***set*** $\rho_{x_t|\mathcal{Y}_{t-1}} := \rho_{x_0}$ ***the density*** of ***the*** initial state x_0. ***Let*** $(A_{\eta_t}, b_{\eta_t}, c_{\eta_t})$ ***be a*** n_{η_t}***-dimensional state space realization*** of ***the*** density summand of ***the*** unnormalized rational ***density*** ρ_{η_t} ***of the*** noise random ***variable*** η_t ***and let*** $(A_{\epsilon_t}, b_{\epsilon_t}, c_{\epsilon_t})$ ***be a*** n_{ϵ_t}***-dimensional state space realization*** *of* ***the density*** summand *of* ***the unnormalized*** rational ***density*** ρ_{ϵ_t} *of* ***the*** noise random ***variable*** ϵ_t, $t \geq 0$.

Let T_1 ***be the unique solution to the equation***

$$(\frac{1}{h_t} A_{\epsilon_t} + iy_t I)T_1 + T_1 A_{x_{t|t-1}} + b_{\epsilon_t} c_{x_{t|t-1}} = 0$$

and let T_2 *be the unique* solution *to the equation*

$$A_{x_{t|t-1}}T_2 + T_2(\frac{1}{h_t}A_{\epsilon_t} + iy_t) + b_{x_{t|t-1}}c_{\epsilon_t} = 0.$$

Then the density summand of the unnormalized density $\rho_{x_t\,|\mathcal{Y}_t}$ has *state space realization*

$$\left(\begin{array}{c|c} A_{x_{t|t}} & b_{x_{t|t}} \\ \hline c_{x_{t|t}} & 0 \end{array}\right) = \left(\begin{array}{cc|c} \frac{1}{h_t}A^*_{\epsilon_t} - iy_tI & c^*_{\epsilon_t}c_{x_{t|t-1}} & T_2^*c^*_{x_{t|t-1}} \\ 0 & A_{x_{t|t-1}} & b_{x_{t|t-1}} \\ \hline \frac{1}{h_t}b^*_{\epsilon_t} & c_{\epsilon_t}T_1 & 0 \end{array}\right).$$

The density summand *of* $\rho_{x_{t+1}\quad \mathcal{Y}_t}$ *has state space* realization

$$\left(\begin{array}{c|c} A_{x_{t+1|t}} & b_{x_{t+1|t}} \\ \hline c_{x_{t+1|t}} & 0 \end{array}\right)$$

$$= \left(\begin{array}{c|c} f_tA_{x_{t|t}} \otimes I_{n_{\eta_t}} + I_{n_{x_{t|t}}} \otimes A_{\eta_t} & b_{x_{t|t}} \otimes b_{\eta_t} \\ \hline c_{x_{t|t}} \otimes c_{\eta_t} & 0 \end{array}\right), \; if\; f_t > 0,$$

$$= \left(\begin{array}{c|c} -f_tA^*_{x_{t|t}} \otimes I_{n_{\eta_t}} + I_{n_{x_{t|t}}} \otimes A_{\eta_t} & c^*_{x_{t|t}} \otimes b_{\eta_t} \\ \hline b^*_{x_{t|t}} \otimes c_{\eta_t} & 0 \end{array}\right), \; if\; f_t < 0.$$

Proof: Since by assumption $h_t > 0$, the density summand of the density $q(x) = \rho_{\epsilon_t}(y_t - h_tx)$, $x \in \Re$, has the realization

$$\left(\begin{array}{c|c} \frac{1}{h_t}A^*_{\epsilon_t} - iy_tI & c^*_{\epsilon_t} \\ \hline b^*_{\epsilon_t} & 0 \end{array}\right).$$

As

$$\rho_{x_t\,|\mathcal{Y}_t}(x) = \text{Pet}\,(y_t - h_tx)\rho_{x_t\,|\mathcal{Y}_{t-1}}(x),$$

$x \in \Re$, by Proposition 5.1 the density summand of ρ has the realization

$$\left(\begin{array}{cc|c} \frac{1}{h_t}A^*_{\epsilon_t} - iy_tI & c^*_{\epsilon_t}c_{x_{t|t-1}} & T_2^*c^*_{x_{t|t-1}} \\ 0 & A_{x_{t|t-1}} & b_{x_{t|t-1}} \\ \hline \frac{1}{h_t}b^*_{\epsilon_t} & c_{\epsilon_t}T_1 & 0 \end{array}\right),$$

where T_1 is the unique solution to the equation

$$(\frac{1}{h_t}A^*_{\epsilon_t} - iy_tI)^*T_1 + T_1A_{x_{t|t-1}} + b_{\epsilon_t}c_{x_{t|t-1}} = (\frac{1}{h_t}A_{\epsilon_t} + iy_tI)T_1 + T_1A_{x_{t|t-1}} + b_{\epsilon_t}c_{x_{t|t-1}} = 0$$

and T_2 is the unique solution to the equation

$$A_{x_{t|t-1}}T_2 + T_2(\frac{1}{h_t}A^*_{\epsilon_t} - iy_tI)^* + b_{x_{t|t-1}}c_{\epsilon_t} = A_{x_{t|t-1}}T_2 + T_2(\frac{1}{h_t}A_{\epsilon_t} + iy_tI) + b_{x_{t|t-1}}c_{\epsilon_t} = o.$$

To obtain a state space formula for the prediction step

$$\rho_{x_{t+1}\;\mathrm{I}\;\mathcal{Y}_t} = \rho_{f_tx_t\quad \mathcal{Y}_t} * \rho_{\eta_t}$$

we use Proposition 5.3. We need to introduce two cases depending on the sign of f_t. If $f_t > 0$ the density summand of $\rho_{f_tx_t\;|\;\mathcal{Y}_t}$ has the realization

$$\left(\begin{array}{c|c} f_tA_{x_{t|t}} & b_{x_{t|t}} \\ \hline c_{x_{t|t}} & 0 \end{array}\right).$$

If $f_t < 0$ the density summand of $\rho_{f_t x_t \mid \mathcal{Y}_t}$ has the realization

$$\left(\begin{array}{c|c} -f_t A^*_{x_{t|t}} & c^*_{x_{t|t}} \\ \hline b^*_{x_{t|t}} & 0 \end{array}\right).$$

The remaining parts of the result now follow by Proposition 5.3. ❑

As the formulae that use Kronecker products show, the dimensions of the state space representation can potentially grow very fast as the number of data points increases. If, however, the density summand corresponding to η_t only has McMillan degree 1, i.e. η_t has Cauchy distribution, then the Kronecker products reduce to standard multiplication and the prediction step does not lead to an increase in dimension. Also, if the density summand corresponding to ϵ_t has McMillan degree 1, i.e. ϵ_t has Cauchy distribution, then the matrix equations can be solved explicitly to give

$$T_1 = -b_{\epsilon_t} c_{x_{t|t-1}} \left((\frac{1}{h_t} A_{\epsilon_t} + iy_t) I + A_{x_{t|t-1}} \right)^{-1},$$

$$T_2 = -\left(\frac{1}{h_t} (A_{\epsilon_t} + iy_t + I) + A_{x_{t|t-1}} \right)^{-1} b_{x_{t|t-1}} c_{\epsilon_t},$$

Note that the inverse exists, since $A_{x_{t|t-1}}$ has all eigenvalues in the open left half plane and $\frac{1}{h_t} A_{\epsilon_t} + iy_t$ has negative real part, because of the stability of $A_{,,}$ and since $h_t > 0$.

From the remark after Proposition 5.1 it follows that the conditional mean $E(x_t|\mathcal{Y}_t)$ and the corresponding conditional variance $E\left((x_t - E(x_t|\mathcal{Y}_t)^2 \mid \mathcal{Y}_t\right)$ exist and can be calculated from the density summand realization $(A_{,,,,}, b_{x_{t|t}}, c_{x_{t|t}})$ using the formulas given in Theorem 4.1.

Note that prediction is also possible using the formulas presented here. For example the unnormalized rational conditional probability density of the output variable at time $t + 1$ given the observations of the output until time t is equal to $\rho_{y_{t+1|t}}(y) = \rho_{h_{t+1} x_{t+1|t}} * \rho_{\epsilon_{t+1}}$, and the spectral summand of this density can be calculated using the formulas of Section 5.

7 Conclusions

State space formulae have been developped for various operations on rational density functions, and it is shown how this can be used to treat the filtering problem in case of a first order linear stochastic model with stochastically independent noise variables with rational probability densities and stochastically independent initial state with rational probability density. This makes such filters easy to program on present day computers, using e.g. a linear algebra package. If the number of observations is not very small, however, the order of the conditional rational densities will tend to grow quickly. Therefore various schemes of order reduction for positive real functions may be of relevance in practical applications. The formulae presented can also be used for further theoretical research in the behaviour of the optimal filter. It follows for example that the conditional mean of the present state given present and past observations, is a rational function of the present and past observations, which could be further investigated. The formula that is presented for realization of the product of impulse response functions appears to be important in its own right.

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