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# Growing super stable tensegrity frameworks

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*Abstract*—This paper discusses methods for growing tensegrity frameworks akin to what are now known as Henneberg constructions, which apply to bar-joint frameworks. In particular, the paper presents tensegrity framework versions of the three key Henneberg constructions of vertex addition, edge splitting and framework merging (whereby separate frameworks are combined into a larger framework). This is done for super stable tensegrity frameworks in an ambient two or three-dimensional space. We start with the operation of adding a new vertex to an original super stable tensegrity framework, named *vertex addition*. We prove that the new tensegrity framework can be super stable as well if the new vertex is attached to the original framework by an appropriate number of members, which include struts or cables, with suitably assigned stresses. *Edge splitting* can be secured in $\mathbb{R}^2$ ($\mathbb{R}^3$) by adding a vertex joined to three (four) existing vertices, two of which are connected by a member, and then removing that member. This procedure, with appropriate selection of struts or cables, preserves super-stability. In $d$ dimensional ambient space, merging two super stable frameworks sharing at least $d+1$ vertices that are in general positions, we show that the resulting tensegrity framework is still super stable. Based on these results, we further investigate the strategies of merging two super stable tensegrity frameworks in $\mathbb{R}^d, (d \in \{2,3\})$ that share fewer than $d+1$ vertices, and show how they may be merged through the insertion of struts or cables as appropriate between the two structures, with a super stable structure resulting from the merge.

*Index terms*— Super-stability, Graph rigidity, Henneberg construction, Tensegrity frameworks

## I. INTRODUCTION

Rigidity graph theory serves as a fundamental mathematical tool to solve a wide range of problems in different fields, such as formation control of teams of mobile robots [1–3], molecular structural analysis in bio-chemistry [4, 5], and construction of stable structures in [6]. A *graph* comprises a set of vertices and edges, in which the edges specify how the vertices are connected. A *framework* is introduced by embedding a graph into some Euclidean space $\mathbb{R}^d$, the process involving the assigning of coordinates to each vertex of the graph. Of particular theoretical and practical interest is a class of frameworks called *tensegrity frameworks*, which realizes the edges of the embedded graph by three different types of *members*: *cables*, only allowed to become shorter; *struts*, only

allowed to become longer, and *bars*, constrained to maintain a fixed length [7]. Because of the use of cables, tensegrity structures may well end up lighter than a similar bar structure, able to support the same load. This property has been well employed in the design and control of tensegrity robots, see e.g. [8].

In many, if not most, application, the framework is expected to be *rigid*. This means the formation shape of the framework can be maintained as long as the distance constraints associated with all the edges are maintained, i.e. for a bar, an exact distance is maintained, for a cable, an upper bound is maintained, and for a strut, a lower bound is maintained. The rigid framework is said to be *globally rigid* if it is uniquely determined up to congruence in the given space in the sense that all shapes consistent with the constraints are congruent, i.e. obtainable from each other using one or more of translation, rotation and reflection. Furthermore, if the rigid framework is also uniquely determined in any higher dimensional space, it is termed *universally rigid*. All *super stable* tensegrity frameworks are universally rigid, but not vice versa [9]. A universally rigid tensegrity framework is able to maintain its shape when placed in higher dimensional space with some additional degrees of freedom [10].

Much attention, especially but not exclusively in the tensegrity literature, has been given to super-stability due to its superior properties in robustness. One surprising fact is that a globally rigid tensegrity framework can be drastically deformed under mild perturbation even at an equilibrium configuration [11]. It turns out that it is generally easier to analyze super stable tensegrity structures as opposed to tensegrity structures that are not super stable, due to the availability of more relevant theoretical foundations. Universally rigid tensegrity structures are often intuitively and easily understandable, for example, we note the concept of *Cauchy polygon* [12]. It is a class of tensegrity frameworks in the plane, where the vertices $1, \cdots, n$ in order form a convex polygon, and the edges $(i, i+1), i = 1, \cdots, n$, are cables and $(i, i+2), i = 1, \cdots, n-2$, are struts with the indices modulo $n$. In [12], it was shown that any Cauchy polygon is super stable. In addition, sufficient conditions were given for general convex polygons to be super stable, and these conditions are cast in terms of scalar variables termed *stresses*, one of which is associated with each member of the framework. Later, the results were extended in [13] for general tensegrity frameworks. This makes it possible to infer super-stability using the stress concept tool.

A framework is said to be *generic* if the vertex coordinates are algebraically independent over the rationals. Also, to avoid certain special cases, for a framework in an ambient $d$-dimensional space, an assumption is often made that the framework is in a *general position*, that is, no $d+1$ vertices are

affinely dependent. Providing foundations to study universal rigidity, [14] investigated global rigidity for tensegrity frameworks that are generic. These results were further extended to universal rigidity in [15]. In addition, [16] presented conditions for frameworks in general position to be universally rigid. In [17], it was demonstrated that universal rigidity can be maintained even under the weaker condition that each vertex and its neighbors affinely span $\mathbb{R}^d$.

From an engineering point of view, a framework may be required to be augmented by adding one or more vertices, or even merging or becoming connected with another framework. More precisely, by *merging* we mean, given two frameworks, the operations of one or both of superimposing some of their vertices and adding additional members joining a vertex pair with the vertices drawn from the two different frameworks. Normally, rigidity of frameworks is aimed to be preserved after adding vertices or merging.

In the plane, it is well known that the *Henneberg construction (HC)* [18] is an efficient technique to grow *minimally rigid* graphs. Recall that a rigid graph is said to be *minimally rigid* if no single edge can be removed without losing rigidity. The constructions of [18] propose two techniques, termed *vertex addition* and *edge splitting*, and due originally to Henneberg [19] whereby a minimally rigid framework (in an ambient two or three-dimensional space) can acquire an additional vertex (in the process that additional members are introduced). Henneberg also proposed a merging procedure for two (minimally) rigid graphs in an ambient two-dimensional space, whereby three members (bars in a normal structure) were inserted to link the two structures. In [20], strategies were developed to create a minimally rigid post-merging framework from two minimally rigid sub-frameworks. To fully cover all the possible cases of merging frameworks, where it is permitted to have one or more of the vertices of one merging framework made coincident with the same number of the other framework, three principles to conduct *optimal merging* of minimally or globally rigid frameworks were proposed in [21] for $\mathbb{R}^2$ and $\mathbb{R}^3$ frameworks. The merging is said to be *optimal* if the number of newly added member for a given number of shared vertices is minimized. Relying on HC operations, [22] investigated optimal growing of rigid frameworks in the sense of $\mathcal{H}_2$ performance. In [23], it has been proved that the extended framework is still generically global rigid if the new vertex is linked to $d + 1$ existing vertices in general positions of a generically globally rigid framework. Motivated by the implications of rigid networks in formation control and localizability, [24] identified the conditions for rigidity-preserving splitting as opposed to merging, under which the corresponding algorithms to perform the partition were also proposed therein.

All these results mentioned above on merging/splitting were for joint-bar frameworks; in contrast, the merging of tensegrity frameworks was first reported in [11], where only two special examples were discussed as demonstrations. Later, the superposition of super stable tensegrity frameworks was briefly discussed in [13]. It has been illustrated by several examples that the resulting tensegrity framework might not be super stable or even rigid if we glue two frameworks along some common vertices. However, for the purpose to ensure super stability after merging, no general principle or systematic analysis has been developed.

In terms of global rigidity, [25] studied how to combine two generically globally rigid frameworks without losing generically global rigidity. In [26], the procedure for growing a rigid tensegrity graph via adding in sequence new vertices was briefly introduced, but there were no discussions on how to assign stresses (and therefore no assignation of type, viz cable or strut) to the new members. More recently, it has been shown that the necessary and sufficient condition for a framework obtained by merging two super stable frameworks that are in general positions in $\mathbb{R}^d$ to be super stable, and without the introduction of new members, is that the number of their shared vertices is no fewer than $d + 1$ [27]. This has implications for tensegrity frameworks.

In spite of the aforementioned efforts made to study merging of tensegrity frameworks, there exists no systematic strategy for augmenting super stable tensegrity frameworks by adding new vertices in sequence. It is also desirable to design strategies for merging super stable tensegrity frameworks when they share fewer than $d + 1$ vertices, indeed possibly no vertices; this requires the introduction of new members.

Tensegrity frameworks, due to their robustness and scalability, have been employed as the virtual framework to solve the formation control problem of multi-agent systems. Starting from one-dimensional space, i.e., a line, [28] introduced a tensegrity-based control law that can exponentially stabilize the agents with prescribed distances. Then the same idea was used to deal with the problem in higher-dimensional space by collinear projections. In [29], the model of an unmanned aerial vehicle was integrated with a virtual cross-tensegrity framework, based on which a decentralized control strategy was designed such that a scalable formation was achieved. As a direct application, a stress-based formation control scheme is proposed to stabilize "affine" formations in [30]. In contrast to a rigid formation, an affine formation allows more transformations besides translation and rotation, such as scaling, shearing and reflection. Recently, we have made a sequence of efforts to explore the application of the stress matrix in formation control [31–33]. In [31], we have shown that the stress-based control law implies global exponential convergence to the target scaled formation. This result was further extended in [32] in the sense that the scaled formation can be achieved only by controlling one pair of agents. To broaden the feasibility of stress-based control schemes in applications, we also investigated how to ensure the connectivity of the underlying graph using distance-based control algorithms proposed in [33].

Motivated by these considerations, the aim of this paper is to first extend the various Henneberg construction steps to super stable tensegrity frameworks in $\mathbb{R}^d$, ($d \in \{2, 3\}$), such that the tensegrity frameworks after the *vertex addition* or *edge splitting* operation are still super stable. We then show that when two super stable tensegrity frameworks in $\mathbb{R}^d$ share no fewer than $d + 1$ vertices, super-stability of the merged tensegrity framework can be guaranteed under the weaker condition that only the shared vertices are in general positions. We further develop strategies to merge super stable

frameworks in the case of sharing fewer than $d + 1$ vertices by introducing new elements in $\mathbb{R}^d$, $(d \in \{2, 3\})$, to bridge the theoretical gap. Our constructions also are underpinned by algorithms for determining whether an introduced member should be a cable or a strut.

The rest of the paper is organized as follows. In Section II, we review some basic concepts of rigidity and sufficient conditions for tensegrity frameworks to be super stable. In Section III, we propose an Henneberg construction on super stable frameworks, including vertex addition and edge splitting operations. The strategies of merging super stable frameworks are presented in Section IV. Conclusions are given in Section V.

## II. PRELIMINARIES

In this section, we introduce some basic definitions on tensegrity frameworks and useful lemmas.

Let $\mathcal{V} = \{1, 2, \cdots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ be, respectively, the vertex set and the edge set of an undirected *graph* $\mathcal{G}(\mathcal{V}, \mathcal{E})$ describing the neighbor relationships between the $n$ vertices. There is an edge $(i, j)$ if and only if vertices $i$ and $j$ are neighbors of each other. The set of vertices that are adjacent to $i$ is denoted by $\mathcal{N}_i = \{j | (i, j) \in \mathcal{E}\}$. We assume that the graphs are finite and simple, i.e., without loops or multiple edges. A *configuration* is a finite collection of $n$ labeled points in the $d$-dimensional Euclidean space $\mathbb{R}^d$, denoted by $q = [q_1, \cdots, q_n] \in \mathbb{R}^{d \times n}$. A *tensegrity framework* $(\mathcal{G}, q)$ is obtained by embedding an undirected graph $\mathcal{G}$ in $\mathbb{R}^d$ and replacing edges of $\mathcal{G}$ by three types of *members*: *cables*, *struts* or *bars*, where cables and struts can only carry tensions and compressions respectively, while bars can carry either tensions or compressions. Equivalently, vertex pairs joined by a cable have a maximum length, pairs joined by strut have a minimum length and pairs joined by a bar have a fixed length in any framework consistent with the constraints.

For a tensegrity framework $(\mathcal{G}, q)$ in $\mathbb{R}^d$ with the fixed configuration $q$, we are interested in its associated configurations $p$ that satisfy the following *tensegrity constraints*

$$\begin{cases} |p_i - p_j| \leq |q_i - q_j|, & \text{when } (i, j) \text{ is a cable,} \\ |p_i - p_j| \geq |q_i - q_j|, & \text{when } (i, j) \text{ is a strut and} \\ |p_i - p_j| = |q_i - q_j|, & \text{when } (i, j) \text{ is a bar.} \end{cases} \quad (1)$$

We say that the tensegrity framework $(\mathcal{G}, q)$ whose shape is determined by the configuration $q$ is *rigid* if any other associated configuration $p$ is always congruent to $q$ whenever $p$ is sufficiently close to $q$ and satisfies the tensegrity constraints (1); furthermore, if the congruency relationship between $p$ and $q$ holds for all $p$ in $\mathbb{R}^{d \times n}$, then we say $(\mathcal{G}, q)$ is *globally rigid*; and even more strongly, if this congruent relationship still holds for all $p$ living in any higher-dimensional space than $\mathbb{R}^d$, we say $(\mathcal{G}, q)$ is *universally rigid* [11, 34].

To distinguish different members in a tensegrity framework, we employ the concept of *stress*. For each member $(i, j)$ of $(\mathcal{G}, q)$, we assign a scalar $\omega_{ij} = \omega_{ji}$, and use $\omega \in \mathbb{R}^{|\mathcal{E}|}$, where $|\mathcal{E}|$ is the number of members of $(\mathcal{G}, q)$, to denote the concatenated vector $\omega = (\cdots, \omega_{ij}, \cdots)^T$. Then $\omega$ is called a

*stress* of $(\mathcal{G}, q)$; if further, each $\omega_{ij}$ satisfies $\omega_{ij} \geq 0$ whenever $(i, j)$ is a cable and $\omega_{ij} \leq 0$ whenever $(i, j)$ is a strut, then $\omega$ is said to be a *proper* stress. Note that for a stress to be proper, there is no restriction associated with a bar. We say that a proper stress $\omega$ is *strict* if the stresses of cables and struts are nonzero. If there exists no member between vertices $i$ and $j$, the corresponding stress $\omega_{ij}$ is set to be zero. In physics, $\omega_{ij}$ is interpreted as the axial force per unit length along the member $(i, j)$. Given a framework $(\mathcal{G}, q)$, if for each vertex $i$, we have

$$\sum_{j \in \mathcal{N}_i} \omega_{ij}(q_j - q_i) = \mathbf{0}, \quad (2)$$

then, we call $\omega$ an *equilibrium stress* with respect to the configuration $q$. The corresponding *stress matrix* $\Omega = [\Omega_{ij}] \in \mathbb{R}^{n \times n}$ is defined by

$$\Omega_{ij} = \begin{cases} -\omega_{ij}, & i \neq j, \\ \sum_{j \in \mathcal{N}_i} \omega_{ij}, & i = j. \end{cases} \quad (3)$$

The following lemma will be used in the sequel at various points, where we combine positive semi-definite stress matrices.

**Lemma 1.** *Given positive semi-definite matrices $X \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{n \times n}$, let $Z = X + Y$. Then for any nonzero vector $\xi \in \mathbb{R}^n$, $\xi \in \ker(Z)$ if and only if $\xi \in \ker(X)$ and $\xi \in \ker(Y)$.*

Next we record conditions to guarantee super-stability of a tensegrity framework.

**Lemma 2.** *[13] Let $(\mathcal{G}, q)$ be a tensegrity framework whose affine span of $q$ is $\mathbb{R}^d$, with an equilibrium stress $\omega$ and stress matrix $\Omega$. Suppose further that*

1) $\Omega$ *is positive semi-definite,*
2) *the rank of $\Omega$ is $n - d - 1$,*
3) *and the stressed directions of $(\mathcal{G}, q)$ do not lie on a quadric at infinity,* [1]

*then $(\mathcal{G}, q)$ is super stable.*

**Remark 1.** *Lemma 2 is known as the fundamental theorem for super-stability. When $\omega$ is a proper equilibrium stress for $(\mathcal{G}, q)$, a stressed direction is the relative position of two connected nodes $i$ and $j$ with $\omega_{ij} \neq 0$, i.e., $q_i - q_j$. From [13], condition 3) of Lemma 2 can be replaced by "the framework $(\mathcal{G}, q)$ is rigid in $\mathbb{R}^d$".*

For the rest of the paper, we only consider tensegrity frameworks whose members are all cables and struts.

## III. HENNEBERG CONSTRUCTION ON SUPER STABLE TENSEGRITY FRAMEWORKS

In this section, we aim at extending the classical *Henneberg constructions (HC)* operating on graphs associated with bar-joint frameworks to super stable tensegrity frameworks in $\mathbb{R}^d$, $(d \in \{2, 3\})$. Two types of operations to grow minimally rigid graphs are reviewed as follows.

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[1]A set of vectors $\{v_1, v_2, \ldots v_k\}$ in $\mathbb{R}^d$ is said to lie on *a quadric at infinity* if for some nonzero symmetric $d \times d$ matrix $Q$, there holds $v_i^\top Q v_i = 0$, for $i = 1, 2, ..., k$.

1) Vertex addition: Adding a new vertex $u$ to the existing graph $\mathcal{G}$ via $d$ new edges between $u$ and $d$ vertices in $\mathcal{G}$.
2) Edge splitting: Removing an edge $(j,k)$, then adding a new vertex $u$ and $d+1$ new edges between $u$ and $d+1$ vertices to $\mathcal{G}$, two of which are $(u,j)$ and $(u,k)$.

It can be checked that for both operations in the plane, the increase in the number of edges at each step to form a new minimally rigid graph is two. Correspondingly, for the spatial graphs, the number will increase by three. We first consider the growing of super stable tensegrity frameworks in the plane. Under this scenario, vertex addition requires three new members; any notion of minimality is destroyed. However, if the three new members are linked to vertices for which a pair already have a member between them, that member can be removed without loss of super-stability by properly adjusting the remaining members' stresses, known as edge splitting, and each additional vertex involves adding $d$ new members. Thus this is a cheaper approach in terms of members than vertex addition.

The tensegrity framework $(\mathcal{G},q)$ to be operated on is assumed to be super stable with $n \geq 3$ vertices, three arbitrary vertices of which are denoted by $i,j$ and $k$. The resulting tensegrity framework after adding the new vertex $u$ and new members of cables and struts, is denoted by $(\bar{\mathcal{G}},\bar{q})$, where $\bar{q} = [q_1,\cdots,q_n,q_u] \in \mathbb{R}^{2\times(n+1)}$. Now, we first consider the vertex addition operation to generate a super stable framework $(\bar{\mathcal{G}},\bar{q})$.

### A. Vertex addition in $\mathbb{R}^2$

The position of the new vertex $u$ to be connected to $(\mathcal{G},q)$ can fall into the following three situations:

(a) not collinear with any two of $i,j$ and $k$;
(b) collinear with two of $i,j$ and $k$;
(c) collinear with all of $i,j,k$. (This situation can be reduced to (b).)

For situation (a), under the assumption that $i,j$ and $k$ are not collinear, there are seven possible regions to place the new vertex $u$, shown in Fig. 1, denoted by region $A,B\cdots,F$, and $H$. Note that the members (cables or struts) need to be inserted between the new vertex $u$ and the vertices in the original tenserity framework $(\mathcal{G},q)$ vary as the position of vertex $u$ changes. But, the necessary condition of the equilibrium stress with respect to vertex $u$ is always

$$\omega_{ui}(q_u-q_i) + \omega_{uj}(q_u-q_j) + \omega_{uk}(q_u-q_k) = \mathbf{0}, \quad (4)$$

where $\omega_{ui},\omega_{uj}$ and $\omega_{uk}$ are the stresses of members $(u,i),(u,j)$ and $(u,k)$, respectively. Here, we associate the new vertex $u$ with three vertices $i,j$ and $k$ rather than only two, since in scenario (a), any two of the three vectors, $(q_u-q_i),(q_u-q_j)$ and $(q_u-q_k)$, are linearly independent, which implies that there is no solution to (4) if we remove any single term on its left-hand side; equivalently, the three stresses must all be nonzero. This immediately means that in the plane, any one of the three vectors can be represented as a
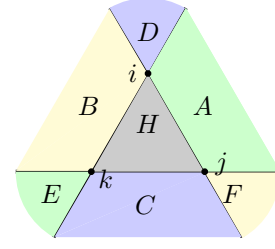


Fig. 1. Possible regions for $u$ to place in scenario (a).

linear combination of the other two. Without loss of generality, we assume

$$q_u - q_k = \kappa_1(q_u-q_i) + \kappa_2(q_u-q_j), \quad (5)$$

where $\kappa_1$ and $\kappa_2$ are nonzero scalars. Using the fact that any two vectors in the vector set $\{(q_u-q_i),(q_u-q_j),(q_u-q_k)\}$ are linearly independent, we have

$$\omega_{ui} + \kappa_1\omega_{uk} = 0, \quad (6a)$$

$$\omega_{uj} + \kappa_2\omega_{uk} = 0. \quad (6b)$$

Now, we record the member assignations (cable/strut) required to meet the equilibrium stress condition with respect to $u$ in different regions.

1) The new vertex $u$ lies in regions outside of $H$, i.e., $A,\cdots,F$, shown in Fig. 1.
First, consider the case when $u$ lies in region $A$ or $E$. In this case, the two scalars $\kappa_1$ and $\kappa_2$ in (5) are both positive, i.e., $\kappa_1 > 0$ and $\kappa_2 > 0$. Then, (6) implies

$$\begin{cases}\omega_{ui}\omega_{uk} < 0\\\omega_{uj}\omega_{uk} < 0 \\\omega_{ui}\omega_{uj} > 0\end{cases} \quad (7)$$

which in turn implies

$$\begin{cases}\omega_{ui} > 0\\\omega_{uk} < 0 \\\omega_{uj} > 0\end{cases} \quad \text{or} \quad \begin{cases}\omega_{ui} < 0\\\omega_{uk} > 0 \\\omega_{uj} < 0\end{cases} \quad (8)$$

Equivalently, members $(u,i)$ and $(u,j)$ are cables with $(u,k)$ being a strut, or members $(u,i)$ and $(u,j)$ are struts with $(u,k)$ being a cable.
Analogously, when vertex $u$ is located in region $B$ or $F$, we know $(u,i)$ and $(u,k)$ are the same type of members, either cable or strut, while $(u,j)$ should be different from them; when vertex $u$ is located in region $C$ or $D$, the two members that are of the same type are $(u,j)$ and $(u,k)$, which differ from member $(u,i)$.

2) The new vertex $u$ lies in region $H$.
In this case, from the geometric relationship, we know both $\kappa_1$ and $\kappa_2$ in (5) are negative, and consequently solutions to (6) satisfy

$$\begin{cases}\omega_{ui}\omega_{uk} > 0,\\\omega_{uj}\omega_{uk} > 0,\\\omega_{ui}\omega_{uj} > 0,\end{cases} \quad (9)$$

which implies all the three stresses have the same sign. In other words, when the newly added vertex $u$ lies within the convex hull spanned by the three existing vertices $i, j$ and $k$, the three new members connecting $u$ and $i, j, k$ are of the same type, which are either cables or struts.

We then consider situation (b) for which the newly added vertex $u$ is collinear with two of the existing vertices, say $i$ and $j$, and thus the new members to be inserted are $(u, i)$ and $(u, j)$. In view of the collinearity between $i, j$ and $u$, we have

$$q_u - q_i = \lambda(q_u - q_j), \tag{10}$$

where $\lambda > 0$ if $u$ lies outside of the line segment with two endpoints $i$ and $j$; $\lambda < 0$, otherwise. Hence, the equilibrium stress condition (4) reduces to

$$\omega_{ui}(q_u - q_i) + \omega_{uj}(q_u - q_j) = \mathbf{0}, \tag{11}$$

where $\omega_{ui}$ and $\omega_{uj}$ are stresses of the new members $(u, i)$ and $(u, j)$, respectively. Consequently, $\omega_{ui}\omega_{uj} < 0$ if $\lambda > 0$; $\omega_{ui}\omega_{uj} > 0$, if $\lambda < 0$. In other words, when the new vertex $u$ is not between $i$ and $j$, the two new members $(u, i)$ and $(u, j)$ are of different types. In contrast, when the new vertex $u$ is between $i$ and $j$, the two new members are of the same type. At the same time, it should be noted that to stabilize three vertices in $\mathbb{R}^1$, the two members incident to the middle vertex should be of the same type, and the other member connecting the two endpoints is of the other type. A sketch will rapidly show these conclusions are intuitively reasonable, if not obvious.

Situation (c) can be reduced to situation (b) by only considering the new vertex $u$ and any two of the three collinear vertices $i, j, k$ in $(\mathcal{G}, q)$. Actually, both (b) and (c) can be regarded as operations in $\mathbb{R}^1$.

The main theorem on vertex addition for super stable tensegrity frameworks in the plane is given as follows.

**Theorem 1.** *Given a super stable tensegrity framework $(\mathcal{G}, q)$ in $\mathbb{R}^2$, after (i) adding a new vertex $u$ and three members between $u$ and three distinct noncollinear vertices $i, j$ and $k$ to $(\mathcal{G}, q)$ when $u$ is not collinear with any two of $i, j, k$, or (ii) adding $u$ and two members between $u$ and two distinct vertices $i, j$ when $u$ is collinear with two vertices of the original framework, there always exist stresses of the new members, such that the newly obtained tensegrity framework $(\bar{\mathcal{G}}, \bar{q})$ is also super stable.*

*Proof.* First, we consider the scenario when the new vertex $u$ is not collinear with any two of the three distinct noncollinear vertices $i, j$ and $k$ in $(\mathcal{G}, q)$. Note that the equilibrium condition (4) can be written as

$$\underbrace{[q_u - q_i, q_u - q_j, q_u - q_k]}_{\triangleq q_r} \begin{bmatrix} \omega_{ui} \\ \omega_{uj} \\ \omega_{uk} \end{bmatrix} = \mathbf{0}, \tag{12}$$

where $q_r \in \mathbb{R}^{2 \times 3}$. Since $rank(q_r) = 2$, the solution to (12) with respect to $\omega$ cannot be uniquely determined. However, for a fixed but arbitrary vector $[a_1, a_2, a_3]^T$ satisfying $a_1 + a_2 + a_3 \neq 0$ in the null space of $q_r$, the solution to (12) is

$$\omega_{ui} = a_1 s, \quad \omega_{uj} = a_2 s, \quad \omega_{uk} = a_3 s, \tag{13}$$

for $s \in \mathbb{R}$ and $s \neq 0$. In view of the non-collinearity of the three vertices, there holds $q_k - q_u = c_1(q_k - q_i) + c_2(q_k - q_j)$ for some nonzero $c_1, c_2$. It follows that $c_1(q_u - q_i) + c_2(q_u - q_j) - (c_1 + c_2 - 1)(q_u - q_k) = 0$. Then one can observe that there always exist vectors satisfying (13).

Assume the stress matrix of the original framework $(\mathcal{G}, q)$ is $\Omega \in \mathbb{R}^{n \times n}$, which is positive semi-definite with rank $n-3$. Then, to derive the new stress matrix $\bar{\Omega} \in \mathbb{R}^{(n+1) \times (n+1)}$ for the framework $(\bar{\mathcal{G}}, \bar{q})$, one seeks to directly augment $\Omega$ by adding a new row and column to $\Omega$ in the form of

$$\hat{\Omega} = \left( \begin{array}{c:c} \Omega & \begin{matrix} 0 \\ \vdots \\ 0 \\ -\omega_{ui} \\ -\omega_{uj} \\ -\omega_{uk} \end{matrix} \\ \hdashline \begin{matrix} 0 & \cdots & 0 & -\omega_{ui} & -\omega_{uj} & -\omega_{uk} \end{matrix} & \hat{\Omega}_{uu} \end{array} \right). \tag{14}$$

However, this $\hat{\Omega}$ is not a stress matrix, since the $(n-2)$th to $n$th row/column sum is not zero. Therefore, to obtain a valid stress matrix based on $\hat{\Omega}$, the values of some entries in the original stress matrix $\Omega$ need to be changed correspondingly. Further, to ensure the new tensegrity framework $(\bar{\mathcal{G}}, \bar{q})$ is super stable, the new stress matrix should be positive semi-definite with rank $n-2$.

Since the new edges might affect the stresses of the edges between vertices $i, j$ and $k$, we look for the new stress matrix $\hat{\Omega}$ with the following form

$$\hat{\Omega} = \underbrace{\left[ \begin{array}{c:c} \Omega & \mathbf{0}_{n \times 1} \\ \hdashline \mathbf{0}_{1 \times n} & 0 \end{array} \right]}_{\triangleq \Omega_a} + \underbrace{\left[ \begin{array}{c:c} \mathbf{0}_{(n-3) \times (n-3)} & \mathbf{0}_{(n-3) \times 4} \\ \hdashline \mathbf{0}_{4 \times (n-3)} & \bar{\Omega}_u \end{array} \right]}_{\triangleq \Omega_b}, \tag{15}$$

where $\Omega_u \in \mathbb{R}^{4 \times 4}$ is a positive semi-definite stress matrix of rank 1 associated with the vertices $i, j, k$ and $u$. Existence and construction of $\Omega_u$ will be demonstrated later. Further, we seek to ensure that $\hat{\Omega}$ satisfies

a) $\hat{\Omega}$ is positive semi-definite.
b) $\hat{\Omega}$ is a stress matrix associated with vertices $1, \cdots, n, u$, whose stresses are in equilibrium with the configuration $\bar{q} = [q, q_u] \in \mathbb{R}^{2 \times (n+1)}$.
c) $rank(\hat{\Omega}) = n - 2$.

For statement a), it is straightforward to check $\Omega_a$ and $\Omega_b$ are both positive semi-definite from (15). So obviously, $\hat{\Omega} = \Omega_a + \Omega_b$ is also positive semi-definite.

For statement b), consider the facts that

$$\sum_{j=1,\cdots,n,(n+1)} \omega_{ij}^a (q_j - q_i) = \mathbf{0}, \quad \forall i, \tag{16}$$

and

$$\sum_{j=(1,\cdots,n-3),n-2,\cdots,n+1} \omega_{ij}^b (q_j - q_i) = \mathbf{0}, \quad \forall i, \tag{17}$$

where $\omega_{ij}^a$ and $\omega_{ij}^b$ are respectively the entries associated with matrices $\Omega_a$ and $\Omega_b$, vertices $i, j$ and $k$ are assigned with the indexes as $(n-2)$, $(n-1)$ and $n$, repectively, and the new

vertex $u$ is labeled as $n+1$ for consistence. Summing up (16) and (17), we get the equilibrium equation

$$\sum_{j=1,\cdots,n+1} \hat{\omega}_{ij}(q_j - q_i) = \mathbf{0}, \quad \forall i, \tag{18}$$

where $\hat{\omega}_{ij} = \omega_{ij}^a + \omega_{ij}^b$.

Furthermore, it can be concluded from Lemma 3 in the Appendix that statement c) also holds.

Hence, the augmented stress matrix $\hat{\Omega}$ through operation (15) is positive semi-definite with the maximal rank $n-2$, and the stresses are in equilibrium with $\bar{q}$. Note that for a general framework $(\mathcal{G}, q)$ that is rigid, through the typical Henneberg operation, the resulted new framework is still rigid. Hence, it can be concluded from Lemma 2 that the new framework $(\bar{\mathcal{G}}, \bar{q})$ is super stable. In the construction, the type of the new members, strut or cable, is determined by the signs of the stresses, which satisfy (12) and (13).

As for the scenario that the newly added vertex $u$ is collinear with two existing vertices in the original framework, the dimension of the stress matrix $\Omega_u$ in (15) will decrease to 3-by-3, since three vertices are sufficient to determine a super stable tensegrity framework in $\mathbb{R}^1$. Moreover, it should be noted that in this case only two new members are required to make the new tensegrity framework super stable. The proof can be conducted following the same argument as above, which is omitted here.

To sum up, we have shown that for a super stable framework in the plane, by vertex addition, the newly obtained tensegrity framework is still super stable. $\qquad\square$

**Remark 2.** *When vertices $i, j$ and $k$ in $(\mathcal{G}, q)$ are collinear, one can always find another vertex $k'$ in the original framework such that $i, j$ and $k'$ are not collinear; otherwise the tensegrity framework will be reduced to $1D$. Then the new vertex $u$ will be connected to vertices $i, j$ and $k'$. Following the same analysis, we know there exist proper stresses of the new members such that the augmented framework $(\bar{\mathcal{G}}, \bar{q})$ is super stable.*

### B. Vertex addition in $\mathbb{R}^3$

For the vertex addition in $\mathbb{R}^3$, the type of new members are also determined by the position of the new vertex $u$ with respect to the four vertices, denoted by $i, j, k$ and $l$, to be connected in $(\mathcal{G}, q)$. In view of their geometric relationship in the space, three cases might arise, namely

(a) The new vertex $u$ is collinear with two of the four vertices;
(b) The new vertex $u$ is coplanar with three of the four vertices;
(c) $u$ and the four vertices are neither collinear nor coplanar.

Cases (a) and (b) can be reduced to $\mathbb{R}^1$ and $\mathbb{R}^2$ respectively, which have been addressed above. For case (c), analogously, the equilibrium stress condition with respect to $u$ implies

$$\omega_{ui}(q_u-q_i)+\omega_{uj}(q_u-q_j)+\omega_{uk}(q_u-q_k)+\omega_{ul}(q_u-q_l) = \mathbf{0}, \tag{19}$$

where $\omega_{ui}, \omega_{uj}, \omega_{uk}$ and $\omega_{ul}$ are the stresses of members $(u,i), (u,j), (u,k)$ and $(u,l)$, respectively. Again from the linear independence relationship, we have

$$q_u - q_l = \kappa_1'(q_u - q_i) + \kappa_2'(q_u - q_j) + \kappa_3'(q_u - q_k), \tag{20}$$

where $\kappa_1', \kappa_2'$ and $\kappa_3'$ are nonzero scalars. Combining (19) and (20), we know

$$\begin{cases} \omega_{ui} + \kappa_1'\omega_{ul} = 0, \\ \omega_{uj} + \kappa_2'\omega_{ul} = 0, \\ \omega_{uk} + \kappa_3'\omega_{ul} = 0. \end{cases} \tag{21}$$

Then, following the same analysis in $\mathbb{R}^2$, one can determine the type of new members by looking at the signs of the stresses, derived from (21). To avoid repetition, we omit the details here. Correspondingly, for case (c), we have the following main result on vertex addition for super stable tensegrity frameworks in $\mathbb{R}^3$.

**Corollary 1.** *For a given super stable tensegrity framework $(\mathcal{G}, q)$ in $\mathbb{R}^3$, adding a new vertex $u$ and four members between $u$ and four distinct vertices in $(\mathcal{G}, q)$, where there exists no collinear or coplanar relationship between $u$ and the four vertices, there always exist stresses of the members incident to the chosen vertices, such that the extended tensegrity framework is also super stable.*

The same strategy employed in the proof of Theorem 1 can be used for proving Corollary 1. We omit it here, again to avoid repetition.

### C. Computation of the stress matrix $\Omega_u$

In this subsection, for completeness, we present the specific form of the matrix $\Omega_u$. Since the techniques used in the computation of the matrix $\Omega_u$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ are the same, we only focus on the scenario of $\mathbb{R}^2$. For the case when $u$ is not collinear with any two of the existing vertices $i, j$ and $k$, the stresses of the newly added members are represented in (13), based on which we will come up with a numerical method to derive the stress matrix $\Omega_u$. Before moving on, we define the sub-configuration matrix with respect to vertices $i, j, k$ and $u$ as

$$Q_u \triangleq \begin{bmatrix} q_i & q_j & q_k & q_u \\ 1 & 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 4}, \tag{22}$$

and note it satisfies

$$Q_u\Omega_u = \mathbf{0}_{3\times 4}. \tag{23}$$

Since $rank(Q_u) = 3$, there exists a nonzero vector $\phi = [\phi_1, \phi_2, \phi_3, \phi_4]^T \in \mathbb{R}^4$ satisfying

$$Q_u\phi = \mathbf{0}. \tag{24}$$

Then matrix $\Omega_u$ can be determined up to scaling through

$$\Omega_u = \phi\phi^T = \begin{bmatrix} \phi_1^2 & \phi_1\phi_2 & \phi_1\phi_3 & \phi_1\phi_4 \\ \phi_2\phi_1 & \phi_2^2 & \phi_2\phi_3 & \phi_2\phi_4 \\ \phi_3\phi_1 & \phi_3\phi_2 & \phi_3^2 & \phi_3\phi_4 \\ \phi_4\phi_1 & \phi_4\phi_2 & \phi_4\phi_3 & \phi_4^2 \end{bmatrix}. \tag{25}$$

Combining (25) and (13), we have

$$\begin{cases} \phi_1\phi_4 = -\omega_{ui} = -a_1 s \\ \phi_2\phi_4 = -\omega_{uj} = -a_2 s \\ \phi_3\phi_4 = -\omega_{uk} = -a_3 s \end{cases}. \tag{26}$$

Furthermore, in light of the fact that the row/column sum of $\Omega_u$ in (25) is zero, we know

$$\phi_4^2 = (a_1 + a_2 + a_3)s. \tag{27}$$

Then, by setting $s$ so that $(a_1 + a_2 + a_3)s > 0$, it follows from (26) and (27) that $\phi$ can be represented in terms of $s$ as follows

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = \frac{1}{\sqrt{(a_1 + a_2 + a_3)s}} \begin{bmatrix} -a_1 s \\ -a_2 s \\ -a_3 s \\ (a_1 + a_2 + a_3)s \end{bmatrix}. \tag{28}$$

Therefore, as long as $s$ is determined, the specific form of $\Omega_u$ can be obtained as well by substituting (28) into (24).

Based on (28), $\Omega_u$ is in the form of

$$\Omega_u = \frac{1}{\Omega_{uu}} \begin{bmatrix} \omega_{ui}^2 & \omega_{ui}\omega_{uj} & \omega_{ui}\omega_{uk} & -\omega_{ui}\Omega_{uu} \\ \omega_{ui}\omega_{uj} & \omega_{uj}^2 & \omega_{uj}\omega_{uk} & -\omega_{uj}\Omega_{uu} \\ \omega_{ui}\omega_{uk} & \omega_{uj}\omega_{uk} & \omega_{uk}^2 & -\omega_{uk}\Omega_{uu} \\ -\omega_{ui}\Omega_{uu} & -\omega_{uj}\Omega_{uu} & -\omega_{uk}\Omega_{uu} & \Omega_{uu}^2 \end{bmatrix}. \tag{29}$$

For the case when vertex $u$ is collinear with at least two vertices, we omit the calculation procedure here due to space limit. It is similar to the computations above.

**Remark 3.** *If the configuration of vertices $i, j, k$ and $u$ is fixed, the values of $\Omega_u$ is unique up to the* affine transformation *of $[q_i, q_j, q_k, q_u]$. We define the affine transformation of $q$ by*

$$\begin{aligned} \mathcal{A}(q) &\triangleq \{p = [p_1, \cdots, p_n] | p_i = Aq_i + b, \\ & A \in \mathbb{R}^{d \times d} \text{ and } b \in \mathbb{R}^d, i = 1, \cdots, n\}. \end{aligned} \tag{30}$$

### D. Edge splitting

In this subsection, the edge splitting strategy on super stable tensegrity frameworks is designed based on the vertex addition of a degree 3 or degree 4 vertex in $\mathbb{R}^2$ or $\mathbb{R}^3$ respectively, together with the removal of a member $(j, k)$ of the original tensegrity framework. To be consistent with the discussions above, the matrix $\hat{\Omega}$ will denote the stress matrix of the new super stable tensegrity framework after the operation of vertex addition. Note that from the perspective of stress, removing a member (following the vertex addition) is equivalent to altering the stress of the corresponding member to be zero without changing the positive semi-definiteness and the rank of $\hat{\Omega}$, as well as the self-equilibrium condition for $\bar{q}$. As mentioned before, the new vertex $u$ can lie in several possible regions. We first consider the case when $u$ is not collinear (coplanar) with any two (three) of the existing vertices $i, j$ and $k$ ($i, j, k$ and $l$) in $\mathbb{R}^2$ ($\mathbb{R}^3$). The main result is given as follows.

**Theorem 2.** *Remove a member $(j, k)$ in the original super stable tensegrity framework $(\mathcal{G}, q)$ in $\mathbb{R}^2$ ($\mathbb{R}^3$), and then add*

to $(\mathcal{G}, q)$ *a new vertex $u$ together with three (four) members incident on $u$, two of which are $(u, j)$ and $(u, k)$. Then, there exist appropriate stresses of the three (four) members such that the new tensegrity framework $(\mathcal{G}', \bar{q})$ is super stable.*

*Proof.* We present the proof only for $\mathbb{R}^2$ for simplicity; it can be straightforwardly extended to the analysis in $\mathbb{R}^3$. The stress matrix after a vertex addition operation is presented in (39). (on the next page)

Notice that in light of (28), the values of the entries of the matrix $\Omega_u$ in (29) is uniquely determined up to the scaling variable $s$. This implies that we have one degree of freedom to set the values of $\omega_{ui}, \omega_{uj}$ and $\omega_{uk}$. The observation motivates us to seek to zero out $\hat{\Omega}_{jk}$ through properly setting $\omega_{uk}$ such that

$$\Omega_{jk} + \frac{\omega_{uj}\omega_{uk}}{\Omega_{uu}} = 0.$$

Then by simple calculation, it follows

$$\omega_{uk} = -\frac{\Omega_{jk}\Omega_{uu}}{\omega_{uj}}. \tag{40}$$

Replacing $\omega_{uk}$ in (39) with (40), we have the matrix $\hat{\Omega}'$ given in (43). (on the next page)

It is obvious that $rank(\hat{\Omega}') = rank(\hat{\Omega})$. Moreover, the positive semi-definiteness, as well as the null space, of the matrix $\hat{\Omega}$ is not altered. Therefore, the new stress matrix $\hat{\Omega}'$ is still positive semi-definite with rank $n - 2$, and at equilibrium with the configuration $\bar{q}$. Recalling that rigidity of a framework can be maintained through typical Henneberg operation, so the new tensegrity framework $(\mathcal{G}', \bar{q})$ is still super stable with the corresponding stress matrix $\hat{\Omega}'$.

$\square$

Note that if $u$ is coplanar with some of the vertices in $\mathbb{R}^3$, then one can fall back on analysis in $\mathbb{R}^2$. Hence, as for the location of the new vertex $u$, we only need to consider another possible scenario that $u$ is collinear with two vertices in $\mathbb{R}^2$. In this case, only three vertices together with three members are involved to construct the stress matrix $\Omega_u$, and the dimension of their configuration has reduced to one. It can be further checked that no one of the three members can be removed without losing super-stability. Hence, for the collinear situation, only when the newly added vertex $u$ is collinear with at least three vertices in the original tensegrity framework $(\mathcal{G}, q)$, can an edge splitting operation be conducted. We have the following result.

**Corollary 2.** *Given a super stable tensegrity framework $(\mathcal{G}, q)$ with three collinear vertices $i, j$ and $k$, add a new vertex $u$ on some member $(j, k)$ and thus replace the member $(j, k)$ by two new members $(j, u)$ and $(u, k)$. Then, there exist appropriate members $(j, u)$, $(u, k)$ and $(u, i)$ to be inserted to $(\mathcal{G}, q)$ such that the new tensegrity framework is still super stable.*

**Remark 4.** *The idea of Corollary 2 is the same as that of Theorem 2, namely, remove some member by altering*

$$\hat{\Omega} = \begin{pmatrix} \Omega_{1,1} & \cdots & \Omega_{1,n-3} & \Omega_{1,n-2} & \Omega_{1,n-1} & \Omega_{1,n} & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Omega_{n-3,1} & \cdots & \Omega_{n-3,n-3} & \Omega_{n-3,n-2} & \Omega_{n-3,n-1} & \Omega_{n-3,n} & 0 \\ \hline \Omega_{i,1} & \cdots & \Omega_{i,n-3} & \Omega_{ii}+\frac{\omega_{ui}^2}{\Omega_{uu}} & \Omega_{ij}+\frac{\omega_{ui}\omega_{uj}}{\Omega_{uu}} & \Omega_{ik}+\frac{\omega_{ui}\omega_{uk}}{\Omega_{uu}} & -\omega_{ui} \\ \Omega_{j,1} & \cdots & \Omega_{j,n-3} & \Omega_{ji}+\frac{\omega_{uj}\omega_{ui}}{\Omega_{uu}} & \Omega_{jj}+\frac{\omega_{uj}^2}{\Omega_{uu}} & \Omega_{jk}+\frac{\omega_{uj}\omega_{uk}}{\Omega_{uu}} & -\omega_{uj} \\ \Omega_{k,1} & \cdots & \Omega_{k,n-3} & \Omega_{ki}+\frac{\omega_{uk}\omega_{ui}}{\Omega_{uu}} & \Omega_{kj}+\frac{\omega_{uk}\omega_{uj}}{\Omega_{uu}} & \Omega_{kk}+\frac{\omega_{uk}^2}{\Omega_{uu}} & -\omega_{uk} \\ 0 & \cdots & 0 & -\omega_{ui} & -\omega_{uj} & -\omega_{uk} & \Omega_{uu} \end{pmatrix}. \tag{39}$$

$$\hat{\Omega}' = \begin{pmatrix} \Omega_{1,1} & \cdots & \Omega_{1,n-3} & \Omega_{1,n-2} & \Omega_{1,n-1} & \Omega_{1,n} & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Omega_{n-3,1} & \cdots & \Omega_{n-3,n-3} & \Omega_{n-3,n-2} & \Omega_{n-3,n-1} & \Omega_{n-3,n} & 0 \\ \hline \Omega_{i,1} & \cdots & \Omega_{i,n-3} & \Omega_{ii}+\frac{\omega_{ui}^2}{\Omega_{uu}} & \Omega_{ij}+\frac{\omega_{ui}\omega_{uj}}{\Omega_{uu}} & \Omega_{ik}-\frac{\omega_{ui}}{\omega_{uj}}\Omega_{jk} & -\omega_{ui} \\ \Omega_{j,1} & \cdots & \Omega_{j,n-3} & \Omega_{ji}+\frac{\omega_{uj}\omega_{ui}}{\Omega_{uu}} & \Omega_{jj}+\frac{\omega_{uj}^2}{\Omega_{uu}} & 0 & -\omega_{uj} \\ \Omega_{k,1} & \cdots & \Omega_{k,n-3} & \Omega_{ik}-\frac{\omega_{ui}}{\omega_{uj}}\Omega_{jk} & 0 & \Omega_{kk}+\frac{\Omega_{jk}^2\Omega_{uu}}{\omega_{uj}^2} & \frac{\Omega_{jk}\Omega_{uu}}{\omega_{uj}} \\ 0 & \cdots & 0 & -\omega_{ui} & -\omega_{uj} & \frac{\Omega_{jk}\Omega_{uu}}{\omega_{uj}} & \Omega_{uu} \end{pmatrix}. \tag{43}$$

*its stress to be zero through properly setting one of the stresses associated with the new members. Hence, the proof of Corollary 2 is omitted here. For the case when the new vertex u is collinear with four or more vertices, only three of them together with the new vertex u are needed to conduct the edge splitting operation.*

## IV. MERGING TWO SUPER STABLE TENSEGRITY FRAMEWORKS

In this section, we aim to investigate the strategies of merging two super stable tensegrity frameworks $(\mathcal{G}_A, q_A)$ and $(\mathcal{G}_B, q_B)$. According to the number of shared vertices between the two tensegrity frameworks before merging, denoted by $|\mathcal{V}_C|$, we consider two sub-scenarios: $|\mathcal{V}_C| \geq d+1$, and $|\mathcal{V}_C| < d+1$. When $(\mathcal{G}_A, q_A)$ and $(\mathcal{G}_B, q_B)$ share no fewer than $d+1$ vertices, we show that the merged tensegrity framework is still super stable if the shared vertices are in general position. This result relaxes the stringent condition that both of the two frameworks need to be in general positions in [27]. For the case when $|\mathcal{V}_C| < d+1$, we summarize the results recording the minimum number of new members required in a table by constraining $d$ to be 2 and 3. The type of these members, i.e. strut or cable, depends on the specific location of the various vertices, and so cannot be recorded.

In the following, we denote the *positive semi-definite (PSD) stress matrices* associated with $(\mathcal{G}_A, q_A)$ and $(\mathcal{G}_B, q_B)$ as $\Omega_A$ and $\Omega_B$, respectively, each of which has nullity $d+1$. The cardinalities of the vertex sets satisfy $|\mathcal{V}_A| = n_A$, $|\mathcal{V}_B| = n_B$, and $|\mathcal{V}_C| = n_C$.

### A. The number of shared vertices is no fewer than $d+1$

To be consistent with the merging of two tensegrity frameworks, we assume that the last (resp. first) $n_C$ rows and

columns of $\Omega_A$ (resp. $\Omega_B$) correspond to the stresses incident on the shared vertices. The merged tensegrity framework is denoted by $(\tilde{\mathcal{G}}, \tilde{q})$ with the stress matrix $\tilde{\Omega} \in \mathbb{R}^{n \times n}$, where $\tilde{n} = n_A + n_B - n_C$. Accordingly, we argument the stress matrices $\Omega_A$ and $\Omega_B$ to form matrices $\tilde{\Omega}_A$ and $\tilde{\Omega}_B$ of size $\tilde{n} \times \tilde{n}$ by adding zeros as follows:

$$\tilde{\Omega}_A = \left[ \begin{array}{c:c} \Omega_A & \mathbf{0}_{n_A \times (\tilde{n}-n_A)} \\ \hdashline \mathbf{0}_{(\tilde{n}-n_A) \times n_A} & \mathbf{0}_{(\tilde{n}-n_A) \times (\tilde{n}-n_A)} \end{array} \right],$$
$$\tilde{\Omega}_B = \left[ \begin{array}{c:c} \mathbf{0}_{(n_A-n_C) \times (n_A-n_C)} & \mathbf{0}_{(n_A-n_C) \times n_B} \\ \hdashline \mathbf{0}_{n_B \times (n_A-n_C)} & \Omega_B \end{array} \right]. \tag{44}$$

Note that the stress matrices $\Omega_A$ and $\Omega_B$ can also be partitioned as

$$\Omega_A = \left[ \begin{array}{c:c} \Omega_{A1} & \Omega_{A2} \\ \hdashline \Omega_{A3} & \Omega_{A4} \end{array} \right], \quad \text{and} \quad \Omega_B = \left[ \begin{array}{c:c} \Omega_{B4} & \Omega_{B2} \\ \hdashline \Omega_{B3} & \Omega_{B1} \end{array} \right], \tag{45}$$

where $\Omega_{A1} \in \mathbb{R}^{(n_A-n_C) \times (n_A-n_C)}$, $\Omega_{A2} \in \mathbb{R}^{(n_A-n_C) \times n_C}$, $\Omega_{A3} \in \mathbb{R}^{n_C \times (n_A-n_C)}$, $\Omega_{A4} \in \mathbb{R}^{n_C \times n_C}$, $\Omega_{B1} \in \mathbb{R}^{(n_B-n_C) \times (n_B-n_C)}$, $\Omega_{B2} \in \mathbb{R}^{n_C \times (n_B-n_C)}$, $\Omega_{B3} \in \mathbb{R}^{(n_B-n_C) \times n_C}$, and $\Omega_{B4} \in \mathbb{R}^{n_C \times n_C}$. Then, the stress matrix of the post-merged tensegrity framework $(\tilde{\mathcal{G}}, \tilde{q})$ can be written as

$$\tilde{\Omega} = \tilde{\Omega}_A + \tilde{\Omega}_B$$
$$= \begin{bmatrix} \Omega_{A1} & \Omega_{A2} & \mathbf{0}_{(n_A-n_C) \times (n_B-n_C)} \\ \Omega_{A3} & \Omega_{A4}+\Omega_{B4} & \Omega_{B2} \\ \mathbf{0}_{(n_B-n_C) \times (n_A-n_C)} & \Omega_{B3} & \Omega_{B1} \end{bmatrix}. \tag{46}$$

Now, we are ready to give another main result.

**Theorem 3.** *Given two super stable tensegrity frameworks in $\mathbb{R}^d$ with the corresponding PSD stress matrices of nullity $d+1$, if they share at least $d+1$ vertices that are in general position, then the merged tensegrity framework $(\tilde{\mathcal{G}}, \tilde{q})$ is still*

*super stable. Moreover, one of the PSD stress matrices of nullity $d + 1$ associated with the new framework is in the form of* (46).

*Proof.* We first consider the case when the two tensegrity frameworks share exactly $d + 1$ vertices, i.e., $n_C = d + 1$. Then, by denoting the configuration of shared $d + 1$ vertices as $q_{C1}, \cdots, q_{C(d+1)}$, one has

$$\tilde{q} = [q_{A1}, \cdots, q_{A(n_A - d - 1)}, q_{C1}, \cdots, q_{C(d+1)}, q_{B(d+2)}, \cdots, q_{Bn_B}].$$
(47)

From Lemma 2, to show that $(\tilde{\mathcal{G}}, \tilde{q})$ is super stable, it is sufficient to prove the synthetic stress matrix $\tilde{\Omega}$ in (46) satisfies the three conditions therein. It is obvious that $\tilde{\Omega}$ is PSD, as $\tilde{\Omega}_A$ and $\tilde{\Omega}_B$ are both PSD from their definitions in (44). In addition, for two rigid frameworks in $\mathbb{R}^d$, if they share no fewer than $d$ vertices, then the framework after merging is rigid [21], which implies that the third condition in Lemma 2 is satisfied. Hence, what is left to show is that the rank of $\tilde{\Omega}$ is $\tilde{n} - d - 1$, namely, the nullity of $\tilde{\Omega}$ is $d + 1$.

Similar to the analysis in the proof of Theorem 1, we consider the solution space of the following equations,

$$\tilde{\Omega}_A x_A = \mathbf{0}, \tag{48a}$$

$$\tilde{\Omega}_B x_B = \mathbf{0}. \tag{48b}$$

Then the solution spaces of (48a) and (48b) are respectively given by

$$
\mathbb{S}_A = \left(
\begin{pmatrix} q_{11}^A \\ \vdots \\ q_{(n_A-d-1)1}^A \\ q_{11}^C \\ \vdots \\ q_{(d+1)1}^C \\ \xi_{11} \\ \vdots \\ \xi_{(n_B-d-1)1} \end{pmatrix},
\cdots,
\begin{pmatrix} q_{1d}^A \\ \vdots \\ q_{(n_A-d-1)d}^A \\ q_{1d}^C \\ \vdots \\ q_{(d+1)d}^C \\ \xi_{1d} \\ \vdots \\ \xi_{(n_B-d-1)d} \end{pmatrix},
\begin{pmatrix} 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ c_{A1} \\ \vdots \\ c_{A(n_B-d-1)} \end{pmatrix}
\right),
$$
(49)

and

$$
\mathbb{S}_B = \left(
\begin{pmatrix} \zeta_{11} \\ \vdots \\ \zeta_{(n_A-d-1)1} \\ q_{11}^C \\ \vdots \\ q_{(d+1)1}^C \\ q_{(d+2)1}^B \\ \vdots \\ q_{n_B 1}^B \end{pmatrix},
\cdots,
\begin{pmatrix} \zeta_{1d} \\ \vdots \\ \zeta_{(n_A-d-1)d} \\ q_{1d}^C \\ \vdots \\ q_{(d+1)d}^C \\ q_{(d+2)d}^B \\ \vdots \\ q_{n_B d}^B \end{pmatrix},
\begin{pmatrix} c_{B1} \\ \vdots \\ c_{B(n_A-d-1)} \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}
\right),
$$
(50)

where for configuration $q$ the superscript denotes the configuration set, and the subscripts, say $(ij)$ in $q_{ij}^A$, represent the $j$th component of vector $q_{Ai}$. $\xi_i \in \mathbb{R}^d, i = 1, \cdots, n_B - d - 1$, $\zeta_j \in \mathbb{R}^d, j = 1, \cdots, n_A - d - 1$, $c_A \in \mathbb{R}^{n_B - d - 1}$, and

$c_B \in \mathbb{R}^{n_A - d - 1}$ are arbitrary real vectors. Following the same line of the proof of Theorem 1, we get

$$null(\tilde{\Omega}) = \mathbb{S}_A \cap \mathbb{S}_B = span\left(\tilde{q}^T, \mathbf{1}_{\tilde{n}}\right), \tag{51}$$

which implies $nul(\tilde{\Omega}) = d + 1$. Therefore, it follows from the relationship between nullity and rank of $\tilde{\Omega}$, $nul(\tilde{\Omega}) + rank(\tilde{\Omega}) = \tilde{n}$, that $rank(\tilde{\Omega}) = \tilde{n} - d - 1$.

The analysis for the scenario when two super stable tensegrity frameworks share more than $d + 1$ vertices is similar to the aforementioned scenario. We omit it to avoid redundancy. This completes the proof of Theorem 3.

$\square$

### B. The number of shared vertices is less than $d + 1$ in $\mathbb{R}^d$ ($d \in \{2, 3\}$)

The aim of this sub-section is to determine the minimum number of both new members and vertices incident to them when merging two super stable tensegrity frameworks in $\mathbb{R}^d$ ($d \in \{2, 3\}$). We refer to this operation as *optimal merging*. Based on Theorem 3 and the HC discussed in Section III, we present iterative procedures to merge two separate tensegrity frameworks.

Before describing the results, let us define $\mathcal{V}_{new}$ to denote a set of vertices satisfying $\mathcal{V}_{new} \subseteq \mathcal{V}_B \backslash \mathcal{V}_A$ and $|\mathcal{V}_{new}| = d + 1 - |\mathcal{V}_C| = n_{new}$. Let $\mathcal{E}_{new}$ be the set of members connecting the vertices in $\mathcal{V}_{new}$ to $(\mathcal{G}_A, q_A)$. We will indicate below how $\mathcal{E}_{new}$ is obtained and determine $|\mathcal{E}_{new}|$ in the process. The situation is akin to linking to globally rigid formations with further edges to ensure the combined formation is globally rigid (see [21]). Then, as a direct extension of Theorem 3, we have the following Corollary.

**Corollary 3.** *Given two super stable tensegrity frameworks* $(\mathcal{G}_A, q_A)$ *and* $(\mathcal{G}_B, q_B)$ *in* $\mathbb{R}^d$ ($d \in \{2, 3\}$), *satisfying* $|\mathcal{V}_C| \leq d$, *if the tensegrity framework* $(\mathcal{G}'_A, q'_A)$ *with* $\mathcal{V}'_A = \mathcal{V}_A \cup \mathcal{V}_{new}$ *and* $\mathcal{E}'_A = \mathcal{E}_A \cup \mathcal{E}_{new}$ *is super stable, in which vertices in* $\mathcal{V}_{new}$ *are in general position, then the tensegrity framework* $(\tilde{\mathcal{G}}, \tilde{q})$ *is super stable, where* $\tilde{\mathcal{V}} = \mathcal{V}_A \cup \mathcal{V}_B$ *and* $\tilde{\mathcal{E}} = \mathcal{E}'_A \cup \mathcal{E}_B$.

Illustrations of Corollary 3 are given in Figs. 2-4, where the merging operation is carried out in $\mathbb{R}^2$. In the plane, three scenarios are considered in terms of $|\mathcal{V}_c|$ as follows.

1) $|\mathcal{V}_C| = 0$.

In this case, $n_{new} = 3 - |\mathcal{V}_C| = 3$.

As Fig. 2 shows, to construct $(\mathcal{G}'_A, q'_A)$, we first add a new vertex $u$ from $\mathcal{V}_B$ to $\mathcal{V}_A$ and three new members $(u, i), (u, j)$ and $(u, k)$ by employing Theorem 1. Then applying Theorem 2, one adds the second new vertex $v$ together with the corresponding members $(v, i)$ and $(v, j)$, noting there is already an explicit or implicit member $(v, u)$. Consequently, the member $(u, j)$ can be removed. Analogously, $w$ and the member $(w, i)$ are added in the last step, in which two explicit or implicit members $(w, u)$ and $(w, v)$ are considered. Again from Theorem 2, the member $(v, i)$ can be removed without losing super-stability. Hence, $\mathcal{E}_{new} = \{(u, i), (u, k), (v, j), (w, i)\}$, and thus $|\mathcal{E}_{new}| = 4$.
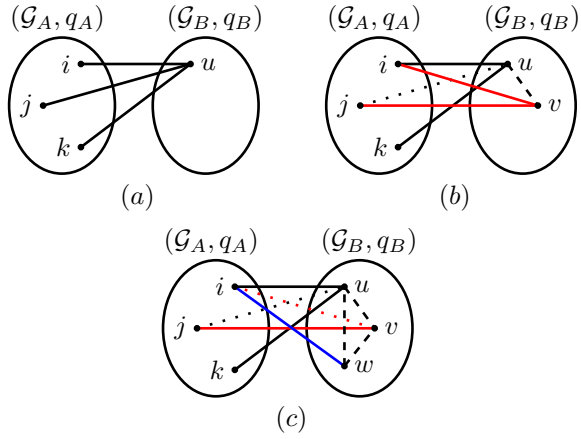
2) $|\mathcal{V}_C| = 1$.

Fig. 2. Three steps of merging two super stable frameworks when $|\mathcal{V}_C| = 0$, where dashed lines and loosely dotted lines represent explicit or implicit members and removed members, respectively.
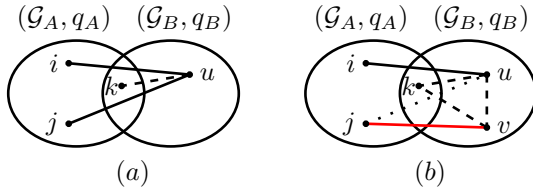


Fig. 3. Procedures of merging two super stable frameworks when $|\mathcal{V}_C| = 1$, where dashed lines and loosely dotted lines represent explicit or implicit members and removed members, respectively.

In this case, $n_{new} = 3 - |\mathcal{V}_C| = 2$.

Vertex $k$ is assumed to be common to $\mathcal{V}_A$ and $\mathcal{V}_B$. Based on Theorem 1 and 2, Fig. 3 shows that two new members, $(u, i)$ and $(v, j)$, are required to construct a super stable tensegrity framework. Hence, we know $|\mathcal{E}_{new}| = 2$.

3) $|\mathcal{V}_C| = 2$.
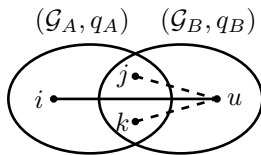
In this case, $n_{new} = 3 - |\mathcal{V}_C| = 1$.



Fig. 4. Merging two super stable frameworks when $|\mathcal{V}_C| = 2$, where dashed lines represent explicit or implicit members.

The common vertices are $j$ and $k$. From Theorem 1, it can be checked that only one member is required to construct a super stable tensegrity framework as shown in Fig. 4, and thus $|\mathcal{E}_{new}| = 1$.

The results for structures defined in $\mathbb{R}^3$ are obtained similarly. Note that whether a new member is a cable or a strut is determined at each step of the addition process in accord with the procedure set out in the earlier section treating vertex addition and edge splitting. To sum up, the optimal merging of two super stable frameworks is listed in Table I and II.

TABLE I
OPTIMAL MERGING OF TWO SUPER STABLE TENSEGRITY FRAMEWORKS IN $\mathbb{R}^2$.

| $|\mathcal{V}_C|$ | $|\mathcal{E}_{new}|$ | $|\mathcal{V}_{new}|$ |
|---|---|---|
| 0 | 4 | 3 |
| 1 | 2 | 2 |
| 2 | 1 | 1 |
| 3 or more | 0 | 0 |

TABLE II
OPTIMAL MERGING OF TWO SUPER STABLE TENSEGRITY FRAMEWORKS IN $\mathbb{R}^3$.

| $|\mathcal{V}_C|$ | $|\mathcal{E}_{new}|$ | $|\mathcal{V}_{new}|$ |
|---|---|---|
| 0 | 6 | 4 |
| 1 | 3 | 3 |
| 2 | 2 | 2 |
| 3 | 1 | 1 |
| 4 or more | 0 | 0 |

The numbers contained in these tables are partially identical with those to be found in [21] for global rigidity. This is not completely surprising, given that super-stability is a specialized form of global rigidity.

## V. CONCLUSION

In this paper, we have addressed the problem of how to grow super stable tensegrity frameworks by adding a vertex or a super stable framework in $\mathbb{R}^d$, ($d \in \{2, 3\}$). We have systematically developed the HC on tensegrity frameworks and a numerical method of calculating stress matrices associated with resultant tensegrity frameworks. In addition, in the case of merging two super stable tensegrity frameworks in $\mathbb{R}^d$, we have shown that super-stability can be maintained if the frameworks share no fewer than $d+1$ vertices in general positions. Finally, to cover all the possible scenarios of merging in $\mathbb{R}^d$, ($d \in \{2, 3\}$), we have presented the detailed steps of optimal merging. The results have been summarized in two tables.

For future research, it is of great interest to study tensegrity frameworks in higher dimensional spaces from theoretical perspective. In addition to the research on super-stability of tensegrity frameworks, it is also essential to investigate the strategies of augmenting rigid or globally rigid tensegrity frameworks systematically. The procedures therein can give more freedom when setting stresses for newly added members. Finally, very few results have been reported in the literature on employing the superior properties of tensegrity frameworks, such as stability, extendability, and robustness, in control engineering. Hence, it is of great interest to make use of tensegrity frameworks in cooperative control for robots, e.g., autonomous formation splitting and merging.

## VI. APPENDIX

### A. Lemma on the rank of the matrix $\hat{\Omega}$ in (15)

**Lemma 3.** *Consider the matrix $\hat{\Omega} \in \mathbb{R}^{(n+1) \times (n+1)}$ defined in (15), where $\Omega \in \mathbb{R}^{n \times n}$ and $\Omega_u \in \mathbb{R}^{4 \times 4}$ are the stress*

*matrices associated with super stable tensegrity frameworks with three common vertices. Then*

$$rank(\hat{\Omega}) = n - 2. \tag{52}$$

*Proof.* We first consider the solution to the following equations

$$\Omega_a x = \mathbf{0}, \tag{53a}$$

$$\Omega_b y = \mathbf{0}, \tag{53b}$$

where $x, y \in \mathbb{R}^{n+1}$. In view of (15), (53a) can be equivalently written as

$$\left[\begin{array}{c|c} \Omega & \mathbf{0}_{n\times 1} \\ \hline \mathbf{0}_{1\times n} & 0 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} \mathbf{0}_{n\times 1} \\ 0 \end{array}\right], \tag{54}$$

where $x_1 \in \mathbb{R}^{n\times 1}$ and $x_2 \in \mathbb{R}$. After simple calculation, (54) can be reduced to

$$\begin{cases} \Omega x_1 = \mathbf{0}, \\ 0x_2 = 0. \end{cases} \tag{55}$$

Since $null(\Omega) = span(q^T, \mathbf{1}_n)$, the solution space of (55) (equivalently, (53a)) is as follows

$$\mathbb{S}_a = span\left(\left[\begin{array}{c} q_{\cdot 1} \\ p_1^a \end{array}\right], \left[\begin{array}{c} q_{\cdot 2} \\ p_2^a \end{array}\right], \left[\begin{array}{c} \mathbf{1}_n \\ c_a \end{array}\right]\right) \tag{56}$$

$$\overset{\Delta}{=} span\left(s_1^a, s_2^a, s_3^a\right),$$

where $q_{\cdot 1} = [q_{11}, \cdots, q_{n1}]^T \in \mathbb{R}^n$ with $q_{i1}$ being the first component of $q_i$, $i = 1, \cdots, n$, and $q_{\cdot 2}$ is defined analogously. $p_1^a, p_2^a$ and $c_a$ are any arbitrary scalars.

Similarly, the solution space of (53b) is given by

$$\mathbb{S}_b = span\left(\left(\begin{array}{c} p_{11}^b \\ \vdots \\ p_{(n-3)1}^b \\ q_{(n-2)1} \\ \vdots \\ q_{(n+1)1} \end{array}\right), \left(\begin{array}{c} p_{12}^b \\ \vdots \\ p_{(n-3)2}^b \\ q_{(n-2)2} \\ \vdots \\ q_{(n+1)2} \end{array}\right), \left(\begin{array}{c} c_{b1} \\ \vdots \\ c_{b(n-3)} \\ 1 \\ \vdots \\ 1 \end{array}\right)\right),$$

$$\overset{\Delta}{=} span\left(s_1^b, s_2^b, s_3^b\right) \tag{57}$$

where $p_{ij}^b$, $i = 1, \cdots, n-3$, $j = 1, 2$, denote the $j$th component of an arbitrary real vector $p_i^b \in \mathbb{R}^2$, and $c_{bi}$, $i = 1, \cdots, n-3$, are arbitrary scalars. In view of Lemma 1, we know

$$null(\hat{\Omega}) = \mathbb{S}_a \cap \mathbb{S}_b. \tag{58}$$

To determine the non-trivial form of $\mathbb{S}_a \cap \mathbb{S}_b$, let

$$\alpha_1 s_1^a + \alpha_2 s_2^a + \alpha_3 s_3^a = \beta_1 s_1^b + \beta_2 s_2^b + \beta_3 s_3^b, \tag{59}$$

where $\alpha_i$ and $\beta_i$, $i = 1, 2, 3$, are scalars, at least one of which is nonzero. Note that $\mathbb{S}_a$ and $\mathbb{S}_b$ share the same entries as follows

$$s_c = \left(\left[\begin{array}{c} q_{(n-2)1} \\ q_{(n-1)1} \\ q_{n1} \end{array}\right], \left[\begin{array}{c} q_{(n-2)2} \\ q_{(n-1)2} \\ q_{n2} \end{array}\right], \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right). \tag{60}$$

Combining (59) and (60), one has

$$(\alpha_1 - \beta_1)\left[\begin{array}{c} q_{(n-2)1} \\ q_{(n-1)1} \\ q_{n1} \end{array}\right] + (\alpha_2 - \beta_2)\left[\begin{array}{c} q_{(n-2)2} \\ q_{(n-1)2} \\ q_{n2} \end{array}\right] + (\alpha_3 - \beta_3)\left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right] = \mathbf{0}, \tag{61}$$

which can be equivalently written as

$$\left[\begin{array}{ccc} q_{(n-2)1} & q_{(n-2)2} & 1 \\ q_{(n-1)1} & q_{(n-2)1} & 1 \\ q_{n1} & q_{n2} & 1 \end{array}\right]\left[\begin{array}{c} \alpha_1 - \beta_1 \\ \alpha_2 - \beta_2 \\ \alpha_3 - \beta_3 \end{array}\right] = \mathbf{0}. \tag{62}$$

Recalling that vertices $i, j$ and $k$ are not collinear, it is equivalent to say that they are in general positions in the plane, which implies

$$rank\left[\begin{array}{ccc} q_{(n-2)1} & q_{(n-2)2} & 1 \\ q_{(n-1)1} & q_{(n-2)1} & 1 \\ q_{n1} & q_{n2} & 1 \end{array}\right] = 3. \tag{63}$$

Then in view of (62), the parameters $\alpha_i$ and $\beta_i$, $i = 1, 2, 3$, in (59) satisfy

$$\begin{cases} \alpha_1 = \beta_1, \\ \alpha_2 = \beta_2, \\ \alpha_3 = \beta_3. \end{cases} \tag{64}$$

From the fact that $\hat{\Omega}$ is a stress matrix associated with configuration $\bar{q}$, we know

$$(\bar{q}_{\cdot 1}, \bar{q}_{\cdot 2}, \mathbf{1}_{n+1}) \subseteq null(\hat{\Omega}), \tag{65}$$

where $\bar{q}_{\cdot 1} = [q_{\cdot 1}^T, q_{(n+1)1}]^T$, and $\bar{q}_{\cdot 2}$ is defined analogously. Since $rank(\bar{q}_{\cdot 1}, \bar{q}_{\cdot 2}, \mathbf{1}_{n+1}) = 3$, we have

$$rank(\hat{\Omega}) \leq n - 2. \tag{66}$$

Then, to prove $rank(\hat{\Omega}) = n - 2$, we need to show that any other vector $v \in null(\hat{\Omega})$ can be represented as a linear combination of vectors $\bar{q}_{\cdot 1}, \bar{q}_{\cdot 2}$, and $\mathbf{1}_{n+1}$, namely, there exist scalars $\gamma_1, \gamma_2$, and $\gamma_3$, such that

$$v = \gamma_1 \bar{q}_{\cdot 1} + \gamma_2 \bar{q}_{\cdot 2} + \gamma_3 \mathbf{1}_{n+1}, \quad \forall v \in null(\hat{\Omega}), \tag{67}$$

where at least one of $\gamma_i, i = 1, 2, 3$, is nonzero. In light of Lemma 1, one has

$$v \in null(\hat{\Omega}) \iff v \in \mathbb{S}_a \text{ and } v \in \mathbb{S}_b, \tag{68}$$

which implies

$$v = \alpha_1 s_1^a + \alpha_2 s_2^a + \alpha_3 s_3^a \\ = \beta_1 s_1^b + \beta_2 s_2^b + \beta_3 s_3^b. \tag{69}$$

It follows from (64) that

$$\left[\begin{array}{c} v \\ v \end{array}\right] = \alpha_1 \left[\begin{array}{c} s_1^a \\ s_1^b \end{array}\right] + \alpha_2 \left[\begin{array}{c} s_2^a \\ s_2^b \end{array}\right] + \alpha_3 \left[\begin{array}{c} s_3^a \\ s_3^b \end{array}\right]. \tag{70}$$

Picking out respectively the first $n$ entries of $s_i^a$ and the last entry of $s_i^b$, $i = 1, 2, 3$, we get

$$v = \alpha_1 \left[\begin{array}{c} q_{\cdot 1} \\ q_{(n+1)1} \end{array}\right] + \alpha_2 \left[\begin{array}{c} q_{\cdot 2} \\ q_{(n+1)2} \end{array}\right] + \alpha_3 \left[\begin{array}{c} \mathbf{1}_n \\ 1 \end{array}\right], \tag{71}$$

equivalently,

$$v = \alpha_1 \bar{q}_{\cdot 1} + \alpha_2 \bar{q}_{\cdot 2} + \alpha_3 \mathbf{1}_{n+1}. \tag{72}$$

Therefore, there exist scalars $\gamma_i, i = 1, 2, 3$, such that any vector $v \in null(\hat{\Omega})$ can be written as a linear combination of $\bar{q}_{\cdot 1}, \bar{q}_{\cdot 2}$, and $\mathbf{1}_{n+1}$. This completes the proof.

$\square$

## References

[1] B. D. O. Anderson, C. Yu, and J. M. Hendrickx, "Rigid graph control architectures for autonomous formations," *IEEE Control Syst.*, vol. 28, no. 6, 2008.

[2] F. Dorfler and B. Francis, "Geometric analysis of the formation problem for autonomous robots," *IEEE Trans. Autom. Control*, vol. 55, no. 10, pp. 2379–2384, 2010.

[3] Z. Han, L. Wang, Z. Lin, and R. Zheng, "Formation control with size scaling via a complex laplacian-based approach," *IEEE Trans. Cybern.*, vol. 46, no. 10, pp. 2348–2359, 2016.

[4] D. J. Jacobs, A. J. Rader, L. A. Kuhn, and M. F. Thorpe, "Protein flexibility predictions using graph theory," *Proteins*, vol. 44, no. 2, pp. 150–165, 2001.

[5] W. Whiteley, "Rigidity of molecular structures: generic and geometric analysis," in *Rigidity Theory and Applications*. Springer, 2002, pp. 21–46.

[6] L. Sarkisov, R. L. Martin, M. Haranczyk, and B. Smit, "On the flexibility of metal–organic frameworks," *J. Am. Chem. Soc.*, vol. 136, no. 6, pp. 2228–2231, 2014.

[7] R. Connelly and W. Whiteley, "Second-order rigidity and prestress stability for tensegrity frameworks," *SIAM Discret. Math.*, vol. 9, no. 3, pp. 453–491, 1996.

[8] C. Paul, F. J. Valero-Cuevas, and H. Lipson, "Design and control of tensegrity robots for locomotion," *IEEE Trans. Robot.*, vol. 22, no. 5, pp. 944–957, 2006.

[9] R. Connelly and S. D. Guest, *Frameworks, Tensegrities and Symmetry: Understanding Stable Structures*, 2015.

[10] S. H. Juan and J. M. M. Tur, "Tensegrity frameworks: static analysis review," *Mech. Mach. Theory*, vol. 43, no. 7, pp. 859–881, 2008.

[11] R. Connelly, "Tensegrity structures: why are they stable?" in *Rigidity theory and applications*. Springer, 2002, pp. 47–54.

[12] ——, "Rigidity and energy," *Invent. Math.*, vol. 66, no. 1, pp. 11–33, 1982.

[13] ——, "Tensegrities and global rigidity," in *Shaping Space*. Springer, 2013, pp. 267–278.

[14] S. J. Gortler, A. D. Healy, and D. P. Thurston, "Characterizing generic global rigidity," *Am. J. Math.*, vol. 132, no. 4, pp. 897–939, 2010.

[15] S. J. Gortler and D. P. Thurston, "Characterizing the universal rigidity of generic frameworks," *Discret. Comput. Geom.*, vol. 51, no. 4, pp. 1017–1036, 2014.

[16] A. Y. Alfakih and Y. Ye, "On affine motions and bar frameworks in general position," *Linear Alg. Appl.*, vol. 438, no. 1, pp. 31–36, 2013.

[17] A. Y. Alfakih and V. H. Nguyen, "On affine motions and universal rigidity of tensegrity frameworks," *Linear Alg. Appl.*, vol. 439, no. 10, pp. 3134–3147, 2013.

[18] T. Tay and W. Whiteley, "Generating isostatic frameworks," *Struct. Topol.*, vol. 11, pp. 21–69, 1985.

[19] L. Henneberg, *Die graphische Statik der starren Systeme*. Leipzig: Teubner, 1911.

[20] T. Eren, B. D. O. Anderson, A. S. Morse, W. Whiteley, and P. N. Belhumeur, "Operations on rigid formations of autonomous agents," *Commun. Inform. Syst.*, vol. 3,

no. 4, pp. 223–258, 2003.

[21] C. Yu, B. Fidan, and B. D. O. Anderson, "Principles to control autonomous formation merging," in *Proc. Amer. Control Conf.*, Minneapolis, Minnesota, USA, Jun. 2006, pp. 762–768.

[22] D. Zelazo and F. Allgöwer, "Growing optimally rigid formations," in *Proc. Amer. Control Conf.*, Fairmont Queen Elizabeth, Montrèal, Canada, Jun. 2012, pp. 3901–3906.

[23] T. Eren, O. K. Goldenberg, W. Whiteley, Y. R. Yang, A. S. Morse, B. D. O. Anderson, and P. N. Belhumeur, "Rigidity, computation, and randomization in network localization," in *Proc. 23rd Annu. Joint Conf. IEEE Comput. Commun. Soc.*, vol. 4, Hong Kong, China, Mar. 2004, pp. 2673–2684.

[24] D. Carboni, R. K. Williams, A. Gasparri, G. Ulivi, and G. S. Sukhatme, "Rigidity-preserving team partitions in multiagent networks," *IEEE Trans. Cybern.*, vol. 45, no. 12, pp. 2640–2653, 2015.

[25] R. Connelly, "Combining globally rigid frameworks," *Proc. Steklov Inst. Math.*, vol. 275, no. 1, pp. 191–198, 2011.

[26] Z. Szabadka, "Globally rigid frameworks and rigid tensegrity graphs in the plane," Ph.D. dissertation, Department of Operations Research, Eötvös Loránd University, 2010.

[27] K. Ratmanski, "Universally rigid framework attachments," *arXiv preprint arXiv:1011.4094*, 2010.

[28] D. Pais, M. Cao, and N. E. Leonard, "Formation shape and orientation control using projected collinear tensegrity structures," in *Proc. Amer. Control Conf.*, Hyatt Regency Riverfront, St. Louis, Mo, USA, Jun. 2009, pp. 610–615.

[29] S. Y. Lau and W. Naeem, "Co-operative tensegrity-based formation control algorithm for a multi-aircraft system," in *Proc. Amer. Control Conf.*, Palmer House Hilton, Chicago, IL, USA, Jul. 2015, pp. 750–756.

[30] Z. Lin, L. Wang, Z. Chen, M. Fu, and Z. Han, "Necessary and sufficient graphical conditions for affine formation control," *IEEE Trans. Autom. Control*, vol. 61, no. 10, pp. 2877–2891, 2016.

[31] Q. Yang, Z. Sun, M. Cao, H. Fang, and J. Chen, "Construction of universally rigid tensegrity frameworks and their applications in formation scaling control," in *Proc. 36th Chin. Control. Conf.*, Dalian, China, Jul. 2017, pp. 8177–8182.

[32] Q. Yang, M. Cao, Z. Sun, H. Fang, and J. Chen, "Formation scaling control using the stress matrix," in *Proc. 56th IEEE Conf. Decis. Control*, Melbourne, Australia, Dec. 2017, pp. 3449–3554.

[33] Q. Yang, M. Cao, H. Fang, and J. Chen, "Constructing universally rigid tensegrity frameworks with application in multi-agent formation control," *IEEE Trans. Autom. Control*, 2017 (Conditionally accepted).

[34] B. Roth and W. Whiteley, "Tensegrity frameworks," *Trans. Am. Math. Soc.*, vol. 265, no. 2, pp. 419–446, 1981.