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Discretized Reality and Spurious Profits in Stochastic Programming Models for Asset/Liability Management

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Discretized Reality and Spurious Profits in Stochastic Programming Models for Asset/Liability Management

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Abstract

In the literature on stochastic programming models for practical portfolio investment problems, relatively little attention has been devoted to the question how the necessarily approximate description of the asset-price uncertainty in these models influences the optimal solution. In this paper we will show that it is important that asset prices in such a description are arbitrage-free. Descriptions which have been suggested in the literature are often inconsistent with observed market prices and / or use sampling to obtain a set of scenarios about the future. We will show that this effectively introduces arbitrage opportunities in the optimization model. For an investor who cannot exploit arbitrage opportunities directly because of market imperfections and trading restrictions, we will illustrate that the presence of such arbitrage opportunities may cause substantial biases in the optimal investment strategy.

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1 Introduction

Several articles in the literature have illustrated that stochastic programming models are flexible tools to describe financial portfolio investment problems under uncertainty with realistic market imperfections and trading restrictions. Examples are Bradley and Crane [2], Kusy and Ziemba [16], Mulvey and Vladimirou [17], Zenios [18], Dantzig and Infanger [5], Hiller and Eckstein [10], Cariiio *et* al. [4], Golub *et* al. [7], and Dert [6]. A problem with the practical application of such models is that only a coarse approximation of the true uncertainty can be included in order to keep the models computationally tractable. In most cases, this approximation is chosen in such a way that asset prices in the stochastic programming model are not arbitrage-free, thereby being inconsistent with the arbitrage-free pricing theory of modern financial economics. As Zipkin notes when he discusses structured bond portfolio models that are used in practice for fixed-income portfolio management [19, p. S157]: "Structured portfolio models have an odd and complex relationship with the normative and descriptive models developed by financial-economic theorists: None of our models is fully consistent with the theory, and some clash sharply with it." Zipkin does not, however, discuss these inconsistencies in detail or analyze the possible consequences for the solution. We will show how the presence of arbitrage opportunities in a stochastic programming model may lead to substantial and unrealistic biases in its optimal solution, even if the investor is not able to exploit arbitrage opportunities directly because of market imperfections and trading restrictions.

We consider in detail two ways in which an approximation of the true asset-price uncertainty will introduce arbitrage opportunities in a stochastic programming model. The first is when the approximate description (itself possibly being arbitrage-free) is inconsistent with observed market prices of assets, which are typically taken as the prices at the initial date in the model. The second way is when the optimization model includes scenarios which are sampled from a larger description of the uncertainty. This larger description may itself be arbitrage-free and consistent with market prices, but this is in general not true for the sample.

Zenios [18], Dantzig and Infanger [5], Hiller and Eckstein [10], Cariiio *et al.* [4], Golub et al. [7] and Dert [6] all use sampling to approximate the asset-price uncertainty in their stochastic programs. Zenios [18], Hiller and Eckstein [10] and Golub *et al.* [7] only consider interest-rate uncertainty, and sample from an arbitrage-free term-structure model. Zenios [18] and Golub *et al.* [7] correct for the difference between market prices and model prices by introducing an option-adjusted spread, while Hiller and Eckstein [10] do not explicitly address this potential discrepancy. The use of an option-adjusted spread, however, contradicts the arbitrage-free pricing theory. Dert [6] recognizes that a sampled set of scenarios may contain arbitrage opportunities, and proves that it is always possible to eliminate them by *constructing* an additional scenario for each node in which a one-period arbitrage opportunity exists. He does not actually implement this in his computational experiments, however, nor does he analyze the consequences of the presence of arbitrage opportunities in his samples.

To illustrate the consequences of the presence of arbitrage opportunities in an optimization model, we consider a simple multi-period asset/liability management (ALM) problem. This problem assumes an investor who faces a sequence of liability payments in the future, and wants to determine a portfolio investment strategy whose payoffs are sufficient to meet these liabilities under a variety of plausible scenarios. We will assume that the investor can neither short sell assets nor borrow money, which implies that he cannot directly exploit arbitrage opportunities if they occur. Moreover, the investor faces proportional transaction costs on all his trades. The optimization results show that the investor's optimal portfolio composition at the initial date in the model will be biased significantly if arbitrage opportunities are present in the optimization model. This is true for any assumption about the investor's preferences, and for any reasonable level of transaction costs.

We formulate the asset/liability management problem mathematically as a multistage stochastic linear program. The uncertainty in this model is described in the form of an event tree. Section 2 introduces notation and terminology that will be used for event trees, and reviews the definition and a characterization of arbitrage-free asset prices in an event tree. In section 3 we will formulate a basic multi-period asset/liability management problem as multistage stochastic linear program. For this general formulation, conditions on the asset prices in the event tree will be derived which preclude spurious profit opportunities, and it will be seen that these conditions are closely related to the absence of arbitrage opportunities in the model. To derive these conditions, however, we have to make a strong assumption about the preferences of the investor. That a violation of these conditions can also bias the optimal solution without assumptions about the preferences of the investor is illustrated in section 4 by means of a numerical example. Section 5 contains the conclusions.

2 Asset-Price Uncertainty in an Event Tree

We describe the uncertainty in future asset prices and returns in the form of an event tree in which events occur at discrete points in time. For simplicity, we assume that these points are equally spaced in time, and they will be referred to by the index $t = 0, 1, \ldots, T$. Period t starts at time $t-1$ and ends at time t. Events in an event tree are also called *states*, and we assume that the number of states at each point in time is finite. A state is characterized by a particular realization of prices and dividends (defined here as all payments to the owner of a security other than the proceeds from selling it) for all securities under consideration.

As an example, figure 1 depicts a binomial event tree with six periods ($T = 6$). The nodes in the tree represent states of the world, while the arcs denote transitions with positive probability. It is a binomial tree because two states can occur at the end of a period for each given state at the beginning of a period. Because the states in the inner part of the tree can be visited by multiple paths, this tree is sometimes called a lattice. A lattice structure

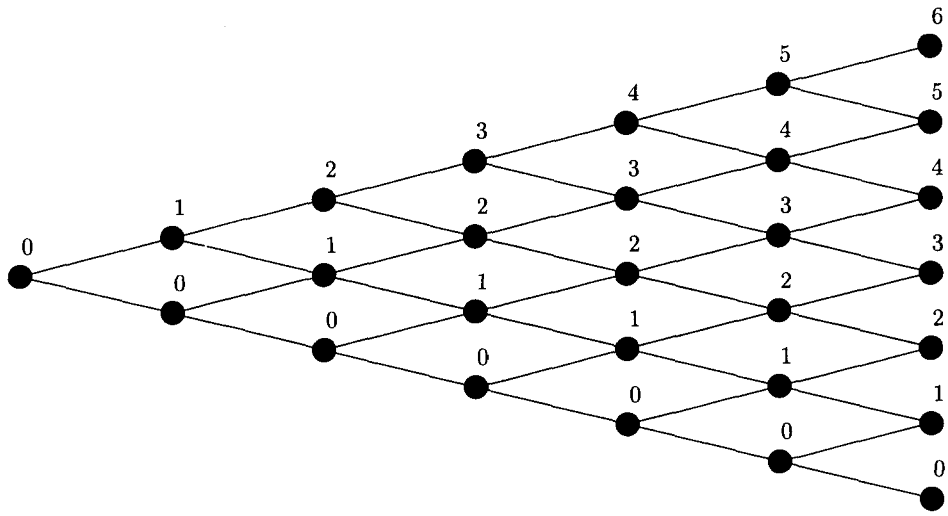

Figure 1: A binomial lattice with six periods.

is often assumed to limit the number of different states at each point in time. However, if securities with path-dependent payoffs, and thus prices, are included in the analysis (e.g., mortgage-backed securities), a state at each point in time can only correspond to one path in the tree; i.e., there would be a one-to-one relation between paths and states in the tree.

We will refer to states of the world by the index n. Given a state n at time $t < T$, each state which can occur with positive probability at time $t + 1$ will be called a successor of state n, and will as such be referred to by the index n^+. For any state n at a trading date $t > 0$, there is at least one state at time $t - 1$ which has state n as its successor. Such a state will be called a predecessor of state n, and is referred to by the index n^-. In the binomial lattice of figure 1, each state at time $t < T$ has two successors, every state in the interior of the lattice has two predecessors, while the two outermost states at each time $t > 0$ have one predecessor each.

In order to get realistic results from a portfolio optimization model which uses an event tree to describe the uncertainty in future asset prices, we will show that it is important that asset prices in the tree do not admit arbitrage opportunities. We will therefore first provide a characterization of an arbitrage opportunity.

2.1 Arbitrage Opportunities

An arbitrage opportunity exists if it is possible to construct a self-financing trading strategy[1] whose payoffs are nonnegative everywhere and strictly positive in at least one state in the event tree, and whose initial investment is nonpositive. Loosely stated, with such a trading strategy it would be possible to create something from nothing. Clearly, if an arbitrage opportunity exists in the market, many investors will try to take advantage of it. (In fact, there is a large group of investors in today's financial markets, called arbitrageurs, whose

[1] A trading strategy is self-financing if it does not require any cash inflows after time 0.

main objective it is to look for and exploit arbitrage opportunities.) This will influence the prices of the securities involved and lead to the elimination of the arbitrage opportunity. A necessary condition for equilibrium in financial markets is therefore that asset prices are such that no arbitrage opportunities exist. In that case, asset prices are said to be *arbitrage-free.*

Although one may argue that financial markets are not in equilibrium in reality and arbitrage opportunities occasionally exist, it is hard to imagine that any investor can predict the occurrence of arbitrage opportunities in the future. It is therefore reasonable to impose that estimates of future asset prices in a portfolio optimization model do not admit arbitrage opportunities.

2.2 A Characterization of Arbitrage-Free Asset Prices

When there are no transaction costs or taxes, securities can be traded in arbitrarily small amounts, interest rates for borrowing and lending are the same, and short sales of assets with full use of proceeds are allowed (collectively referred to as *perfect market conditions),* Harrison and Kreps [8] have provided the following important characterization of arbitrage-free asset prices (see also Huang and Litzenberger [12]).

Theorem 1 (Harrison and Kreps) *Asset prices in an event tree are arbitrage-free if and only if there exists a positive probability measure on the event tree such that the expected one-period* return *in* each given *state with respect to this probability measure is identical for all assets.*

Let $D^n_{i,t}$ denote the dividend payment on security i at the end of period t if state n occurs, and $S^n_{i,t}$ its ex-dividend price in state n at time t. Let π denote a probability measure on the event tree, and $\pi^{n/n+}_{t/t+1}$ the corresponding conditional probability of a transition from state n at time t to its successor state n^+ at time $t+1$. The theorem says that there are no arbitrage opportunities in the event tree if and only if there exists a probability measure π such that

$$\frac{\sum_{n+}\pi^{n/n^+}_{t/t+1}\left(S^{n^+}_{i,t+1}+D^{n^+}_{i,t+1}\right)}{S^n_{i,t}} = \frac{\sum_{n+}\pi^{n/n^+}_{t/t+1}\left(S^{n^+}_{j,t+1}+D^{n^+}_{j,t+1}\right)}{S^n_{j,t}} \tag{1}$$

for all assets i, j and in every state n at each time $t \in \{0,\ldots,T-1\}$ in the event tree. The summations in (1) are over all successor states n^+ of state n.

Condition (1) is equivalent to the condition that a quantity P^n_t exists for every state n at each trading date $t = 0,\ldots,T-1$ such that

$$S^n_{i,t} = P^n_t \sum_{n+} \pi^{n/n^+}_{t/t+1}\left(S^{n^+}_{i,t+1}+D^{n^+}_{i,t+1}\right) \tag{2}$$

for all assets i. The quantity P^n_t can be interpreted as the price in state n at time t of a riskless one-period security that pays one dollar at time $t+1$, and nothing at other times. It thus has the characteristics of a riskless one-period bank deposit.

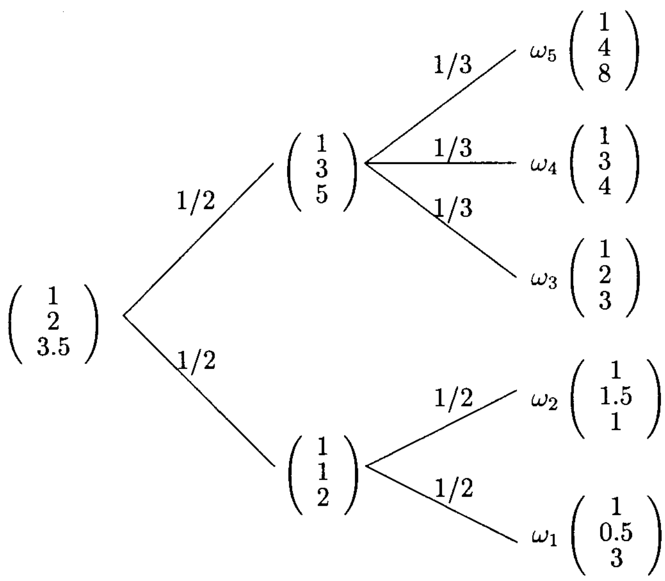

Figure 2: A two-period economy with three securities.

The probability measure in theorem 1 is often called *risk-neutral* probability measure, and does not have to represent the probability beliefs of any investor. We will denote a risk-neutral probability measure in the sequel by $\hat{\pi}$.

Figure 2 illustrates the characterization of theorem 1 for a two-period economy with three assets. At the end of the second period, five different states can occur, labeled ω_1 through ω_5. The vectors at the end of the arcs in the tree are the prices of the three assets (assume no dividend payments). The numbers on the arcs are conditional probabilities such that condition (1) is satisfied, and the asset prices in the event tree are therefore arbitrage-free. Moreover, the conditional probabilities in this example are unique. This is due to the fact that in every state the number of securities with linearly independent payoffs in the successor states equals the number of successor states. Financial markets are then said to be dynamically complete. If only the first two assets had been present in the example, the conditional probabilities of states ω_3, ω_4 and ω_5 in the second period could have been chosen differently without violating condition (1).

3 Stochastic Programming Models for Asset/Liability Management

When an event tree serves as description of the uncertainty in a portfolio optimization model, the points in time at which events occur correspond to the dates at which the investor can rebalance his portfolio *(trading dates).* Obviously, what is optimal depends on the objectives of the investor. We consider an investor who wants to manage a portfolio of assets over time so that the payoffs from this portfolio are sufficient to meet a sequence of target payoffs (liabilities) in the future. This type of investment problem is generally known ***as*** an ***asset/liability management*** (ALM) problem.

Depending on the nature of the target payoffs and the length of the investment horizon, asset/liability management problems arise in many contexts in practice. Examples are a pension fund or an insurance company which have to pay pensions and insurance claims, respectively, in the future. Another example is the selection of a portfolio strategy which provides certain return characteristics over time (e.g., index tracking). Hedging is still another application, where the target payoffs equal the expected depreciation in the asset which has to be hedged.

3.1 Assumptions

The liability payments are allowed to be stochastic, and we assume that they can be written as a function of the states in the event tree which describes the stochasticity in the asset prices and dividends. L_t^n denotes the required liability payment in state n at time t. We do not allow shortfalls in meeting the liabilities, but any excess cash flow from the portfolio can be invested in the traded assets. We furthermore impose the restriction that the investor cannot end up with a deficit at the model horizon.

In meeting the stream of future liabilities, the investor faces a trade-off between the initial investment and the remaining portfolio value at the model horizon. This trade-off is captured in the objective function: the initial portfolio investment is minimized, but a positive final portfolio value is credited to the objective.

We assume that a riskless one-period security (bank deposit) exists in every state, and furthermore that markets are dynamically complete in the approximate reality of the event tree. Let I denote the total number of securities that can be traded in each state.

The investor faces a transaction cost rate c, which applies to the value of the assets traded (bought or sold). We assume that the transaction cost rate does not apply to investments in the riskless one-period security. It is furthermore assumed that the investor can neither borrow money nor short sell assets at any point in time. A relaxation of this assumption will be discussed in section 3.4.

Note that no assumptions are made about what exactly defines a state in the event tree. Thus, each state can represent the realization of several different stochastic variables. We have only assumed that asset prices and dividends as well as liabilities can be written as functions of the states in the event tree.

3.2 Model Formulation

In our notation we have written all data (asset prices and dividends, liabilities) as functions of the states in the event tree. However, an optimal portfolio strategy is in general not only dependent on the current state, but on the sequence of past states as well[2]. The decision variables at time t therefore depend on a path in the event tree between time 0 and time t. However, the trading strategies are not allowed to depend on what will happen in the future. Thus, when two paths in the event tree between time 0 and time T share the same history up to a certain date $t < T$, the optimal trading strategy up to time t must be identical on both paths. The trading strategies are then said to be non-anticipative.

A path in the event tree between time 0 and time t will be called a scenario at time t, and referred to by the index s. The set of all possible scenarios at time t is denoted by $\mathcal{S}_t$. A scenario $s \in \mathcal{S}_t$ visits one node in the event tree at each trading date between time 0 and time t, and such a node will be referred to by the index n(s).

For each scenario s at time t there is exactly one scenario at each time $\tau < t$ which follows the same path in the event tree between time 0 and time τ. This scenario is called the predecessor at time τ of scenario s, and will as such be denoted by s^-. Furthermore, each scenario s at time t is the predecessor of one or more scenarios at time $t + 1$, which are called successors of scenario s. They will be referred to by the index s^+.

Let $\bar{x}_{i,0}$ denote the current holding of asset i in the portfolio of the investor (a known number). The variables $xb^s_{i,t}$ and $xs^s_{i,t}$ denote the units of asset i that are bought and sold, respectively, in scenario s at time t, and $xh^s_{i,t}$ the portfolio holding of asset i in scenario s at time t *after* asset purchases and sales (i.e., the portfolio holdings during period $t + 1$). The holding in the riskless one-period asset during period $t+1$ if scenario $s \in \mathcal{S}_t$ occurs is denoted by the separate variable y^s_t, while y^s_T denotes the final portfolio value in scenario $s \in \mathcal{S}_T$. The unconditional probability of scenario s at time t is denoted by π^s_t.

We emphasize that we do not assume here that π^s_t represents a risk-neutral probability. In fact, a risk-neutral probability measure may not exist as we have not assumed that asset prices in the event tree are arbitrage-free. Instead, we will show in the next section that our model formulation itself strongly suggests that asset prices should be arbitrage-free in order to prevent spurious profit opportunities in the model. When asset prices are arbitrage-free indeed, it is natural to set π^s_t equal to the risk-neutral probability because of our assumption of dynamically complete markets in the event tree. This will be further explained in the next section.

In its basic form, the ALM problem can be formulated mathematically as the following stochastic linear program (the ALM model):

minimize

$$(1+c)\sum_{i=1}^{I} S_{i,0}xb_{i,0} - (1-c)\sum_{i=1}^{I} S_{i,0}xs_{i,0} + P_0 y_0 - \lambda \sum_{s\in\mathcal{S}_T} \pi^s_T \left(\prod_{t=0}^{T-1} P_t^{n(s^-)}\right) y^s_T \quad (3)$$

subject to

[2]When assets with path-dependent payoffs and prices are considered, there is a one-to-one relation between states and scenarios in the event tree. In that case, state-dependence is identical to path-dependence.

$$-xs_{i,0} + xb_{i,0} - xh_{i,0} = -\bar{x}_{i,0} \qquad \forall\, i = 1,\ldots,I \quad (4)$$

$$xh^{s^-}_{i,t-1} - xs^s_{i,t} + xb^s_{i,t} - xh^s_{i,t} = 0 \qquad \forall\, i = 1,\ldots,I, s \in \mathcal{S}_t, t = 1,\ldots, T-1 \quad (5)$$

$$\sum_{i=1}^{I} D^{n(s)}_{i,t} xh^{s^-}_{i,t-1} + y^{s^-}_{t-1} + (1-c) \sum_{i=1}^{I} S^{n(s)}_{i,t} xs^s_{i,t} - (1+c) \sum_{i=1}^{I} S^{n(s)}_{i,t} xb^s_{i,t}$$

$$-P^{n(s)}_t y^s_t = L^{n(s)}_t \qquad \forall\, s \in \mathcal{S}_t, t = 1,\ldots,T-1 \quad (6)$$

$$\sum_{i=1}^{I} \left(D^{n(s)}_{i,T} + S^{n(s)}_{i,T} \right) xh^{s^-}_{i,T-1} + y^{s^-}_{T-1} - y^s_T = L^{n(s)}_T \qquad \forall\, s \in \mathcal{S}_T \quad (7)$$

$$xs^s_{i,t},\ xb^s_{i,t},\ xh^s_{i,t} \geq 0 \qquad \forall\, i = 1,\ldots,I,\ s \in \mathcal{S}_t,\ t = 0,\ldots, T-1 \quad (8)$$

$$y^s_t \geq 0 \qquad \forall\, s \in \mathcal{S}_t,\ t = 0,\ldots,T \quad (9)$$

The first three terms in the objective function represent the cost of *additional* investments at time 0. These additional investments consist of asset purchases (including transaction costs) and investment in the riskless one-period security, while proceeds from the sale of assets (net of transaction costs) are subtracted. The last term in the objective is the credit received from the expected final portfolio surplus, weighted by the scalar λ. In this term, $\pi^s_T\left(\prod_{t=0}^{T-1} P^{n(s^-)}_t\right)$ is the probability-weighted present value at time 0 of a dollar received in scenario s at time T, where the present value is calculated by using the one-period riskless returns along the scenario path in the event tree. The scalar λ reflects the investor's preference in the trade-off between the initial portfolio investment and a surplus at the model horizon.

To facilitate the subsequent solution analysis, the objective function does not incorporate risk aversion, i.e., the value attached to an extra unit of surplus is independent of the level of the surplus. Although it would be more realistic to include risk aversion in the objective function by using a concave utility function, for example, it will be shown in section 4 that the nature of the results does not change when this is done.

We distinguish two types of constraints in the model, *portfolio-balance* constraints and cash-balance constraints. The portfolio-balance constraints link portfolio holdings between successive periods (i.e., before and after rebalancing) in each scenario and for each asset. The portfolio-balance constraints are given by (4) for all assets at time 0, and by (5) for all assets in each scenario after time 0.

The cash-balance constraints make sure that sufficient cash flow is generated to meet the liability payment in each scenario and at each time. For a scenario at time $t < T$, this constraint is given by (6). At the end of a period, the investor receives dividend payments on his asset holdings and the return on his investment in the one-period riskless security (represented by the first two terms on the left-hand side of (6)). The next two terms reflect rebalancing of the portfolio: revenues are generated by selling assets, and money can be invested by buying assets, where both are adjusted for transaction costs. The final term on the left-hand side is the investment in the riskless one-period security during the next period.

The cash-balance constraints (7) at time T define the final portfolio value in each scenario. The first two terms on the left-hand side determine the final portfolio value before meeting the liability: the portfolio holdings are converted at the current market prices and the return on the investment in the riskless one-period security is added. The difference between this portfolio value and the liability payment in a scenario $s \in \mathcal{S}_T$ is the final portfolio surplus y_T^s.

The nonnegativity restrictions on $xh_{i,t}^s$ and y_t^s prevent short sales of assets and borrowing, respectively.

3.3 Solution Analysis

The following lemma will play an important role in our analysis.

Lemma 1 *If $\lambda = 1$ in the ALM model, then the only possible value for the dual variable φ_t^s on the cash-balance* constraint *for scenario s* at *time t in an optimal solution is:*

$$\varphi_t^s = q_t^s \qquad with \qquad q_t^s \equiv \pi_t^s \left(\prod_{\tau=0}^{t-1} P_\tau^{n(s^-)} \right) \tag{10}$$

for all $t = 1, \ldots, T$ and $s \in St$.

The proof of this lemma follows from the constraints in the dual formulation of the ALM model which correspond to the variables y_t^s (see appendix A).

The quantity q_t^s in (10) can be interpreted as the probability-weighted present value at time 0 of a dollar received in scenario s at time t. The assumption $\lambda = 1$ in the lemma corresponds to an investor who is indifferent between one dollar now and a random payment at time T whose expected present value at time 0 equals one dollar.

Notice that the lemma does not make a statement about feasibility of the dual problem. That is, the dual problem may not be feasible, implying that the primal problem is unbounded from below (as there is no upper bound on investments in one-period riskless securities, the ALM model is always feasible). An unbounded solution to the ALM model is obviously an undesirable outcome, as it means that the investor would generate unlimited *benefits* (as measured by the objective function) from meeting his liabilities.

One way in which the ALM model will have an unbounded solution is if $\lambda > 1$, which can be seen directly from the proof of the lemma. The next proposition states that the solution will be unbounded as well if $\lambda = 1$ and if there is an asset whose expected one-period return in some state in the event tree (net of transaction costs that are incurred by buying the asset at the start of the period and selling it at the end) exceeds the riskless one-period return.

Proposition 1 *When $\lambda = 1$, the solution to the ALM model is unbounded if there is an asset i and* a *state n at some time $t \in \{0, \ldots, T-1\}$ in the event tree so that the following inequality is violated:*

$$(1+c)S_{i,t}^n \geq P_t^n \sum_{n^+} \pi_{t/t+1}^{n/n^+} \left((1-c)S_{i,t+1}^{n^+} + D_{i,t+1}^{n^+} \right) \tag{11}$$

where the summation *is over all successor states* n^+ of state n in *the event tree.*

Appendix A contains the proof of this proposition.

If inequality (11) is violated for some asset i^* in state n* at time t^* in the event tree, then the following trading strategy strictly improves the objective function of the ALM model:

1. Take $1 for investment at time 0, and invest it in riskless one-period securities until time t^*.

2. For all scenarios at time t^* which visit node n^*, invest the accumulated money in asset i^* during period $t^* + 1$. For all other scenarios, invest the accumulated money in the riskless one-period security during period $t^* + 1$.

3. For all scenarios at time $t^* + 1$, take the accumulated money and invest it in riskless one-period securities until time T.

The expected present value of the payoffs from this strategy at time T, which is credited to the objective, exceeds the initial cost of this strategy (one dollar). As no limit has been imposed on how much the investor can invest at time 0, an arbitrarily large amount can be spent on this trading strategy, causing the solution to the ALM model to be unbounded. Note that this trading strategy is independent of the liabilities.

Proposition 1 states a necessary condition for a bounded solution to the ALM model if $\lambda = 1$. A sufficient condition is that asset prices in the event tree are arbitrage-free and that the probability π_T^s in the objective function equals the risk-neutral probability $\hat{\pi}_T^s$ of scenario $s \in S_T$. This is easy to see from the proof of the proposition. The expected present value of a final portfolio surplus is then computed as the arbitrage-free value at time 0 of a security with a single (scenario-dependent) cashflow at time T which equals the portfolio surplus. To set π_T^s equal to $\hat{\pi}_T^s$ makes sense as our assumption of dynamically complete markets in the event tree implies that any cashflow pattern in the event tree can be replicated by a trading strategy in the available assets.

If asset prices in the event tree are arbitrage-free and if there are no transaction costs ($c = 0$), then we can also prove the following properties of the optimal solution to the ALM model.

Proposition 2 *If $\lambda = 1$ and $c = 0$, if asset prices* in *the* event *tree are arbitrage-free, and if π_T^s is the risk-neutral probability of scenario $s \in S_T$, then the value of the optimal solution to the ALM model is the difference* between *the arbitrage-free* value at *time 0 of a security whose payoffs exactly match the stream of future liabilities and the value of the investor's initial portfolio. Furthermore, every feasible solution is* an optimal solution *in this case.*

The proof of this proposition can be found in appendix A.

3.4 Extensions of the ALM Model

Allowing for Borrowing and Short Sales of Assets

In its simplest form, one-period borrowing and short sales of assets can be introduced in the ALM model by removing the nonnegativity restrictions on the variables y_t^s and $xh_{i,t}^s$, respectively. If the cost of borrowing is higher than the return on the one-period riskless security, then separate borrowing variables must be introduced in the model. Similarly, if there are additional costs associated with short sales of assets, separate variables for short positions have to be used.

In the ALM model without borrowing and short sales, the investor cannot directly exploit prevailing arbitrage opportunities as this always involves a short position in at least one asset and/or borrowing. The assumption $\lambda = 1$ on the preferences of the investor was therefore necessary to prove that arbitrage opportunities could lead to an unbounded solution. However, if the possibility of borrowing and/or short sales of assets enables the investor in the model to directly exploit an arbitrage opportunity, then the solution to the model will be unbounded (assuming no upper bounds on borrowing and short sales) for any investor who assigns a positive value to a final portfolio surplus ($\lambda > 0$). It is not difficult to prove this from the dual formulation of the ALM model.

In the following section we will show by means of a numerical example that even without the possibility of borrowing and short sales, the optimal solution to the ALM model may be biased towards arbitrage opportunities in the model for values of λ between zero and one.

Adding constraints

In practical applications, additional constraints may be added to the ALM model. Examples are a budget constraint at time 0, and upper bounds on individual asset holdings or asset mixes. Such side constraints may prevent that the solution to the model will be unbounded if the situation of proposition 1 occurs. The results in the next section show, however, that they may not prevent biases in the optimal solution.

4 A Numerical Example

Consider an investor who faces a required liability payment of $1000 after 3, 6, 9 and 12 months from the current date. He currently owns a portfolio which consists of four discount bonds, each with face value of $1000, and with respective maturities of 3, 6, 9 and 12 months. These discount bonds will be referred to as bond 1 through bond 4, respectively, and this initial portfolio as the discount-bond portfolio. Clearly, holding each of the discount bonds in his initial portfolio to maturity enables the investor to exactly match his future liabilities. However, the investor wants to find out whether he can meet his liabilities in a cheaper way by including other securities in his portfolio.

To be able to analyze the biases in the optimal solution to the ALM model for this problem when arbitrage opportunities are present in the underlying event tree, we first derive the optimal solution when no arbitrage opportunities exist. In that case, proposition 2 tells that every feasible solution is an optimal solution to the ALM model if no transaction costs are present (c = 0) and if $\lambda = 1$. As holding each of the discount bonds in the discount-bond portfolio to maturity defines a feasible portfolio strategy in our example, the discount-bond portfolio is an optimal portfolio at time 0 under these assumptions (with optimum objective value zero). When $\lambda < 1$, the optimum objective value can never improve (decrease) as compared with the solution for $\lambda = 1$. As the objective value of the discount-bond portfolio is the same for all $\lambda \leq 1$, the discount-bond portfolio is also an optimal portfolio at time 0 in the ALM model when $\lambda < 1$. Furthermore, when the transaction cost rate is positive, the objective value can never improve in comparison with the model without transaction costs. As holding the discount-bond portfolio until maturity does not involve transaction costs for the investor, leaving the objective value unchanged, this is an optimal portfolio strategy as well in the ALM model with transaction costs.

In conclusion, holding each of the discount bonds in the discount-bond portfolio to maturity is an optimal portfolio strategy for the investor for any value of $\lambda \leq 1$ and transaction cost rate c ≥ 0 if asset prices in the ALM model are arbitrage-free.[3] We will show below that the model may suggest differently if asset prices are not arbitrage-free. When this happens, however, the investor will not be able to realize the profits which the model suggests in reality if not exactly the same arbitrage opportunities occur in practice as are present in the model. Obviously, this is highly unlikely.

Sofar, no assumptions were made about what defines a state in the event tree. In this example, we assume that interest-rate uncertainty is the only source of uncertainty for the investor. Each state in the event tree therefore represents a particular realization of the term structure of interest rates. The current term structure is assumed to be flat with a (continuously compounded) yield of 8% for all maturities. This implies the market prices of the discount bonds which are given in table 1.

We assume that three call options and three put options are available in the market (in addition to the four discount bonds), all of them written on bond 4. The first two columns in table 1 specify the time to maturity and the strike price of each option. The strike price equals 99.9% of the forward price[4] of bond 4 for the maturity date of the option. The

[3] Actually, the discount bond portfolio is also optimal when a more realistic utility function is used to credit a portfolio surplus at the model horizon to the objective function in the ALM model, as long as all asset prices are arbitrage-free. This is due to the fact that the cashflows from the discount bond portfolio exactly match the liabilities, and in an arbitrage-free world no other trading strategy can therefore exist whose cashflows also meet the liabilities, but for which the initial investment is lower than for the discount bond portfolio. The only required condition on the utility function is that no scenario exists in which the utility of a dollar at the model horizon exceeds its present value at time 0 (which is comparable to $\lambda > 1$).

[4] At time 0, the time-t forward price of a discount bond which matures at time T equals $[P_0(T)/P_0(t)]$ times the face value of the bond, where $P_0(t)$ is the price at time 0 of a discount bond which matures at

	Maturity (months)	Strike price	Market price
Bond 1	3	-	980.20
Bond 2	6	-	960.79
Bond 3	9	-	941.77
Bond 4	12	-	923.12
Call 1	3	940.82	1.34
Call 2	6	959.83	1.31
Call 3	9	979.22	1.06
Put 1	3	940.82	0.42
Put 2	6	959.83	0.38
Put 3	9	979.22	0.14

Table 1: Data for options and discount bonds.

derivation of the market prices for the options will be discussed shortly.

4.1 Modelling the Interest-Rate Uncertainty

In the financial literature, several models have been proposed which describe the uncertainty in the term structure of interest rates in the form of an event tree. Examples are Black, Derman and Toy [1], Brennan and Schwartz [3], Heath, Jarrow and Morton [9], Ho and Lee [11] and Hull and White [13]. Their primary use in financial economics has been to calculate arbitrage-free values of interest-rate derivative securities. Given the prices of one-period riskless securities in all states in the event tree (which are derived from the current term structure and assumptions about possible changes in the term structure), the arbitrage-free value at time 0 of a security can be calculated in a recursive manner, starting with the values of the security in the event tree at its maturity date and recursively using relation (2).

A natural extension is to use these term-structure models as description of the uncertainty in the ALM model: they describe the uncertainty in the form of an event tree, and the recursive valuation procedure provides prices of interest-rate derivative securities in the nodes of the tree which are arbitrage-free. As we will illustrate in the next section, however, the number of time steps that is required in the event tree to obtain model prices which are close to observed market prices is generally much too large to include in a stochastic programming model. We will also show that inconsistency between model and market prices which is caused by choosing a small number of time steps can lead to substantial biases in the optimal solution.

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time t.

Several authors (e.g., Zipkin [19], Zenios [18], Hiller and Eckstein [10], Golub et al. [7]) have suggested to sample scenarios from a large term-structure model in order to reduce the size of the stochastic program. Using our ALM example, we will show below that this effectively results in the introduction of arbitrage opportunities in the optimization model, and that this severely biases the investor's optimal portfolio composition at time 0.

Although we present our results in the next sections for the case that arbitrage-free term-structure models are used to approximate the interest-rate and asset-price uncertainty in the ALM model, it should be clear that many other ways of describing the uncertainty (which are often much less structured) have the risk of introducing arbitrage opportunities, and may therefore cause biased solutions.

4.2 Inconsistencies between Market Prices and a Term-Structure Model

Because of its simplicity, we have chosen the term-structure model of Ho and Lee [11] to obtain the numerical results in this and the next section, but any other arbitrage-free term-structure model in discrete time could have been used to obtain similar results. The Ho and Lee model describes the term-structure uncertainty in the form of a binomial lattice (see figure 1). To construct the lattice, it uses the current term-structure, the length of a time step (or, equivalently, the number of time steps within a given horizon), an estimate of the volatility of the one-period interest rate, and the risk-neutral probability of an upward move in the lattice (which is assumed to be the same in every state) as inputs. A detailed description of the Ho and Lee model can be found in appendix B.

Figure 3 depicts the arbitrage-free value at time 0 for the three call options on bond 4 in the example as a function of the number of time steps in the Ho and Lee model (with an horizon of 12 months, the maturity of bond 4). The interest-rate volatility and the risk-neutral probability of an upward move were chosen as 0.58% per year and 0.5, respectively.[5] It follows from the figure that close to 180 time steps are needed before all prices have converged to within a precision of one dollar cent.[6] The dependence of the arbitrage-free values at time 0 on the number of time steps in the term-structure model is similar for the three put options on bond 4, and does not change qualitatively for different assumptions about the current yield curve, the interest-rate volatility or the strike prices of the options.

In a stochastic programming model, one usually takes the observed market prices of securities as the prices at time 0 in the model. If a term-structure model is used to describe the interest-rate uncertainty in the ALM model, the arbitrage-free security values at time 0 that follow from this model should therefore equal the market prices in order to keep prices

[5] The chosen interest-rate volatility is such that interest rates cannot become negative for any number of time steps that is used in figure 3; see also appendix B.

[6] Note that 180 time steps in the model corresponds to 45 periods before the maturity of call option 1, 90 periods before the maturity of call option 2, and 135 periods before the maturity of call option 3. The security with the shortest maturity will therefore often determine the minimum number of time steps that is required for the convergence of all values to within a certain precision.

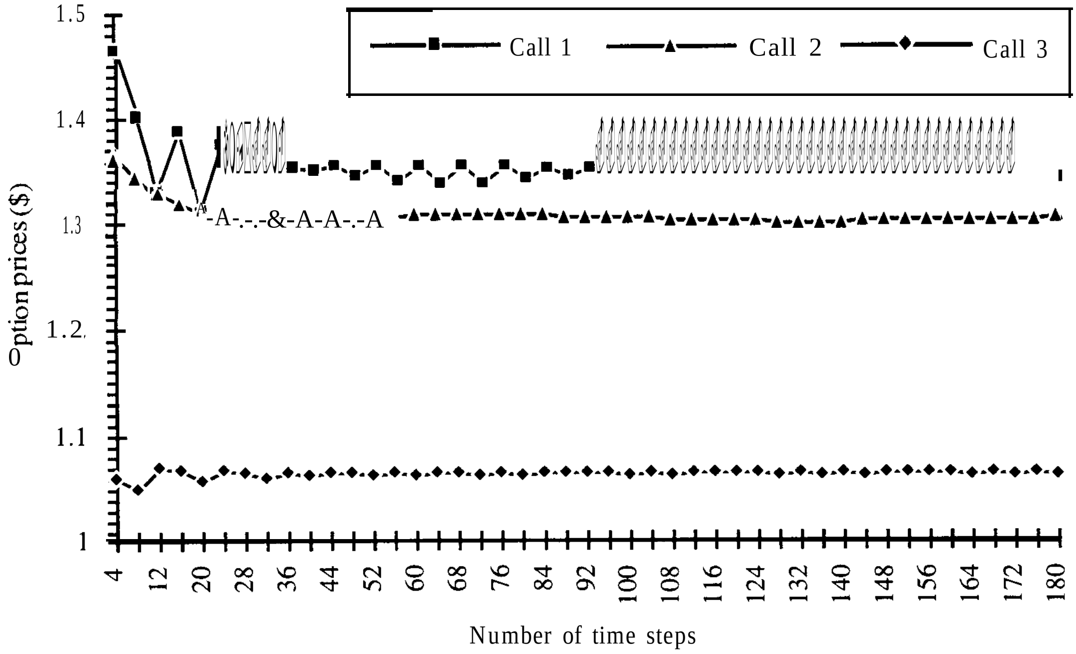

Figure 3: Arbitrage-free values at time 0 for the three call options on discount bond 4 as a function of the number of time steps in the Ho and Lee model.

in the ALM model arbitrage-free. For discount bonds, the arbitrage-free value at time 0 only depends on the current term structure. Because the Ho and Lee model takes the current term structure as input, it always prices discount bonds correctly. For other interest-rate derivative securities, however, the arbitrage-free value at time 0 not only depends on the current term structure but also on the other model parameters.

When looking for a term-structure model that is consistent with observed market prices, we want the implied arbitrage-free security values to be independent of an increase in the number of time steps. Suppose that the market prices of the options in the ALM example equal (within a precision of one dollar cent) the arbitrage-free values in the Ho and Lee model with 180 time steps and other parameter values as mentioned earlier (the market prices are given in table 1). Ideally, we would like to include the binomial lattice with 180 time steps as description of the uncertainty in the stochastic programming formulation. However, a binomial lattice with 180 time steps defines a total of 2^{180} different paths, obviously much too many to include in the ALM model.

We will analyze the effects on the optimal solution if a version of the Ho and Lee model with 8 instead of 180 time steps is used in the ALM model, while the other parameter values are the same. This version of the Ho and Lee model will be referred to as the restricted Ho and Lee model. Table 2 compares the market prices of the options with the arbitrage-free

	Call 1	Call 2	Call 3	Put 1	Put 2	Put 3
Market price	1.34	1.31	1.06	0.42	0.38	0.14
Time-O value	1.40	1.34	1.05	0.48	0.42	0.13
Value/Price	104.4%	103.0%	98.5%	114.2%	110.1%	90.5%

Table 2: Market prices and arbitrage-free values at time 0 in restricted Ho and Lee model for options on discount bond 4.

values that follow from the restricted Ho and Lee model. As we have assumed that the market prices of all securities are used as the prices at time 0 in the ALM model while the prices at all future points in time are derived from the restricted Ho and Lee model, arbitrage opportunities exist during the first period.

We have implemented the ALM model for this example in the modelling language GAMS on a Sun SparcStation 2. MINOS 5.3 was used to optimize the model. The stochastic program has over 2200 variables and 1000 constraints. Table 3 reports the optimum objective values for different values of c and λ. To prevent unbounded solutions, we have limited portfolio holdings for the options to at most 1000 units at any point in time.[7] The negative numbers in the table suggest that the investor can save on his initial portfolio (the value of his initial portfolio is $3805.87, the sum of the prices of the four discount bonds). Obviously, the size of the suggested savings depends on the size of the upper bounds on the option holdings. The results show that the transaction cost rate has to be at least 3% before the bias in the optimal solution value disappears, and up to 15% if λ lies between 0.9 and 1.

To see the nature of the bias in the optimal solution itself, table 4 shows the optimal portfolio composition at time 0 for λ = 0.91 and different values of the transaction cost rate c. This table also shows the corresponding trade-off between the portfolio value at time 0 ("Portf. value") and the expected present value of a final portfolio surplus ("EPV surplus") before applying the weight λ. The results show that the holdings in the discount bonds are partially replaced by investments in the seemingly underpriced options (i.e., for which the arbitrage-free value exceeds the market price) when $c \leq 2\%$, while the investment in put option 1 is added to the discount-bond portfolio when c = 3%.

The optimal portfolio at time 0 when $\lambda < 0.91$ is almost identical to the optimal portfolio for $\lambda = 0.91$ if $c \leq 2\%$, and equal to the discount-bond portfolio if $c \geq 3\%$. When λ increases from 0.91 to one, investments in the seemingly underpriced options increasingly take place in addition to the holdings in the discount-bond portfolio. This corresponds to the trading strategy that was outlined in section 3 for the case λ = 1, and which would lead to an unbounded solution to the ALM model in the absence of upper bounds on the option holdings.

[7] Without these upper bounds, the solution would have been unbounded instead of negative for $\lambda \geq 0.87$.

	$0 \leq \lambda \leq 0.9$	λ = 0.91	λ = 0.95	λ = 1.0
$c = 0\%$	-146.65	-146.68	-154.85	-195.62
c = 1%	- 91.86	-91.86	-91.94	-161.55
c = 2%	- 36.64	-36.64	-40.92	-127.06
c = 3%	0	-3.74	-28.89	-93.10
c = 4%	0	0	-20.88	-71.67
c = 5%	0	0	-14.50	-57.84
c = 6%	0	0	-10.30	-49.82
c = 7%	0	0	-6.11	-41.81
c = 8%	0	0	-1.92	-33.79
c = 9%	0	0	0	-25.77
c = 10%	0	0	0	-17.76
c = 11%	0	0	0	-13.27
c = 12%	0	0	0	-9.08
c = 13%	0	0	0	-4.89
c = 14%	0	0	0	-0.70
c = 15%	0	0	0	0

Table 3: Optimum objective function values in the ALM example for different values of the transaction cost rate c and the surplus weight λ.

λ = 0.91	$c = 0\%$	$c = 1\%$	c = 2%	c = 3%	c > 3%
Bond 1	1.00	0.00	0.00	1.00	1.00
Bond 2	0.80	0.00	0.00	1.00	1.00
Bond 3	0.00	0.87	0.90	1.00	1.00
Bond 4	0.00	1.00	1.00	1.00	1.00
Call 1	829.76	830.60	833.85	0.00	0.00
Call 2	0.00	0.23	0.06	0.00	0.00
Call 3	0.00	0.00	0.00	0.00	0.00
Put 1	1000.00	1000.00	1000.00	1000.00	0.00
Put 2	1000.00	1000.00	1000.00	0.00	0.00
Put 3	0.00	0.00	0.00	0.00	0.00
Lend	0.00	0.00	0.00	0.00	0.00
Portf. value	3658.27	3674.38	3690.20	4224.99	3805.87
EPV surplus	0.20	0.03	0.12	478.50	0.00

Table 4: Optimal portfolio at time 0 when λ = 0.91 and for different values of the transaction cost rate c.

We have thus shown that the optimal solution to the ALM model in which the description of the asset-price uncertainty is inconsistent with observed market prices may be severely biased towards the resulting arbitrage opportunities in the model, despite the fact that the investor could not exploit these opportunities directly because of borrowing and short sale restrictions. Furthermore, these biases were shown to exist for any value of the final-portfolio weight λ, in particular also for $\lambda = 0$. This in turn shows that biases would have been present as well if another criterion had been used to value a final portfolio surplus in the objective function, e.g., a concave utility function. Although the biases were shown to disappear when transaction costs were sufficiently high, even higher transaction costs would have been necessary if the investor's initial portfolio had not been the true optimal portfolio.

4.3 Sampling Interest-Rate Scenarios

An alternative way to restrict the number of scenarios in the ALM model is by sampling interest-rate paths from a large term-structure model which is consistent with observed market prices. In the ALM example and under the assumptions of the previous section, this would mean sampling from the Ho and Lee model with 180 time steps. Limited computer resources, however, did not allow us to include more than a few interest-rate paths with 180 time steps before the resulting ALM model exhausted the available memory. We have therefore slightly modified the assumptions to illustrate the biases in the optimal solution due to sampling.

Suppose that the market prices of all assets are equal to the arbitrage-free values at time 0 in the Ho and Lee model with 32 time steps and other parameter values as in the previous section. (The hypothesized market prices thus differ somewhat from the ones in table 1; see figure 3.) As this Ho and Lee model defines 2^{32} (more than four billion) interest-rate paths, it is clear that only a subset can be included in the ALM model. We will present the results when 64 interest-rate paths are included, sampled randomly from the complete set using the risk-neutral probability measure as sampling distribution. Each of these paths is assigned equal probability, i.e., the probability π_T^s in the objective of the ALM model is set to $(1/64)$ for each sample path. The resulting ALM model has about 9,000 constraints and 25,000 variables.[8]

The biases in the optimal solution to the ALM model with sampled interest-rate paths are much more severe than in the previous section. When $\lambda = 0$ the average optimum objective value of ten optimizations of the ALM model, each with a different sample of 64 interest-rate scenarios, equals -\$2771.31 if c = 3%, and -\$1876.68 if c = 10% (in the presence of upper bounds of 1000 units on investments in the options). These numbers suggest that the investor can save at least half the value of the discount-bond portfolio

[8] As interest-rate paths with the same history up to a certain point in time use the same variables in the ALM model up to that point (to ensure non-anticipativity of the portfolio strategy), the size of the ALM model is somewhat dependent on the particular sample.

($3805.87) by restructuring his portfolio, despite the high transaction costs. The reason for these enormous biases will be explained by looking at the optimal portfolio strategy for one particular interest-rate path in one of the samples when $\lambda = 0$ and c = 3%.

The data for the interest-rate path are given in table 5. The first column lists the number of the node which the path visits in the binomial lattice at each of the 32 points in time. The nodes in the lattice are numbered as in figure 1, i.e., the number of a node at a given point in time equals the number of upward moves in the lattice up to that point in time. The path in the table thus approximately runs through the center of the lattice. The second column in table 5 shows the number of other paths in the sample with the same history up to each point in time. The third column shows the value of the optimal portfolio along the path, and the last column the composition of the optimal portfolio. A liability payment of $1000 is due at $t = 8$, $t = 16$, $t = 24$ and $t = 32$, which explains the drop in portfolio value at these points in time.

The second column in the table shows that after six periods there is no other path in the sample with the same history as the particular path we are looking at. This implies that the variables in the ALM model which correspond to portfolio decisions on this path at $t = 6$ and later do not relate to any of the other paths in the model. The optimal values of these variables are thus fully determined by the situation along this path. Security prices along this path, however, are not arbitrage-free as the return in each period on the path may be different per security, and the optimal values of the variables are chosen to take advantage of that. For example, between $t = 6$ and $t = 7$ the path makes a downward move in the lattice, which corresponds to an increase in the yield on zero-coupon bonds in the Ho and Lee model. As a consequence, the return on the put options between $t = 6$ and $t = 7$ on this path is (significantly) higher than the risk-free rate (except for put option 1, which has no value in this part of the lattice). This causes the radical shift in the portfolio composition at $t = 6$ from call option 1 to put options 2 and 3. The opposite effect can be seen to happen whenever the path makes an upward move in the lattice after $t = 6$. Because no options are traded after $t = 24$, all money is invested in the one-period riskless security ("lending") from that point on to save transaction costs.

Sofar we have only presented results for the case $\lambda = 0$, i.e., when the investor attaches no value to a portfolio surplus at the model horizon. It is clear that the biases would even have been larger for $\lambda > 0$, or when a positive and increasing utility function would have been used to value the portfolio surplus, as this would have presented the investor with an extra incentive to change his portfolio from the discount bond portfolio.

Thus, although asset prices in the event tree from which the interest-rate paths are sampled are arbitrage-free, the fact that not all interest-rate paths are included in the ALM model essentially creates arbitrage opportunities in the model. The resulting biases in the optimal solution persist for any reasonable level of transaction costs. Note that the biases in the optimal solution are primarily caused by the fact that from a certain point in time onwards one is able to trade with perfect foresight on each scenario in the optimization

t =	node	joint paths	portfolio value ($)	portfolio composition (in units, except for lending)
0	0	63	928.97	bond 1: 0.47, bond 4: 0.15, call 1: 240.83
1	0	33	803.45	bond 1: 0.47, call 1: 178.84, put 1: 272.46
2	0	24	820.63	bond 1: 0.47, call 1: 271.86, put 1: 223.18
3	1	10	797.96	bond 1: 0.41, call 1: 349.46, put 1: 223.18
4	2	4	865.93	bond 1: 0.41, call 1: 374.03, lending: $23.60
5	3	2	1054.87	bond 1: 0.41, call 1: 116.64, lending: $451.85
6	4	-	1122.82	bond 1: 0.41, put 2: 1000, put 3: 1000, lending: $614.81
7	4	-	1189.38	bond 1: 0.41, put 2: 1000, put 3: 1000, lending: $616.31
8	4	-	275.46	call 2: 244.62
9	5	-	352.41	put 2: 1000, put 3: 1000, lending: $235.24
10	5	-	432.75	put 2: 1000, put 3: 1000, lending: $235.82
11	5	-	563.53	put 2: 1000, put 3: 1000, lending: $236.41
12	5	-	733.76	call 2: 1000, call 3: 615.33
13	6	-	1082.22	call 2: 1000, call 3: 615.33
14	7	-	1520.53	put 2: 1000, put 3: 1000, lending: $1476.60
15	7	-	1555.45	put 2: 1000, put 3: 1000, lending: $1480.29
16	7	-	611.68	put 3: 1000, lending: $511.77
17	7	-	656.52	call 3: 1000, lending: $276.80
18	8	-	798.09	put 3: 1000, lending: $727.11
19	8	-	836.84	call 3: 1000, lending: $493.94
20	9	-	1004.87	call 3: 1000, lending: $495.23
21	10	-	1220.50	call 3: 1000, lending: $496.50
22	11	-	1470.16	call 3: 1000, lending: $497.76
23	12	-	1724.49	call 3: 1000, lending: $499.00
24	13	-	980.07	lending: $980.07
25	13	-	982.46	lending: $982.46
26	13	-	984.89	lending: $984.89
27	13	-	987.35	lending: $987.35
28	13	-	989.86	lending: $989.86
29	14	-	992.40	lending: $992.40
30	14	-	994.92	lending: $994.92
31	15	-	997.47	lending: $997.47
32	16	-	0.00	-

Table 5: Optimal portfolio strategy along a sampled interest-rate path (λ = 0; c = 3%).

model. The assumption that the underlying tree from which interest-rate paths are sampled is arbitrage-free is somewhat irrelevant in this respect; sampling from another tree would have caused similar biases.

Even if one does not allow portfolio rebalancing in a dynamic model (e.g., Hiller and Eckstein [10]), proposition 1 indicates that sampling may still bias the optimal portfolio at time 0. This is due to the fact that the (risk-neutral) probability π_T^s in the objective function of the ALM model is replaced by the sample probability (l/M) for each sample path, where M is the number of sampled paths. Specifically, deviation of the sample probabilities from the risk-neutral probabilities may result in a violation of inequality (11) (in which the probabilities now equal the sample probabilities), which in turn may bias the optimal solution.

5 Conclusions

We have shown, both mathematically and by means of a numerical example, that the presence of arbitrage opportunities in a description of the asset-price uncertainty which is used in a stochastic programming model may substantially bias its optimal solution. This was shown to be the case for an investor who could not borrow money or short sell assets, and therefore could not directly exploit the arbitrage opportunities. The solution biases suggest profit opportunities for the investor, but the investor will not be able to realize these profits if not exactly the same arbitrage opportunities will occur in reality as are present in the stochastic program. Obviously, this is highly unlikely. Although the mathematical results required a strong assumption about the investor's preferences regarding the trade-off between the initial portfolio investment and a final portfolio surplus, the computational results showed that biases can occur under any preferences assumption.

The occurrence and size of solution biases when arbitrage opportunities are present will depend on several factors. It obviously depends on the magnitude of the arbitrage opportunities and the level of market imperfections and trading restrictions (transaction costs, position limits) for the investor. To a certain extent, it was also shown to depend on the assumption about the investor's preferences. In addition, biases are more likely to occur if the number of traded assets is large in comparison with the number of arcs which emanates from each node in the event tree that describes the uncertainty in the stochastic program. Although biases in the optimal solution are therefore not an automatic consequence of the presence of arbitrage opportunities, the results in this paper should serve as a warning.

For the numerical example in this paper we have shown that arbitrage opportunities may be present in a stochastic programming model, and thereby biases in its optimal solution, if the event tree which describes the asset-price uncertainty in the model is inconsistent with observed market prices and/or if only a sample of scenarios from the event tree is included. Of course, it may be hard (if not impossible) to construct an event tree for practical applications which is at the same time arbitrage-free, consistent with all market prices and concise enough for computational purposes. When an event tree can be found which satisfies the first two properties but not the third, Klaassen [15] describes state and time aggregation methods which can be used to reduce the size of the event tree by combining states and time periods in such a way that the tree remains arbitrage-free and consistent with all market prices.

Although arbitrage-free asset pricing models from the financial literature can be used to construct an event tree in which asset prices are arbitrage-free, it will usually not be possible to obtain full consistency with observed market prices for all assets which one wants to include in practical portfolio optimization models. For example, none of the proposed arbitrage-free term-structure models may be able to fully explain observed market prices of all interest-rate-derivative securities. Typically, values for the parameters in these models are chosen such that the differences between market prices and arbitrage-free model prices are as small as possible. To study the effects which the use of market prices has on the optimal solution when these market prices do not exactly equal the arbitrage-free values which follow from the model, one could solve the stochastic programming model twice: once with the market prices and once with the arbitrage-free model prices as security prices at time 0.

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A Proofs

PROOF OF LEMMA 1: The result follows from the constraints in the dual problem of the ALM model (3)–(9) that correspond to the variables y_t^s for all $t = 0, \ldots, T$ and $s \in \mathcal{S}_t$. These dual constraints are (the corresponding primal variable is listed at the beginning of each constraint) :

$$y_0 \quad : \quad \sum_{s \in \mathcal{S}_1} \varphi_1^s \leq P_0 \tag{12}$$

$$y_t^s \quad : \quad -\varphi_t^s \, P_t^{n(s)} + \sum_{s^+} \varphi_{t+1}^{s^+} \leq 0 \tag{13}$$

$$y_T^s \quad : \quad -\varphi_T^s \leq -q_T^s \tag{14}$$

where we have used the definition of q_T^s from (10) and $\lambda = 1$ in the last constraint.

By recursive substitution and the fact that $q_{t+1}^{s^+} = q_t^s P_t^{n(s)} \pi_{t/t+1}^{n(s)/n(s^+)}$ for each successor s^+ of $s \in \mathcal{S}_t$, it follows directly from (13) and (14) that dual feasibility implies $\varphi_t^s \geq q_t^s$ for all $t = 1, \ldots, T$ and $s \in \mathcal{S}_t$. We will show by induction that equality must hold.

Because $\varphi_1^s \geq q_1^s$ for all $s \in \mathcal{S}_1$, it follows from constraint (12) that:

$$\sum_{s \in \mathcal{S}_1} q_1^s \leq \sum_{s \in \mathcal{S}_1} \varphi_1^s \leq P_0 \tag{15}$$

By definition, $q_1^s = P_0 \pi_1^s$. Because $\sum_{s \in \mathcal{S}_1} \pi_1^s = 1$, it follows that equality must hold throughout in (15), and thus $\varphi_1^s = q_1^s$ for all $s \in \mathcal{S}_1$. This establishes (10) for $t = 1$.

To show that the equality holds for all $t > 1$, we will prove that $\varphi_{t+1}^s = q_{t+1}^s$ must hold for each $s \in \mathcal{S}_{t+1}$ if $\varphi_t^s = q_t^s$ for all $s \in \mathcal{S}_t$. If the latter equality holds, then constraint (13) implies for an arbitrary scenario $s \in \mathcal{S}_t$:

$$\sum_{s^+} q_{t+1}^{s^+} \leq \sum_{s^+} \varphi_{t+1}^{s^+} \leq \varphi_t^s P_t^{n(s)} = q_t^s P_t^{n(s)} \tag{16}$$

where we have used the earlier result that $\varphi_{t+1}^{s^+} \geq q_{t+1}^{s^+}$ for all successors s^+ of s. Because $q_{t+1}^{s^+} = q_t^s P_t^{n(s)} \pi_{t/t+1}^{n(s)/n(s^+)}$ for each s^+ by definition, and as $\sum_{s^+} \pi_{t/t+1}^{n(s)/n(s^+)} = 1$, we see that equality must hold throughout in (16). This implies $\varphi_{t+1}^{s^+} = q_{t+1}^{s^+}$ for each s^+. Because scenario $s \in \mathcal{S}_t$ was chosen arbitrarily, it follows that $\varphi_{t+1}^s = q_{t+1}^s$ for all $s \in \mathcal{S}_{t+1}$ if $\varphi_t^s = q_t^s$ for all $s \in \mathcal{S}_t$.

$\mathcal{QED}$

PROOF OF PROPOSITION 1: We prove the proposition by considering the dual of the ALM model (3)–(9). By linear programming duality, the ALM model has a bounded solution if and only if its dual is feasible.

Let φ_t^s be the dual variable for the cash-balance constraint in scenario s at time t, and $\mu_{i,t}^s$ the dual variable for the portfolio-balance constraint of asset i in scenario s at time t. The constraints in the dual with respect to the primal variables $xs_{i,t}^s$, $xb_{i,t}^s$ and $xh_{i.t}^s$

for $t = 0, \ldots, T-1$ and $s \in \mathcal{S}_t$ are (the corresponding primal variables are listed at the beginning of the constraints):

$$xs^s_{i,t} \quad : \quad \varphi^s_t\,(1-c)S^{n(s)}_{i,t} - \mu^s_{i,t} \leq 0 \tag{17}$$

$$xb^s_{i,t} \quad : \quad -\varphi^s_t\,(1+c)S^{n(s)}_{i,t} + \mu^s_{i,t} \leq 0 \tag{18}$$

$$xh^s_{i,t} \quad : \quad -\mu^s_{i,t} + \sum_{s^+}\left(\varphi^{s^+}_{t+1}\,D^{n(s^+)}_{i,t+1} + \mu^{s^+}_{i,t+1}\right) \leq 0 \qquad \text{if } t \leq T-2 \tag{19}$$

$$xh^s_{i,T-1} \quad : \quad -\mu^s_{i,T-1} + \sum_{s^+}\varphi^{s^+}_T\left(D^{n(s^+)}_{i,T} + S^{n(s^+)}_{i,T}\right) \leq 0 \tag{20}$$

where $\varphi_0 \equiv 1$.

Consider a scenario s $\in \mathcal{S}_t$ with $t \leq T-2$. By adding for each asset i constraint (18), constraint (19) and the dual constraints that correspond to the variables $xs^{s^+}_{i,t+1}$ for all successor scenarios s^+ of scenario s, it follows that the dual of the ALM model can only have a solution if

$$-\varphi^s_t\,(1+c)S^{n(s)}_{i,t} + \sum_{s^+}\varphi^{s^+}_{t+1}\left(D^{n(s^+)}_{i,t+1} + (1-c)S^{n(s^+)}_{i,t+1}\right) \leq 0 \tag{21}$$

for all assets i. As $\lambda = 1$, we know from lemma 1 that $\varphi^s_t = q^s_t$ is the only possible solution to the dual problem. Substituting this in (21), and using the identity $q^{s^+}_{t+1} = q^s_t P^{n(s)}_t \pi^{n(s)/n(s^+)}_{t/t+1}$, it follows that the dual of the ALM model only has a solution if

$$(1+c)\,S^{n(s)}_{i,t} \geq P^{n(s)}_t \sum_{s^+}\pi^{n(s)/n(s^+)}_{t/t+1}\left(D^{n(s^+)}_{i,t+1} + (1-c)S^{n(s^+)}_{i,t+1}\right) \tag{22}$$

for all assets Notice that the summation over all successor scenarios s^+ of scenario s in this expression is identical to the summation over all successor nodes n^+ of node n = n(s). Thus, the dual of the ALM model can only have a solution if inequality (11) is satisfied for all assets i in node n = n(s) at time $t \leq T-2$. As scenario s was chosen arbitrarily, this inequality must hold in all nodes n at times $t \leq T-2$.

That it must also hold in all nodes at time $T-1$ follows in a similar manner by adding the dual constraint (18) for $t = T-1$ to (20) for each scenario $s \in \mathcal{S}_{T-1}$.

$\mathcal{QED}$

Proof of Proposition 2: When c = 0, inequality (11) is both necessary and sufficient for a bounded solution to the ALM model. Because asset prices are assumed to be arbitrage-free while π^s_T is the risk-neutral probability, inequality (11) is satisfied in the event tree. Thus, the ALM model is bounded, and its dual must therefore have a feasible solution.

As $\lambda = 1$, we know from lemma 1 that $\varphi^s_t = q^s_t$ in a solution to the dual of the ALM model. Let $\mu_{i,0}$ denote the dual variable on the portfolio-balance constraint for asset i at time 0. For c = 0, the dual constraints with respect to the variables $xs_{i,0}$ and $xb_{i,0}$ imply that $\mu_{i,0} = S_{i,0}$ in a dual solution (see equations (17) and (18) in the proof of proposition 1). The dual solution is therefore unique, and the objective value in the dual problem equals

$$\sum_{t=1}^{T}\sum_{s\in\mathcal{S}_t}\varphi^s_t L^{n(s)}_t - \sum_{i=1}^{I}\mu_{i,0}\bar{x}_{i,0} = \sum_{t=1}^{T}\sum_{s\in\mathcal{S}_t}\pi^s_t\left(\prod_{\tau=0}^{t-1}P^{n(s^-)}_\tau\right)L^{n(s)}_t - \sum_{i=1}^{I}S_{i,0}\bar{x}_{i,0}$$

which is the difference between the arbitrage-free value at time 0 of the stream of liabilities and the value of the investor's initial portfolio.

To prove the second part of the proposition, note that the assumptions of arbitrage-free asset prices and c = 0 imply that inequality (11) is satisfied with equality for all assets in all states in the event tree. From the proof of proposition 1 it follows directly that this in turn implies that all constraints in the dual of the ALM model will be satisfied with equality. Thus, complementary slackness holds for every feasible solution to the ALM model, and each feasible solution is therefore optimal.

QED

B The Ho and Lee Term-Structure Model

In the term-structure model of Ho and Lee [11], the short rate is the single determinant of changes in the term structure of interest rates. Instead of specifying the process for the short rate directly, however, Ho and Lee describe the evolution of the prices of default-free zero-coupon bonds with different maturities, and derive from that the process of the short rate. We will follow their development in this appendix[9].

Consider a multiperiod economy with a finite horizon H, equally spaced trading dates $t = 0, \ldots, T$ $(T = H)$ and a finite number of possible states at each time. Let $P_t^n(\tau)$ denote the price in state n at time t of a default-free zero-coupon bond that has τ time periods left tomaturity $(\tau = 0, \ldots, T-t)$. $P_t^n(\tau)$as a function of τ is called the *discount* function, and Ho and Lee model the changes of this discount function over time. They assume that the complete discount function at time 0 is known".

In a world of certainty (only one possible state at each trading date), it must be true that $P_{t+1}(\tau) = P_t(\tau+1)/P_t(1)$ to prevent arbitrage opportunities (note that $P_{t+1}(\tau)$ and $P_t(\tau+1)$ represent prices of the same bond, but at different points in time). Under uncertainty, Ho and Lee assume that the discount function can change in two directions in each period. They describe the possible changes for each discount bond as perturbations from the required change in a world of certainty:

$$\text{Upstate:} \quad P_{t+1}^{n+1}(\tau) = \frac{P_t^n(\tau+1)}{P_t^n(1)} h(\tau) \tag{23}$$

$$\text{Downstate:} \quad P_{t+1}^{n}(\tau) = \frac{P_t^n(\tau+1)}{P_t^n(1)} h^*(\tau) \tag{24}$$

The perturbation functions $h(\cdot)$ and $h^*(\cdot)$ are assumed to depend only on the remaining time to maturity of the bonds, and satisfy h(0) = h*(O) = 1 and $h(\tau) > 1$, h*(r) < 1 for $\tau > 0$.

The first restriction which Ho and Lee impose on the perturbation functions is that the discount bond prices in the event tree do not admit arbitrage opportunities. They show that this is equivalent to the requirement that

$$\pi h(\tau) + (1 - \pi) h^*(\tau) = 1 \tag{25}$$

for some constant π, independent of τ. Using the definition of the perturbation functions,

[9] For general one-factor term-structure models, Hull and White [14] show the relationship between the specification of price processes for default-free zero-coupon bonds, processes for the instantaneous forward rates, and the process for the short rate.

[10] This is equivalent to knowing the complete term structure of interest rates at time 0. If $r_t^n(\tau)$ denotes the continuously compounded yield in state *n* at time *t* on a zero-coupon bond with τ periods left to maturity, then $r_t^n(\tau)$ and $P_t^n(\tau)$ are related by

$$r_t^n(\tau) = \frac{-\ln P_t^n(\tau)}{\tau\Delta}$$

where A $\equiv$ ***H/T;*** $r_t^n(\tau)$ as a function of τ is called the term structure of interest rates (or yield curve) in state *n* at time *t*. Ho and Lee assume A = 1, i.e., interest rates are defined per period.

this can be rewritten as $P_t^n(\tau+1) = P_t^n(1)\left[\pi P_{t+1}^{n+1}(\tau) + (1-\pi)P_{t+1}^n(\tau)\right]$, and the constant π can therefore be viewed as the risk-neutral (also called implied) binomial probability.

As the second restriction on the perturbation functions, Ho and Lee require that the price processes are path independent. That is, if the price of a discount bond in state n at time t follows an "upstate" and a "downstate" in the next two periods, respectively, then its price at time $t+2$ must be the same as when it would have followed a "downstate" and an "upstate", respectively. The number of different states at time t is therefore limited to $(t+1)$, which are indexed as $n = 0, \ldots, t$, and the changes in the discount function over time can be represented by a binomial lattice (see figure 1). Ho and Lee show that the path-independence condition, together with condition (25), leads to the following expressions for the perturbation functions:

$$h(\tau) = \frac{1}{\pi + (1-\pi)\delta^\tau} \qquad \text{and} \qquad h^*(\tau) = \delta^\tau \cdot h(\tau) \qquad \forall \tau \geq 1 \tag{26}$$

where δ is some constant between 0 and 1 ($\delta = 1$ is the certainty case). The parameters π and δ, together with the initial discount function, thus completely define the term-structure model of Ho and Lee.

The short rate r_t^n in state n at time t is defined as the continuously compounded interest rate on the zero-coupon bond with one period left to maturity: $r_t^n = -\ln P_t^n(1)$. Equations (23) and (24) enable us to write:

$$P_t^n(1) = \frac{P_0(t+1)}{P_0(t)} \frac{\delta^{t-n}}{\pi + (1-\pi)\delta^t}, n \leq t.$$

The process for the short rate can therefore be written as

$$r_t^0 = \ln\left[\frac{P_0(t)}{P_0(t+1)}\right] + \ln\left[\pi\delta^{-n} + (1-\pi)\right],$$

$$r_t^n = r_t^0 + n\ln\delta \qquad \text{for } n \leq t.$$

We note that $r_t^{n+1} - r_t^n = \ln\delta$, independent of n and t. Thus, the volatility of the short rate is independent of the state at a given time. To prevent the short rate from being negative anywhere in the binomial lattice, we need that $r_t^t \geq 0$ for all t (note that $\ln\delta \leq 0$), or equivalently,

$$\delta > \left[\frac{\frac{P_0(t+1)}{P_0(t)} - \pi}{1-\pi}\right]^{\frac{1}{t}} \qquad \forall\, t = 0, \ldots, T. \tag{27}$$

We have imposed this restriction on δ throughout this paper.

The expectation and the variance of the short rate at time t, $\tilde{r}_t$, evaluated at time 0 and with respect to the risk-neutral binomial probability π, can thus be written as:

$$\mu_t \equiv E_0\{\tilde{r}_t\} = \ln\left[\frac{P_0(t)}{P_0(t+1)}\right] + \ln\left[\pi\delta^{-t} + (1-\pi)\right] + t\pi\ln\delta \tag{28}$$

$$\sigma_t^2 \equiv E_0\{(\tilde{r}_t - \mu_t)^2\} = t\pi(1-\pi)(\ln\delta)^2 \tag{29}$$

It is clear that there is a one-to-one relation between σ_t and δ for all t. Notice that the lower bound on δ in (27) defines an upper bound on σ_t for each t.

B.1 Asset Prices in the Ho and Lee Model

One of the explicit requirements in the construction of the Ho and Lee model was that the prices of the discount bonds in the binomial lattice are arbitrage-free. This section describes how the model can be used to determine arbitrage-free prices of general interest-rate-dependent securities.

It was shown in section 2 that security prices in an event tree must satisfy relation (2) to prevent arbitrage opportunities. For the Ho and Lee model, this relation can be written as

$$S_{i,t}^{n} = P_t^n(1)\left[\pi(S_{i,t+1}^{n+1} + D_{i,t+1}^{n+1}) + (1-\pi)(S_{i,t+1}^{n} + D_{i,t+1}^{n})\right] \tag{30}$$

Notice that this is a recursive relation: from the dividends and ex-dividend prices at time $t+1$, the risk-neutral probability π and the discount function, we can calculate arbitrage-free security prices at time t in the lattice. Thus, by starting at the maturity date of a security and working backwards in the lattice, we can derive the arbitrage-free security prices at all times and in all states before the maturity date. In particular, we derive in this manner the arbitrage-free security price at time 0. Under the assumption that financial markets are in equilibrium, we would like that this arbitrage-free price equals the current market price.