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Ph.D. Thesis

# THE POINT MASS AS A MODEL FOR EPISTEMIC REPRESENTATION.

## A HISTORICAL & EPISTEMOLOGICAL APPROACH

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*To Martina and Giada*

If our highly pointed Triangles of the Soldier class are formidable, it may be readily inferred that far more formidable are our Women. For, if a Soldier is a wedge, a Woman is a needle; being, so to speak, *all point*, at least at the two extremities. Add to this the power of making herself practically invisible at will, and you will perceive that a Female, in Flatland, is a creature by no means to be trifled with.

But here, perhaps, some of my younger Readers may ask *how* a woman in Flatland can make herself invisible. This ought, I think, to be apparent without any explanation. However, a few words will make it clear to the most unreflecting.

Place a needle on the table. Then, with your eye on the level of the table, look at it side-ways, and you see the whole length of it; but look at it end-ways, and you see nothing but a point, it has become practically invisible. Just so is it with one of our Women. When her side is turned towards us, we see her as a straight line; when the end containing her eye or mouth – for with us these two organs are identical – is the part that meets our eye, then we see nothing but a highly lustrous point; but when the back is presented to our view, then – being only sublustrous, and, indeed, almost as dim as an inanimate object – her hinder extremity serves her as a kind of Invisible Cap.

[E. A. Abbott, *Flatland. A Romance in Many Dimensions,* 1882]

Mathematical truth, unlike a mathematical construction,
is not something I can hope to find by introspection. It does not exist in my mind.
A mathematical theory, like any other scientific theory is a social product.
It is created and developed by the dialectical interplay of many minds, not just one mind.
When we study the history of mathematics we do not find a mere accumulation of new definitions,
new techniques, and new theorems.
Instead, we find a repeated refinement and sharpening of old concepts and old formulations,
gradually rising standard of rigor, and an impressive secular increase in generality and depth.
Each generation of mathematicians rethinks the mathematics of the previous generation,
discarding what was faddish or superficial or false and recasting what
is still fertile into new and sharper form.

[N. D. Goodman 1979, p. 545]

**Table of Contents**

## Introduction

Any casual philosopher of science writing in the twentieth century cannot help but showcase the recurrent presence of Newton's science of mechanics in introducing her diverging views on the nature of (physical) science. Newton's *Principia* embodies a powerful and coherent research program for linking mathematical representations with real-world structures. A crucial passage where Newton himself expresses clearly what he is up to in his *Principia* occurs in a *scholium* to section 11 of the first book:

> In mathematics we are to investigate the quantities of forces with their proportions consequent upon any conditions supposed; then, *when we enter upon physics* [my emphasis], we compare those proportions with the phenomena of Nature, that we may know what conditions [or laws] of those forces answer to the several kinds of attractive bodies. And, this preparation being made, we argue more safely, concerning the physical species, causes and proportions of the forces.[1]

The first two books of his treatise deal with forces treated abstractly; they contain purely mathematical exercises in determining the implications of the laws of motion under different conditions. The investigation of all sorts of systematic relations which occur in physical phenomena is constructed by the means of mathematical models,[2] which function as instruments of theoretical measurement, and which in turn allow the determination of parameters that characterise forces from parameters that characterise motion. In a later stage, or what Newton describes as "enter[ing] upon physics", these models are put to use in the third book to measure the characteristics of the forces that can be found in our solar system.

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[1] Newton (1687) 1726, p. 218.
*In mathesi investigandae sunt virium quantitates et rationes illae, quae ex conditionibus quibuscunque positis consequentur: deinde, ubi in physicam descenditur conferendae sunt hac rationes cum phaenomenis; ut innotescat quaenam virium conditiones singulis corporum attractivorum generis competant. Et tum demum de virium speciebus, causis et rationibus physicistutius disputare licebit.*

[2] Hereafter, the terms 'mathematical model' and 'mathematical object' are used interchangeably.

This powerful methodology has the scope to function as an example for emphasising how the world could be formally defined with mathematical models, which can additionally be used for deductive reasoning, simplifying computations and mathematical proofs. Moreover, these models can also be used in physics for representational, explicative and predictive purposes. Examples of them include numbers, equations, functions, relations, and so on. Geometry too has its own mathematical objects, such as points, lines, planes, and solid figures.

The philosophy of science deals with the ontological commitments relating to the nature of those objects. Are they mind-dependent objects arbitrarily produced by the scientific community for practical and speculative purposes? Or are they rather mind-independent objects, which have their own autonomy and existence?

The fact that mathematics represents the best way we have today of representing the world in which we are embedded, allows us to use those objects – no matter what is the nature of their existence – as a model for representing physical phenomena in the natural world. They are also considered to be useful for mathematical manipulation and clarification, but they represent only idealized states which, strictly speaking, do not exist ontologically. To clarify this point, I suggest to the reader an example, which will also be the case study throughout the research presented in this dissertation.

Let us look at the point mass[3] as a mathematical entity which is used as a model for epistemic representational purposes of the phenomena under observation. First, let me define this entity with reference to its current definition taken from modern rational mechanics.[4]

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[3] The term 'point mass' can be found in the literature under several different expressions, including 'material point', 'mass point', 'point particle' and 'point charge'. Hereafter the terms 'material point' and 'point mass' will be used synonymously in order to indicate either small bodies or their mathematical idealizations with the geometrically unextended point, or a point of a continuum model of matter. Of course the smallness in question refers to the extension of the space considered; for instance at the astronomical scale, the planets could be considered as punctiform. Moreover, I think that the expression 'material point' is not anachronistic, because it helps to refer to the materiality of the object, whilst at the same time the term 'point mass' is appropriate because of its explicit reference to mass.

[4] The term 'rational mechanics' is used here in distinction from the mere mechanics of the past, in order to ensure that when we speak about modern mechanics, we are aware that statics, dynamics and kinematics

> The material point – regarding its kinematic aspects (i.e. position, trajectory, velocity, acceleration, etc.) – could be considered as a geometrical point. However, under the action of a force it still should be considered as a natural body. Using this schematic representation it might be possible, on the one hand, to obtain the fundamental dynamic laws which underlie its behaviour; but on the other hand the material point's dynamics will give us the fundamentals of mechanics. Therefore the laws of motion of any kind of body could be established just considering that body as an aggregate of material points.[5]

In this respect, the point mass is a theoretical and abstract entity which has dimensions that can be considered negligible when compared to a problem under examination. Not only that, all the forces acting on it can theoretically have the same site of application. The point mass in this sense is a useful mathematical and physical idealization for the analysis of various physical phenomena.

The aim of this dissertation is twofold. Firstly, I will aim at showing the role that models play in the philosophy of science as epistemic instruments for representing the dynamic and variegated aspects of the phenomena in the physical world. Secondly, I will intend to support this first aim by providing extensive analysis of a single case study, namely the point mass – which is one of the fundamental models of today's mathematical physics – by pursuing a hypothesis concerning its possible development. In this respect my thesis aims to show that some mathematical models – in this particular case, the point mass – derive from a process involving the objectification of scientific practice, instead of being considered as abstractions from natural properties or

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are all part of the same discipline, they only have a different approach to the study of bodies. Moreover, it is a discipline of *indisputable* and *highly formalised* principles. Throughout this approach, the natural world is mapped onto a mathematical environment to perform calculations, after which these results are reported to the physical world by an inverse process of mapping.

[5] Cf. Levi-Civita and Amaldi 1929, p. 314: "Il punto materiale, per ciò che riguarda i caratteri puramente cinematici (posizione, traiettoria, velocità, accelerazione, ecc.), andrà, per la sua stessa definizione, considerato come un punto geometrico; ma, di fronte all'azione delle forze, non cesserà di comportarsi come un corpo naturale. La semplicità schematica degli aspetti cinematici dei moti di un punto materiale ci permetterà di coglierne le leggi dinamiche fondamentali; e la Dinamica del punto fornirà la base di tutta la meccanica, in quanto [...] le leggi del moto di ogni altro corpo, di cui non sia lecito trascura- re le dimensioni (rispetto a quelle della ragione spaziale in cui ha luogo il moto), si possono stabilire, considerando codesto corpo come un aggregato di punti materiali." The accompanying English trans. is my own.

objects. The outcome of this thesis will be to emphasise the idea that the point mass is a useful idealization used in mathematical physics as a model for epistemic representation.

The first objective is to define what an epistemic representation actually is. Following Contessa (2007) and Swoyer (1991), we can maintain that a scientific model is an epistemic representation insofar as it gives a representation of any particular set of physical phenomena under observation, even though the knowledge drawn about it does not correspond to what we directly observe in nature. There is, in fact, an epistemic gap between the conclusions that we infer from the model and what we directly observe in nature. In this respect, the model of a certain system is an epistemic representation of reality, insofar as it allows the user – by following a certain set of rules – to infer some conclusions and acquire some knowledge concerning that particular system of reality. Therefore, those conclusions need to be "translated", before one can judge them in a certain respect. The model is first built in such a way that it is easier to study than the target-system, and this factor therefore allows us to derive results. Second, it is assumed to represent its target system, in which representation stands for something like a *licence to draw inferences*. Representation allows us to 'carry over' results obtained in the model to the target-system, and hence to learn something about that system by studying the model.

This approach aims at emphasising the cognitive role played by the scientific community, in the sense that every conclusion drawn about the model is only partial and should be interpreted by taking every feature of the model as a denotation that stands for a piece of the modelled system. More precisely, every user takes the model to stand for the target system, and, moreover, the user is also able to perform valid surrogative inferences from the model to the target. This requires that the user interpret the surrogative inferences in order to acquire knowledge about the phenomenon under observation. Every user is a cognitive agent, who is able to develop representations of the world and make judgements about both the world and their representations of it.

In an attempt to understand what it takes to develop a mathematical representation of nature, the historiographical purpose of this dissertation – to which Chapters 2-4 are dedicated – is invoked to support the idea of considering the point mass as part of the phenomenon that Enrico Giusti has labelled the 'objectification of procedure'.[6] His hypothesis states that mathematical objects are obtained from idealizations of procedures through three stages: i) 'as' investigative tools 'within' demonstrative procedures; ii) by becoming common elements that are used for solving problems, for which reason they also became objects of study, depending on a specific practical context; and iii) by becoming new and abstract mathematical objects (e.g. as mediators between the world and the scientific theories or models of intelligibility) which deserve to be studied on their own.

First, I should point out that the sequence of these stages is not always clearly determined, and it is not always possible to distinguish between them. However, as we shall see in the arguments in favour of this methodological approach, our case study fits almost perfectly within this threefold development. The claim is not to establish *who* was able to identify this mathematical object, but rather to find an answer to the question: 'When and in which demonstrations does the idea of the point mass first become conceptualised as such?'.

Traditionally, mathematical objects or models emerge from two different approaches, which are commonly labelled the Newtonian approach – that we have already seen above in the introductory example – and the Baconian approach. In the first of these, the so-called 'top-down'[7] approach, we start with an assumed mathematical model of a natural or technical system, and deduce its behaviour by solving the corresponding dynamic equations under certain initial conditions. Then, after one has established the main general foundational principles that are deemed to be valid in our field of research, we turn to apply that model to the piece of world under observation, proceeding by means of approximations.

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[6] The original Italian expression is *oggettivizzazione delle procedure*: see Giusti 1999.

[7] That is, one starts with a concept, and then one figures out what this concept means or how it fits together by breaking it down.

The second approach, or so-called 'bottom-up'[8] approach, is mainly used in scientific practice, because physicists, chemists, biologists or engineers start by mining data in an unknown field of research, and only at a second stage reconstruct the behaviour of the system from these data, in order to guess the type of its dynamic equations.

My dissertation's research intends to show that our case study does not derive from either of these two classical approaches, and that therefore not all the mathematical models derive from an act of abstraction from natural objects, any one of which describes certain physical (and qualitative) properties. Rather, from my perspective, some mathematical objects – and my case study of the point mass – formalize the practical enquiry and the heuristic or mechanical demonstrative procedures carried out by the scientific community.

This dissertation is divided into four chapters. The first chapter focuses on the role of models in science, which are seen not solely as tools that provide a means for interpreting a formal system, but also as tools for representing the world. After this, each of the three subsequent chapters – Chapters 2, 3 and 4 – is devoted to one of the stages of the objectification of procedure. However, none of them pretends to deliver an exhaustive introduction to the historical period under observation, but each respectively sketches out only a historiographical perspective. For example, my discussion of Archimedean mechanics in Chapter 2 does not follow a straightforward chronological approach, because I wish specifically to pursue the claim that I made use of a portion of treatises that have been recently discovered (e.g. *The Method*, which was rediscovered only at the end of the nineteenth century). However, this aspect should not be seen as a limitation and defect of my reconstruction, because, although Renaissance mathematicians did not have direct access to this last Archimedean treatise, it can be maintained – and I will prove this assertion throughout my dissertation – that there exists a direct methodological affiliation between the two lines of thought (i.e. those in

[8] That is, one starts with the observation of behaviour or events, and then one notices a pattern and builds up to a model or broader metaphor.

the Archimedean and the Renaissance eras). The only point of dissonance between the Archimedean outcomes and the Renaissance works was the purpose that they had in mind when they were carrying out their research.

Let me now turn to provide an overview of the structure of this dissertation.

The first chapter begins by aiming to examine the various perspectives relating to the structure of scientific theories. It will, however, only give an overview of the philosophical context in which the notion of the model assumes a decisive role in science. We shall focus on the different kinds of theoretical models (e.g. abstraction, idealizations and analogies) commonly used in scientific practice in order to represent physical reality and simplify computations. Only in the last section of the first chapter will we emphasise the surrogative role fulfilled by a model; in fact, models are more often used as surrogates of states of affairs which are taken into account for a certain purpose, i.e. as a source from which every user can infer conclusions specifically relating to that model. These conclusions in turn need to be translated from the users in advance, in order to reach some scientific conclusions about that part of reality in itself.[9]

In Chapter 2, we shall see that the first stage of the objectification of procedure relates to ancient Greek geometry, where its main impact lay in those demonstrations in which the notion of the centre of gravity – which is of central importance – shares some essential properties with the modern notion of the point mass. The point mass, as a mathematical entity having an algebraic meaning, is not yet present in Greek geometry; rather it is the centre of gravity that functions as a demonstrative tool detached from some of its geometrical features. This stage corresponds to the enquiry made within the field of Archimedean equilibrium concerning the application of the law of the lever, according to which geometrical objects stand in equilibrium at distances that are inversely proportional to their extensions. This constitutes one of the foundational principles of statics imported into a geometrical context. My emphasis focuses on the

[9] For the approach, see further Contessa 2007 and 2010.

fact that Archimedes should be held among the first authors to give importance to the heuristic and mechanical meaning of geometrical demonstrations. His heuristic practice can be read as a 'physicization' of mechanics, which allows us to consider *only the quantitative relations between bodies*, without considering some of their attributes (e.g. mass and spatial dimension).

Chapter 3 next introduces the restoration phase of the ancient Greek mathematical tradition and the legacy of Archimedes between the Middle Ages and the Renaissance period. In particular, this Chapter is dedicated to the School of Urbino and its rediscovery of the ancient Archimedean treatises, not only from a philological but also from both a mathematical and a speculative point of view. In order to show the different approach of the Renaissance towards mechanics, and the use that scholars made of the notion of the centre of gravity, our focus will lie with the practical and theoretical contributions that Renaissance scientific humanism provided: i) in consolidating the new way of doing and conceiving mechanics; and ii) in the second stage of the objectification of procedure, namely that, as can be seen in several treatises published between the fifteenth and sixteenth centuries by Urbino School mathematicians – such as Federico Commandino and Guidobaldo dal Monte – it was possible to switch attention onto the practical use we can make of geometrical elements, such as the centre of gravity. This notion became fundamental in various practical contexts linked to the *scienza de ponderibus* (science of weights), the aim of which was to solve and formalize static problems concerning heavy bodies, with particular reference to those hanging from a balance. Moreover, the concept of the centre of gravity is also useful to solve the so-called "Equilibrium Controversy", which addresses the question whether or not a deflected balance will return to its horizontal position, a controversy which, though it had already seen its birth during the Medieval period, was only during the Renaissance, and following the rediscovery of all the Archimedean corpus on statics and hydrostatics, applied to the science of machines and to other purely practical contexts. It is only at the end of the Renaissance, at the turn of the seventeenth century, that the work of Luca Valerio promoted the introduction of a philosophical, almost epistemological, debate over the meaning of physical properties applied in a formalized mathematical context. The Renaissance mathematicians worked towards the foundational programme of constructing an epistemological debate about the analysis of the conditions under which mathematical principles can and should be considered to be true of physical things. A purely mathematical treatment of physical reality was achievable due to the awareness that, in contrast, a solely empirical and mechanical approach is insufficient to understanding the ways in which the physical world behaves.

Chapter 4 is devoted to modern rational mechanics. In pursuing the aim of understanding what it takes to develop a mathematical representation of nature by means of an idealized and abstract model, we will see that seventeenth-century mathematical physics represents the framework for the completion of the third stage of the objectification of procedure, insofar as it was at this time that the centre of gravity became an independent object of study *as* the point mass , i.e. as an idealized entity used to represent physical objects and to which we can ascribe natural properties, such as volumetric extension, mass and forces. The point mass is still a representational geometrical point whose features are idealized, but it now assumes the role of being an independent object of research, which is useful for building the foundational principles of rational mechanics. We shall see how Galileo was able to shift the investigative methodology used in the field of mechanics to a more theoretical and physical perspective, by showing how the purely practical operations carried out through machinery could be used to interpret the working principles behind nature. Or, in other words, within Galileo's research we can observe that he uses machines as aids in a bid to confer a discursive structure onto the phenomenal world.

Within the second part of this chapter, I will examine the more metaphysical contributions of Thomas Hobbes concerning the rule of human imagination in building our representational model of physical phenomena and the point mass. Finally, in the last part of this chapter, I will examine the way in which the model of the point mass began to be used from a purely mathematical point of view, in order to represent, with purely algebraic language, a series of states of affairs which were not only made up of simple rigid bodies, but which were also typified by a higher level of complexity. This represents the climax of the procedure of objectification of our centre of gravity. Now it is no longer a geometrical tool shifted along the arms of a real mechanical balance, but instead a mathematical entity, investigative tool and a model of intelligibility – which is explicitly introduced for the first time in Newton's and Euler's research – that has two main characteristics: i) it is the centre of mass of any rigid body, no matter how it is shaped; and ii) it is the point of applicability of all the forces – gravity, works, pressure and so on – acting on a stationary or a moving body, be it travelling in uniform motion, in a state of acceleration or in parabolic motion.

The analysis of this scientific practice allows us to reach the conclusion that models serve to understand how the world works and not to help us argue about the ontological claims of what we can observe or think about the nature of the physical world itself. Models have an epistemic value that is strongly dependent on the interpretation that we – as users – attribute them with respect to their specific and (crucially) context-dependent rules.

# Chapter 1

## Models for Scientific Representation

In mechanics we investigate the velocity of a moving body or the length of the path covered by a moving body by neglecting or omitting a series of quantitative or qualitative properties, because we deem them superfluous to the purpose at hand. Population biologists study the evolution of a species which procreates at a constant rate by considering it as an isolated ecosystem. These descriptions are descriptions of a modelled system, and scientists use models to represent aspects of the real world's system.

Most philosophers of science today seem to agree on two basic points. The first is that models play a central role in science, in that they are one of the primary means by which scientists represent the world in a mathematical form. The second is that models are not truth-apt, or, in other words, they are not capable of being true or false due to the fact that they are not linguistic entities, and therefore scientific models are much more like portraits or representations than something capable of being true or false. Indeed they can be defined only as more or less accurate or faithful to reality. However, this has not always been so. In fact, up until the 1960s, models did not play a central role in what we now standardly call the philosophy of science. Rather, scholars only began to acknowledge the importance of models with increasing attention after the decline of the so-called *Received View* of theories.[10] This line of argument was more emphatically considered a mature version of logical empiricism, according to which the proper characterisation of a scientific theory consisted of an axiomatisation in first-order logic. In other words, scientific theories are portrayed as semi-interpreted formal systems, or as sets of axioms which house empirical rules of interpretation. As Morgan and Morrison write, "[t]he theory itself was explicated in terms of its logical form with the

[10] See further Putnam 1962.

meanings or semantics given by an additional set of definitions, the correspondence rules. That is to say, although the theory consisted of a set of sentences expressed in a particular language, the axioms were syntactically describable."[11] One way to work out these problems was to provide a semantics for a theory (*T*) by specifying a model (*M*) for the theory, that is, by specifying an interpretation on which all the axioms of the theory are true. Instead of formalising the theory in accordance with first-order logic, one defines the intended class of models for a particular theory. Since the 1960s, the *Semantic View* of theories, which is considered nowadays the mainstream view concerning the structure of scientific theories, has become dominant.[12] By 'semantic', we refer to the fact that a model provides a realisation through which the theory is satisfied. This is so because the notion of a model is defined in terms of truth. In other words, the claims made by a theory are true in the model, and in order for *M* to be a model, this condition must hold. This also brings us back around to the second item on which philosophers of science seem to agree, namely that models are not truth-apt. As Giere puts it:

> Semantic notions such as *meaning* and *truth* entered the logical empiricist picture of theories only indirectly by means of the interpretative rules employed. A minimal sort of interpretation serves to introduce the important semantic notions of "structure" and "model". Imagine some axioms formulated in a simple first-order language, *L*. A *structure* for *L* is a set of objects, *O*, and a function that assigns subsets of *O* to one-place predicates of *L*, ordered pairs of objects to two-place relations, and so on. A *model* of a theory *T*, expressed as axioms in *L*, is any structure in which the axioms of *T* are true. The concept of model, being defined in terms of truth, is therefore a semantic, as opposed to syntactic, concept. Whether any given structure is indeed a model of *T* requires independent determinations of the truth of axioms. This may be done mathematically if *O* is a set of mathematical entities, such as integers, or empirically if *O* is a set of physical objects, such as planets. In any case, models are *non-linguistic* entities – sets of objects, not set of statements.[13]

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[11] Morgan and Morrison 1999, p. 2.

[12] Historically one of the first scientists who emphasised the role of models in science was R. B. Braithwaite in his book, *Scientific Explanation. A study of the function of theory, probability and law in science* (1968). We will return to this reconstruction in the next section.

[13] Giere 1988, p. 47. Giere's view follows in Suppes's footsteps, in stipulating that models are not linguistic entities.

Let me clarify some points to this approach. First, it was presented in three different variations, all of which related to the assumption that models are not truth-apt: i) Suppes 1962; ii) van Fraassen 1967 and 1970; and iii) Giere 1988.[14] Secondly, this view still allows for axiomatisation, provided that one can state a set of axioms whereby the models of these axioms are exactly the models in the defined class. One could still formulate axioms in a first-order language in the manner of the syntactic view; the difference, however, is that now it is the model – rather than the correspondence rule – that provides the interpretation for the axioms.

Moreover, there is another commonly held view concerning the value of models, which dates back to the end of the last century. This is the so-called "models-as-mediators" view, which focuses on what a model *is* in itself and *how* it is able to function in an autonomous way. Giving importance only to the relationship between theory and models draws our attention away from the processes of constructing models and manipulating them, both of which are nevertheless crucial in gaining information about the world, theories and the model itself.[15] In addition, this account focuses on the ways that models and theories interact, in an attempt to understand the dynamics of modelling and its impact on the broader context of scientific practice. In other words, models hereby work as instruments of investigation, in mediating between abstract theories and a concrete target system.

To sum up, models since the second half of the twentieth century have started to play a crucial role in the philosophy of science. Since models are *non-linguistic entities*, according to the general view advocated by semanticists, they can be designated as mind-independent objects and thus as part of the physical reality. However, according to the "models-as-mediators" account, the Semantic view does not provide a satisfactory account of the role provided by models in scientific practice. The question that arises at this point is: *In virtue of what is a model to be considered a representation*

[14] We will return to consider this tripartite division in the next section. Here I simply introduce the Semantic approach from a general perspective, in order to help the reader to focus the role that models play in the philosophy of science, and in no way do I claim the aim of completeness.
[15] Morgan and Morrison 1999, p. 8.

*of something else?* Or in other words: *What is it for an object* (that is not a word or a sentence) *to represent a phenomenon scientifically?* The last century's literature has shown widespread disagreement over *what* it means to say that a certain model represents a certain state of affairs, as well as over *how* a certain model represents a certain system. As far as philosophers of science are concerned, the increased interest in the subject of scientific representation has disappointingly not been accompanied by an increase in our understanding of how models represent any particular target system of reality. The willing student in fact collides with a cluster of interpretations which all have different labels, such as the 'inferential conception' [16], the 'isomorphistic conception'[17], the 'similarity conception'[18], and the 'structural conception'.[19] And so it seems very hard to find, even nearly a half-century later, a well-defined set of worked-out solutions to the questions posed above. There is only one aspect that seems to connect all these different interpretations, namely the way in which scientists usually connect the data of physical reality with the abstract features of models, a process that requires at least two steps. These two steps are: i) *interpretation*, with the intention that elements of an abstract model are provided with general physical interpretations, such as mass, position and velocity; and ii) *identification* or *denotation*, namely of elements of a representational model which are identified with elements of the real system. In scientific practice, these two acts are not usually well distinguished, and hence one of the first issues to address should be whether or not we, as theorists of science, need to give a more detailed account of this dichotomy. In other words, are representational models directly related to empirical or observational data, or are they related to models of data?[20] Even more precisely, does the interpretation of a model give us a specific

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16 Suárez 2004.
17 van Fraassen 1980.
18 Giere 1988.
19 Suppes 1960-62.
20 See Frigg and Hartmann 2010, p. 5, who writes: "A model of data is a corrected, rectified, regimented, and idealized version of the data we gain from immediate observation, the so-called raw data. Characteristically, one first eliminates errors (e.g. removes points from the record that are due to faulty observation) and then present the data in a 'neat' way, for instance by drawing a smooth curve through a set of points. These two steps are commonly referred to as 'data reduction' and 'curve fitting'. When we investigate the trajectory of a certain planet, for instance, we first eliminate points that are fallacious from

account of some part of physical reality in itself, or do we need a certain set of context-dependent rules, in order to interpret and translate the information derived from models to physical reality?

There are at least two different lines of thought here, which try to give an answer to these questions. On the one hand, some theorists account for a theory of representational models by which models stand for the physical reality in itself, and so in this respect models are considered as spatio-temporally independent entities. These theorists suggest that this line of argument recalls a realistic stance, inasmuch as their theories are based on the idea that there is an intrinsic relation between the vehicle and the target. By assuming that a model has a counterpart in physical phenomena, we are allowed to interpret knowledge relating to the model as knowledge regarding the target system in itself.

On the other hand, other scientists account for a different interpretation of the representational role of models. This viewpoint, which is known as the 'agent-based construction', has been most strongly advocated by Ronald Giere, who argues the following:

> […] [I]t is tempting to think that there is a binary representational relationship between the model and a system in the real world that it represents. I [sc. R. Giere] agree[s] with Suárez (2003), however, that, whether based, for example, on either similarity or isomorphism, no such binary relationship exists. We need to introduce agents who consciously use the models to represent things. And once we have agents, we must consider the purposes for which they are doing the representing.[21]

This second line of argument evokes firstly the idea that the model and the real situation are distinct, and secondly that, since the user can draw conclusions about the target from the observation and the examination of the vehicle, it is therefore symptomatic that a

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the observation records and then fit a smooth curve to the remaining ones. Models of data play a crucial role in confirming theories because it is the model of data and not the often messy and complex raw data that we compare to a theoretical prediction." For further see Suppes 1962.

[21] Giere 2008, pp. 129-130.

vehicle is an epistemic representation of that target (for that user).[22] This entails that the representational role of models and their epistemic functions derive from a triadic relation that involves a vehicle (the model), a user (i.e. an agent such as the scientist) and a target.

Despite the fact that, in the last twenty years or so, this issue has been one of the most discussed issues in the literature, and besides the fact that it is also one of the few issues on which theorists seem to agree[23], this triadic relation has not been sufficiently taken into account in the present dissertation. To be exact, at least the cognitive role played by the user is neglected. However, our strategy shows affinities with both R. Giere and G. Contessa, inasmuch as it aims at looking at: i) the epistemological meaning of the representational role fulfilled by models, specifically the role played by the model of the point mass for representational purposes; and ii) the scientific practice, namely consideration of the point mass as a mind-dependent object, which exists only in the mind of the scientists and every user, and which is defined as a social construction, i.e. as the product of a deep theoretical analysis of the practice relating to the development of mechanics, physics and mathematics within a certain cultural context and a certain scientific community. In order to clarify the epistemological meaning of models as forms of representation, and the fact that the model building practice is rooted in scientific practice, our strategy is based on two factors. On the one hand, our approach will focus on the way in which a particular physical notion – the centre of gravity – has been used to solve geometrical and mechanical issues; and on the other hand, it will look at the way in which the model-building practice has become an accustomed *modus operandi* that takes its origins from the scientific practice. More precisely, our main hypothesis is that the model of the point mass takes its origins from an act of abstraction relating to the practical and mechanical techniques of weighing.

Our investigation converses with a significant part of this scientific practice, extending from ancient geometry to modern mathematical physics. It is only by looking

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[22] Thanks for this point go to G. Contessa, who has clarified it in his doctoral dissertation, entitled *Representing Reality: The Ontology of Scientific Models and their Representational Function*, published in 2007.

[23] See e.g. the work of Suárez 2002 and 2003; Frigg 2002; Giere 2004.

at the scientific practice, intended as it is as the technical development of a series of mechanical devices and instruments of precision, that we can describe the development of such a notion through three stages. We will call our working hypothesis the 'objectification of procedure', a process which follows a historical reconstruction which will be presented in Chapters 2, 3 and 4.

The result of this analysis, as it is presented in this chapter, is that, broadly speaking, a model of scientific representation is given by a cluster and an overlapping of model-building practices, despite the fact that idealization, abstraction, approximation, and so on, really have different meanings and functions. As we will see more clearly below, it will become evident that in the model-building, these processes can coexist and are not necessarily mutually exclusive.

Thus, the purpose of the remaining three chapters is twofold. Firstly our aim is to proceed through the three stages of the objectification of procedure, in order to find an answer to the following question: *When and in which demonstrations does the idea of the material point – or point mass – stand out?* This question is motivated by the attempt to undertake a plausible reconstruction of the point mass's theoretical development. In this attempt, the idea is not to see in the geometrical notion of the centre of gravity – or barycentre – a *precursor* of the mathematical model of the point mass, but rather to look at the way in which the conception of the point mass has been treated in other fields of study, such as mechanics and physics, especially for practical purposes. Secondly, we shall review the scientific practice and its historical scientific purposes, in order to enquire into the theoretical origins of the 'idealizing' technique. Again here we are not looking for precursors; I do not mean to prove that there has to be an 'inventor', or that, *given* that idealization is usually labelled a 'Galilean idealization', this automatically means that it was invented by Galileo. In fact, we shall see that, in the work of Galileo, idealizations played an important role, but several affinities with his contemporaries' theoretical techniques can be outlined. By highlighting the different performative components developed between the sixteenth and seventeenth centuries, we will gain a more complete picture of the different levels at which a representative model functions.

The first item on the agenda is reviewed in § 1.1, namely a brief examination of the various perspectives relating to the structure of scientific theories. Directly related to this question, there is a second issue which represents the main interest of this chapter, that is the investigation of the role attributed to models, which are nowadays considered one of the principal instruments of modern science. More precisely, in § 1.2, we shall see the different kinds of modelling techniques (e.g. abstractions, idealizations and analogies) which are commonly used in scientific practice in order to represent physical reality and simplify computations. After this, § 1.3 will focus on the roles attributed to models, be it representational, explanatory or predictive. Here we will argue that models are used as surrogates of states of affairs which are taken into account for a certain purpose, i.e. as a source from which every user can infer conclusions specifically related to that model. These are conclusions which need to be translated by the users in advance, in order to reach some final scientific judgment about that part of reality in itself.[24]

## 1.1 The Structure of Scientific Theories

How does science represent the world? How does a scientific theory link itself to the world? What is the best characterisation of the composition and function of scientific theories? Which tools can and should be employed in order to describe and remodel scientific theories? Is the understanding of practice and the application of those tools necessary for a comprehension of the core structure of scientific theories?

In the last century there has been a notable increase in interest in questions of this kind. Within the philosophy of science, three families of perspective on the structure of scientific theory were operative: the *Syntactic View*, which is perhaps better known as the *Received View*, the *Semantic View*, and the *Pragmatic View*. The baptism of each of

[24] For further details, see Contessa 2007 and 2010.

these schools of thought is a result of how the three different aspects of language – the grammar, the meaning and the use of the whole language's tools – are perceived to be dominant: each school stands respectively for theory as syntactic logical reconstruction (the Syntactic View), theory as semantically meaningful mathematical models (the Semantic View), and theory structured as complex and as closely tied to theory pragmatics in its function and context (the Pragmatic View).

According to the Syntactic View, which was mainly developed by Rudolf Carnap[25] and Carl Hempel[26], later influenced by among others Hans Reichenbach[27] and Otto Neurath[28], and which emerged from the work of the Vienna Circle and Logical Empiricism[29], the structure of a scientific theory is conceived as a reconstruction in terms of sentences cast in a meta-mathematical language. The axiomatic method for building the foundations of science includes various theoretical instruments, e.g. predicate logic, set theory, and model theory. The Syntactic View questions the logical language that should be used to recast any particular scientific theory. As Carnap puts it:

> […] [P]hilosophy is to be replaced by the logic of science – that is to say, by the logical analysis of the concepts and sentences of the concept and sentences of the science, for the logic of science is

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[25] Above all Carnap 1939 and 1966 [reprinted in 1972].
[26] See e.g. Hempel 1958, pp. 142-163.
[27] Reichenbach 1938.
[28] Neurath 1932 [reprinted in 1983, pp. 91-99].
[29] Logical Empiricism is a philosophical movement that flourished between the 1920s and 1930s in Europe and in the 1940s and 1950s in the United States. It is acknowledged as the primary source for current investigation on this subject. Logical empiricism was explicitly "foundationalist" with regard to a different point. First of all, it was foundationalist in the sense of the logicist program of *Principia Mathematica*, published in 1910-12 by A. N. Whitehead and B. Russell; for them, logical notions are more basic and clear than mathematical ones. After this, Carnap, with his *Logical Construction of the World* (1928, 1967), having been inspired by Russell himself, attempted to develop a foundational program for the arithmetical dimension of every subject, such as geometry, physics, biology or psychology, and even sociology. They did not seem to doubt either that science needed philosophical foundations, or that they possessed an adequate method to the task. Moreover, they did not show any interest in a descriptive account of how science works. Rather their main aim was to provide science with logical and epistemological foundations. For an historical and concise overview of Logical Empiricism, see the entry in the Stanford Encyclopedia: R. Creath, "Logical Empiricism", The Stanford Encyclopedia of Philosophy (Spring 2014 Edition), E. N. Zalta (ed.);
On line: https://plato.stanford.edu/archives/spr2014/entries/logical-empiricism/.
In addition to this brief overview, see Giere 1988, pp. 22-28.

nothing other than the logical syntax of the language.[30]

Logical languages allow us to reconstruct theories which – by definition – are sets of sentences in a given logical domain language.[31]

The structure of a syntactic scientific theory consists of terms, sentences, and languages. The vocabularies for these separate components are of three kinds: theoretical, logical and observational. The theoretical concepts can be classificatory, comparative or quantitative. Logical terms include logical quantifiers and logical connectives. Predicates and relations are considered observational terms.

Terms are bound together into a triple set of possible sentences: theoretical sentences, correspondence sentences, and observational sentences. $T_s$ is defined as the set of theoretical sentences, including axioms, theorems, and laws of the theory.[32] $C_s$ is defined as a set of correspondence sentences connecting theoretical sentences to observable phenomena or to the system of reality. Finally, $O_s$ is defined as the set of observational sentences which contains only observational vocabulary, and which may be further restricted in its logical strength, for example to first-order logic, or molecular sentences.[33]

The whole language's domain of science consists of two kinds: the theoretical and non-observational language, $L_T$, and the observational language, $L_O$. This central distinction can be found in almost all the works published by Carnap on the Received View.[34] The former includes theoretical vocabulary, while the latter involves observational terms. Both of the languages contain logical terms. Moreover, the

[30] Carnap 1937, xiii.

[31] See for example Campbell 1920, p. 122; Hempel 1958, p. 46 and Carnap 1928, § 156.

These views have been denominated in many ways: the Received View by Putnam (1962), Hempel (1970) and Craver (2002); the Syntactic or Axiomatic View by van Fraassen (1970 and 1989 respectively); the Syntactic View by Wessel (1976); the Standard Conception by Hempel (1970); the Orthodox View by Feigl (1970); and finally the Statement View by Moulines (1976, 2002) and Stegmüller (1976).

[32] Theoretical sentences include the laws of Newtonian mechanics and the laws of the Kinetic Theory of Gases, all suitably axiomatised. Primitive theoretical sentences (e.g. axioms) can be distinguished from derivative theoretical sentences.

[33] Carnap 1956, p. 41.

[34] See for example Carnap 1923, pp. 99-107; 1966, ch. 23.

theoretical language includes and is constrained by the logical calculus of the axiomatic system adopted and labelled as *Calc*. This calculus specifies the grammatical nature of sentences as well as appropriate deductive and non-ampliative inference rules, such as the *modus ponens*, which is pertinent to theoretical sentences; *Calc* can itself be written in theoretical sentences. A scientific theory is thus taken to be a syntactically formulated set of theoretical sentences (axioms, theorems, and laws) together with their interpretation by means of correspondence sentences. However, Carnap[35] maintained that the bipartition between theoretical and observational language is not fixed because the choice of the observational domain is partially context-dependent.

The classical Carnapian view, however, assumes that there are at most countably many observable objects, and his suggestion was to map them "injectively to the natural number, and [to] trea[t] theoretical terms as applying only to the natural numbers and objects that can be constructed from the natural numbers with the help of Cartesian products or powerset."[36]

The theoretical syntactic language aims at describing the structure of the world by a set of correspondence rules $C_s$, whose functions are: i) to interpret the theoretical terms of the theories, and ii) to reduce the objects of physical reality to theoretical terms. However, in order to understand the vitality and the heterogeneity of physical reality, we need to understand the axiomatic system of physics. At this point, scientific theories require observational interpretation through correspondence rules, which are defined as a part of the scientific theories' structure which serves as the *glue between theories and observations*.

From 1965 to 1969, Carl Gustav Hempel – who was also one of the main supporters of the Received View – had criticized and gradually abandoned the Received View not only in its theories but also due to its reliance on syntactic axiomatization, in a series of talks which were only published posthumously.[37]

Before entering into a detailed analysis of the reasons for the decline of the

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[35] Carnap 1932, p. 224 [reprinted in 1987, pp. 457-470].
[36] Lutz 2014, p. 5.
[37] The main supporters of the Semantic View regarding the structure of a scientific theory are Hempel 1970 and Suppe 2000.

Syntactic View, let me first provide some historical remarks on the notion of ‘model’. The beginning of the tradition relating to models can be traced back to the second half of the nineteenth century, in particular to the works of scientists such as William Thomson, James Clerk Maxwell, Heinrich Hertz (especially), and Ludwig Boltzmann. Boltzmann’s article in the *Encyclopedia Britannica* is commonly associated with the *belle époque* of this ‘modelling attitude’, which has done nothing but continue to flourish and inform much scientific practice during the twentieth century (despite its philosophical detractors such as Pierre Duhem, who famously disparaged it in Duhem 1906). Within the philosophy of science, this traditional approach mainly constituted opposition to logical empiricist reconstructions of knowledge that pursued this modelling tradition, and in its place continued to emphasise the essential role of models, model-building, and analogical reasoning in the sciences. Thus, for example, Norman Campbell’s masterly *Physics: The Elements* (1920) had a very considerable influence in advancing the case for modelling among some British scholars of the second half of the last century, such as Max Black (1954) and Mary Hesse (1962). During the 1950s, the Campbellian idea of models as an alternative interpretation of a certain calculus, i.e. that “if, for instance, we take the mathematics used in the kinetic theory of gases and reinterpret the terms of this calculus in a way that makes them refer to billiard balls, the billiard balls are a model of the kinetic theory of gases”[38], was re-proposed by Braithwaite in his *Scientific Explanation* (eds. 1953 and 1968), in which he tries, above all, to explain the parts played by mathematical reasoning and by theoretical concepts and ‘models’ in the organisation of scientific theory. Braithwaite argues that what is needed is a preparedness to think explicitly about modes of thinking, and the way in which scientific language and symbolism is used in the expression of a scientific theory. He maintains that it signifies a great deal to show why theoretical concepts – i.e. atoms, electrons, forces acting on punctiform bodies or fields of force and so on – find their way into scientific theory and exactly how mathematico-logical deductions are

[38] R. Frigg and S. Hartmann, “Models in Science”, The Stanford Encyclopedia of Philosophy (Spring 2017 Edition), E. N. Zalta (ed.), p. 11;
On-line at: https://plato.stanford.edu/archives/spr2017/entries/models-science/.

involved, in order both to give a precise meaning to the notion of a model for a scientific theory, and to discuss the proper and the improper use of such models. In the hierarchical structure delineated by Braithwaite, propositions of both the theory and the model are correlated via a one-to-one correspondence. Or to put this in other words, propositions which are logical consequences of propositions of the theory have correlates in the model which are logical consequences for the correlates in the model of these latter propositions in the theory, and *vice versa*. However, the theory and the model have different epistemological structures, inasmuch as, in the model, the prior logical premises determine the meaning of the terms that occur in the representation in the calculus of the conclusions; whereas in the theory, the logically posterior consequences determine the meaning of the theoretical terms that occur in the representation in the calculus of the premises. In order to clarify this point, Braithwaite uses the metaphor of the zip-fastener: the calculus is attached to the theory at the bottom, and the zip-fastener moves upwards; the calculus is attached to the model at the top, and the zip-fastener moves downwards. According to Braithwaite,[39] to think about a scientific theory by thinking about a model represents an alternative to thinking about the theory by explicitly thinking about the calculus that represents it. For to think about the model is to think about an interpretation of the calculus which functions with respect to the order in which the interpretation is effected, as well as with respect to the order of deduction in the same direction as the order-deriving formulae within the theory. The model is a straightforward interpretation of the calculus, and has its own advantages because it avoids the complications and difficulties involved in having to think explicitly about the language or other forms of symbolism by which the theory is

[39] In this respect Braithwaite refers to Hertz, who says that "we make for ourselves internal pictures or symbols of external objects, and we make them of such a kind that the necessary consequences in thought of the pictures are always the pictures of the necessary consequences in nature of the object pictured [...] when on the basis of our accumulated previous experience we have succeeded in constructing pictures with the desired properties, we can quickly derive by means of them, as by means of models, the consequences which in the external world would only occur in the course of a long period of time or as a result of our own intervention." This quotation comes from Hertz 1894, the English trans. of which dates back to 1899, p. 1.

represented.[40] Braithwaite, following in Hertz's footsteps, stood among the advocates of the idea that the correctness or incorrectness of images – or models, which are a product of our mind – are both contained in the results of experience, which is in turn used to build up the images itself. In other words, the correctness of the model can be tested from this perspective only *a posteriori* to an empirical investigation.

The philosophers of science in the 1960s do not desist in thinking in terms of models, and the reason for the decline of the Received View relies on the fact that the advocates of the opposite Semantic View asked themselves the question, *Why should we imprison mathematics and mathematical scientific theory in a syntactically defined language, when we could directly investigate the mathematical objects, relations, and functions of scientific theories?*[41] Moreover, according to the semanticists, the main gap in the syntacticists' thought was that they did not consider methods of data acquisition, experiments, and measurements to be philosophically interesting. To state this point more exactly, Frederick Suppe[42] summarized the main lines of the Positivistic Received View's failure as follows:

1) its observational-theoretical distinction is untenable;
2) correspondence rules constitute a heterogeneous confusion of relationships of meaning, experimental design, measurement, and causal relationships, some of which are not properly parts of theories;
3) the notion of partial interpretation that is associated with more liberal correspondence rules is incoherent;
4) theories are non-axiomatic systems;
5) symbolic logic is an inappropriate formalism;
6) theories are non-linguistic entities and thus theories are individuated incorrectly.

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[40] Braithwaite 1968, pp. 88-93.
[41] On which see further Suppe 2000, pp. 102-115.
[42] *Ibid*.

Therefore, the main objection which attracted them towards the Carnapian stance was that, although they had already reached an extremely high level in syntactical interpretation, its semantic account "was a mere impressionistic story, not yet described with any formal rigor."[43] Indeed, the only linguistic component in this account that received a direct semantic interpretation was – according to Suppe and the semanticists in general – the Observational Vocabulary $O_s$. Hence, according to Suppe the main limit of the syntacticists was that they usually adjusted their logical analysis according to the observable reality they wanted to express by means of the theoretical language's components. However, Fano has expressed the idea that the Received View still represents the best approach we have for understanding the empirical meaning of theoretical models, insofar as it plays a normative role in defining objectively which scientific concept (or concepts) we – as users – should assimilate.[44]

However, for those who supported the Semantic View, the basic question remained unchanged: *Which mathematical tools* – currently called models – *are actually used in science?*

Broadly speaking, the Semantic View "identifies theories with certain kinds of abstract theory structures, such as configurated state space, standing in mapping relations to phenomena. Theory structures and phenomena are referents of linguistic theory-formulations."[45] According to the semanticist, the idea is that the whole set of a theory has a structure identifiable with suitably connected families of models. Some proponents of the Semantic View move the meta-mathematical apparatus from predicate logic to set theory. Others insist that the structure of scientific theory has to be purely mathematical in nature. The main points of agreement have been the following: i) that models are non-linguistic entities in the sense specified earlier; and ii) that, for an empirical understanding of the world, we should analyse theory structures by employing models rather than predicate logic. The semanticist's efforts mainly consist in analysing

[43] Lutz 2014, p. 8.
[44] Fano 2005, pp. 71-72.
[45] Suppe 2000, p. 105.

the mathematical structures, functions, objects and relations at the very basis of any scientific theory.

To state this in greater detail, as we have already said in the previous section, the Semantic View comes in three formulations: i) Suppes (1962), ii) Van Fraassen (1967 and 1970) and iii) Giere (1988). All of these formulations aim at understanding and promoting a plausible theory in reply to the following problem: *How do we connect theory and data by means of observation and experimental and measuring techniques?* Each author attempts this task by using different strategies of characterising and comprehending a theory's structure from a semanticist point of view, respectively the *set/model-theoretical* approach, the *state-space* approach, and the *pragmatic and cognitive* approach. Let us survey what was accomplished by each of these approaches.

On the one hand, the set-model theoretical approach derived predominantly from the work of Tarski and was later articulated by Suppes and his associates.[46] Set theory denotes a general language for formalizing mathematical structures as a cluster or collection of abstract objects. Model theory investigates the relations between formal axioms, theorems, and laws of a particular theory, as well as models (i.e. mathematical structures) which provide an interpretation of that theory, or otherwise which make theory's axioms, theorems and laws true.

With regard to the second program, there are a few items that deserve our attention, which can help us to understand 'why' and 'how' the Syntactic View was eventually replaced. Firstly, the semantic conception abandoned the corresponding rule in favour of an enquiry into the epistemology of experimental design, data analysis, instrumentation, and calibration. Secondly, the goal of the semanticists was to map the relations between a configured state-space and a system within the theory's scope. Thirdly, they investigated the semantic relations between theories, their formulations, and the reality without making any excessive ontological commitments. In this way, a leading semanticist such as van Fraassen could argue in favour of the idea that not all the elements constituting a theory have a direct relation with an ontological state of

[46] van Fraassen 1989, p. 67.

affairs in physical reality. With regard to this point, van Fraassen in 1967 undertook to find a solution by using a method which he labelled ‘semi-interpreted languages’, according to which “languages are interpreted as referring to logical spaces providing full semantic interpretations for theorising languages. Ontological commitments are left unconstrained, being a matter of which logical-space points one wishes ontologically to commit.”[47] Fourthly, the semantic view generated the so-called realism/antirealism controversy.[48] Fifthly, thanks to the semantic view, a better understanding of modality in science has now been reached. Thus, according to the proponents of the semantic view, scientific theory can be reduced to meta-mathematical language. The state-space approach emphasises the mathematical models of actual science, and draws a clear line between mathematics and meta-mathematics. For this reason, the structure of a scientific theory is identified with a cluster of mathematical models which constitute it, rather than with any meta-mathematical axioms which are “yoked to a particular syntax”.[49] To be precise, a state-space is defined as a *N*-dimensional space, where each of the variables of a theory corresponds to a single dimension and *each point in that space represents a possible state* of a real system. An actual, real system can take on, and change, states according to different kinds of laws, namely the *laws of succession*, which determine possible trajectories through that space the *laws of co-existence*, which specify the permitted regions of the total space; and the *laws of interaction*, which combine multiple laws either of succession or of co-existence, or of both. For the advocates of the state-space approach, meta-mathematics has to be considered as a part of mathematics.

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[47] Suppe 2000, p. 106.

[48] The realist/antirealist debate concerns the nature of the relationship between theory structures and the world, and it involves ontological commitments associated with theories. The realist account claims that some sorts of similarities exist between models and phenomena in the real world; moreover, those similarities are restricted only to the observable aspects of the world. More precisely, the realist stance identifies *Loc* functions onto every state variables, and maintains that a theory is *empirically* true only in the case that a theory structure allows state transitions to be identical to those possibly occurring in the actual world. There has to be a direct relation between the *Loc* functions and the observables. Moreover, the semantic theory requires the compliance that a theory has to be empirically adequate: “If W is that portion of reality to which one attaches Loc functions, the image M* of W is among the models comprising the theory.” See van Fraassen 1980, 1989 and Suppe 1989, 1998.

[49] In the words of van Fraassen 1989, p. 366.

Thirdly, Giere's account also emphasises the non-linguistic character of models, but it differs in construing them in slightly less abstract terms. From his point of view, the idealized systems described in mechanics texts, such as the simple harmonic oscillator, constitute a model. As such the model perfectly satisfies the equations of motion for the oscillator in the way that the logicians' model satisfies the axioms of a theory. Models come in varying degrees of abstraction: for example, the simple harmonic oscillator has only a linear restoring force, while the damped oscillator incorporates both a restoring and a damping force. These models function as *representations* in "one of the more general senses now current in cognitive psychology".[50]

Each of these approaches aims at analysing the way in which theories posit a relationship with the physical world, or how the theory fulfills its epistemological commitments: How do we face the problem of explaining the relationship between models and theories? With regard to this demand, three types of analysis of theory interpretation deserve our attention: i) *a hierarchy of models*, which is mainly supported by Patrick Suppes[51] and Frederick Suppe[52]; ii) the *isomorphistic* approach, which is advocated by Bas van Fraassen[53], Steven French and James Ladyman[54] and iii) the *similarity* approach, which is promoted by Ronald Giere[55] and Michael Weisberg.[56] Here I will only point out the main advantages of each interpretation, without entering deeply into this debate.

The hierarchy of models presents an internal structure which includes axioms and models of theory, models of experiments, and models of data. The similarity analysis of theory interpretation combines both semantic and pragmatic dimensions.[57] In particular, Giere argues that the interpretation of a theory is mediated by theoretical hypotheses

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[50] Giere 1988, p. 80.
[51] Suppes 1962, pp. 252-261 [reprinted in 1969, pp. 24-35].
[52] Suppe 1977.
[53] van Fraassen, 1980 and 1989.
[54] French and Ladyman, 1997, 1999, 2003.
[55] Giere 1988; *Id*., 2004, pp. 742-752; and Giere, Bickle, and Mauldin 2006.
[56] Weisberg 2013.
[57] Giere 1988; *Id*., 2004; *Id.,* 2010; Giere, Bickle, and Mauldin 2006; and Weisberg 2013.

which posit representational relations between a model and a relevant part of the physical system. Such a relation may be stated as follows: *S*, as a scientist, uses *X*, as a model-vehicle, to represent *W*, as a piece of the world, for a certain purpose, *P*. This approach also takes into account the cognitive role performed by the research group: what the scientist writes and says, the different aspects of her scientific epistemology, the historical developments, purely theoretical features, as well as sociological and cultural attributes. Finally, the isomorphistic-theoretical interpretation aims to give a one-to-one bi-jective mapping between two structures or sets, that is between observable phenomena and empirical substructures, which are *themselves* isomorphic with one or more theoretical models.

Finally, the Pragmatic View outlines the idea that a scientific theory has both an internal and an external complex structure: mathematical components, while often present, are neither necessary nor sufficient for characterising the core structure of scientific theories. A theory also consists of a rich variety of non-formal components such as analogies and natural kinds. Thus, the pragmatists argue that a proper analysis of the grammar – or syntax – and of the meaning – or semantics – of theories must pay attention to the complexity of scientific theories, as well as to the various assumptions, purposes, values, and practices which inform them. A central question for adherents of this view is: *Which theoretical components and which models of theorising are present in scientific theories, as they are found across a variety of disciplines?* Following here the lead of Nancy Cartwright, models are held to present the appropriate level of investigation for philosophers who are trying to understand science. Cartwright also claims that the laws of nature hardly ever turn out to be true – as if they were semantic objects – and, moreover, that these laws are considered to be epistemically weak: "to explain a phenomenon is to find a model that fits it into the basic framework of the theory and that thus allows us to derive analogues for the messy and complicated phenomenological laws which are true of it."[58] According to the pragmatist's position, in

[58] Cartwright 1983, p. 152.

order to have a better understanding of the physical reality, we should focus on models rather than theories, on model practice or on the process of model building.

For these reasons, they consider models as mediators between the theory and the world:

> […] [M]ediation models always stand between theory and the physical world. Their main function is to enable us to apply scientific theories to natural phenomena. A mediation model often involves a novel conception of a particular physical phenomenon that facilitates the application of some established physical theory to such a phenomenon.[59]

In her *Mediating Models,* Margaret Morrison has identified three main stipulations of this mediating theory: i) the construction of models is not theory-driven; ii) the model does not necessitate a physical counterpart in the domain of empirical data; and iii) mediating models differ from other types of models because they can *replace* the physical system under consideration as the central objects of scientific enquiry.[60]

Then, over the years and especially since the 1960s onwards, an increased interest has been registered in models, model-building practice, and a whole series of connected theoretical, ontological and metaphysical issues.

While the purpose of this opening section was to give a brief overview of the different approaches applied to the structure of scientific theories, in the next section we will adhere to the model-building practice, focusing on models rather than theories. We have seen that the problem of scientific theory structure is a rich and yielding topic, because, although these three views seems to be in competition with one another and to be mutually exclusive, they indeed aim at the same broad issue: to *give a plausible description of the world. They simply try to focus on different aspects and questions relating to the structure of scientific theories*. In this respect, their goals can be considered complementary and overlapping.

For my purpose relating to scientific practice, it seems that only Giere and

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[59] Suárez 1999, p. 169.
[60] See further Morgan and Morrison 1999 and Suárez 2016.

Cartwright have a clear understanding of what should be the epistemic role of models. Specifically, according to Giere, the axiomatic account fails to capture (let us say for example) the correct structure of classical mechanics. In other words, general laws of physics like Newton's laws or the Schrodinger equation are not descriptions of real systems, but rather part of the characterisation of models, which can in turn represent different kinds of real systems. But a law such as $F=ma$ does not by itself define a model of anything; we need in addition specific force functions, boundary conditions, approximations, and so on. Only when these conditions are added – by the user – can a model be compared with a real system.[61]

## 1.2 Models and Model-Building Practice

"Science represents the phenomena, and it does so by providing a mathematical form."[62] Models are of central importance in theoretical science for practical purposes, in order to make predictions and explanations. They are – broadly speaking – abstract, idealized, approximated or simplified physical tools, or, as they have been otherwise defined, they are surrogate structures that can potentially represent the variety of real-world phenomena. Many different things can be employed in model building, such as physically constructed scale models (which can also be called heuristic and mechanical models), model organisms, and mathematical objects such as sets of trajectories through a state-space. In addition to model building, some other questions merit further investigation: i) Are models "autonomous agents"?[63] ii) Are they complements of theories? iii) Should models be constructed prior to theories? iv) Are they useful to adopt when theories become too complex to handle? and v) Do models yield results to problems in which theories remain silent?[64] Model building is sometimes considered an art and not a mechanical or mathematical procedure, insofar as in model-building, we

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[61] Giere 1988.
[62] van Fraassen 2004, p. 1.
[63] For this question, cf. further Morgan and Morrison 1999, pp. 38-65.
[64] *Ibid*.

can recognise at least two different moments: firstly, on the basis of pure observation, one can build up by means of a bottom-up approach a model having general features; and secondly after this, one can integrate the model in the theory. Only upon the completion of this second phase does the theory produce new models with a reverse approach, i.e. top-down.

As we have seen in the previous section, in the first half of the last century, physical reality was interpreted mostly by giving attention to the structure of scientific theories. But in the last fifty years, philosophers of science have become newly devoted in their efforts in this new direction. Despite the abundance of questions and models seen in the approaches above, there is something on which scientists can unanimously agree, namely that the purpose of models is the same: *they are new theoretical tools used to build knowledge on physical phenomena, and are considered to be one of the leading instruments of science*. Some well-known examples of this purpose include the billiard ball model of gas, the Bohr model of atom, the Lotka-Volterra model of predator-prey interaction, the double helix model of DNA, and so on. Models consist of a compound of elements belonging to different ontological categories and deriving from an overlapping of different procedures.

In 1960 Patrick Suppes published *A Comparison of the Meaning and Uses of Models in Mathematics and the Empirical Sciences*, whose aim was to point out that the meaning and the use of models can be interpreted as being the same in the empirical sciences as it was in mathematics, and, more precisely, as it was in mathematical logic. It may be that it is impossible to place the several uses of the world 'model' under a single concept, as exhibited by the quotations given by Suppes in his essay. However, according to the author, one common pre-existing tendency is "to confuse or to amalgamate what logicians would call the model and the theory of the model."[65] Not only this, but the lack of a homogeneous interpretation of the nature and role of models in science also entails that it would be a challenge to argue that all occurrences of the term 'model' have exactly the same sense. Our aim, therefore, is first of all to provide a

[65] Suppes 1962, p. 3.

clear and thorough conceptual definition of the point mass, and thereafter to clarify to what extent we consider it to be a model for epistemic representation. Then, in Chapters 2, 3 and 4, we will be able to show that the theoretical notion of the point mass corresponds to what we call an idealized model, i.e. idealized in the sense that it is a geometrical point having 0-dimension, to which we can ascribe certain physical features (e.g. mass and forces acting upon it), and which itself derives from a process of abstraction as a result of the scientific practice. Thus, by looking at the use that the Greeks made of the geometrical centre of gravity for heuristic purposes, and the use to which Renaissance mathematicians put the centre of gravity for mechanical and practical aims, we will discover that in both Newton's and Euler's rational mechanics, the centre of gravity fulfils the role of the point mass as a model for representing not only simple physical and moving bodies but also more complex physical states of affairs.

Hence, despite this heterogeneity of approaches highlighted by Suppes, the essential purpose of models remains unchanged: *they are vehicles for learning about the world*. Significant parts of scientific investigations are carried out through models rather than through reality itself, because, by studying a model, we can discover features of, and ascertain facts about, the system for which the model stands. In other words, models allow for *surrogative* reasoning.[66] Moreover, there are neither any fixed and exact rules or recipes for model building, nor any singular activity of figuring out what fits together and how an opportunity presents itself to learn about the model. However, by following this commitment to the model-building practice, and by emphasising the point laid down by Gabriele Contessa, I will hereafter argue in favour of the idea that once the model is built, we no longer learn about the properties of reality in itself, but rather we have to use and manipulate the model in order to elicit its secrets and discover

[66] See further Swoyer 1991, pp. 449-508. In Contessa [2007, p. 51] we find the following definition: "[S]urrogative reasoning is the expression introduced by Chris Swoyer (1991) to designate those cases in which someone uses one object, the *vehicle* of representation, to learn about some other object, the *target* of representation. A good example of a piece of surrogative reasoning is the case in which someone uses a map of the London Underground to find out how to get from one station on the London Underground network to another."

something new about ontology[67] and the realistic commitments relating to the physical world.[68]

Now, the several different categories of models which are currently in circulation are not mutually exclusive, since, for instance, some idealized models can also be qualified as abstract models. In recent years many authors have attempted to find a strict criterion by which to draw a dividing line between idealization, abstraction, approximation, and so on. Indeed, in what follows I will define the different kinds of models, and our efforts will focus not on producing a regimented set of these definitions on which these philosophers of science do not stand in unison, but rather on promoting an overview of the modelling techniques in order to support our aims of developing a systematic account of scientific practice. In fact, in order to show the scientific practice that underlies the conceptual development of the point mass model, the scheme presented in the next three chapters captures a significant part of the scientific practice of mechanics. From ancient Archimedean geometry, which is characterised by an heuristic and mechanical stance rather than a synthetic Euclidean approach, we will pass through the Renaissance equilibrium controversy, the chief aim of which was to regiment the foundational principles of statics by focusing on the operative principles of machines, to conclude with modern Newtonian mathematical physics, characterised as it is by the application of purely mathematical methods to problems (esp. problems of a kinematic and dynamic nature) in physics. In this way, the epistemological debate will come to the fore concerning the model-building practice together with the epistemological meaning we attribute to such mathematical entities as the point mass as a model of representation.

The differences in meaning among the variety of models arises from a range of different questions, both semantic (i.e. what is the representational function that models perform?), ontological (i.e. what kind of things are models?) and epistemological (i.e. how do we learn and gain knowledge from models?) in nature. We must also encounter

[67] The problem of the ontology of models has been widely discussed: see recently in particular Contessa 2007 and 2010.

[68] Among other advocates of this idea, see also Giere 1988.

at this stage more general claims, such as the kinds of connections which stand between models and reality.

In order to reply to these challenges and show the variety of model-building practices, let us now turn to see the main types of models that are exhibited in the scientific literature.

*Idealizations*

Firstly, we have *idealized models*, sometimes called 'theoretical models', which are built with the help of theoretical principles. Ronald Giere suggests that we take as an example of a theoretical model the use of the term 'body' introduced by Newton, when he asked whether this term referred to empirical objects, such as cannon balls and planets, or to abstract objects. Now, these objects satisfy the laws of motion, the laws of universal gravitation, and so on. Moreover, the generalization of the laws allows us to say that these objects function as general objects and are built on the basis of common elements in the target system of the real world. For example, Newton's bodies are considered as point masses. No real objects can be *point masses*. Any real classical objects with mass must be somewhat extended. So in order to apply Newtonian models to real objects, one must treat their mass as being concentrated at their centre of mass, which is ideally a geometrical point with 0-dimension. This also supports the interpretation of Newton's laws as defining idealized abstract objects, rather than as describing real objects.[69]

The main object of study in this approach concerns the Galilean idealizations, which are those that involve "deliberate distortions".[70] Physicists build models consisting of point masses – as in the case of Newtonian bodies – before moving on frictionless planes, while economists assume that agents are omniscient, biologists study

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[69] Giere 1999, p. 50. This line of argument is indeed the one that we will follow in the rest of this dissertation.

[70] This deliberative operation seems to evoke Giere's and Contessa's attempts to define a model as the product of a triadic relation that involves the idealized model, a user and the modelled system. However, in the literature that refers to Galilean idealization, there are no traces of this cognitive or interpretational approach.

isolated populations, and so on. It was a characteristic of Galileo's approach to science – as we will see extensively in Chapter 4 – to use simplifications of this sort, in the cases in which a real situation was too complicated to tackle it.

Idealization can be defined as "a real or fictitious idealizing system, distinct from the target system, whose properties provide an inexact description of the target system".[71] Idealization usually *neglects* pragmatic considerations which scientists judge to be relevant or essential for the simplicity of the description or the "intelligibility of the idealizing system". According to Martin Jones, idealization requires first and foremost the misrepresentation of the physical system. However, since not all misrepresentations can properly be called idealizations, we should find a way to codify the use of this term by supplying some necessary conditions that can help us to recognise to what extent an idealization differs from an abstraction or an approximation.

Jones gives an example to argue in favour of a distinction between idealization and abstraction. Imagine that there is a certain cannon that is wheeled onto an open plain and fired. In the attempt to 'predict' where the cannonball will land, or perhaps alternatively to 'explain' why it lands where it does, we might construct a model of the system along the following lines. We assume that the path of the cannonball can be contained within a Cartesian system, which takes as its *x-axis* the level along the ground and has its *y-axis* pointing vertically upwards. In our assumption, we also suppose that the cannonball moves solely under the influence of gravity, which exerts a force vertically downwards with a magnitude of *mg* throughout the motion. By following a very straightforward calculation on the basis of Newton's second law, we can calculate the time of flight of the cannonball and the distance from the cannon at which it will land. The model delineated by Jones contains several idealizations; it actually neglects all the forces which act on the cannonball except the earth's gravitational effect, whereas in reality other massive bodies, such as the moon and the sun, exert a force on

[71] Norton 2012, p. 3. Norton specifies that this is not a definition of the process of idealization, and that actually very few theorists seem to be interested in this topic. However, as Norton himself says, idealization (as well as approximation) is considered to be something that specifies important properties of the target we want to model. This said, it neglects to identify, for example, how inexact a description may become, before we cease to admit it as an approximation of some target system.

the cannonball. Thus, we here spot "discrepancies between the way the model in question represents the model[l]ed system as being, and the way the system really is."[72] The misrepresentation given by the idealized system is generally related to some features which the system has, which the model in turn represents it as *not* having, or rather some features that the system does not have, which the model represents it as having. For example, a point mass is in this respect an idealization, because we ascribe to a simple geometrical point some physical features in order to confer upon it a representational function. This suggests that a misrepresentation can be considered an approximation to the truth,[73] but idealization aims to be a simplification of the modelled system, and from simplicity derives tractability. By tractability, we mean that the model has to be a misrepresentation of some of the relevant features of the reality. These relevant features are ones with an explanatory or predictive power. The aim of a model is to have a high level of applicability, with the result that the less complex the model is, the more applicable it will be. To explain this in greater detail, if a model has a high range of application, it means that it has a very general structure, which in turn allows the scientist to use it in order to represent or adapt it for prediction on a greater number of different phenomena.

*Abstractions*

Abstractions – sometimes labelled approximations – are simplifying assumptions which aim at introducing a numerical, but not an ontological, distortion. An approximation does not introduce any distortions which seriously misrepresent the kind of thing the system is[74]; in this respect, it can be defined as the omission of truth. In any particular physical object, there are innumerable features that the abstract model omits, without misrepresenting or distorting the specific part of its reality we want to represent. To go

[72] Jones 2005, p. 182.
[73] Since a model is understood as a symbolic system or a representation of a certain physical system, we cannot attribute to it a truth value as if it were a linguistic entity. A symbol of this kind is neither true nor false, since its value is determined by its usefulness; the approximate truth of a symbol embodies an idea such as its being "reliable within certain limits for certain purposes". See Duhem 1977.
[74] Teller 1979, p. 349.

back to the example of the cannonball, there is no mention of the material composition, the internal and external features of either the ball or the cannon, no information about the colour or temperature of the ball, or about the environment in which the experiment is carried out. The model is simply silent in all these respects. Or, as Jones says, "a model of a particular system involves an abstraction in a particular respect only when it omits some features of the model[l]ed system without representing the system as lacking that features."[75]

Again, there are some specific necessary – but not sufficient – conditions that allow us to elect a model to serve as an abstraction of a certain physical system. The abstraction usually concerns the relationship between the model and the *actual* features of the modelled system, not the relationship between the model and the holistic features we take the system to have. These aspects persuade us to think that the abstraction's conditions are arbitrarily defined. Thus, whether or not an abstraction approximates the truth is a rather trivial point, because when a model is said to contain an abstraction with respect to a certain set of features, it is entirely silent on its either having or lacking that particular set of features. Moreover, it is also worth noting that any omission of features related to the internal or external structure of the physical system entails a simplification, which thus would not help us to define a simplification as an abstraction. Given that these conditions are necessary but not sufficient to define a model as something abstract, it seems unlikely to find a way of regimenting the dividing line between the abstract and the other types of model-building techinques. Even by looking at the role of relevance, or the use of abstractions – and models in general – in scientific practice, as Giere would suggest, this attempt cannot be performed. Following McMullin and Cartwright[76], it rather seems that a model will contain abstractions only with respect to features which we deem irrelevant for the purpose at hand; they both suggest that in constructing a model we can make assumptions about the presence or absence of a certain set of relevant features of the modelled system. In the modelled system of the cannonball as mentioned above, we cannot postulate the absence of air

[75] Jones 2005, pp. 184-185.
[76] McMullin 1985 and Cartwright 1983.

resistance, but we might be said to have constructed that model by 'assuming' the omission of air resistance, simply because air resistance is the gravitational force that the earth exerts on the cannonball which is taken into account in Newton's second law, a law which relates mass and acceleration to the total force acting on a particular body.

Idealizations and abstractions are not mutually exclusive. Indeed it is sometimes the case that they often come together, an issue to which we shall return presently before moving on the next section.

*Analogies*

After this, one further category of models are the *analogical models*. These are, for example, the well-known billiard ball model of a gas, or the computer model of the mind, and the liquid drop model of the nucleus. Historically speaking, the first thinker who gave an account of model as analogies was the English physicist N. R. Campbell, in his book *Physics, the Elements* (1920). His encouragement to take models and model building seriously was based on two main points. First, he suggests that "we require to be intellectually satisfied by a theory if it is to be an explanation of phenomena, and this satisfaction implies that the theory has an intelligible interpretation in terms of a model, as well as having mere mathematical intelligibility and perhaps the formal characteristics of simplicity and economy".[77] The second point regards the dynamic character of theories. A theory in its scientific context is not a static museum piece, but is rather always being extended and modified to account for new phenomena. Thereby, he concludes that:

> [...] [A]nalogies are not "aids" to the establishment of theories; they are an utterly essential part of theories, without which theories would be completely valueless and unworthy of the name. It is often suggested that the analogy leads to the formulation of the theory, but that once the theory is formulated the analogy has served its purpose and may be removed or forgotten. Such a suggestion is absolutely false and perniciously misleading.[78]

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[77] This quotation is taken from Hesse 1963, p. 4.

[78] Campbell 1920, p. 129. In particular Campbell developed his approach in the attempt to deal with those theorists who advocated the idea that *models were mere aids to theory-construction that can be easily*

This approach, which was mainly restricted to mechanical models, was augmented from a theoretical viewpoint in the 1960s by Mary B. Hesse,[79] who distinguished between different types of analogies on the basis of the kinds of similarity-relations into which two objects entered. One of the first types of analogy is based on the notion of shared properties. Yet the sameness of properties does not constitute a necessary condition for representing the distinction among physical bodies or objects. Secondly, an analogy between two objects can also be based on relevant similarities between their properties. For example, with regard to this second "more liberal" sense, we can say that there is an analogy between sound and light,[80] because echoes are similar to reflections, loudness is similar to brightness, pitch to colour, detectability by the ear to detectability by the eye, and so on. Thirdly, analogies can also be based on the resemblance of relations between parts of two systems rather than on their monadic properties.

According to Hesse's theory of building analogies, we shall find two sorts of common features, or 'dyadic relations', either between two analogies or between an analogy and its target system. These common features are: i) *horizontal relations,* or one-to-one relations of either identity or difference between a property of one of the analogues and a corresponding property of the other (i.e. or, as we would say, of the target system); and, ii) *vertical relations*, or the relations between properties of the same analogue which are also properties of the same object, together with causal relations between these properties.

By pointing out these types of analogies (which can themselves be differentiated between positive, neutral and negative types), we can establish these relations securely. Moreover, in virtue of the recognition of positive and neutral analogies, we can on the

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*thrown away when the theory has been developed*. The main supporter of this idea is Duhem, who in 1914 published his *La Théorie Physique*, in which he distinguished two kinds of theories in physics: i) systematic theories and ii) theories using familiar mechanical models. Both these models should be interpreted as mechanical gadgets – having a purely heuristic value – or as psychological aids in order to build theories. So Duhem's main objection to models was that they are incoherent and superficial and tended to distract the mind from the search for logical order. In this respect, Campbell aimed at maintaining that without models theories cannot fulfil all the functions traditionally required of them, and in particular that they cannot be genuinely predictive. For more analysis, see Hesse 1963, pp. 3-5.

[79] *Ibid*.

[80] For a complete description of the example given by Hesse, see Hesse 1963, pp. 10-14, 59-61.

one hand make new predictions about the target system – or the modelled system – under examination, and on the other hand reach a total interpretation of a deductive system.[81]

Although it seems that we can give a precise definition of which model is which, we cannot argue in favour of an accurate regimentation of the different uses of these models or the different processes of model building, because while some of the features of the target system will be straightforwardly omitted in the model, others will be simplified in order to build that model. Therefore, it follows that with regard to *a particular feature* of a certain real state of affairs, a given representation can contain either an idealization or an abstraction, or neither, but it cannot contain both. This could be so, but only if the features that undergo the abstraction on the one hand are different from the features that undergo the idealization on the other hand, in such a way that some of the features are abstracted, while some other features are idealized in the final model obtained.

From this it derives that most of the models commonly used in scientific practice capture *both* part of the meaning of the idealization procedure *and* part of the abstraction, and so on. As clearly stated in the 1960s by Suppes, the theoretical notion and the practical use of a model are given by a cluster of model-building practices.[82] There are so many types of model that a single account probably cannot do justice to them all. A precise organisation of these competing types of model seems thus difficult to pin down.

## 1.3 One Model for many purposes

Scientific representation[83] is a blooming field of study which has only received

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[81] *Ibid.,* pp. 58-64.
[82] Besides Suppes 1962, see also Jones 2005, pp. 173-217.
[83] See further Suárez 2016, p. 4, who maintains: "[t]here are many different types of representations in the

considerable attention in the last twenty years. After a peak of interest in representation throughout the 1960s, attention on models and representation waned to a large extent, only to re-emerge at the end of the 1990s with the movement known as the 'mediating models movement'.[84] The ordinary meaning of the word 'model' suggests that theoretical models are intended to be not merely exemplars to be used in the construction of other scientific structures such as laws and theories, but above all aim to be models *of something*. That is to say, they function on the one hand as descriptions and representations, and on the other hand as tools with predictive and explanatory power.

Whether or not almost all the current branches of the philosophy of science agree with the characterisation of science as an activity striving at representing parts of the physical world with the aid of scientific models, the scientific community strongly disagrees on two factors: i) what exactly should be the role performed by the model in representing phenomena?; and moreover ii) whether or not the model system is something that actually exists in nature as an independent spatio-temporal entity. Every scientist offers us a model which aims to be suitable not only for the enquiry at hand, but indeed to be sufficiently wide-ranging to be employed for a large variety of physical situations which possess in some respect a certain degree of 'similarity'. Moreover, as my example of the previous section show, every model leaves out a series of unspecified features which have been deemed by the user to be irrelevant for the purpose at hand. However, this does not change the fact that these features could still be added back in for a later enquiry and a new purpose.

For this reason, we remain unsure about the possibility of undertaking a full-scale regimentation of the differences between the purposes that a model should fulfil. Indeed it is not my task to point out a suitable description of a model or model-building

sciences, in areas as diverse as engineering, mathematical physics, evolutionary biology, physical chemistry, economics. Modelling techniques in these areas also vary greatly as do the typical means for a successful application of a model. This is *prima facie* a thorny issue for a theory of representation, which must provide some account of what all these representations have in common."

[84] This was a movement of scholars based at the London School of Economics, the Tinbergen Institute and the *Wissenschaftskolleg* in Berlin, who developed and advanced a case for models and their role in scientific enquiries during the 1990s. See now Suárez 2016.

practice, which either perfectly predicts phenomena or perfectly explains or represents a certain system of reality. Even if we were to find out that models simply function as representations, they cannot be accounted as an epistemically faithful representation of phenomena. The claim of this section is to provide an overview of the "powers" that *are* fulfilled by models. Whether or not a particular idealized or abstract model performs a specific role in predicting or explaining a phenomenon is not the whole story; the model – either idealized or abstract – is only a *part* of the account of our explanation and prediction, and is not in itself a faithful representation of the system of reality. The model of a certain system is rather an epistemic representation of reality, insofar as it allows the user, by following a certain set of rules, to infer some conclusions and acquire some knowledge concerning that particular system of reality. This denotes knowledge that has to be "translated"[85] before it is judged in some respect. The model is first built in such a way that "it is easier to study than the target-system and therefore allows us to derive results. Second, it is assumed to represent its target system, and representation is something like a *licence to draw inferences*. Representation allows us to 'carry over' results obtained in the model to the target-system and hence it enables us to learn something about that system by studying the model."[86] Every user aims at building a representational model which stands for a certain aspect of reality and which should help her in her reasoning and mathematical computations. These representations play such a crucial role in science because they can *stand for* phenomena which may be difficult to observe, understand and manipulate. Thereby they allow the user to draw a reasoned conclusion concerning the simplified model instead of the complex real state of affairs. In this sense, reasoning, inferences and conclusions are generally directly related to the model as a vehicle, instead of the phenomenon that the model represents. The conclusive inferences can thus be studied independently from the empirical investigations.

[85] In order to understand what it means to "translate" knowledge from the model, Swoyer [1991, p. 487] offers the following example: "for instance, to our translating facts involving lengths into their numerical surrogates, engaging in mathematical reasoning, then making the return trip to a conclusion about our original object."
[86] Frigg 2010, p. 98.

The first item on this section's agenda is to ensure clarity concerning the difference between description and representation on the one hand, and denotation and representation on the other hand. First, by following Suárez, we can say that "a critical difference between description and representation concerns the applicability of semantic notions such as truth, which are built into descriptions but seem *prima facie* ill-suited for representations.[87] In focusing on the role that language plays in science, the logical empiricists and their successors may thus have implicitly privileged theoretical description."[88] Second, the meaning of 'denotation' in this context can be explained by means of a simple example: the circular red white and blue logo of the London underground is simply a sign that denotes every London Underground station; but the map represents the entire London underground network. The map is simply a model used to provide a representation on a different scale from that of the real network. The value of the model as a representation of a phenomenon in the physical world was emphasised respectively by van Fraassen's[89] version of the semantic view in terms of state-space, Ronald Giere's theorisation[90] in terms of cognitive models and most recently by Gabriele Contessa. All these authors maintain that models are suitable tools for representation rather than simply instruments of description or denotation. Historically speaking, the most important route to this notion of representation can be traced back to the tradition of the so-called 'modelling attitude'. This represents the efforts of philosophers and scientists to understand the practical role of model building, or in other words, the role that images, metaphors and diagrams play in modelling.[91]

In contrast, contemporary discussion on representation emerges from two distinct currents of thought: the semantic approach and the 'mediating models' movement.

[87] This point is well emphasised in Giere 1988, ch. 4.
[88] Suárez 2016, p. 2.
[89] van Fraassen 1980, ch. 3.
[90] Giere 1988, ch. 3.
[91] We have already pointed out in the previous chapter that the beginning of the tradition relating to models can be traced back to works of scientists such as Thomson, Maxwell, Hertz, and Boltzmann. I found in Hertz's *The Principles of Mechanics presented in a new form* an interesting reflection on the role performed by images in science. I think this historical aspect on the development of the concept of model as representation deserves further independent scrutiny.

Although some supporters of the 'mediating models' movement reject the semantic view, mainly on the account of its construal of what models are, both movements agree on the idea that models are genuinely representational and that *their predictive and explanatory rules are merely marginal*. One of the most significant works in today's scholarship is *Model and Representation*, written in 1997 by R. I. G. Hughes, which represents the result of exposure to both movements. Hughes was additionally one of the leading advocates of the semantic view, before going on to become a prominent contributor to the mediating models movement.[92]

To explain more clearly the main lines of the representational role fulfilled by models, let us refer to the aforementioned classical example of the billiard ball model of gases, which has been commonly recognised as a central analogy in the kinetic theory of gases since its scientific development in the second half of the nineteenth century:

> When we take a collection of billiard balls in random motion as a model for a gas, we are not asserting that billiard balls are in all respects like gas particles, for billiard balls are red or white, and hard and shiny, and we are not intending to suggest that gas molecules have these properties. We are in fact saying that gas molecules are *analogous* to billiard balls, and the relation of analogy means that there are some properties of billiard balls which are not found in molecules. Let us call those properties we know belong to billiard balls and not to molecules the *negative analogy* of the model. Motion and impact, on the other hand, are just the properties of billiard balls that we do want to ascribe to molecules in our model, and these we can call the *positive analogy* [...] There will generally be some properties of the model about which we do not yet know whether they are positive or negative analogies [...] Let us call this third set of properties the *neutral analogy*.[93]

In the first place, let us clarify some terminology used in model-building practice: the system of billiard balls is defined as the *source*, and the system of gas molecules as the *target*. We can therefore say that billiard balls *represent* gas molecules *if and only if* the system of billiard balls is a representational source for the target system of gas

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[92] Hughes 1997, pp. 325-336. According to Suárez, it is important to bear this dual heritage in mind since it goes some way towards explaining some of the inner tensions and open disagreements that one finds nowadays in this area.

[93] See further Brush 2003. The illustration quoted here is taken from Hesse 1963, p. 8.

molecules. “The extensions of ‘source’ and ‘target’ are then picked out implicitly by this claim, i.e. any pair of objects about which this claim is true is a <source, target> pair. We can then list the properties of the source object as $\{P^s_1, P^s_2, \ldots, P^s_i, \ldots, P^s_n\}$ and those of the target object as $\{P^t_1, P^t_2, \ldots, P^t_i, \ldots, P^t_n\}$.”[94] The claim is thus that there is a relation of identity between some of these properties, namely: $P^s_1=P^t_1$, $P^s_2=P^t_2$, …, $P^s_i=P^t_i$. As a result, the set made up of $\{P^t_1, P^t_2, \ldots, P^t_i\}$ contains the positive analogy between the source and the target.

On the other hand, the remaining properties are properties that are omitted, if the model in question is an abstraction, or otherwise misrepresented, if we are talking about an idealized model, and so on. Thus for instance, the set $\{P^s_1+1, \ldots, P^s_j\}$ is made up of properties that are not identical to any of the properties of the target gas molecules $\{P^t_1, P^t_2, \ldots, P^t_i, \ldots P^t_j, \ldots, P^t_n\}$. These properties of billiard balls constitute rather the negative analogy. Moreover, the other remaining properties, i.e. $\{P^s_j+1, \ldots, P^s_n\}$, constitute the neutral analogy.[95]

Whether or not the relations of morphism – i.e. isomorphism – hold, advocates of this positions presuppose that there is just a single one-to-one correspondence between the elements of the target and the elements of the source. Thus, when the user draws inferences on the target, she is trying to say something about the phenomena at hand, and she is also claiming that this information does not require any interpretation. However, at this stage we would adopt Contessa’s framework, defined as the interpretational conception for the scientific representation of models, according to which: i) “scientific representations are nothing but epistemic representations that scientists use in the pursuit of their research”; and ii) “a vehicle (or a source) is an epistemic representation of a certain target for a certain user if and only if: a) the user takes the vehicle to stand for the target and b) the user is able to perform valid

[94] Suárez 2016, p. 5.

[95] Suárez suggests that this is a purely epistemic criterion and we may suppose that all properties of billiard balls are objectively either in the positive or in the negative form of the analogy. They are either really shared by both balls and molecules, or they are not, regardless of how accurate our knowledge of these facts is.

surrogative inferences from the vehicle to the target."[96]

In this dissertation we will not analyse the cognitive or imaginative (as they have been previously designated) processes which both users and the scientific community tend to adopt in order to achieve the most suitable scientific models to be used. Rather, as we have already stated, we will look at the practice that scientists use to achieve a new theoretical layer for the model under examination, namely in our case the point mass.

To be more accurate, the billiard ball models and the map of the London Underground system mentioned in the previous paragraph are considered *epistemic representations* of the gas molecules and the underground network respectively. In support of Contessa, we can argue that, if scientific models play a crucial role in science, and if scientific models relate to the world by representing aspects or portions of it, so then by understanding how models work in representing the world will we configure a thorough understanding of how science in itself functions.

Moreover, it is in virtue of the fact that a vehicle – in our case study, the point mass – can epistemically represent a body detached from some of its physical features (e.g. the force(s) acting on it, mass and charge) that, to borrow the terminology introduced by Chris Swoyer in 1991, the theoreticians – and every competent user in general – can perform "surrogative inferences" from the target to the source. One can also infer conclusions about a physical body from considerations with the help of a point mass model.[97]

According to Contessa, the intuitive idea behind the notion of valid surrogative inferences is that an inference of this kind is valid only if it is in accordance with a systematic set of rules, as they have been defined and followed by the scientific

[96] Contessa 2007, ch. 2, p. 48.

[97] Cf. here the words of Contessa [2007, pp. 24-25]: "The fact that a user can perform a surrogative inference from a certain object, the vehicle, to another, the target system, is a symptom of the fact that, for that user, that vehicle is an epistemic representation of that target, a symptom that allows one to distinguish case of epistemic representation from cases of mere denotation. So, it is important that it is not necessary that the conclusions the user draws about the target are true in order for the vehicle to be an epistemic representation of the target. In other words, if a user is able to perform inferences from a certain vehicle to a certain target, the vehicle is an epistemic representation of that target for that user, independently of whether or not the conclusions are true of the target."

community. Thus, by following this idea, an inference is of a surrogative kind only if every user is able to translate facts about the vehicle of the point mass into *presumed and acceptable* facts about the physical body or the modelled system of reality. However, in this light we should pay attention to the fact that the content of the representational vehicle is not knowledge concerning the modelled system in itself. Or, as Contessa would say, the fact that a vehicle is an epistemic representation for many people or even for everyone does not imply that it is an epistemic representation in and of itself. In fact, the representational content of the epistemic representational vehicle is made up of a set of propositions from which it is valid to infer the vehicle in accordance with a given set of rules. The inferential process from a vehicle bring us, as users, thereby closer to understanding and knowledge about the physical world, but it cannot give us knowledge about the world in itself.

More precisely, let us take again the example given by Contessa of the map of the London Underground network, which is considered to be an epistemic representation of the real underground network. If we look at the map, and we find out that between Holborn and Bethnal Green there are only five centimetres on the map, this information cannot be translated in the form of direct knowledge to the real London Underground network, since there will never exist a distance of five centimetres between two stations in the real state of affairs. Rather, we should apply some specific set of rules standardly associated with that vehicle in order to draw some conclusions which will help us to define the real distance between Holborn and Bethnal Green, and therefore the time it will take us to reach our destination. In this sense, the map or vehicle gives us some information that users need to translate by using the scale of the map; these conclusions are the so-called surrogative inferences *from* the vehicle *to* the target.

As Contessa says, "[w]henever I talk of the representational content of a map, I intend 'map' to refer to an epistemic representation not to the material object that is the vehicle."[98] Otherwise, in other words, the map will never be small enough to being put into the pocket of our coat.

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[98] Contessa 2007, p. 33.

To offer a preview of the research continued in further chapters of this dissertation, I now turn to analyse an example to understand more clearly what we mean by the 'representational role fulfilled by the point mass'.

Suppose we have a car affected by a certain conservative force: the car is moving at constant velocity *v* along a 1 km road, until at the end of the path it will hit a wall.

In order to perform this task, we may use a very simple model from classical mechanics, namely the model of the point mass. In our example, the car is represented by a point mass which is moving towards the end of the path on the road parallel to the horizon. Let us suppose that already at time $t_0$ the car is in the position $x_0$ with a velocity $v_c=v$. Then, by following the Euler-Lagrange equation, we can determine the trajectory of the moving point mass by referring it to a Cartesian co-ordinate system. In this case the Lagrangian *L* will correspond only to the Kinetic energy *T* of the point mass, which is $T=1/2mv^2$. There is no friction on the road surface, and the only forces acting on the point are the gravitational force ($F=mg$), the normal force *N* perpendicular to the road surface, and a constraint reaction which helps to keep the velocity constant.

If we forget for a moment both the consequences of the impact and the fact that the car will probably bounce back upon the point of impact, let us seek to calculate the average velocity of the car during its route. We have the mass $m$=100 kg of our body, and we presuppose to know the Kinetic Energy $T$=30000 J. In order to calculate the velocity of the car, we can use the following formula:

$$v = \sqrt{\frac{2L}{m}} = \sqrt{\frac{2T}{m}}$$

From the model of the point mass, we can infer that the constant velocity of the car along its road will be approximately 24.5 m/s. A competent user will take this conclusion with a pinch of salt, because the point mass model cannot be a faithful epistemic representation of the car driving into the wall. One of the reasons to suspect this is that some of the features that affect the velocity of the car have no counterparts in the model. Some other features are neglected and are thus deemed to be useless for the specific purpose at hand. The car, or any other differently shaped body will likewise fit the model presented above, but there will always be some physical properties that will never affect the result of our equation. Thus, it is also important, as Contessa maintains, that we distinguish what the user directly infers from the model from what her background knowledge tells her about the conclusion drawn from the model. What she infers – i.e. by following certain rules – will never be different from what every other user – who has followed the same rules – will infer from the model. Moreover, the user will also be aware that the model can only be *approximately* true, because the value of the velocity of the car will be close to, but not exactly the same as, that of the velocity of the point mass. The model can hence be put forward as the generator of a hypothesis about the system whose truth and falsity need to be empirically investigated. As Contessa writes, "[i]n the model proposed above, so as in every other model, scientists do not commit themselves to the model being a completely faithful epistemic representation of the target system. It is only through an investigative process that our competence in using a certain model as a faithful epistemic representation of the system increases. This process consists in determining to what extent the valid surrogative inferences from the model to the system are sound."[99]

This model is used to show that any other model cannot be used as a faithful epistemic representation, but is rather employable to perform inferences about a certain modelled system, which only at a later time is to be translated into knowledge about real-world phenomena.

Let us now turn to the second aspect related to the role of models in science, namely

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[99] In his doctoral dissertation, G. Contessa provides two examples of this: (i) the inclined plane model, and (ii) the Thomson and the Rutherford Models of the Atom, in order to convey the idea that in some cases, some of the inferences that are considered to be valid by the scientific community in the interpretation of the model may be found to be *unsound* when that aspect of the system is empirically investigated. However, this invalidity of reasoning could also function the other way around: inferences whose conclusions are known to be incorrect may be shown to be valid when inferred from the model. See further Contessa 2007, § 1.2.6, pp. 35-45.

their explanatory power. There are two conditions that make deductive arguments explanatory, in the sense that they are able to show the causes underlying the evolution of a certain phenomenon. These conditions are respectively: i) the statements in the *explanans*, i.e. the premises laid out in the argument, which include relevant scientific laws and some background conditions;[100] and ii) the statements that fall under commonly used patterns of argumentation.[101] Another conventional approach argues that an explanation gives a true causal or counterfactual story relevant to the occurrence of the *explanandum,* i.e. the phenomenon that is (needing) to be explained.[102] One thing that philosophers do generally agree on, however, is that statements in the *explanans* are true, whether they are about some feature either of the system or a relevant natural law, or they are related to a causal or counterfactual relation which performs some explanatory function. It seems a reasonable requirement that the statements adduced for the purposes of explanation should describe laws, regularities, causal relations, properties, structures, and so on, which obtain in the physical system exhibiting the *explanandum*.

However, all cases of what physicists take to consider as scientific explanations fail to satisfy even the basic requirements I have just articulated. Explanation in physics relies essentially on Galilean idealizations, or on abstractions of a certain physical system, within which the explanations themselves contain false statements about both explanatorily relevant features of the physical system and the phenomenon to be explained.

One question at this point is the following: *Can idealization and abstraction be used only as forms of explanation?* Some scientists say “yes”, whereas others argue that an idealization can account for an explanation *only* where the premises in the *explanans* are both approximately true of, and are fully corrigible with, the target system, at least in principle. In a certain sense this ideal model of the pendulum system can be seen as an epistemic representation of the real pendulum, but not as a *faithful* epistemic

[100] Hempel 1965.
[101] Kitcher 1989, pp. 410-505.
[102] Salmon 1984.

representation.

Historically, Galileo developed a range of idealizing techniques which were aimed at predicting and explaining natural phenomena. Galileo's "idealized construct" of a pendulum hides the *assumptions* that the pendulum is not subject to air resistance, the wire is mass-less and inelastic, and so on. Moreover, he also hypothesized that an ideal – or a general – pendulum would continue to oscillate indefinitely with the same amplitude and period and that it would obey his pendulum law. Galileo himself says:

> […] [A]s to the ratio of the times of oscillation of bodies hanging from strings of different lengths, those times are as the square roots of the string lengths.[103]

Now, Galileo was aware of the fact that this failed to describe and predict *accurately* the behaviour of *any* of the real pendulums he used in his extended and meticulous experimental work. So again, this inference has to be taken with a pinch of salt, because the oscillations of real pendulums become smaller and smaller over time and they are not isochronous, as his pendulum law ideally requires. On the other hand, since Galileo and a whole generation of physicists after him have taken the pendulum law to be part of the explanation of the behaviour of physical pendulums, they were evidently not aware of the negative aspect related to idealization. In other words, "none of the standard philosophical accounts of explanation canvassed at the outset makes sense of this sort of explanatory practice."[104]

With regard to this problem, one of the first authors who argued in opposition to the exclusive explanatory power fulfilled by the Galilean idealization was Ernan McMullin. His doubts towards the Galilean idealizations and their explanatory incompleteness lies in the fact that the idealized model *approximates* the target system, and, more importantly, that reverse techniques exist which are complementary to idealization, but which are nevertheless used for *de-idealizing* the model either by eliminating simplifying assumptions or by adding back physical details into the modelled system.

[103] Galilei 1989, p. 97.
[104] Wayne 2011, p. 833.

As Robert Batterman writes, "Galilean idealizations thus have an intrinsic "self-correcting" feature such that they can (at least in principle) be brought in ever closer agreement with empirical observations in a theoretically justified, non-*ad hoc* way."[105]

Later on, Laurence Sklar and Robert Batterman made a similar distinction between what they call *controllable* and *uncontrollable* idealizations. An idealization denotes a controllable means whereby it is possible, through appeal to theory, to compensate in some way for the idealizations; whereas uncontrollable idealizations typically involve singular limits and *preclude* explanation.

These, and other scholars, have argued in favour of this position, simply by saying that the so-called Galilean idealizations fail to deliver an answer sufficient for giving the explanation of a phenomenon. But as Andrew Wayne maintains, they are taken to *support* scientific explanation, because these idealizations achieve a kind of *common-sense representational success*.[106]

Furthermore, according to Eleanor Knox, almost the same reasoning fits for abstractions and the explanatory power commonly attributed to them. She has recently recognised the fundamental role played by abstraction in explaining the evolution of phenomena, whilst also showing awareness that phenomena cannot be explained only by the resources of more fundamental theory. In agreement with Batterman, she suggests that there is something more that gives to us the explanation of a phenomenon. Besides the omission or approximation of the modelled system's features, we might then ask whether abstraction is really the whole story:

> […] [C]an we account for the explanatory utility of the higher level explanation purely by noting that it is a distant abstraction of the more fundamental explanation? Should we model all explanations as situated along a sliding scale of abstraction, with higher level explanations deriving their considerable explanatory power from their ability to concisely summarize relevant information from some fundamental picture?[107]

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[105] Batterman 2005, p. 235. For further details on this interpretation, see also Sklar 2000, p. 44 and Wayne 2011, pp. 830-841.
[106] Wayne 2011, p. 834.
[107] Knox 2012, p. 22.

Knox argues that abstraction is *not* the whole story as it is, at least for the issue of idealization, according to McMullin, Sklar and Batterman. However, abstraction *is* an important part of the explanation:

> […] [T]he abstraction that we deem appropriate is highly sensitive to changes in variables that the explanations given by one theory may be deemed novel with respect to another theory. When we properly understand the role of abstraction, we appreciate that *explanatory value may be irreducible*, even where theoretical reduction is possible.[108]

To this extent, it seems that a model does not suffice to give a satisfactory or complete explanation of the dynamic aspects of physical reality. Abstraction, or even idealization, is a part of the explanation, but each of these, even taken together, is not the only components. As Margaret Morrison has put it, "the explanatory role is a *function* of the representational features of the model."[109] That is to say, *all idealizations and abstractions involve a series of approximations or omissions whose explanations apply only to the idealized and abstract models respectively, and yet are not too far off when applied to the modelled system of interest*. Moreover, it is possible to refine the model systematically – i.e. by adding back in the contingent ('real-world') properties – in order to bring it closer and closer to the target system, to such a point that the statements in the explanation eventually become true of the physical system as well.

Finally, there is one more aspect that deserves our attention: the predictive power of models, and more precisely that of the mechanical model, an account which is commonly analysed in connection with the heuristic value of a model. The contention here is that predictions do not refer directly to the so-called course of nature, but rather to our theoretical models.[110] In fact, the heuristic power of a model does not guarantee a direct connection to a real physical system, even though it enables us to deduce some knowledge and conclusions concerning the latter. Thus, to test the model means to

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[108] *Ibid*., pp. 2-3.
[109] Morgan and Morrison 1999, p. 64.
[110] Meyer 1951, p. 113.

compare the conclusions deriving from the model's features with observational phenomena. These processes are built on shared mathematical formulae and sets of rules which have been arbitrarily established by the scientific community. In other words, heuristic models *could* help us to focus our attention on the mechanical consequences of a modelled system in question, but they cannot predict what will exactly happen in the real system. As Knox and Wayne have argued in assigning explanatory power to models, given that models tend to simplify and organise the "complicated and messy" real world, we will lose this particular variety of the real world.

A well-known example of a mechanistic model is our point mass, which is found only in its explicit version in Newton's *Philosophiae Naturalis Principia Mathematica*. Newton's aim in the *Principia* was to work out the mathematical principles of physical science. He did so by means of geometrical representations of moving point masses, i.e. points in which the mass of moving bodies is deemed to be concentrated.

Certainly, before coming out with an independent and explicit theoretical model, a lot of work was required, both in acknowledging the existence of the phenomenon under scrutiny and in understanding the benefits of models and the advantages of the model-building practice. To make this point clear, it is important to be aware that physicists throughout history have imagined all sorts of mechanistic models, but far fewer of them have stood the test of experiential evidence.

In classical physics, theorists assert that models represent more or less accurately what is actually going on in nature.[111] This is also what we read in Galileo's *Two New Sciences* (1638), in which Galileo marked himself as one of the first advocates of the reliable predictive and explanatory power of idealization. His claim stood in favour of

[111] It is likewise common to find theoreticians who argue in favour of the idea that heuristic or mechanistic models offer more than a merely human and imaginative way of explaining phenomena. In fact they think that mathematics expresses some sort of "metaphysical truth" with reference to the properties of the modelled system. We have seen in the beginning of this section that the scientific community itself shares the idea that the model does not stand for the modelled system in itself, but rather that the user interprets and denotes what the model stands for.

the idea that a model *does not depart* from reality too much, due to its power of self-correction and to the practice of "adding back in the properties" that have been previously omitted. According to Galileo, we can "de-idealize" the model in adopting this technique in order to bring it closer again to the modelled system; or, in other words, we can make the model more specific, and adapt it to several different real states of affairs, by eliminating simplifying assumptions. In this way:

> […] [T]he model then serves as the basis for a continuing research program. This technique will work only if the original model idealizes the real structure of the object. To the extent that it does, one would *expect* the technique to work. If simplifications have been made in the course of formulating the original model, once the operations of this model have been explored and tested against experimental data, the model can be improved by gradually adding back the complexities.[112]

Galileo's defence, when he takes e.g. the pendulum which is not subjected to friction, or when he uses the geometrical diagram of the inclined plane to calculate the velocity of a body moving down a slope or to perceive the forces acting on it, is to argue that the departure from truth is *imperceptibly small*. Moreover, Galileo also claims that the idealization enables, and makes more economical, a calculation that would otherwise be impossibly complicated. The same can be argued for the Archimedean geometrical representation of a vessel placed in the sea, which, in the Archimedean treatises, is simply shaped as a paraboloid floating in a liquid. By using this extremely straightforward geometrical sketch, Archimedes was representing a simplified idea of a real vessel and by analysing the components in the model, he was able to translate this "geometrical knowledge" onto a real physical object shaped like a vessel. Of course, the gap between the model – or, in this case, the geometrical diagram – is ample enough that the user needs to be careful when she translates the knowledge from the model to the target.

The gap between the source and the target additionally represents a limit for our faithful understanding of the world, but it is not a hindrance to the aim of representation,

[112] McMullin 1985, p. 261.

explanation or prediction. It is seen as an obstacle only if we want to replace our knowledge concerning physical reality with the knowledge on representational models in itself. As McMullin writes, "[e]very theoretical model idealizes, simplifies to some extent, the actual structure of the *explanandum* object(s). It leaves out of account features deemed not to be relevant to the explanatory task at hand. Complicated features of the real object(s) are deliberately simplified in order to make theoretical laws easier to infer, in order to get the process of explanation under way."[113] Having said that Galileo pretends to adopt a realistic stance about the conclusions drawn on the model, and thereby that knowledge concerning the model is considered knowledge concerning reality, no epistemological commitments need to be taken into account. This contention shall be in favour of the idea that Galileo and of course all preceding scientists were not aware either of the epistemological consequences of these epistemic concerns or of the ontological commitments implied by their realistic stances.

By contrast, we have seen that some authors, such as McMullin, became aware of the difficulties of this unsophisticated conception of the world, because, as we have emphasised at the beginning of this section, statements about predictions or conclusions are made on the properties of our model, *not* directly about the course of nature itself.[114] Let us recall the example of the car driving into the wall. There we know just by making some computation that its average velocity is more or less 24.5 m/s, the user will take this solution with a pinch of salt; she will perhaps say that this figure is the approximate velocity of the car moving toward the end of the road, because in the real target system, most of the factors that contribute to the deceleration of the car are deemed to be irrelevant by the user.

The way in which we may transfer idealized events into elements of mental construction is an epistemological problem of the greatest importance, but it is of too intricate a nature to attempt a full solution here. All the models shown in these examples are responsible for giving us information about the world, but perhaps they are – albeit deliberately and arbitrarily – too simplified to give complete information (or even

[113] *Ibid.*, p. 258.
[114] This is also Simplicio's idea, the Aristotelian spokesman in Galileo's work.

faithful knowledge) about the dynamic nature of real-world phenomena. Their goal is rather only to achieve at least a partial understanding of the target system. Moreover, as in the case of extremely accurate idealizations or abstractions, there are always some features that are left out (as we have adequately said above), so that the aim of using these models is not "simply to escape from the intractable irregularity of the real world into the intelligible order of Form, but to make use of this order in an attempt to grasp the real-world form which the idealization [sc. or all the other types of the aforementioned models] takes it origins."[115]

The main constituent of this process has to be the user of the model in question. Now that we have established that a certain vehicle represents a certain state of affairs, we can infer conclusions, make predictions or give explanations relating to it. But in order to translate this knowledge accurately, there are sets of rules – usually context-dependent – that we have to follow. Thus, conclusions about the model are always partial.

Explanatory, predictive and representational powers have nothing to do with semantic truth. As stated by Duhem and Cartwright, models are not truth-producing or truth-apt pieces of equipment. We can only obtain some *partial* prediction, representation and explanation that is directly related to the model, but not a faithful representation about the modelled system (i.e. of a part of physical reality); scientific knowledge is a mental operation *sui generis* which is constructed by using scientific models. Moreover, our scientific knowledge may be restricted to the representational layer of physical phenomena, not to the physical reality itself. This scepticism concerning the ontological tractability of physical reality also seems to undermine the realistic stance.

To simplify and sum up our claim, we can make a final comparison between our model and a bicycle.[116] The latter does not derive from Newton's laws of motion, but obeys them. We as users build the bicycle because we need it to travel from one place to another. Only once we have the first prototype of a bicycle can we schedule a

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[115] McMullin 1985, p. 248.
[116] I use this example of the bicycle by way of *hommage* to my Dutch inclinations!

performance agenda by looking at the Newton’s laws of motion: we can think how to improve the bicycle by making predictions on the basis of Newton’s law, or we can think how to improve it in order to make it easier to ride, e.g. by making it faster and lighter. The model in question undergoes a series of inspections over time concerning the certain purpose for which it is ostensibly employed; by using it for practical purposes, the user is able to modify the model in order to understand better the dynamic character of the physical reality, and to draw a model which, by the means of a sophisticated theoretical layer, allows us to represent epistemically, predict and explain the course of nature. Even if this model can effectively manage this, it nevertheless has only partial power, because it is a tool from which the user, by necessarily following certain rules, can draw some surrogative inferences, not conclusions relating to the reality in itself.

# Chapter 2

# Ancient Mathematics

## 2.1 A General Introduction to Ancient Mathematics: the methods and tools of Greek mathematicians

The birth of Greek mathematics owes its earliest impetus to the influence of some of its near neighbours, especially from Egypt. During the 26th Dynasty of Egypt (ca. 685-525 BCE), the ports of the Nile were opened to Greek traders for the first time and important Greek figures such as Thales and Pythagoras visited Egypt, bringing back with them new skills and knowledge. Ionia was also exposed to the culture and ideas of Mesopotamia through its neighbour, the kingdom of Lydia. Some centuries later, during the Hellenistic period, Greek astronomy flourished after Alexander the Great's conquest of the East. The astronomical knowledge of the Babylonians and Chaldeans became available to the Greeks who profited by exploiting it systematically. This led to the advance of many Greek mathematical tools and methods.

What the Greeks derived from Egyptian mathematics were mainly rules of thumb with specific applications. Egyptians knew, for example, that a triangle whose sides are in a 3:4:5 ratio has a right angle. The statement of the now commonly called 'Pythagorean Theorem' was discovered on a Babylonian tablet (ca. 1900-1600 BCE), whether first by Pythagoras (ca. 560-480 BCE), or someone else from his school, who was the first to discover that its proof cannot be claimed with any degree of credibility.[117]

[117] On the basis of the extant fragments and later ancient sources, it appears that the Pythagorean movement was considered as a religious movement, a political party more than a philosophical school, and even a scientific movement with pronounced interest in mathematics. This multiplicity of projected images is a reflection of the different characterisations of the figure of Pythagoras himself, who is depicted by the later tradition as a political man, philosopher, mathematician, shaman, the head of a religious sect, and a figure next to divinity. Even the most recent reconstructions of Pythagoras and Pythagorean oscillate, paradoxically, between the characterisations proposed. A conciliatory but eclectic

Despite the Greeks' attempts to systematize earlier Eastern scientific knowledge, their mathematics still remained limited to purely practical geometry. Only with the advancements made chiefly by Euclid – having been anticipated by the Pythagoreans Hippasus, Archytas and Hipparchus of Nicea – did geometry undergo a transformation from being a mere practice to becoming an exact theoretical science; thereafter geometrical entities were abstracted[118] and began to be considered as entities independent from any material and physical content. In this connection, the purpose of this research is exactly to find the way that one of the most common idealized entities of actual mathematical physics, the point mass, has reached this status as a purely abstract entity. By following the three stages of this thesis' methodology, which we have labelled "objectification of procedure", we will analyse in this chapter the stage in which the geometrical notion of the centre of gravity was used, regarding it not as a precursor to the modern notion of the material point, but rather as an investigative geometrical tool for practical purposes aiming at engineering works. This stage corresponds to the enquiry into Archimedean equilibrium concerning the law of the lever, and in particular to the use he made of the centre of gravity as a crucial notion in statics (§ 2.1). Following this, and for the sake of completeness, we will provide an introduction to the Euclidean geometry dealing with points, lines and surfaces in their standard geometrical usage (§ 2.2). Next, we will present the main differences between the Euclidean and Archimedean approaches to geometry, emphasising the fact that Archimedes should be

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position is bound to encounter insurmountable difficulties, or in any case, even recognising that there might be an ounce of truth in all of these characterisations, it is not possible to make a simple summation. The solution of accepting contradiction as an index of the inscrutability of Pythagoras seems to lead to the renunciation of clarifying the various aspects of this biographical tradition. To isolate a nucleus of historical truth in the face of a legend that has hardened over the centuries is an extremely difficult task, but it is nevertheless indispensable to try to determine which of these characterisations the historical Pythagoras was closest to, thus identifying the nature of that movement that is usually described, with inevitable margins of imprecision in terms of the school, sect, community, brotherhood, and party besides. Ultimately, it is important to understand what it meant in antiquity to be a"Pythagorean"; therefore the issue to define a criterion of classification in order to define what a Pythagorean is, is relatively old-fashioned, and it is not completely solved at all. However, it is important to bear in mind that there is a well-founded tradition of the master's doctrines' succession. See on this Bruno Centrone's introduction to the Pythagorean, esp. pp. 3-12 and pp. 102-104 [Centrone 1996].

[118] In the context of ancient geometry, we may use, following Enrico Giusti, the expressions 'abstract' and 'idealized' entities synonymously, if only in order to emphasise the idea that they are not dependent upon any real and material objects.

held among the first authors to give importance to the heuristic and mechanical meaning of geometrical demonstrations (§ 2.3). Thereafter, in §§ 2.4-2.6 we will focus on the most important treatises dealing with the science of the barycentre, introducing the most important demonstrations and methods used to find a "method" for calculating the centre of gravity in the case of one-dimensional, two-dimensional and three-dimensional figures. In all these sections we will seek to show the affinities between the standard definition of the centre of gravity as it is presented in Greek mathematics and the modern idealized notion of the material point.

Let us begin with an overview of the role and status fulfilled by the notion of the geometrical point in Euclid's *Elements*, before turning to an analysis of the two methods of enquiry typically used in ancient geometry, namely the analytical and the synthetical method respectively.

Following the Pythagoreans' discovery of incommensurable lines, which dates back to ca. 500 BCE, we witness the so-called *process of annihilation of the point*. Incommensurable lines, such as the diagonal line of a square, are defined as such because the ratio between the side and the diagonal of a square cannot be expressed by a rational number **R**. Conversely, lines are defined as commensurable when they can be measured and the ratio between them can be expressed by a rational number. The discovery of these magnitudes led to the abstract conception of geometrical entities, in which a point is held to be an entity without any dimension, a line is held to be a one-dimensional magnitude without breadth, and a surface is held to be a two-dimensional magnitude without height.

Lines in pre-Hellenistic geometry were considered to be material entities made up of thousands upon thousands of infinitely small grains (or points) and possessing of dimension. However, this definition of lines, combined with the discovery of incommensurable lines, had paradoxical consequences, since, assuming the physical existence of infinitely small grains, the ratio between two different sizes, even if they were incommensurable, may always be reduced to a common size, namely a grain, which is common to both the side and the diagonal of the square.

The only way to overcome this paradox is to *annihilate the point*. While previous geometry was anchored to matter, we now see the advent of a new abstract geometry, in which points are considered to be entities without length, breadth and height. New theoretical components are introduced; the idealized entities are now the new objects of study of geometry. In this way, the infinite enters the previously earthly scope of geometry. Since the point has no dimensions, every segment of a line is conceived as being composed of many infinitely small points. A line becomes seen as a continuous magnitude because between any two arbitrary points one may always find a third one. Every line is infinitely greater in size when compared to a point that is infinitely smaller than the line.[119]

Another significant aspect of pre-Euclidean mathematics is the method of enquiry. Traditionally before then, natural philosophers had proceeded by means of the analytical method, which consisted of the analysis of complex hypotheses through common and empirically verifiable intuitions, proceeding gradually towards simpler propositions from which the given complex statements might be inferred as logical consequences. Only at a later stage did the synthetic method mature into a method which uses postulates and axioms to deduce more complex propositions and demonstrate theorems.

The most valuable preservation of the use of the synthetical method is Euclid's *Elements*. In Book XIII one finds the terms 'analysis' and 'synthesis' defined in the following way:

> Analysis is an assumption of that which is sought as if it were admitted through its consequences to something admitted (to be) true […] whereas synthesis is an assumption of that which is admitted through its consequences to the finishing or attainment of what is sought.[120]

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[119] See further Acerbi 2010, pp. 183-195 and Frajese 1964, pp. 26-37. I return to this point in the next section (2.2), with regard to Euclid's approach to mathematical entities.
[120] Translations are taken from Heath 1908.

The analytical method can be seen as a supplementary enquiry, or, as Pappus maintained,[121] a special body of doctrine to be studied by those who, after finishing the *Elements*, desired to acquire the knowhow of solving problems which may be set before them involving the construction of lines, since it is useful for this alone. On the one hand, we assume in analysis that which is sought as if it had already been sought, and we enquire what it is from what is its result, and again what is the antecedent cause of the latter result, and so on, until, by retracing our steps, we come upon something which is already known or which belongs to the class of first principles. On the other hand, synthesis consists of the inverse process, insofar as we take as already sought that which was last arrived at in the analysis, and, by arranging in their natural order as consequences what were previously antecedents, and connecting them successively with one another, we arrive finally at the construction of what was sought.[122]

The methods of synthesis, together with the abstract and idealized conception of mathematical objects, are two of the main features of standard Euclidean geometry, in the sense that this geometry still prevails today. Having offered an introduction to the dawn of geometry as a discipline, let us now turn to Euclid's own context and intellectual background for the composition of his *Elements*.

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[121] Pappus 1876-78, Vol. 2, VII, pp. 635-636 and Jones 1986, p. 83.

[122] Heath 1908, Vol. 1, pp. 137-140. One should also note that analysis itself divides up in two directions: i) the theoretical kind, which is directed at searching for the truth; and ii) the problem-oriented kind, which is directed at finding what we are told to find. In the first kind, we assume that what is sought is also existent and true, after which we advance through its successive consequences, as if they too were true and established by virtue of our hypothesis, to something which is admitted. In which case a), if that something admitted is true, that which is sought will also be true and the proof will correspond in the reverse order to the analysis; but also b), if we come upon something that is admittedly false, then that which is sought will also be false. In the second kind, we assume that which is propounded is also knowable, after which we advance through its successive consequences, as if they too were true, up to the point that something is admitted. If then a) what is admitted is possible and obtainable, that is, what mathematicians call "given", what was originally proposed will also be possible, and the proof will again correspond in a reverse order to the analysis, but b) if we come upon something admittedly impossible, the problem will also be impossible to solve.

## 2.2 The standard Euclidean geometry

Very little can be said with certainty about the biography of Euclid himself.[123] Most of what we know about him is contained in a passage of Proclus' commentary.[124] However, we can infer from Proclus' and Stobaeus' works that Euclid lived in the time of the first Ptolemy, and thus flourished in ca. 300 BCE. It is highly likely that Euclid received his mathematical training in Athens from the pupils of Plato, and that he not only taught at Alexandria, but also founded a school there.

Although Euclid's *Elements*[125] were not the first work of algebraic method, they seem to have quickly become a point of reference for later writers, with innumerable commentaries, editions and translations being produced over the centuries. For a long time they were an active source of new mathematical theorems; in due course they became canonised, upheld by some as a fine tool for mathematical education[126], and today they represent, thanks to their highly theoretical nature, a typical elementary treatise that is widely used for training students, mainly because they avoid any reference to practical and heuristic procedure. The purely geometrical constructions employed in the *Elements* are restricted to those which can be achieved using a straight-rule and a compass; empirical proofs by means of measurement are strictly forbidden. Most of the theorems appearing in the *Elements* were not discovered by Euclid himself but were the work of earlier Greek mathematicians such as Pythagoras (and his school), Hippocrates of Chios, Theaetetus of Athens and Eudoxus of Cnidos. However, Euclid is generally credited with arranging these theorems logically and systematically, so as to demonstrate that they necessarily follow from the first five simple axioms (i.e. the well-

[123] Euclid the geometrician is sometimes confused with the philosopher Euclid of Megara, who lived *ca.* 400 BCE.
[124] That is, using the editions Proclus 1970 and 1978.
[125] Although the *Elements* are the most famous treatise deriving from ancient Greek mathematics, Euclid was also the author of other works, such as the *Pseudaria*, *Data*, *Divisions of Figures*, *Porisms*, *Surface-loci*, *Conics, Phaenomena* and the *Optics*. Among the commentators there are additionally traces of two treatises on the *Elements of Music*. Some of these works have been lost during the centuries, while others are contained in and have been transmitted through other scientists of antiquity such as Pappus, Heron, Phorphyry, Simplicius and the well-known Proclus. A detailed study of the Euclidean treatises and commentators can be found in Heath 1908, Vol. 1, pp. 20-45.
[126] Grattan-Guinnes 1996, p. 1.

known five postulates of Euclid).[127]

The treatise consists of thirteen books and consists of i) axioms or postulates, ii) common notions, iii) definitions, iv) propositions and v) theorems. By their own definition, postulates do not require proofs or demonstrations, since they are self-evident. However, over the centuries, some scholars came to believe that axioms and postulates were provable and derivable only from definitions.[128] The axioms and postulates that are derived from these definitions are the statements which are properly employed in the proofs, and which therefore affect the deductive structure of the *Elements*. Below I present a summary of the main contents of each book.

Book 1 outlines the fundamental propositions of plane geometry, including the three cases in which triangles are congruent, various theorems involving parallel lines, the theorem regarding the sum of the angles of a triangle, and the Pythagorean theorem. Book 2 is commonly said to deal with 'geometrical algebra', since most of the theorems contained within it have simple algebraic interpretations. Book 3 investigates circles and their properties, and includes theorems on tangents and inscribed angles. Book 4 is concerned with regular polygons inscribed in, and circumscribed around, circles. Book 5 develops the arithmetical theory of proportion. Book 6 applies this theory of proportion to plane geometry, and contains theorems on similar figures. Book 7 deals with elementary number theory (e.g. prime numbers, greatest common denominators, etc.). Book 8 is concerned with the geometric series. Book 9 contains various applications of results in the preceding two books, and includes theorems on the infinite

[127] These five postulates are:

1: to draw a straight line from any point to any point;

2: to produce a finite straight line continuously in a straight line;

3: to describe a circle with any centre and distance;

4: that all right angles are equal to one another; and

5: that, if a straight line falling between two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, will meet on the side on which the angles are smaller than the two right angles.

[128] See e.g. the Renaissance commentary by Clavius, who gave demonstrations for all mathematical principles with the exception of definitions. It consequently became a point of controversy whether or not it was useful to prove the axioms at all; many French authors during the seventeenth century claimed that Clavius's exercise had been completely futile. See further de Risi 2016, p. 11.

nature of prime numbers as well as the sum of a geometric series. Book 10 attempts to classify incommensurable (i.e. irrational) magnitudes using the so-called method or 'principle of exhaustion'.[129] Book 11 deals with the fundamental propositions of three-dimensional geometry. Book 12 calculates the relative volumes of cones, pyramids, cylinders and spheres using the method of exhaustion. Book 13 lastly deals with the five so-called Platonic solids (since they refer back to Platonic mathematical theory expounded in the *Timaeus*).

For centuries the *Elements* of Euclid embodied the very model of scientific and deductive reasoning, and its diffuse influence throughout Europe was matched only by the Bible and by a few other writings of the Church Fathers. They were translated, edited and commented upon hundreds of times, and these editions and commentaries shaped the scientific tools, methodological standards and mathematical language of many centuries.[130] Euclid's theorems were used to build further and more audacious mathematical theories, and applied in the physical sciences, while the deductive structure of the proofs was studied by mathematicians, logicians and epistemologists as the ideal of reason itself. A special historical role was additionally played by the principles, which were employed as the foundational elements of the entire construction.

Euclid's *Elements* drew together materials from an older mathematical tradition, and had presumably already undergone several changes and modifications during the Hellenistic Age. Around the turn of the fourth century CE, the mathematician Theon of Alexandria prepared an edition of the *Elements* that had great importance in the subsequent textual tradition. But the oldest existing copies of the *Elements* are Byzantine manuscripts dating back to the beginning of the ninth century; later Greek manuscripts were in scholarly circulation during the Renaissance and in fact provided the Greek text for the first editions of Euclid in the Western world. One further

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[129] Further details on the meaning of this heuristic procedure will be given in the sections concerning Archimedes of Syracuse below, from § 2.3 onwards.

[130] Some of the main Medieval and Renaissance translations and adaptations will be analysed in the next chapter, see § 3.1.

important manuscript was only discovered at the beginning of the nineteenth century by François Peyrard, and it apparently contained a version of the *Elements* older than any other, possibly even predating Theon's edition. It seems that its author had access to both Theonine and non-Theonine manuscripts. This edition is still considered the best that we possess for the original form of Euclid's work, and has also been employed by the Danish philologist Johan Ludvig Heiberg[131] to prepare the only critical edition of the *Elements* (1883-1916), which is still currently used for research purposes.

These Greek manuscripts show many definitions prefacing several books of the *Elements*, and a set of fourteen or fifteen other principles (depending on the manuscripts) at the beginning of Book 1. The latter principles were usually considered by modern scholars to be the only statements on which the mathematical construction of Euclid was to be based, and are divided into a first list of 'postulates' (αἴτηματα) and a second list of 'common notions' (κοιναι ἔννοιαι) or, as they were later called, axioms. The rationale behind this division of the principles into 'postulates' on the one hand and 'common notions' on the other was already a subject of debate and controversy in antiquity, and several philosophers and mathematicians advanced their own opinions on the epistemological status of this distinction. Some principles among the fourteen or fifteen contained in the oldest manuscripts, however, came to be considered spurious already in late antiquity, and Proclus (6th century CE, and the author of a crucial commentary on the *Elements*' first book) informs us that a number of them were in fact later interpolations. Modern philology inclines to accept Proclus' opinion and attempts to prove that some of the principles must have been added in the ancient transmission of the text. A largely (albeit not unanimously) endorsed opinion among scholars is that Euclid originally had only ten principles (five postulates and five common notions), while the other axioms are Hellenistic additions. Some doubts are usually raised about

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[131] Heiberg's edition constitutes the most valuable and thorough work ever published on the history of the text. Fortunately it is also available in English translation [Heath 1908]. In particular the introduction ranges widely from simple notions related to Euclid's life to probing analysis of earlier treatises on elementary mathematics and geometry, as well as all the later commentaries and reworkings across Europe, even accounting for circulation in the Arabic-speaking world. See Heiberg and Menghe 1883-1895 and Heath 1908, Vol 1, pp. 14-151.

two further common notions that some scholars wish to regard as interpolations as well: if this latter view holds true, then Euclid originally had only eight principles. We know in any case that several ancient mathematicians discussed the Euclidean system of principles, and began to add, change or remove a number of axioms and postulates; this seems to have been so, for instance, of Apollonius, Geminus, Ptolemy and Pappus. In particular, Heron of Alexandria (1st century CE) wrote a commentary on the *Elements* in which he addressed certain gaps in the Euclidean proofs and tried to fill them by introducing new assumptions (which he may, however, have regarded as provable). Even though Heron's commentary is now lost, it was still known in the Islamic Middle Ages.[132] A few fragments of it were included in Proclus' commentary to the *Elements*, whose text enjoyed only a very limited circulation in the Middle Ages, but was rediscovered in the Renaissance. Similarly, Simplicius' commentary of the *Elements,* which dates back to the 6th century, was extensively read in the Arabic-speaking world, but was later lost. A few of the axioms added in the Middle Ages and the Renaissance in fact derive from these Greek commentaries, and from Heron in particular, since the philosophers Proclus and Simplicius did not dare add their own principles to the "most perfect" work of Euclid.

More can certainly be said on the circulation and the transmission of one of the most important and well-known geometrical treatises ever published, but only in the next chapter will we advance to a more detailed exposition of its dissemination during the Middle Ages and in the Renaissance Italian courts. In spite of all the adaptations and translations from antiquity, it is impossibly difficult to find how the original Euclidean edition of the *Elements* was structured. However, nowadays every scholar who refers to this classic treatise uses Heiberg's edition, which originates on the one hand from a long tradition of redrafting, adding and making alterations to the original order with the aim of improving its clarity, and on the other hand from changes in some demonstrations which seek to remove some demonstrative procedure, such as the *reductio ad absurdum*.

[132] See further Brentjes 1997-1998, pp. 55-117, and the introductory essay in Acerbi and Vitrac 2014.

From the analysis of Euclidean thought, we can maintain that the objects of classical Greek mathematics were always particular objects given by an axiomatized construction procedure. Euclid's axioms allow the existence of straight lines and circles, and an implicit procedure (cutting a given cone with a given plane) admits, for instance, the existence of Apollonius' conic sections. Thus the main goal of ancient geometry was to study these objects and determine their properties. This approach is considerably different from that which is pioneered in the modern post-Cartesian era. From the development of modern geometry onwards, an ellipse is the locus of the zeros of a particular quadratic equation, or the locus of the points, such that the sum of the distances from two fixed points is constant. As Napolitani and Saito explain, "the property precedes the object, which, *a priori*, might not even exist. For the Greeks the ellipse was the object determined by a plane cutting a cone, meeting all its generatrices; this curve therefore existed, but all its properties were unknown, and needed to be investigated. It is not by chance that the foci of the ellipse and the hyperbola are introduced by Apollonius only at the end of the third book of *Conics*. The object precedes its properties."[133] The immediate consequence of this conception of mathematical objects, which highlights the limitations of classical Greek geometry, is that a general object could not exist. Nothing like our curves existed in classical geometry. Various curved lines existed, but no single conceptual operational category under which a series of different curves could be included. *Each curve has its own special procedure that defines it*. Consequently if a general object did not exist, then a general method could not exist either. The method for the determination of areas and volumes were always presented on an *ad hoc* basis, in connection with the practical problem to be solved; for example, the quadrature of the parabola is dealt with in a radically different way from the squaring of the circle, and the methods developed for the study of spirals were altogether different from those devised for conoids and spheroids.[134]

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[133] Napolitani and Saito 2004, pp. 68-69.
[134] *Ibid.*, pp. 69-70.

In general, the Greeks did not use definitions as foundations for their demonstrations, but rather as a means of clarifying the nature of the objects under analysis. As a result, their objects of study do not derive from a process of idealization or abstraction with respect to concrete objects shaped in a certain way: the circle does not derive from the act of abstracting a spherical concrete object, but rather from practical purposes and surveying operations. Stakes, for example, stand for a point, and an outstretched rope used to connect the two stakes represents a geometrical line. Whether geometrical definitions in the *Elements* stand for definitions of features belonging to practical operation or not, axioms and postulates stand for a translation of empirical and mechanical procedures. From the abstraction of this practical procedure is derived what we now call abstract mathematical and geometrical entities.

Regardless of whether this approach could not be clearly and explicitly seen in Euclid's treatise, the contrary holds for the mechanical and heuristic approach adopted by Archimedes in his geometrical and mathematical methods. In fact, it is not the aim of this dissertation to embark upon a deep and exhaustive analysis of the *Elements*: we present only the main features in order to supply the theoretical background and mathematical environment in which Archimedes, the first main character of this reconstruction, grew up and worked. Accordingly, we suggest that it is also important to present the debate concerning the notion of the geometrical point and the use that Euclid made of it. In fact, although the purpose is to outline the theoretical development of the notion of material point, we believe that the first item on the agenda is necessarily to present the simple geometrical notion of the point. In ordinary geometry the point is usually defined in the following way: *a point is that which has no parts*.[135]

We are aware that behind this definition lies a deeply protracted and paradoxical debate, which has been the subject of hundreds of different interpretations, in the first place stemming from linguistic confusion. In ancient Greek there are two different terms with which to express our modern idea of 'point'. The first, used by Euclid, is

[135] Heath 1908, Vol. 1, p. 155.

σημεῖον, which literally means 'sign' and which alludes to the unperceivable existence of the point. In other words a point can be interpreted as a geometrical construct.

It is not the same for pre-Euclidean philosophy of nature, because in the fragments of earlier treatises we find the term στιγμή to refer to the point produced by a sharpened object, lit. a 'puncture'. In the Pythagorean fragments[136], a point is defined as a monad having position or with position added; this is by far the first definition we hear of among the pre-Euclidean philosophers. The same meaning is given by Aristotle in *Metaph.* 7016b 24, where he says that what is indivisible in respect of magnitude and *qua* magnitude, but has no position, is a monad, while something similar that is indivisible but has position is a point. Thus, according to this pre-Euclidean theory, a point could be defined as that which is indivisible and has position.

Along the same line of argument, Plato seems to be in disagreement with the Aristotelian and Pythagorean definitions, disapproving – as Aristotle himself mentioned in *Metaph.* 992a 20 – "to this genus [viz. that of a point indivisible and with position] as being a geometrical fiction, and called a point the beginning of a line, while again frequently spoke of *indivisible lines*". However Aristotle replies again that even these indivisible lines must have extremities, for example two end-points, so that the same argument which proves the existence of lines also proves the existence of points.

In several Aristotelian treatises one can find the objection against the unscientific Platonic definition of point as extremes of the line.[137] According to Heiberg's interpretation, this debate leads to the replacement of the previous notion of point as στιγμή with the current Euclidean and post-Euclidean meaning expressed by the term

[136] Proclus 1788, p. 95 and *Id.* 1978, p. 94

[137] Aristotle's conception of a point as that which is indivisible and has position is further illustrated by his observation that a point is not a body [*De Caelo* II.13, 296a 17] and has no weight [*Ibid.* III.1, 299a 30]. Also in the *Physics* he makes no distinction between a point and the place where it is [*Physics* IV.1 209a 11]. He finds the usual difficulty in accounting for the transition from the indivisible to the finite or divisible magnitude. If a point is indivisible, no accumulation of point, however far it may be carried, can give us anything divisible, whereas of course a line is a divisible magnitude. Hence, Aristotle holds that points cannot make up anything continuous like a line. A point is like the now in time: now is indivisible and is not a part of time, it is only the beginning or end, or a division, of time. And similarly a point may be an extremity, a beginning or division of a line, but it is not part of it or of magnitude. It is only by motion that a point can generate a line and thus be the origin of magnitude. For further details, see Heath 1908, Vol. 1, p. 156, and see also Heiberg 1904 and Mugler 1958.

σημεῖον. Thus, it was due to Plato's influence that there occurred a shift from the earlier notion of a puncture or sting towards the regular term used by Euclid and Archimedes later on, the latter term probably being considered more suitable than the former to claim greater reality for a point.

In contrast with Heiberg's interpretation, Vita[138] in the 1980s promoted a new conjecture claiming that after Aristotle's death we assisted a process of furthr specialization of different fields of knowledge in such a way that each discipline assumed its own specific range of application and range of problems. Philosophy lost its inclination towards the interpretation of natural phenomena, while scientific research acquired its own autonomy and emancipation from philosophy. In this cultural milieu the term σημεῖον was chosen. In 1999, on the basis of the evidence that ancient Greek is characterised by an absence of nuances and that the ancient Greek lexicon is constituted of words clearly marked off from each other and not of a continuous spectrum of words shading into each other, Netz – also following Vita – theorised that the Greek lexicon operates on the principle 'one-concept/one-word', due to which Greek practitioners tended to formulate an economical system based on simplicity. Therefore in this context the word σημεῖον was chosen instead of στιγμή.[139]

Furthermore, the Euclidean definition is perfectly compatible with the earlier geometrical developments of Parmenides and Zeno of Elea (both 5th century BCE), whose concepts still occupy a place in modern geometry, according to which points are merely unextended objects of study and thus differ from any tangible object. In Euclidean doctrine the point is not the only thing without parts, since Euclid also mentioned the factors of *now in time*[140] and the *unit in number*.[141] The only voice to disagree with this statement is Proclus, who declares that the point is the only thing in

[138] Vita 1982.
[139] Vita 1982 and Netz 1999, pp. 108-114.
[140] According to Aristotle, time contains something indivisible, and this is what we call the now. See *Physics* VI, 234 a, 23.
[141] A unit is that by virtue of which each of the things that exist is called one. Heath 1908, Vol. 2, Book VII, definition 1, p. 277.

the field of geometry that is indivisible.[142] Simplicius says that a point is the beginning of magnitude and that from which it grows, and also that it is the only thing which, having position, is not divisible. Like Aristotle, Simplicius adds that it is by its motion that a point can generate a greater magnitude, such as a line (i.e. Simplicius has a dynamic conception of a line).

In the third definition of Book 1, Euclid says that *the extremities of a line are points*. However, in some other definitions that Euclid used, such as the *infinite* straight line, the circle or the ellipse, lines do not possess extremities. These definitions conceal a question that more than any other had given rise to a fierce controversy, which continued across all further centuries between mathematics and physics. The 'point' controversy is clearly analogous to the controversy between atomism and continuism, as well as the debate surrounding Zeno's paradoxes. The spectacular intervention of Zeno of Elea in the evolution of mathematical thought seems to have been largely caused by the embarrassment into which the human intellect had been thrown by the mathematical continuum; it constituted perhaps the most powerful trigger of the famous crisis of principles, which disturbed the incremental progress of mathematics around the turn of the fourth century BCE.

Are lines made up of a finite number of points, and are there thus indivisible parts in geometrical extensions, as atomists argue? Or are rather those lines infinitely divisible, that is having the property of being divisible into parts that can themselves be further divided, so that the process never terminates in indivisibles? These, and not only these, are the questions arising from the centuries-old debate between atomists and continuists.

According to the main tenets of atomism, the Greek adjective *atomos* literally means 'uncuttable'; in this light the history of atomism is not only the history of a theory relating to ancient philosophy but involves over time physics, mathematics and metaphysics right up until the contemporary turn. In fact, a wide variety of ontological questions concerning the nature of space, time and matter originate therefrom, involving

[142] Proclus 1788, pp. 92-96 and *Id*. 1978, pp. 87-98.

the notion that there are indivisible parts in *any* kind of magnitudes, geometrical extension, time, and so on. This debate originates in response to paradoxes such as those of Zeno of Elea about the infinite divisibility of magnitudes. According to Aristotle (*On Generation and Corruption* 1.8), it is likely that the first postulation of indivisible bodies is given as a response to a metaphysical puzzle about the possibility of change and multiplicity. Parmenides had argued that any differentiation or change in Being implies that "what is not either is or comes to be". Furthermore, by means of indivisible bodies, the atomists were also thought to answer Zeno's paradoxes about the impossibility of motion.[143] Zeno had argued that, if magnitudes can be divided an infinite

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[143] Traditionally the problem of divisibility has reached us through different, but closely related, paradoxes. On the one hand stand the group of paradoxes on plurality which argue in favour of monism; on the other hand the second group relate to the paradoxes of motion.

Throughout the argument for denseness, Zeno attempts to show that there could not be more than one thing, on pain of contradiction. Assume then that there are many things; he argues that they are both 'limited' and 'unlimited', which is a contradiction. Firstly, he says that any collection must contain some definite number of things; but if you have a definite number of things, you must have a finite (limited) number of them. But second, imagine any collection of 'many' things arranged in space. Between any two of them, he claims, there is a third thing; and in-between these three elements another two; and another four between these five; and so on, endlessly. Thus the 'limited' collection is also 'unlimited', which is a contradiction. But, why are there 'always others between the things that are'? As I have already said above (§ 2.1), we can argue in favour of the existence of an infinite number of physical grains, that are physically separated between each other, even if this is just so by air, contained in an unlimited size; or rather one might hold that any body has parts that can be densely ordered. Of course material and physical point-parts (or grains) are not densely ordered. However, according to modern geometry, familiar geometric points are densely ordered in a finite line because there are no common idealized entities without any dimension.

A step forward for this line of thought is the argument for finite size. Zeno argues that if those point-parts are imagined without dimensions at all, since they are so indefinitely small, with a series of additions of a point-particle to another one it would not make it any bigger. Moreover, if, when it is subtracted, the other thing is no smaller, or if it does not increase when something is added, clearly the thing being added or subtracted is nothing.

The second group of paradoxes argue in favour of the non-existence of motion. The most famous paradoxes are *The Dichotomy* and *Achilles and the Tortoise* (the others are *The Arrow* and *The Stadium*). Since both of these two paradoxes resemble each other I will briefly analyse only the first one. Even if all those arguments were developed for claiming the non-existence of motion, our humble point is closely related to those kinematic problems which raise a number of physical and methaphysical issues about the divisibility of space and time. Imagine a runner needs to run for the train. Before she reaches the station she must run half-way. Suppose she is running with constant speed, so she takes a half of the time to run half-way there and a half of the time to run the rest of the way. Now she must also run half-way to the half-way point—i.e. a quarter of the total distance—before she reaches the half-way point, but again she is left with a finite number of finite lengths to run, and plenty of time to do it. And before she reaches a quarter of the way she must reach a half of a quarter = a 1/8 of the way; and before that a 1/16; and so on. There is no problem at any finite point in this series, but what if the halving is carried out infinitely many times? The resulting series contains, for any possible first distance, a half-way distance. Thus, the series number of times, it would be impossible for motion to occur. The problem seems to be that a moving body would have to traverse an infinite number of spaces in a finite time. By supposing that the atoms form the lowest denominator for division, the atomists escape from this dilemma: a total space traversed has only a finite number of parts. Since it is unclear whether the earliest atomists understood the atoms to be physically or theoretically indivisible (or divisible), one wonders whether they may not have made this distinction. The dilemma concerning (in)divisibility, added to the paradoxes posed by Zeno, seems to lead to one rational solution: in order to overcome all these paradoxes we must deny the *actual infinite* and accept the *potential infinite*. Traditionally both theories are attributed to Aristotle in his *Physics*, Books 3 and 6 respectively.[144] The potential parts doctrine argues that the parts into which a body can be metaphysically[145] divided are not distinctly existent entities prior to their being actualized by some operation of division. Prior to the act of division that actualizes or creates the parts, the whole is best described as containing possible or potential parts. Those potential parts represent the way in which the whole could be broken down, and consequently any talk of potential parts is really just talk about the modal properties of the whole original. Moreover, these parts are not created *ex nihilo*. The whole matter from which they are made existed beforehand, and thus the whole is not a composite or

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of distances that the runner is required to run is: … 1/16 of the way, then 1/8 of the way, then 1/4 of the way, and finally 1/2 of the way. There is clearly a problem, for this description of her run has her travelling an *infinite* number of *finite* distances, which, Zeno would have us conclude, must take an infinite time, which is another way of saying that it is never completed. Consequently, Zeno's conclusion is in favour of the non-existence of motion at all.

However, Euclid's postulate 1 ("to draw a straight line from any point to any point"), postulate 2 ("to produce a finite straight line continuously in a straight line") and postulate 3 ("to describe a circle with any centre and distance") could be considered a valuable reply to Zeno's refutation of motion and divisibility, because in drawing lines and circles, a sort of motion is required. I think that this suggestion requires further and independent scrutiny.

[144] Aristotle (ed.) 1973.

[145] Throughout the history of ancient Greek philosophy, four different kinds of divisibility were recognised: i) matter is physically divisible (p-divisible) if and only if (iff) it can be broken apart by natural processes; ii) matter is said to be metaphysically divisible (m-divisible) iff it is logically possible that its spatially distinct parts could exist separately from one another; iii) matter is said to be formally divisible iff it has parts that can be distinguished by their spatial properties (i.e. because of the relation with their adjacent parts); and finally iv) matter is intellectually divisible iff a mind could represent it in thought as conteining different parts, regardless of whether these parts are p-divisible or m-divisible and regardless if the parts are spatially distinct (f-divisible). For further remarks, see Holden 2004, pp. 9-16.

aggregate structure, a construction from so many distinct actual parts. Nor then is the whole an ontologically derivative entity, depending for its existence on the ontologically prior existence of component parts. In fact it is a metaphysical thesis about the ontological status of the parts of material bodies, not a thesis about their internal physical structure.[146]

On the other hand, the actual parts doctrine states that the parts of bodies are each fully-fledged distinct entities. As Holden writes, "[t]his implies that the whole gross extended body is a compound or composite, a structured aggregate of these *pre-existing*, independently existing parts. Since each actual part is a distinct entity, the whole must be conceptualised as a composite structure, a compound aggregated from ontologically prior concrete elements."[147]

All those paradoxical arguments were only definitively solved in the nineteenth century. However it is not the aim of this section to explain how this was achieved or to offer an exhaustive analysis of the debate arising from the geometrical notion of point. While the latter had given rise to such a debate, the same cannot be said for the notion of the point mass. Despite its complicated aspects, mathematics and physics were not intimidated by this centaur-like problem[148], which has been treated – be it implicitly or explicitly, as we shall see in the subsequent chapters – somewhat casually over the centuries.

Throughout this section we have aimed, first to locate the context in which Archimedes, who clearly gave importantce to the notion of centre of gravity, lived, grew and developed his research. Secondly, between Euclid and Archimedes, we have sought to locate a turn in the approach adopted towards geometry, insofar as the synthetic method and the claim of looking for foundational principles was substituted for a heuristic and mechanical approach aiming at purely practical purposes. With

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[146] *Ibid.*, pp. 91-92.
[147] *Ibid.*, p. 88.
[148] The reason we define the material point as *centaur* derives from the metaphor used by George Israel: the material point is like a centaur, in that in it coexist two heterogeneous notions. On the one hand we have the notion of matter-mass that come from physics, and refers to extension; on the other hand we have the mathematical notion of point, which refers to a body that has no parts. Israel 2007.

respect to this second point we shall examine the development of the first stage of the objectification of procedure, when the procedure of weighing is adopted in a purely engineering context, in particular within the context of marine engineering and in building fortifications and weapons for defending the city of Syracuse.

The first kernel of abstraction and idealization hidden behind the modern notion of the material point can be traced back to Archimedes, because he used the centre of gravity of planes and solid figures, even though he did not question the epistemological menaning of this procedure. In fact, we have to wait the end of the Renaissance period for such a turn, in order that the whole figure is represented in itself. Archimedes imagined that an entire triangle for example could have been reduced to its centre of gravity and be shifted along the arms of the balance or of the lever. Let us proceed by presenting an overview of the Archimedean context of research.

## 2.3 Towards a Heuristic tradition

Archimedes is considered one of the high-water marks of the mathematical culture of Greek antiquity. As in the case of Euclid, so for Archimedes our knowledge is very scanty regarding his life and personality.[149] The year of his birth cannot be stated with certainty: it is usually dated to 287 BCE, on the evidence of the Byzantine polyhistor Tzetzes, who reported that Archimedes was 75 when he was killed in 212 BCE during the Roman conquest of Syracuse. Archimedes was also a leading figure in the defence of Syracuse against the Romans in one of the defining moments of the Second Punic War.[150] It is probably this special role as a scientist, at such a pivotal moment of history, which gave Archimedes fame during his lifetime.[151] As a leading mathematician in

[149] In antiquity there existed a biography of Archimedes written by Heraclides. The latter was also mentioned once by Eutocius in his commentary on the *Measurement of a Circle* [*Opera* III, 228] and once by Archimedes himself in the introduction of the treatise *On Spirals* [*Opera* II, 2], yet there are no certain references a man called Heraclides. Most of the essential information on Archimedes' personal and social life can be found in Dijksterhuis 1987.
[150] I.e. the Great World War of the classical Mediterranean.
[151] Netz 2004, p. 10.

Syracuse he evidently made a deep impression on the classical world, in particular due to his engineering talent; he built[152] several technical and mechanical devices such as the choclias, the planetarium and the hydraulic organ, acquiring fame through his devotion to the engines of war.

According to Cicero and Silus Italicus it is likely that he was born into a poor and humble family, although Plutarch[153] reports intimate connections between his family and King Hieron II of Syracuse. He was the son of the astronomer Pheidias, whom Archimedes refers to in a passage of *The Sand-Reckoner* or *Arenarius*. What can be dated with greater precision is the fact that Archimedes spent some time in Egypt, above all because he is mentioned by Diodorus of Agyrium, who declares that Archimedes made his hydraulic machine in Egypt, and, in the second place, because in Archimedes' prefaces to his treatises appear many references to a friendship with various scholars from Alexandria.[154]

Even the attribution of Archimedes' works is a difficult subject. The corpus surviving in Greek – which, following Reviel Netz, includes Eutocius' commentaries as well – includes the following works:

*SC* I: *On the Sphere and the Cylinder* (Book 1)

*SC* II: *On the Sphere and the Cylinder* (Book 2)

*SL*: *Spiral Lines*

*CS*: *Conoids and Spheroids*

*DC*: *Measurement of the Circle* (*Dimensio Circuli*)

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[152] Several writers of the classical world (among others, Cicero, Aratus of Soli, Sextus Empiricus and Cassiodorus) report the technical achievements of Archimedes, but from those writings it may be inferred that he was credited with the construction of such an instrument, not that he was held to be its inventor.

[153] In the *Tusculanae Disputationes* Cicero calls Archimedes *humilem homunculum*; Silus Italicus in his epic *Punica* mentions Archimedes' share in the defense of Syracuse and calls him *nudus opum*; Plutarch in his *Viae Marceli* XVI calls him *Ἱέρωνι τῷ Βασιλεῖ συγγενὴς καὶ φίλος*. Dijksterhuis 1987, p. 10, fn. 1-3.

[154] For more information, see Favaro 1923 and Dijksterhuis 1987, pp. 8-14.

*Aren.*: *The Sand Reckoner* (*Arenarius*)

*PE* I, II: *Planes in Equilibrium*

*QP*: *Quadrature of the Parabola*

*Meth.*: *The Method*

*CF* I: *On Floating Bodies* (*de Corporibus Fluitantibus*) (Book 1)

*CF* II: *On Floating Bodies* (*de Corporibus Fluitantibus*) (Book 2)

*Bov.*: *The Cattle Problem* (*Problema Bovinum*)

*Stom.*: *Stomachion*

Some works may be ascribed to Archimedes because they start with a letter by Archimedes himself, introducing and contextualizing the work in question: assuming that these are not forgeries (and their sober style suggests authenticity), they are the best evidence for ascription. Even more useful to us are the prefatory letters, which often connect the introduced works with other previous works by Archimedes.[155] In the third century BCE, knowledge of the works of Archimedes would entail an exposure to a complex web of correspondence between Mediterranean intellectuals. As for all the other ancient treatises (not just those of Archimedes), no one in this time seems to have known the works in the same arrangement as that of any of the surviving manuscripts. Books from late antiquity very rarely survive, and we can only guess that, during the fifth and sixth centuries (i.e. in Byzantium's first period of glory) several such collections containing works by Archimedes were made. In particular, it appears that an important collection was made by no less than Isidore of Miletus, the architect of Hagia Sophia. In late antiquity most of the books were transformed from papyrus rolls into parchment codices.[156] Most of the evidence concerning these ancient scholars only

[155] Further remarks on the autenticity of Archimedes' treatises can be found in Netz 2004, pp. 10-13.
[156] A codex by definition holds a collection of treatises.

emerged in the late ninth century CE: Byzantine culture initiated a process of renaissance by producing a substantial number of copies of ancient works. At least three codices[157], namely codex A, codex 𝔅 and codex C containing all of Archimedes' works, were produced during the ninth and tenth centuries. Codices A, founded in 1884, and 𝔅, which is no longer extant today, both had an important role to play in the history of Western science, for it was in Western Europe that they performed their historical service, having being removed there following Western Europe's first colonizing push. The culmination of this push was reached in 1204, when Constantinople itself was sacked by Venice and its allies, its old territories were parceled out to Western knights, and many of its intellectual treasures were looted. Codices A and 𝔅, among such looted works, soon made their way to Europe, and by 1269 were in the papal library in Viterbo, where William of Moerbeke used them for his own choice of collected works by Archimedes translated into Latin. Moerbeke's translation was not altogether unknown at the time, but it was the mathematical Renaissance of the fifteenth and sixteenth centuries that delivered to Archimedes greater intellectual prominence.[158]

Currently, as we have already said for Euclid, even if many translations and restorations are available, the most widely used is Heiberg's reconstruction. Heiberg had studied the manuscript tradition of Archimedes for over 35 years, starting with his doctoral dissertation, *Quaestiones Archimedeae* (1879), his First Edition (1880-81) and,

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[157] The three codices are divided and labelled as follows. i) Heiberg's codex A contains the works in the following sequence: *SC* I, II, *DC*, *CS*, *SL*, *PE* I, II, *Aren*., *QP*; Eutoc. *In SC* I, II, *In DC*, *In PE* I, II, and a work by Hero. This codex was prepared by Isidore of Miletus in Constantinople in the sixth century CE, and then was copied either by Leo the geometer or his associates, once again in Constantinople, in the ninth century CE. Later on it was lost, and then rediscovered again by Heiberg in Jerusalem at the beginning of the twentieth century. In turn the palimpsest disappeared again until in 1998 it was sold at a public auction at Christie's. The anonymous buyer granted the exhibit to William Noel, the director of the Walters Art Museum in Baltimore and the curator of the palimpsest. Dr. Reviel Netz of Stanford University, Nigel Wilson and Dr. Abigail Quandt constituted a team appointed to produce a transcription of the text, filling in gaps in Heiberg's account with the figures founded in codex A. ii) Around 975 CE another such codex was made, to be named by Heiberg codex B. It was probably a copy of a late ancient book. This codex B seems to have contained the following works: *PE* (I?), II, *CF* I, II, *Meth*., *SL*, *SC*, *DC*, *Stom*. Finally, iii) codex C (the classification given by Heiberg again) is a third Byzantine collection including works on mechanics and optics. In particular this was made up of the following works: *PE* I, II, *CF* I, II, *QP*.

[158] For further information, see Rose 1974, in particular ch. 10. I return to this topic in chapter 3.

after numerous articles detailing new discoveries and observations, proceeding to the Second Edition (1910-15). He considerably refined his views throughout this process, and the final position reached in 1915 seems to be proven solidly. Still, his final choice of letters reflects the circuitous path which got him there and is somewhat confusing.[159] Since Heiberg's Archimedes in its original version is in Latin, the main sources for comprehending demonstrations of the proposition that will be presented below are taken from Dijksterhuis' 1956 edition[160], which is considered to be one of the most thorough-going sources for studying and approaching Archimedean thought.

It is not the aim of this study to analyse in their totality all the contents of the numerous treatises published during all over the centuries. For our purposes we will focus the attention on several proofs and demonstrations concerning only the centre of gravity.

## 2.4 Archimedes and the Centres of Gravity

Archimedes' reflection on the centres of gravity can be understood by analysing his most important work on this topic, the *Planes in Equilibrium* (hereafter *PE*). Whereas in all other mathematical treatises Archimedes builds upon foundations established long ago by other scholars, in this work he deals with an investigation relating to the very foundations of physics, leaving the domain of pure mathematics to consider natural science from a mathematical viewpoint. As Dijksterhuis says, "he sets forth certain postulates in which he bases a chapter from the theory of equilibrium, and he is thus the first to establish the close interrelation between mathematics and mechanics, which was to become of such far-reaching significance for physics as well for mathematics".[161] The text is divided into two books. The first is organised in fifteen propositions with seven

[159] There is a long traditon explaining the use of symbols and sigla in Heiberg; he even used symbols of different kinds A, B and C, for codices that are similar in nature, in order to distinguish independent Byzantine manuscripts from the ninth to tenth centuries. Netz 2004, pp. 17-18.
[160] The Dutch original edition dates back to 1956, rather than the English translation which dates to 1987.
[161] Dijksterhuis 1987, p. 286.

postulates, and deals with the so-called static geometrical phenomena; from the postulates of the scale with equal arms, the law of the lever is stated. Later, the centres of gravity of the parallelogram, the triangle and the trapezoid are calculated. In particular, Archimedes demonstrates in the fifteen propositions of the first book that the centre of gravity of a triangle lies on the median line. The culmination of arguments about statics and the core of Archimedes' argumentation is represented by proposition 13, followed by the corollary proposition 14. In both of these, which will be analysed below, the centre of gravity is *intuitively* understood as the point in which the weight of the body is fully concentrated.

The second book, made up of ten propositions, starts with a demonstration of the law of the lever applied to a particular case, namely the situation in which two portions of segments are hung at the extremities of a balance. The book continues with the statement of the centre of gravity of the segment of the parabola (orthotome) and the truncated parabola. The most important proposition in the second book is the fourth one, which clearly deals with the demonstration of the centre of gravity of a segment of a parabola which lies on the diameter. The method used to attain this proposition is a combined application of the principles of the barycentric theory and the theory of the quadrature of the orthotome (which has been earlier dealt in Book 1); it is further based on the properties of this curve.

Although the text deals with the study of barycentres, it is particularly interesting that in both books there is no definition of the term 'barycentre', literally 'centre of weight'. The term appears for the first time only in the fourth postulate[162] of the treatise at hand, but without providing any definition. Since I have no intention to enter into the

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[162] The reasons that Archimedes refers to the centre of gravity without giving any precise definition can be traced to the following two explanations. First, it is possible that Archimedes, when writing the treatise on the equilibrium of planes, could assume the theory of the centre of gravity to be familiar to a certain extent, because this theory had already been developed either by earlier students of mechanics or by himself in a treatise now lost. Second, it is possible that the work on the equilibrium of planes is an entirely autonomous treatise, and that the definition of the concept of the centre of gravity is to be conceived of as being implied in the postulates on which this work is built. Dijksterhuis maintains that both these points of view are valid; Vailati 1996-97 upholds the palusibility of the first hypothesis. Toeplitz and Stein advoctes the second hypothesis, for which see Stein 1930. For further detailed study on the notion of centre of gravity, see Dijksterhuis 1987, pp. 289-304.

debate on the reasons for this absence, let us say simply that the notion of centre of gravity has a connection with the concept of the *centre of suspension* or *point of suspension*. The centre (or point) of suspension is the point in which, given two weights, we must suspend the rod of a balance in order to maintain its equilibrium. Archimedes himself in *PE I*, propositions 6 and 7, analyses the concept of equilibrium[163] in terms of the centre of suspension, and provides the following account: "Commensurable magnitudes are in equilibrium at a distance reciprocally proportional to the weights."[164] We can assume, therefore, that the concept of centre of gravity also derives from the use of the balance and has been refined geometrically and physically[165] over time, starting from Archimedes' works themselves.[166]

In *PE I*, 6-7, for example, in order to prove that the centre of gravity of a figure which possesses a diameter or axis falls on the diameter or axis, it is sufficient to have the approximants[167] with their centre of gravity on the axis.[168] In the case of the parabola, its relation with the triangle seems immediate, since the latter may be inscribed into the segment of the parabola, in order to reach such an approximation that the surface

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[163] Pappus has given the following definition: "the centre of gravity of any body is a point situated within which, and such that, if the body is imagined to be suspended from it, the weight will be at rest as it hangs and will keep its original position." [Pappus 1876-78, VIII, prop 1, p. 1031] Heron of Alexandria instead wrote that "the centre of gravity or inclination is a point which is such that the weight, if it is suspended in it, will be divided into equal parts." [Heron, *Mechanics*, 1, 24] Again in contrast, Eutokios of Ascalona defines it as "the centre of inclination of a plane figure the point from which it is to be suspended to stay parallel to the horizon, and by the centre of inclination or gravity of two more planes the one in which the balance is hung to be parallel to the horizon" [Cf. *Archimedis opera omnia cum commentariis Eutocii*, ed. J. L. Heiberg, 3 vols, B. G. Teubner, Leipzig 1880-1881]. Among other works on the law of the lever it is necessary to remember the *Liber Euclidis de ponderoso et levi,* attributed to Euclid, in which we find a geometrical proof of the law of the lever which is completely independent of Aristotelian dynamics. This proof also refers to the thesis according to which the effect of a weight *W* on the end of an arm of the lever (with length *L*) is given by the product *W*L*. See further Drachmann 1963 and Migliorato 2013, p. 86.

[164] Dijksterhius 1987, p. 289. Again in proposition 7 he says: "Even if the magniturde are incommensurable, they will be in equilibrium at distances reciprocally proportional to the magnitudes". See also *Ibid*. p. 305.

[165] Migliorato 2013, pp. 84-90.

[166] At the same time other scientists were working on the same topic, such as for example Pappus of Alexandria.

[167] Approximating figures are figures that can be inscribed and circumscribed to the figure whose centre of gravity one wishes to find out. Following this technique, the difference between the inscribed and the circumscribed figures may be made arbitrarily small. We will specify in the next section how to use the approximating figures when will we deal with the method of exhaustion.

[168] Napolitani 2001, p. 45.

difference between the major and the minor shape grows smaller until they almost touch.

For the sake of clarification let us observe the proof of the two above-mentioned propositions 13 and 4 of Books 1 and 2 respectively.

Proposition *PE I*-13: *In any triangle the centre of gravity lies on the straight line which joins any vertex to the middle point of the base.*

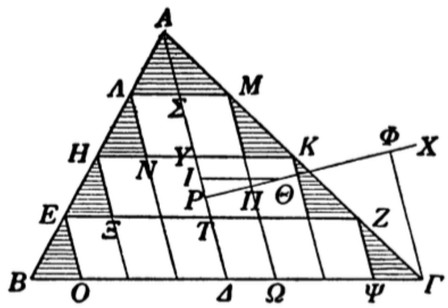


Figure 2.1

In Figure 2.1, with reference to the previous proposition, let $\Delta$ be the middle point of the base $B\Gamma$ of triangle $AB\Gamma$. Suppose that the centre of gravity $\Theta$ of the triangle $AB\Gamma$ does not lie on $A\Delta$. We know from the postulate VII[169] only that the centre of gravity lies *within* the triangle $AB\Gamma$. Draw the straight line through $\Theta$ parallel to $B\Gamma$ until the parts thus obtained (each equal to $\Delta\Omega$) are fewer than $\Theta I$; through the points of division draw straight lines parallel to $A\Delta$, and divide the triangle in the manner indicated in the figure into parallelograms ($MN$, $K\Xi$ etc.) and triangles ($\Lambda\Sigma M$, $Z\Psi\Gamma$, etc.). In proposition 9[170] the centre of gravity of the parallelograms all lie on $A\Delta$, and consequently the centre of gravity $P$ of the figure $X_\Pi$, consisting of all the parallelograms, also lies on $A\Delta$ (prop. 6).

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$P$ of the figure $X_\Pi$, consisting of all the parallelograms, also lies on $A\Delta$ (prop. 6). Join $P$

[169] In any figure whose perimeter is concave in the same direction, the centre of gravity must be within the figure.

[170] Proposition 9 states that the centre of gravity of any parallelogram lies on the straight line joining the middle points of opposite sides of the parallelogram. Dijksterhuis 1987, p. 307.

straight line meet the straight line $\Omega M$ in $\Pi$ and the straight line drawn through $\Gamma$ parallel to $A\Delta$ in $\Phi$. *$\Pi$ lies between P* and $\Theta$, and $\Phi$ lies outside the triangle.

Then it must be ascertained how the centre of gravity of the figure $X_\Delta$, consisting of all the shaded triangles, must be situated with respect to the points $P$, $\Theta$ and $\Phi$.

For this we first compare the area of the triangle $AB\Gamma$ with the sum of the areas of the shaded triangles. By similarity we have:

$$(\mathrm{A\Delta\Gamma}, \mathrm{\Lambda\Sigma M}) = [\mathrm{T(A\Gamma)}, \mathrm{T(AM)}], \text{etc.}$$

Whence:

$$(\mathrm{A\Delta\Gamma}, \mathrm{\Lambda\Sigma M} + \cdots + \mathrm{Z\Psi\Gamma}) = [\mathrm{T(A\Gamma)}, \mathrm{T(AM)} + \cdots + \mathrm{T(Z\Gamma)}] =$$
$$= [\mathrm{T(A\Gamma)}, \mathrm{O(AM, A\Gamma)}] = (\mathrm{A\Gamma}), \mathrm{T(AM)}$$

Likewise:

$$(\mathrm{A\Delta\Gamma}, \mathrm{\Lambda\Sigma\Lambda} + \cdots + \mathrm{EOB}) = (\mathrm{AB}, \mathrm{A\Lambda}) = (\mathrm{A\Gamma}, \mathrm{AM})$$

Therefore:

$$(\mathrm{A\Delta\Gamma}, \mathrm{X_\Delta}) = (\mathrm{A\Gamma}, \mathrm{AM}) = (\mathrm{\Delta\Gamma}, \mathrm{\Delta\Omega}) = (\mathrm{P\Phi}, \mathrm{P\Pi}) > (\mathrm{P\Phi}, \mathrm{P\Theta})$$

And thence:

$$(\mathrm{X_\Pi}, \mathrm{X_\Delta}) > (\mathrm{\Theta\Phi}, \mathrm{\Theta P})$$

Now determine a point $X$ on the straight line $P\Phi$ such that:

$$(X_\Pi, X_\Delta) = (\Theta X, \Theta P)$$

then $\Theta X > \Theta\Phi$, therefore $X$ lies on $P\Phi$ produced.

In proposition 8[171], $X$ is now the centre of gravity of the figure $X_\Delta$, consisting of the shaded triangles, which is impossible because all these triangles are on the opposite side of the straight line drawn through $\Gamma$ parallel to $A\Delta$ from $X$. The latter conclusion is not based on postulate VII, for the perimeter of the figure $X_\Delta$ is not concave in the same direction. It should rather be imagined that it is made in view of the consideration that, if the centre of gravity is found of a figure whose component parts all lie on the same side of the straight line, after having combined with the aid of prop. 6 two parts, i.e. by combining their combination with a third part, etc., the centre of gravity of the whole figure must lie on the same side of the straight line on which all the parts also lie.[172]

Let us turn now to the Proposition *PE* II-4: *the centre of gravity of any segment comprehended by a straight line and an orthotome lies on the diameter of the segment.*

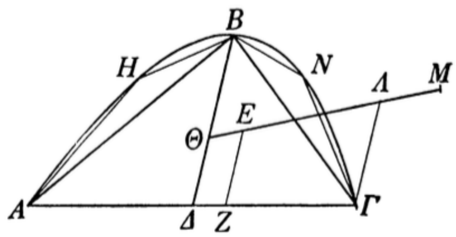


Figure 2.2

[171] Proposition 8 states that if another magnitude be taken away from a magnitude which does not have the same centre as the whole, when the straight line joining the centres of gravity of the whole magnitude and the magnitude that is taken away is produced towards the side where the centre of the whole magnitude is situated, and when from the produced part of the line joining the said centres, a segment is cut off in such a way that it has to the segment between the centres, the same ratio as the weight of the magnitude which is taken away has to the remaining magnitude, and the extremity of the segment cut off will be the centre of gravity of the remaining magnitude. Dijksterhuis 1987, p. 306.

[172] Dijksterhuis 1987, pp. 310-311.

Suppose that the centre of gravity $E$ of the segment $\Sigma$ $(AB\Gamma)$ does not lie on the diameter $B\Delta$ and the straight line through $E$ parallel to the diameter meets the chord $A\Gamma$ in $Z$. Inscribe in the segment $AB\Gamma$ the triangle $AB\Gamma$, and now take a magnitude $K$ such that:

$$(\Gamma Z, \Delta Z) = (\text{triangle } AB\Gamma, K)$$

Now inscribe in the segment $\Sigma$ in the recognised manner the figure $\Omega$, such that the sum of the remaining segment $\Sigma - \Omega$ is less than $K$. Let $\Theta E$ be met in $\Lambda$ by the straight line through $\Gamma$ parallel to the diameter of the segment. Since $\Omega > AB\Gamma - triangle$ and $\Sigma - \Omega < K$, we have:

$$(\Omega, \Sigma - \Omega) > (AB\Gamma - \text{triangle}, K) = (\Gamma Z, \Delta Z) = (\Delta E, E\Theta)$$

The centre of gravity $M$ of $\Sigma - \Omega$ is now found from the centre of gravity $E$ of $\Sigma$ and $\Theta$ of $\Omega$ by means of the relation (Book 1, prop. 8):

$$(ME, \Theta E) = (\Omega, \Sigma - \Omega)$$

Thence:

$$(ME, \Theta E) > (\Lambda E, \Theta E) \text{ so that } ME > \Lambda E$$

$M$ is therefore on the opposite side of $\Gamma\Lambda$ to $E$, which is impossible because the whole segment lies on the same side of $\Gamma\Lambda$ with $E$.[173]

Those two proofs can be considered equivalent in their methodology, firstly because both are based on the well-known technique of *reductio ad absurdum,* and secondly

[173] This reconstruction is taken from Dijksterhuis 1987, pp. 348-349.

because the mechanical demonstrative procedure is based on the following common idea, namely that "to demonstrate that the centre of gravity of a figure having a diameter (or an axis) lies on the diameter, it is sufficient to construct approximating figures whose centres of gravity are on the diameter."[174] To show where is the centre of gravity of a certain figure, Archimedes uses the basic idea that we have called 'the expulsion of the centre of gravity' of the remaining figure, which in turn is given by the difference between the circumscribed and the inscribed figure. By 'expulsion' we mean that the figure we want to measure (or weigh[175]) is first of all considered as if it were represented by its own centre of gravity – as a point with a volume and other neglected physical features – and secondly the centre of gravity is imagined to be shifted along the arm of a virtual balance represented (as in the proposition presented above) by the segment $\Theta M$, where $\Theta$ is the centre of gravity of both the parabolid and the triangle, and $M$ is the centre of gravity of the figure inscribed in the paraboloid and circumscribed to the triangle.

The method of expulsion and the weighing technique will become clearer in the next section, where we will analyse in detail a proposition from the most famous treatise attributed to Archimedes, *The Method*, in which the mechanical procedure of the virtual balance is used in a particularly confident way.

Finally, the procedure used for drawing the circumscribed and inscribed approximating figures was already stated by Archimedes in some of the propositions demonstrated in *Conoids and Spheroids*.

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[174] Napolitani and Saito 2004, p. 77.

[175] Let me clarify why we use the term 'weight' instead of 'mass', at least when we refer to Ancient and Renaissance mathematics. In antiquity the term 'weight' was used as a qualitative feature, just like colour and temperature, and it was used in reference to these properties related to single bodies. Directly related to it is the concept of lightness – a complementary qualitative feature of bodies. According to the Peripatetic school – and before them, Aristotle – weight is an intensive magnitude, so it cannot be related to the concept of *quantitas materiae* (quantity of matter) which in turn constitutes a quantitative attribute of rigid bodies. In fact, in nature there also exist some elements, such as fire and its derivative, which are naturally light. Only in Platonic thought will we facilitate a metaphysical identification between matter and space, therefore at this time we have the formulation of the first kernel of the theoretical quantitative determination of matter in the defining of the volume and (quantity of) space occupied by the body.

For the formulation of the modern Newtonian notion of mass, we must wait until the second half of the seventeenth century. For further information on this last issue see chapter 4.

## 2.5 The Method of Mechanical Theorems

As sketched above (fn. 157), the *Method* has travelled all around Europe and the Middle East for centuries. For our purposes, the most important refining of the concept of the centre of weight appears only in Archimedes' *The Method of Mechanical Theorems* or (in its briefer title) *The Method*. According to the latest historiographical research, the *Method* can be dated back to the last years of Archimedes' scientific activity. This treatise is thus a mature work, in which Archimedes sums up his mathematical activity, declaring to the addressee of the work Eratosthenes of Chios his intention to illustrate in detail his mechanical procedure, which he had already used in other essays and was useful for his illustrative purposes. It is clear that Archimedes' intention was to give a mechanical interpretation to those propositions which were purely geometrical. Furthermore, the mechanical nature of his work highlights the heuristic dimension of his arguments. For example, in the letter to Eratosthenes, Archimedes writes:

> Since, as I said, I know that you are diligent, an excellent teacher in philosophy, and greatly interested in any mathematical investigations that may come your way, I thought it might be appropriate to write down and set forth for you in this same book certain special methods, by means of which you will be enable recognize certain mathematical questions with the aid of mechanics. I am convinced this is no less useful for finding the proofs of these same theorems. For some things, which first became clear to me by the mechanical method, were afterwards proved geometrically, because their investigation by the said method does not furnish an actual demonstration. For it is easier to supply the proof when we have previously acquired, by the method, some knowledge of the questions than it is to find it without any previous knowledge.[176]

Euclidean mathematics is characterised by a special case of the form of mathematical argument, i.e. a kind of exaggeration of the synthetic structure requiring an irrefutably persuasive sequence of logical conditions. In contrast, Archimedes, already in the prefatory letter to Eratosthenes, states that "before he knew how to prove his theorems,

[176] English trans. taken from Dijksterhuis 1987, p. 314.

one needs to become convinced of their empirical truth." The *Method* is an elementary essay written in order to train an audience unprepared on the topic; Eratosthenes is the pupil needing of education. Compared with *PE I* and *II*, the *Method* represents the *chef d'oeuvre* of the barycentre's science of solid figures, the acme of Archimedes' theoretical works. At the base of this exposition there are three mathematical techniques: the *idealized lever* or *virtual balance*, the *process through infinitesimal sections* and the *principle* or *method of exhaustion*. An extensive discussion of these procedures exceeds the boundaries of this dissertation, but we wish to show, through the analysis of some Archimedean demonstrations, that the notion of centre of gravity and the material point share some essential features. For our purposes, therefore, it is enough to outline the mechanical and heuristic values attributed to the centre of gravity as a *material point*, paying attention to the combined application of these techniques.

Let us analyse briefly these procedures by looking at the construction and the measurement of the volume of the so-called cylinder hoof,[177] as shown in Figure 2.3 below.

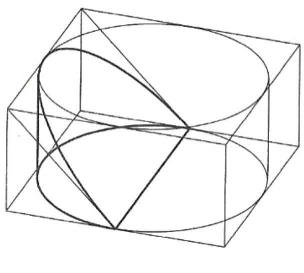

Figure 2.3

[177] The four propositions (12-15) that deal with the construction of the cylinder hoof cannot be found in Archimedes' original treatise. Rather they represent a modern reconstrution made after the rediscovery of the palimpsest. The reason I have decided to introduce this demonstration in order to present the three mathematical techniques is that in its modern reconstruction it appears clearly how these techniques are used independently from one another, and what the basic assumptions are on which they rely. Moreover, proposition 15 does not make use of the notion of barycentre, hence it is not indispensable in my reconstruction in aiming to show the affinities between the geometrical notion of centre of gravity and the mathematical object of the point mass.

When we read Archimedes' postulates or propositions with an unbiased mind, we often have the striking impression that the author is thinking of a lever, or balance, in its fulcrum (the lever being idealized as a straightforward straight line) on which thin plates (idealized as planimetric figures, which clearly represent idealized physical weights) are attached in their centres of gravity. In other words, this virtual machinery is assembled in the following way*:* first of all, we build a geometrical figure whose axis of symmetry is linked to the arm of the balance and its fulcrum is placed in the middle point of this axis. On its ends are shifted sections of figures that need to be weighed (or measured). Secondly, we consider the ratios of the different sections, in order to establish the equilibrium between them. Basing ourselves on this proportion and the fundamental law of the lever (previously demonstrated by Archimedes) we can state that geometrical objects are in equilibrium at inversely proportional distances to their *volumetric* extensions,[178] even when these mechanical operations are calculated with the virtual balance. Once we have completed these steps, we put the figures back together around their own respective barycentre, extending the possibility of the application of the law of equilibrium to the reconstructed figures: *if the single sections are in equilibrium, even the original figures* (made up of these single sections) *are supposed to be in equilibrium.*

Let us now turn to proposition 12 which reads as follows: *If in a right prism with square bases a cylinder be inscribed which has its bases in the squares facing each other and the surface of which touches the four other faces, and a plane be drawn through the centre of the circle which is the base of the cylinder and a side of the opposite square, it is recognised by this method that the solid cut off by the plane thus drawn is one-sixth of the whole prism.*[179]

Now by looking at the Figure 2.4 A and B below, it may appear clearer what the constituents of the virtual geometrical balance are. The arm of the balance is the segment $\Xi\Pi$, having the point $\Xi$ and $\Pi$ as ends; the fulcrum is the point $\Theta$; and the two weights hanging at the ends are the right prism and the half-cylinder respectively.

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[178] Acerbi, Fontanari and Guardini 2013, p. 65.
[179] Dijksterhuis 1987, p. 331.

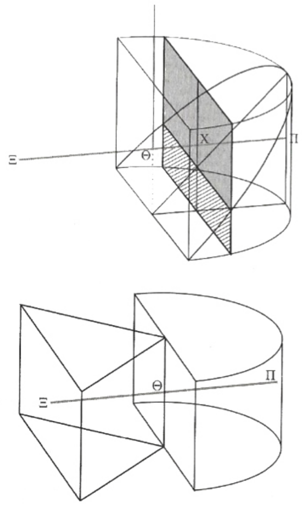


Figure 2.4 A and B

The *process through infinitesimal sections* is firstly stated in the preface cited after the prefatory letter to Eratosthenes. This paragraph asserts as follows:

> If in two series of magnitudes those of the first series are, in order, proportional to those of the second series and further the magnitudes of the first series, either all or some of them, are in any ratio whatever to those of a third series, and if the magnitudes of the second series are in the same ratio to the corresponding magnitudes of a fourth series, then the sum of the magnitudes of the first series has to the sum of the selected magnitudes of the third series the same ratio which the sum of the magnitudes of the second series has to the sum of the correspondingly selected magnitudes of the fourth series.[180]

This technique connects portions of figures which shows the *iterated proportionality* between them. This proportionality also exists among the reconstructed figures. Once again this technique is used in proposition 15, which displays the procedure used to build the *cylinder hoof*. The *modus operandi* is the following: consider a prism with square base in which is inscribed a cylinder. Now cut the prism by a plane passing

[180] Trans. taken from Heath 1912, p. 15. The same statement is also asserted in Heath 1912, proposition 11, pp. 163-166; and *On Conoids and Spheroids*, lemma and proposition 1, pp. 105-106, for which see also Heath 1897.

through the centre of the circle of one of the cylinder's bases and through one of the sides of the square on the opposite face of the prism. In the figure below, the shaded triangle represents the section that will be iterated in order to measure the magnitude of the cylinder hoof.

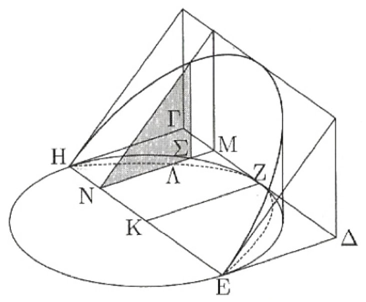


Figure 2.5

Finally, there is the *principle of exhaustion*, also called *indirect passage to the limit* by Dijksterhuis. Although the first formulation of this can be traced back to Eudoxus of Cnidus, one of the first examples of its application is only presented in *Elements* XII, 2[181] and in the *Method*. According to the modern reconstruction, the *principle of exhaustion* consists of the approximation of a figure by a sequence of figures inside it; this allows us to determine the volumetric expansion of a two-dimensional or three-dimensional figure. The important point of this principle is that the sequence of approximations can be made, such that the difference between the original figure and the inscribed figure decreases by at least half at each step of the sequence. In other words, at the very core of this procedure, there is the so-called axiom of Archimedes, sometimes also termed the postulate of Eudoxus – which was already used in Euclid V – which states: "[m]agnitudes are said to *have a ratio* to one another which can, when multiplied,

[181] In this specific case (Elements, XII, 2) it is used to prove that "circles are to one another as the squares on their diameters" (see the demonstration of proposition 4).

exceed one another.”[182] Thanks to this axiom it is possible to express the process called the ‘indefinite approach’. Here again we provide two figures in order to understand better how the procedure works geometrically; having built a figure bigger than the portion of hoof – having the shape of a parabola in its two-dimensional drawing – we want to measure another figure smaller than the hoof; then these two are the approximations of the figure in question, and their difference gives us the magnitude of the cylinder hoof.

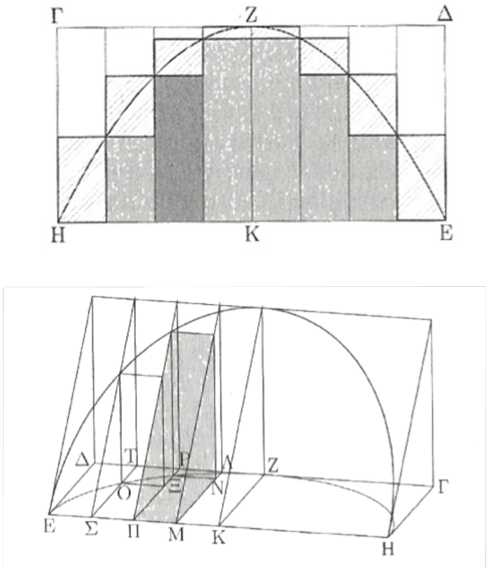


Figure 2.6 A and B

[182] Cf. the online source: http://aleph0.clarku.edu/~djoyce/java/elements/bookV/defV5.html; also http://aleph0.clarku.edu/~djoyce/java/elements/bookX/bookX.html.

This axiom can be found in Euclid, Book X, proposition 1 as follows: “Two unequal magnitudes being set out, if from the greater there is subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, then there will be left some magnitude less than the lesser magnitude set out. And the theorem can similarly be proven even if the parts subtracted are halves.” This proposition is considered as the foundation of the method of exhaustion of Book XII. It is not used in the rest of Book X and would perhaps be better placed at the beginning of Book XII. This method is used in the propositions concerning areas of circles and volumes of solids. At the same time it is also used in Archimedes, *On the Sphere and Cylinder* in the following guise: “With regard to two unequal magnitudes *a* and *b* ($a<b$) it is assumed that there exist a number *n* having the property that $n*a>b$.” In this last case it can be considered interpretation of the already-mentioned definition 4 of Euclid V.

Turning now our attention to Archimedes' demonstration, let us now analyse the way in which Archimedes combined the three mathematical techniques, focusing attention on proposition number 4 of the treatise at hand, a proposition of central importance due to its clarity in showing the essence of Archimedes' heuristic 'method'. We shall reconstruct some passages of Archimedes' presentation that are useful to understand the development of the notion of the *material point*. First we will offer the original geometrical version of proposition number 4, and second we will show its algebraic reconstruction.

Proposition 4: *Any segment of a right-angled conoid* (i.e. a paraboloid of revolution) *cut off by a plane at right angles to the axis is* $1\frac{1}{2}$ *times the cone which has the same base and the same axis as the segment.*

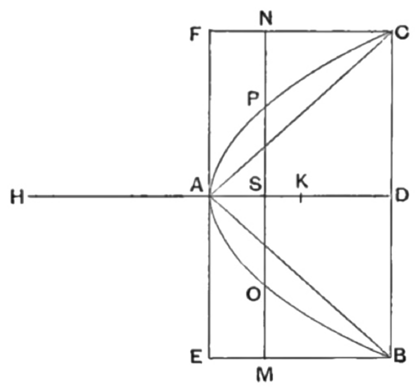


Figure 2.7

Let a paraboloid of revolution be cut by a plane through the axis in the parabola $BAC$; and let it also be cut by another plane at right angles to the axis and intersecting the former plane $BC$. Produce $DA$, the axis of the segment, to $H$, making $HA$ equal to $AD$. Imagine that $HD$ is the arm of a balance, $A$ being its middle point – i.e. the fulcrum. The base of the segment being the circle on $BC$ as diameter and in a plane perpendicular to $AD$, imagine i) a cone drawn with the latter circle as base and $A$ as vertex, and ii) a cylinder with the same circle as base and *AD* as axis. In the parallelogram *EC* let any straight line *MN* be drawn parallel to *BC*; and through *MN* let a plane be drawn at right angles to *AD*; this plane will cut the cylinder in a circle with diameter *MN* and the paraboloid in a circle with diameter $OP$. Now, $BAC$ being a parabola and $BD$, $OS$ ordinates:

$$DA\colon AS = BD^2\colon OS^2$$

or:

$$HA\colon AS = MS^2\colon SO^2$$

Therefore:

$$HA\colon AS = (\text{circle, rad. MS}) : (\text{circle, rad. OS}) =$$
$$= (\text{circle in cylinder}) : (\text{circle in paraboloid})$$

Now we can establish the *equilibrium between the sections*. As already said, for the fundamental law of the lever, geometrical objects are in equilibrium at distances inversely proportional to their extension. We can deduce that the diameter circle *MN* (which constitutes a section of the cylinder), just in the place where it is, is in a state of equilibrium with respect to the point *A* (fulcrum), with the diameter circle *OP* (section of the parable) translated in $H$ (which is also its centre of gravity).[183]

Finally, the figures are *reassembled*, starting from their circular sections: the cylinder comes to coincide with the original one, and the segment of the parabola is not rebuilt around the centre of gravity $H$, but is shifted and set to be disjointed with respect to the other geometrical component of the figure, which stands for the virtual balance. The circles that compose the parabola are superimposed upon one another, so that the

[183] This mathematical reconstruction is taken from Heath 1912, pp. 24-25.

centre of gravity of their aggregate coincides with the centre of gravity of a single circular section which is located in $H$. Since the individual sections are in equilibrium, so too are the original figures.

The core of this procedure is represented by the geometrical argument of *proportionality*. For each figure a proportion is established between its sections and the sections of an adequate auxiliary figure, as well as between two straight lines which coincide with the arms of the balance.[184] This technique is used in all propositions of the *Method* with the cases' variations, which are more or less complex depending on the figures involved in the demonstration. In fact, sometimes the topic gets more complex when a third auxiliary figure is introduced. This figure is divided in an infinite number of sections and used as a match to the original one (in most cases it is a cone). The richness of Archimedes' *Method* has no equal in the history of ancient mathematical thinking, and he shows a justly famous ability when applying these heuristic and mechanical arguments of proportionality in the already-mentioned case of the *cylindric hoof* (propositions 12-15).

To clarify Archimedes' method, let us explain proposition number 4 in a more familiar mathematical vocabulary.[185]

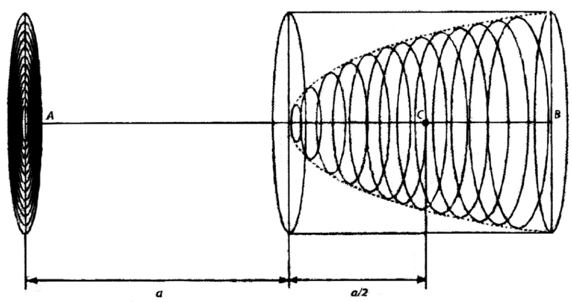


Figure 2.8

Paraboloid of revolution is divided into infinitely many concentric circular-sections and circumscribed by a cylinder with equal height. In order to determine the volume of the paraboloid, imagine that all the

[184] *Ibid.*, pp. 66-67.
[185] Both the figure and the reconstruction are taken from Napolitani 2001 pp. 49-53.

circular sections are gathered together around the point *A* of one of the arms of the balance.

Let us take the paraboloid *P*, with axis measuring *a* and the circumscribed cylinder *C*, with a height of *a* and whose radius measures *b*. We proceed by cutting the solid with a plane in a generic point *x* of its axis: the circular-section *Px* of the paraboloid has radius *y*, while the circular-section *Cx* of the cylinder has radius *b*. The circles are related to each other as the squares of the radii; that is:

$$P_X : C_X = y^2 : b^2$$

Here *y* and *b* are the ordinates corresponding to the abscissaes *x* and *a* respectively. In the parabola the squares of the ordinates are proportional to the abscissas, that is:

$$y^2 : b^2 = x : a$$

By the rule of transitivity we will have:

$$P_X : C_X = x : a$$

Therefore: *a generic circular-section of the parabola is to the circular-section of the cylinder as the abscissa of the point section is to the axis of the parabola*. So far, Archimedes has applied some well-known theorems, combined with the principle of proportionality, that were also already used in *PE I* and *II*. Now we are going to see how his mechanical techniques are applied in order to find the volumes of the figures involved. We are also going to study the application of the method of the virtual balance in the case in which the arm of the balance has length *2a* (twice as much the axis of the parabola) with *A* and *B* as ends. Let us imagine that around the end *A* of the balance are concentrated all the circular sections of the parabola. Each of them stands in equilibrium with a single circular section of the cylinder. For this reason all the circular sections are supposed to be in equilibrium with the entire cylinder. They will be in equilibrium with the entire cylinder if and only if we imagine that its weight is fully concentrated around its centre of gravity (where the centre of gravity of a cylinder is located in the middle of its axis). Here we have two quantities hung at the two ends of the balance: at the point *A* we have the mass of the circular sections of the parabola, which is located at *a* distance from the fulcrum. At the point *C* we have the cylinder (assuming that its weight is concentrated in the centre of gravity), which is located at a distance *a/2* from the fulcrum. Given the equilibrium of the system and according to the law of the lever, the two quantities stand in inverse proportion to the distances at which they are located with reference to the fulcrum.[186]

Thanks to this analysis it is possible to explain the connection between the concept of the centre of weight and the concept of the point mass. It is moreover vital to bear in mind that in modern physics and mechanics, a point mass is a point-shaped body in which we can neglect the geometrical dimensions with reference to its movements and its possible rotations.[187] The identification of the barycentre is important to treat a series of theoretical problems of the equilibrium for two-dimensional and three-dimensional figures. Moreover, although this notion is related to geometrical problems, it is calculated by a mechanical procedure; triangles, parabola, cylinders, cones and cubes are seen as three-dimensional bodies whose weight (as volumes) are supposed to be concentrated in a single point, which takes the name of the *centre of weight*. This mechanical technique is considered to be a preparatory step towards an ultimately geometric proof; Archimedes himself at the end of proposition 1 of *The Method* trains Eratosthenes in encountering a technique which is only heuristic in nature and will be useful later on, in the search for the geometric demonstration. Archimedes' argument means that the aforementioned conclusion has not therefore been proved by the above demonstration, but a certain impression has been created that the conclusion is true. Since we thus see that the conclusion has not yet been proved, but we suppose that it is true, we shall mention the previously published geometrical proof, which we ourselves

[186] Napolitani 2001, pp. 50-51.
[187] Nolan 1996.

have found for it, in its appointed place.

Since the *Method* ends with proposition 15, we do not know if Archimedes ever sent to Eratosthenes supplementary sections with the geometric illustrations. The nucleus of Archimedes' technique is the shift from the study of an *individual* section of two three-dimensional objects being compared to the analysis of a *set* of sections, which are supposed to gather around a centre of gravity. Archimedes' intuition is to consider the parabola as a body divided properly in endless concentric sections. This intuition stood contrary to classical Greek geometry: expressions such as 'division in infinite sections' may be considered problematic because they resemble the problem of the geometric continuum.

The *mechanical abstraction* used to calculate the volume of the three-dimensional objects, which stands in a certain ratio to one another, and which are weighed as if they were material bodies, is an absolute innovation. This technique allows mathematics to consider only the quantitative relations between the bodies, without considering some of their physical properties. The three-dimensional objects are imagined to be represented only by their own centre of weight, or rather by means of a geometrical point devoid of any physical content. Although the model-building practice is still yet to develop properly, and in Archimedes' time there is no awareness of the meaning and practice of idealization and abstraction, it seems reasonable to examine this mechanical procedure as if any rigid body is treated like the so-called *material point*, that is to say like abstract bodies that can be freely shifted along the arms of a balance and weighed. This idealized lever – as we have already stated before – is realized with two-dimensional geometric elements. In this way, the centre of gravity becomes a useful abstract instrument to represent geometrical bodies in a simplified way. In fact, in the final step of the weighing procedure, we *do not* find any longer the two three-dimensional figures of the beginning, but instead we have the volume of the circular sections of the parabola in point $A$ and the cylinder in point $C$, with its weight supposed to be concentrated in the centre of gravity.

Before moving on to the next section I wish to consider the arguments in favour of the idea that behind Archimedes' *Methods* lies hidden an early *method of indivisibles*. The belief that the heuristic method used by Archimedes could be seen as a precursor of the procedure attributed to Bonaventura Cavalieri (1598-1647) is widely discussed in Heiberg, Heath, Dijksterhuis and Netz, and the aim of this paragraph is to show how anachronistic this idea is if we carefully read Archimedes' demonstrations without properly contextualizing them. According to these authors, traces of the method of indivisibles are related to the well-known exhaustion method which consists – as already said and shown in the previous demonstrations of the fourth proposition of *The Method* – in weighing *infinitesimal* elements of *X* (with or without the addition of the corresponding elements of another figure *C*) against the corresponding elements of a figure *B*, *B* and *C* being such figures that their areas or volumes, and the position of the centres of gravity of *B*, are known beforehand. For this purpose the figures are first placed in such a position that they have, as a common diameter or axis, one and the same straight line; if then the infinitesimal elements are sections of the figures made by parallel planes perpendicular (in general) to the axis and cutting the figures, the centres of gravity of all the elements lie at one point or another on the common diameter or axis. The diameter or axis is produced and is imagined to be the arm of a balance. It is sufficient to take the simple case where the elements of *X* alone are weighed against the elements of another figure *B*. The elements which correspond to one another are the sections of *X* and *B* respectively by any one plane perpendicular to the diameter or axis, and which cut both figures; the elements are spoken of as straight lines in the case of planes and as plane areas in the case of solid figures. But are those planes or solid figures really made up of infinitesimal elements, as we have said above? Did Archimedes ever even mention the word 'infinitesimal'? Was he aware of this 'modern' procedure widely used after Bonaventura Cavalieri? Although Archimedes calls the elements straight lines and plane areas respectively, we cannot argue that they are *of course* (as Heiberg and Heath did in their translations of the treatise), in the first case, indefinitely narrow strips (areas) and, in the second case, indefinitely thin plane *laminae* (solids). I am aware that from the modern visual reconstruction which we provide above, it appears that the number of the elements in each figure is infinite, but Archimedes has no need to say this; he merely says that figures such as $X$ and $B$ are *made up of all the elements* in them respectively, i.e. of the straight lines in the case of areas and of the plane areas in the case of solids. The target of Archimedes was thus to arrange the balancing of the elements in such a way that the elements of $X$ are all considered at *one point* of the lever, while the elements of $B$ are placed at different points, namely where they actually are in the first instance. The reason that *The Method* is so relevant is because Archimedes contrives therefore to *move* the elements (or the circle sections of the figures) of $X$ away from their first position and concentrate them at one point on the lever, while the elements of $B$ are left where they are, and so act at their respective centres of gravity. This mechanical method is not to be seen as integration, which would naturally be used to find the area or volume required directly. Archimedes deals with moments about the point of suspension of the lever, i.e. the products of the elements of area or volume with the distances between the point of suspension of the lever and the centres of gravity of the elements respectively. Archimedes was bound to experience a great deal of doubt and uncertainty with regard to the application of the method of indivisibles, for here he touches upon a question which in the preceding centuries had given rise to keen controversy more than most other questions in Greek mathematics. It was the profound question of atomism or continuity on which, though originating from physics, opinions were also divided in mathematics, and which finds its clearest expression in the *aporia* that worried Democritus: if the circular sections that can be made in a cone parallel to the base are congruent, how can the cone differ from a cylinder; and if they grow smaller towards the vertex, is not then the curved surface, which should be smooth, and scalariform?[188]

[188] For the details on this debate, see Dijksterhuis 1987, pp. 318-21.

## 2.6 Floating Centres of Gravity

The treatise titled *On Floating Bodies* (hereafter *FB*) is devoted to hydraulics, hydrodynamics and hydrostatics. It is composed of two books: the first (consisting of 9 propositions) is purely geometrical in nature and contains an examination of the principles which underlie the flotation of bodies, whereas the second (10 propositions) deals with the application of the theorems previously stated, and so Book 2 is entirely dedicated to hydrostatics regarding the construction of vessels. In fact, the main purpose of Archimedes' study was exactly marine engineering, and once again the ambition was to emphasise the heuristic and mechanical approach relating to his work.

The Greek text has only been known since 1899. Before this time scholars always had to make do with William of Moerbeke's Latin translation. It is in this treatise that one can finally find the definition of the centre of weight, or, to put this more exactly, this definition becomes clear as soon as we have apprehended the meaning and results of the proofs of some essential propositions, in particular those in which Archimedes examines the different behaviour of bodies with different (specific)[189] weights with respect to the liquid on which they are floating.

The entire treatise relies on postulate 1,[190] which states the following: *Let it be supposed that a fluid is of such a character that, its parts lying evenly and being continuous, that part which is thrust the less is driven along by that which is thrust the more; and that each of its parts is thrust if the fluid be sunk in anything and compressed by anything else*. Thus, two *adjacent sections of fluid* cannot stand in equilibrium if they are pressurized differently. Let us now turn to the outcomes deriving from the application of this postulate.

The first case, as we can see in Figure 2.9 below, envisages a material body and a medium with the same (specific) weight. The body will sink down in the fluid until the

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[189] The specific weight, also known as the unit weight, is the weight per unit volume of a meterial. The reason 'specific' is included in brackets is that Archimedes never used that term, although it is clear in contemporary physics that in problems concerning hydrostatics we refer to it in order to explain the reason for the equilibrium of bodies immersed in fluids.

[190] We will say more on this postulate below, but for now let us take it for granted.

surfaces no longer project above that of the fluid, and they will not be driven down any further (proposition 3).

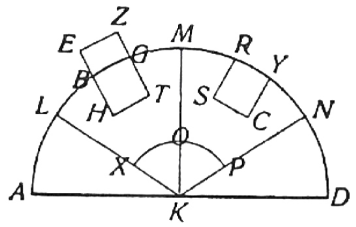


Figure 2.9

In the second case (Figure 2.10) the solid has a (specific) weight that is lighter than the medium, so that when the body is thrown into the fluid it will not be completely submerged, but a portion of it will project above the surface of the fluid. From here Archimedes derives that a solid that is lighter than the fluid, when thrown into the fluid, will sink until a volume of the fluid equal to the volume of the immersed portion has the same weight as the whole solid. Further, *when solids lighter than the fluid are forcibly immersed in the fluid, they are thrust upwards by a force equal to the weight by which a volume of the fluid equal to the solid exceeds that solid* (propositions 4, 5 and 6).

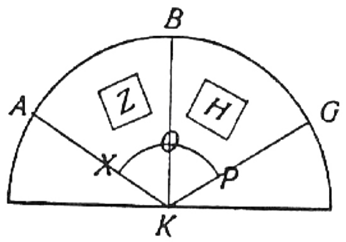


Figure 2.10

The third case (Figure 2.11) asserts that when a solid has a (specific) weight that is greater than the medium and is thrown into the fluid, it will be driven down as far as it can sink, and it will be lighter in the fluid by the weight of a portion of the fluid having the same volume as the solid (proposition 7).

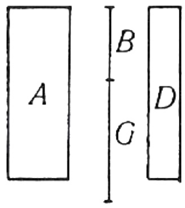


Figure 2.11

It is now time to raise the question of the real meaning of Archimedes' hydrostatic arguments. Two items in particular deserved to be emphasised: i) the definition of the barycentre and ii) the main outcomes represented by the so-called 'principle of Archimedes', which is not properly a principle but rather a theorem because it is demonstrated from postulate 1 of Book 1, and only at a later time is a proper principle derived from it.

Regarding point (i), the propositions mentioned above allow one to derive postulate 2 which reads as follows: *Let it be granted that bodies which are forced upwards in a fluid are forced upwards along the perpendicular [to the surface] which passes through their centre of gravity*.[191] Therefore, Archimedes is saying here that the body is forced to move upwards (but the same is also true if the body is forced to move downwards) by a force which exerts its intensity along the perpendicular, and passes through the centre of gravity of the body, rising or sinking in the fluid. However, the force is exerting its intensity uniformly on the whole surface of the body sinking in the fluid. By following postulate 2, we can maintain that it is only a matter of simplification and representation that a body is imagined, first of all, as devoid of its volume, and it is represented through its centre of gravity, and secondly, on this simplified point, that the force exerts its own intensity.

[191] Heath 1897, p. 261.

According to a modern point of view, the point at which all the parallel verticals along which the heavy bodies tend upwards (or downwards) is one in which a unique force is fully concentrated, i.e. as a force resulting from the sum of all the upward (or downward) forces acting on a body, towards which that body will rise (or sink). In other words this leads to the definition of the *centre of pressure*, or rather a point where the total sum of a parallel pressure field acts on a body, causing a force to act through that point.

With respect to point (ii), the well-known principle of Archimedes, which is demonstrable by considering only postulate 1 at the very beginning of the treatise, can, as has been already mentioned above, be derived proposition 1: *If a surface be cut by a plane always passing through a certain point, and if the section be always a circumference [of a circle] whose centre is the aforesaid point, the surface is that of a sphere.*[192] This often overlooked demonstration[193] gives us a notable geometrical demonstration of the roundness of the Earth as if it were a fluid spheroid.

The extraordinary beauty of Archimedes' demonstrative procedure and the outcomes reached also derive from his ability to adopt the methodology of the virtual balance in the analysis of hydrostatic problems, merely by thinking of the portions of fluids as if they were sections of material bodies moving along the arms of an idealized hydrostatic balance. Later on, those fluid bodies move up or downwards because of a resultant strength acting along the parallels passing through their centres of gravity.

In general, Archimedean issues ought to be a matter for statics, but in the treatises analysed above, and as I have tried to show, he clearly deals with the kinematic and dynamic behaviour of geometric figures, physical bodies and fluids. Archimedes moves these objects along the arms of his idealized balance. For this reason it can be argued that Archimedes' subject of study is definable as a science of quasi-statical dynamics: the framework in which all these proofs take place conceive conditions of equilibrium between all the forces acting on a body, such that Archimedes' bodies are considered in

[192] *Ibid.*, p. 262.
[193] One of the main discussions on this demonstration can be found in Russo 2013.

the case in which the kinetic energy is negligible with respect to the potential energy.

Archimedes' procedure should be considered a truly important insight, but it must not be seen as an anticipation of a modern method. As declared at the beginning of the argument on Archimedes' methodological approach, our purpose is not to identify precursors, but rather to explain when and during which demonstrations those ideas appeared, which led to the discovery of the corresponding mathematical objects (sometimes thanks to the works of other scientists, who have been attracted by the efficacy of the new method). My claim is that such developments occurred during the Renaissance era of mathematical humanism, a period marked by the rediscovery of Archimedes' works and in which the abstractions of the concept of the point mass received a new upsurge in attention. In fact, during the Italian Renaissance, the work of translating, editing, reviewing and extending the intellectual limits of ancient scientific writings, in particular those of Archimedes, was very important. It is not possible here to give a complete analysis of the mechanical works of authors such as Leonardo da Vinci (1452-1519), Francesco Maurolico (1494-1575), Federico Commandino (1509-1575), Guidobaldo del Monte (1545-1607), Luca Valerio (1552-1618) and Galileo Galilei (1564-1642), but in the next chapter we will take a closer look at Guidobaldo del Monte's *Mechanicorum Liber*, which is usually considered the most influential treatise of the Renaissance period, as well as Galileo's hints on the idealization technique which gave rise to a prolonged debate about the epistemological meaning on model-building practice.

In this direction, I will now turn to argue in favour of a direct connection between the Renaissance reception of Archimedes and the development of the material point.

# Chapter 3

## The Renaissance Turns

The Renaissance was a period of immense cultural transformation within Europe, not the least of which involved a major shift in the European educational system. During this period, a new way of thinking came to the fore which proposed a different form of pedagogical training, one which would provide students with the skills for life besides those which were required by their occupation. This view was championed by humanists[194] who established schools and institutions in order to implement their ideas. However, with regard to mathematical education, it is impossible to separate the work and diligence of the humanists from the approach of the mathematicians. In effect they collaborated to reach the same goal – the recovery of ancient Greek mathematics – and stood jointly at the forefront in the translation of treatises from Greek into Latin. While, on the one hand, the technical and theoretical competence of several mathematicians was protected by the Courts, on the other hand, the interest of humanists in preserving earlier cultural heritage was seen as a continuation and derivation from ancient times. Hence, throughout this refurbishment of ancient culture, they also oversaw the renewal of the Renaissance educational system.

There are several closely related, and perhaps not altogether distinguishable, keys for understanding the significance of the Renaissance from the point of view of mathematics. These include:

i) the need for a renovation of mathematical science: a restoration, repair, restitution, revival, instauration, or renaissance of mathematics (as they variously called it); in connection with this, the humanistic obsession for

[194] A humanist is simply any kind Renaissance who pursued and disseminated the study and understanding of the ancient Greek and Roman cultures, placing an emphasising on secular (versus inherited religious-traditional) and critical thought.

significant educational reform;

ii) directly connected to this demand, there existed a keen request for the renewed study of Medieval Arabic sources: through the invention of the printing press (1455), accurate dissemination and translation of the original texts became possible, something which seemed previously inconceivable in an age where manuscripts represented the sole means of circulating the written word; and

iii) the decrease in interest for a purely speculative Aristotelian *philosophia naturalis,* counterbalanced by an increase in interest for the Archimedean *heuristic and mechanical* tradition.[195] This general shift in perspective led to the heightened appreciation of mathematics as a purely speculative and self-reliant discipline, as is witnessed above all by the School of Urbino.

Certainly, other more minor factors – as we will see – are directly related to the three key factors just mentioned, so the whole review which is delineated in this chapter orbits around them. The Renaissance world would be unimaginable without the emergence of novelty and the proliferation of new knowledge, which reflected the dynamics of the early modern economy and shattered the foundations of the traditional feudal system; this was a fragmented world in which urban centres and feudal courts were competing directly with one another. In this environment, classical antiquity served as an alternative model for shaping individual lives and collective culture in a way that mastered the challenges of society and nature. Hence, the desire to match the stable achievements of antiquity with the purpose of keeping up with the times in which a rapid expansion of economy, technology and knowledge flourished[196], progressed at the same pace. This was the period in which the evolution of both the economy and society was also strongly influenced by three much-needed technological discoveries: the circumnavigability of the earth, gunpowder, and printing. Each of these aspects

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[195] During the sixteenth century, first Guidobaldo del Monte and later Bernardino Baldi tried to unify the Aristotelian mathematical approach with the Archimedean physical tradition. A detailed analysis of this will be presented in §§ 3.7 and 3.7.1 below.

[196] Renn and Damerow 2010, p. 8.

developed independently, but at the same time each of them contributed to the development of the other fields, as well as to the improvement of the system of knowledge holistically. The emergence of the engineer as an intellectual technician educated in sciences is furthermore a peculiarity of these centuries. Indeed this is the main feature of science, where the reduced creativity of 'pure speculative scientists' was counterbalanced by the great creativity of 'applied scientists.[197]

It was exactly in this varied and intercultural environment that, I suggest, the second stage of the process of objectification of procedure for the material point developed, providing the chance for the geometrical notion of the centre of gravity to reach a full theoretical status. This second stage shows that scientific practice is related directly to the development of mechanics as an independent field of study, which is itself accompanied by both a practical and a speculative component. Following further analysis of the three main items stated above, the historical, cultural and social framework of the Renaissance will firstly be delineated, before a thorough investigation of the theoretical refinement of mechanics, thanks to some of the most important mathematicians of the fifteenth and sixteenth centuries. In particular, I will examine the life and work of the mathematicians who were active at the court of Urbino, such as Federico Commandino, Guidobaldo del Monte, and a few other scholars operating in a more isolated cultural context, such as Francesco Maurolico and Luca Valerio.

This chapter will follow the proceeding structure. § 3.1 will attempt to give a general introduction to the restoration phase of the ancient Greek mathematical tradition and the legacy of Euclid between the Middle Ages and the Renaissance period, in order to highlight the different reactions of Renaissance scholars towards the two main authors of ancient Greece, Euclid and Archimedes. § 3.2 is dedicated to the School of Urbino, and in particular to the rediscovery of the ancient Archimedean treatises: a wide introduction to the ducal milieu will be presented, giving importance to those scholars

[197] To offer but an outline of the numerous names involved in this process, we can mention the works of Mariano di Jacopo, called Taccola (1381-1458), Leon Battista Alberti, Francesco di Giorgio Martini (1439-1501), Leonardo da Vinci, Vannoccio Vincenzio Austino Luca Biringuccio (also known as Vannuccio Biringuccio, 1480-1539), Francesco de' Marchi (1504-1576), Giovanni Battista Bellucci (1506-1554) and Daniele Barbaro (1513-1570).

who advanced the recovery of Greek mathematical knowledge, and in order to show the different approach of the Renaissance towards mechanics, and the use that scholars made of the notion of the centre of gravity. In this section I will put side-by-side the elements characterising the two Hellenistic mechanical traditions – the Aristotelian and the Archimedean – with the new mechanical approach which began from the Renaissance onwards. Thus from § 3.3, our focus will lie with the practical and theoretical contributions that Renaissance scientific humanism gave: i) in consolidating the new way of doing and conceiving mechanics, and ii) in the second stage of the objectification of procedure, namely that, as is seen in several treatises published between the fifteenth and sixteenth centuries by the mathematicians of the *Urbino School* – such as Federico Commandino (§ 3.4) and Guidobaldo del Monte (§ 3.7) – it was possible to switch the attention onto the practical use we can make of geometrical elements, such as the centre of gravity. This notion became fundamental in various practical contexts linked to the *scienza de ponderibus* (science of equilibrium, § 3.6), the aim of which is to solve and formalize static problems concerning heavy bodies with particular reference to those hanging from a balance. Moreover, the concept of the centre of gravity is also useful to solve the so-called ‘Equilibrium Controversy’, which addresses the question whether or not a deflected balance will return to its horizontal position. This is a controversy which, though it had already has its birth during the Medieval era, was only during the Renaissance, and following the rediscovery of all the Archimedean corpus on statics and hydrostatics, applied to the science of machines and to other purely practical contexts. However, we have to wait until the end of the sixteenth century, and in particular the research into indifferent equilibrium advocated by Guidobaldo del Monte in his *Mechanicorum Liber* (1577), to see the controversy solved, mainly thanks to the conjunction of the Aristotelian mechanical conception with regard to cosmology (based on the notion of the centre of the world) with the Archimedean barycentric theory. Thus, the second part of the chapter will be focused on the Equilibrium Controversy, its origins, its claims and the scientific protagonists involved. At the same time, great attention is dedicated to del Monte’s attempt to treat the idea of finding a wide theoretical application of the centre of gravity as a geometrical point devoid of purely physical content.

### 3.1 The 'Restoration'

At the end of the Middle Ages, mathematics was essentially taught at universities and at Abacus Schools, as part of the *quadrivium* (the four subjects consisting of arithmetic, geometry, astronomy and music) of the Faculty of Arts that, while maintaining their autonomy, were instrumental to the training of future Renaissance physicists and technicians.[198] All the leading Italian mathematicians of the fifteenth and sixteenth centuries were obsessed with the need for a renovation of the mathematical sciences, and, at the same time, humanists were obsessed with the need for significant educational reform.

One of the founders of the new education in the Renaissance was Vittorino da Feltre (1373/78-1446), whose purpose it was that each student should both leave school with a basic understanding of all the available subjects, and also have received the time and support to study those subjects at which they excelled in greater depth. However, only a selected number of students were encouraged to focus their efforts on the field of mathematics, *the practice of which was not promoted by many other educators*. The study of mathematics was treated in low regard by many educationalists because of its strong association with trade and commerce.[199] Up until this time merchants and master craftsmen living in many areas in Europe were not accorded the same level of respect or deference as they were in Germany. This meant that sons of the merchant class were taught only in those subjects which would aid them in their efforts to become statesmen and politicians. What little mathematics was taught in the merchant schools became therefore highly theoretical and divorced from practical application in the real world.

[198] For the role of European universities in the fifteenth century see de Ridder-Symoens 2003 and Ruegg 2004. For the development of the Italian universities, see *The Annals of the History of Italian Universities*, Bologna, CLUEB.
[199] See for example Tartaglia 1556-1560.

In order to achieve the goal of renewing the educational system, the so-called *Scuola d'Abaco*[200] was founded in Florence and was attended by those who wanted to improve their skills in commercial areas. It hence offered courses in arithmetic, algebra, astronomy, book-keeping, and practical geometry, which grew in importance due to recent advances in navigation.[201] Technical advancements were being made in other sciences and technologies, in particular thanks to the invention of the printing press, with the result that a profound transformation of education was taking place. This allowed for a rapid dissemination of scientific knowledge, and besides the translation of important texts such as Euclid's *Elements* into German, French and Italian, an increased demand for astronomical charts and commercial tables was recorded. A series of didactic textbooks[202] spread across Europe, which were mainly dedicated to common men seeking to improve their understanding of such subjects as the Hindu-Arabic

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[200] The *Schools of Abacus* were born in the thirteenth century with the spread of *Liber Abaci* by Leonardo Pisano, better known as Fibonacci (1170-1250). Some of these schools were subsidized by the municipalities, others by private organisations. The practical mathematics that emerged from the abacus treatises had so many characteristics that it can be considered quite clearly different from traditional Euclidean axiomatic-deductive mathematics. The main features of the abacus treatises were the use of the vernacular, mercantile writing, a great number of examples, and the presence of important drawings for illustrative purposes. The treatises on the abacus had different levels of quality, which reflected the skills of teachers who had drawn them up: some were very simple and neglected those parts of mathematics that were not immediately applicable in the art of the merchant (e.g. algebra, practical geometry, and speculative arithmetic). For further on this, see Pisano and Bussotti 2013; Pisano and Bussotti 2015 and Ulivi 2002, pp. 121-159.

[201] See fn. 195.

[202] In England, Robert Recorde wrote what is thought to be the first series of textbooks in English: *The Grounde of Artes* was first printed in 1540, but was reprinted over fifty times in nearly a hundred and sixty years. Recorde's three other major works – *The Pathwaie to Knowledge*, *The Castle of Knowledge* and *The Whetstone of Witte* – were not so popular. This is most likely due to their less practical and more advanced contents. Recorde's texts were very close to the teaching style used during the Medieval Ages, which was obviously beginning to be recognised as inadequate by the author himself. Later on, around the middle of the sixteenth century in France, the humanist Petrus Ramus proposed that the arts courses taught at universities should return to the seven classical liberal arts, with the syllabus based to a greater degree on applied topics. In keeping with this approach, Ramus wrote a series of textbooks on logic and rhetoric, grammar, mathematics, astronomy and optics. John Dee, one of the editors for *The Grounde* after graduating from Cambridge with both a BA and an MA and after later lecturing at the University of Paris on sections from Euclid's books, emphasised the need to improve the status of mathematics in education. He argued that mathematics should be studied not only for its practical use but also for its ability to "lift the heart to the heavens", a phrase which is reminiscent of Pythagorean beliefs. He proposed translating currently available mathematical texts into English, in order to aid the spread of knowledge to those who do not study Latin at school and University and who found studying the texts in the original language difficult. Dee himself helped to translate Euclid's *Elements* into English, and this was then published in 1570.

numeral system, conversions between weights and coins, and computations with prototypical counters which would aid their work in trade and commerce.

Noticeable precedents for the Renaissance of mathematics may be found in the great cultural awakening of the twelfth and thirteenth centuries. The two major schools of translation in the twelfth century were the Spanish school, which concentrated on translations from Arabic versions of ancient mathematics, and the Sicilian school, which was involved solely in translations directly from ancient Greek. The head of the Spanish school was Gerard of Cremona (1114-1187), who is usually heralded for his translation of the *Almagest* (1175). On the other hand, the main translators of the Sicilian school were the Admiral Eugenius (1130-1203) and Henry Aristippus of Calabria (1105-1162). At a later time, the major translator of the thirteenth century was William of Moerbeke (1215-1285/86), the pupil of Albertus Magnus (1206-1280) in Cologne. After a series of philosophical translations of Aristotle, Proclus and Galen, Moerbeke tackled a group of mathematical translations, perhaps as a result of his friendship with the mathematicians Campanus of Novara (1220-1269) and Erazmus Ciolek Witelo (1230-1280/1314), both of whom were attached to the papal court. Despite Moerbeke's translations being extremely literal and rough, he played an important role among the other translators of his time, firstly because he managed to recover directly from the ancient Greek texts, and secondly because his complete translation of Archimedes was among the first ever done.[203]

Moerbeke and his contemporaries played an important role in the later revival of mathematics for two reasons: first because they ensured continuity with the scientific heritage of ancient Greece, and second because in dealing with these translations they also took into account the neglected technical development of mathematics made by the works and inventions of Leonardo Pisano (1175-1235) – better known as Fibonacci – and Jordanus Nemorarius (thirteenth century), both of whom enjoyed a second life from the fourteenth century onwards. As Rose writes, "[t]he combined activity of translators and mathematicians in the Middle Ages enables us to speak of a medieval renaissance

[203] Rose 1975, pp. 76-89.

of mathematics. Yet one should be careful of assuming that one renaissance led without interruption into another. In fact, after 1300 the Medieval renaissance waned, eclipsed by the popularity of Scholastic [Aristotelian] physics."[204] From a mathematical viewpoint the Scholastic physics has been seen as a continuation of the Medieval renaissance movement, due to the fact that the former attempted to appropriate some features from the latter, for instance the Arabo-Latin Euclid, the Ptolemaic astronomy, the theory of proportions or Oresme's work on infinity and incommensurability. However from a general point of view the two movements had quite different aims and purposes.

Following the line of argument presented in the section 2.2 dedicated to the legacy of Euclid in the Latin and Arabic world, and before moving on the next step of the objectification of procedure related to the head of the Urbino's School, in the rest of this section we will offer a survey of the continuation of this legacy between the Middle Ages and the Renaissance. We have already said that it was due to the invention of the printing press that the translated texts were spread among the wider public, and that the most renowned treatise was Euclid's *Elements*, whose success was matched only by the Bible. The first printed edition of the *Elements* can be dated to 1482 and was drafted on the *recensio* of the *Elements* made by Campanus of Novara between 1255 and 1261. Campanus used the *Elements* of Robert of Chester (1110-1160), which itself dates back to the twelfth century, and which was joined with several Arabic and Latin texts, among which was Johannes de Tineume's (eleventh and twelfth century) edition. Campanus' work was not restricted to a literal translation, but also included new mathematical and geometrical outcomes using Jordanus Nemorarius' *Arithmetica* and Anaritus Nazirius' *Elements* (865-922). The humanists were highly critical of Campanus' edition, in particular because it was heavily influenced by the medieval tradition, and because he did not improve the translation from either philological or linguistic points of view. On the contrary, he sought to bring out the philosophical interest of the *Elements* by elucidating its axiomatic structure. Moreover, in order to adapt Euclid further to the

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[204] *Ibid*., p. 76.

didactic needs of the school, Campanus moved away from geometrical proofs to an emphasis on arithmetical proofs instead.[205]

In 1505 in Venice Bartolomeo Zamberti (1473-1539) published the entire Euclidean corpus, including the *Phaenomena*, *Optica*, *Catoptrica* and *Data*. His main purpose was to restore Campanus' edition using the original Greek version of the treatises. Campanus' *recensio* and Zamberti's edition represent two different, but complementary, ways of translating an ancient text. While Campanus' main task was to give mathematical coherence to the *Elements*, Zamberti's purpose was to restore the text from both philological and linguistic points of view.

Between these two translators one can place the figure of Luca Pacioli (1445-1517), a mathematician who was also the author of two different editions of Euclid's *Elements*. The first was a scholarly edition directly translated from Greek to Latin, while the second was rather addressed to an untrained audience. Meanwhile at the same time, one finds traces of both official and unofficial vernacular translations. Almost all the official translations were based on both Campanus' and Zamberti's *Elements*. Campano's publication and its axiomatic system were even used several times in the following century. For example, Niccolò Tartaglia (1499-1557)[206] inaugurated the translations of Euclid into modern languages (altering a few axioms in the process) and was followed by Guilielmus Xylander's (1532-1575) publication into German (1562), Pierre Forcadel's (fourteenth century) into French (1564), Sir Henry Billingsley's into English (1570), Zamorano's into Spanish (1576) and Dou's into Dutch (1606).

Beyond the Alps, the German astronomer Regiomontanus[207] (1436-1476) deserves a special mention as the first European mathematician to undertake a complete restoration

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[205] *Qui constat multos Euclidis locus tum praeteriisse, tum non commode interpretatum et sua non satis examinate subdidisse, in multis tamen fatemus acute interpretatum, sed errorum nunc non bene dictorum nobis esse cura debet* [*De expetendis et fugiendis rebus opus*, Lib. XI, Ch. 3 by Giorgio Valla, 1501 postumo]. Bartolomeo Zamberti, another translator of the Euclidean corpus, defined Campanus as an "*interpres barbarissimus*".

[206] *Euclide megarense philosopho: solo introduttore delle scientie mathematice: diligentemente reassettato et alla integrita ridotto per il degno orifessore di tal scientie Nicolo Tartarea Brisciano*, Venezia 1543. Nicolò Tartaglia received no formal education, except for a period of fifteen days in a "scuola per scrivere", in which he learned to read Latin but wrote in a not very elegant Italian.

[207] This is the pseudonym of Johannes Müller from Königsberg.

of the ancient treatises on mathematics and astronomy, whose translations were disseminated more widely thanks both to his friendship with Cardinal Basilios Bessarion (1403-1472) and his connection with other Italian humanists. Owing to the patronage of Bessarion, he renovated the *Almagest*, clarified and solved several problems relating to the Ptolemaic system, and on his watch astronomy became the pearl of mathematics because of its connection with geometry and arithmetic.[208] Mathematics and mathematicians enjoyed a special role during the Renaissance because – as Regiomontanus says – at that time it seems that humanities, or liberal arts, were floundering in uncertainty, a state of affairs usually attributed to philosophers. So in order to reform this state of uncertainty, one must begin with those arts which are constructed upon the most certain principles, i.e. those which cannot be distorted either by the passage of time or by the clever arguments of an eloquent professor. In the mathematical sciences, and especially in astronomy, truth is certain, immutable and eternal. If any errors creep in, they may be ascribed to the ignorant mistakes of translators and copyists, or to the wilfulness of a commentator. To repair the damage, Regiomontanus has made a large number of astronomical observations and compared them with the figures given in the works of classical authors.[209]

An edition that truly stands out for its originality and its mathematical autonomy is that edited by Francesco Maurolico (1494-1575) from Messina and published in 1530. His text differs from the previous ones because, thanks to his mathematical and geometrical skills, Maurolico was able to suggest many solutions to geometrical problems.

After the proliferation of so many translations, we must wait until the second half of the sixteenth century for a complete reception and a reliably restored version of the *Elements*. Then in 1572 Federico Commandino (1509-1575) published his Latin (and Italian) version, which was based on a variety of Greek manuscripts and was in many respects more correct than all of its predecessors. Whilst Commandino had clear

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[208] For further details on the Euclidean tradition, see De Risi 2016; Gavagna 2009, pp. 97-124 and Rose 1975.

[209] Regiomontanus 1514, pp. 211-214.

philological aims, he was also an outstanding mathematician and endorsed several changes in the axiomatic system, and added for the first time certain axioms in Books V and X of the *Elements*. The remarkable outcome of this is an edition which was trusted thereafter for its philological accuracy, and on which almost all subsequent editions of the seventeenth and eighteenth centuries would be based.

Only two years later in 1574, the Jesuit mathematician Christoph Clavius (1538-1612) published another influential edition of Euclid in Latin. Clavius' edition was by far the longest and most complete edition of the *Elements* ever produced, and was accompanied by a lengthy commentary that collected almost everything that had been published on the Euclidean text in previous centuries. For this reason it quickly became the reference point for all later discussion of the foundations of geometry in the proceeding two hundred years. Clavius believed that the original Greek text was important and had to be preserved, but that mathematical clarity was a higher aim. Clavius himself envisaged a few new axioms which he thought would be useful to strengthen the mathematical proofs of the *Elements*; given the enormous success of this edition (and the fact that it immediately became an example for all the further Jesuit editions of Euclid), these new Clavian axioms were successively endorsed by almost the whole community of mathematicians.

Nonetheless, who were the mathematicians during these centuries? In the *Quattrocento* and *Cinquecento*, the term 'scientist' did not exist at all, having only become used in its current meaning from the second half of the nineteenth century. A mathematician during the Renaissance, on the other hand, was an intellectual, with competence in (among other subjects) astronomy, astrology and medicine. The medical faculties of the early Renaissance were usually those in which mathematics had a central role. In fact, mathematics was connected to the study of astrology, which required the students to have rudimentary knowledge of Ptolemaic astronomy and early cosmology[210] together with elements of geometry and arithmetic. Therefore, the place

[210] See further Kusukawa 2012. At the University of Padua, for example, the study of astronomy and cosmology was considered a preliminary and preparatory stage for those who wanted to study mathematics; the introduction of mathematics to the undergraduate curriculum preceded that of

occupied by mathematics was still marginal, and, with the exception of some outstanding teachers, the level of mathematical knowledge was limited to what was indispensable for the practice of astrology. In fact, mathematics did not cover the study of many Greek sources which were already available at the time in Latin, and which derived from twelfth-century Arabic translations.[211] It is likely that the figure of the mathematician had the same academic status as a philosopher, theologian or jurist. But within the technical and practical environment, it was only with the Renaissance rediscovery of ancient mathematics and geometry that the mathematician began to enjoy greater treatment and respect. In fact, we know from the Renaissance treatises that mathematics was applied to the construction industry and agricultural engineering: the task of an architect, for example, was not restricted to construction engineering, since he was also expected to make both specific contributions during military attacks and more trivial arrangements in organizing banquets or celebrations. The increase in value of this technical and mechanical tradition and its relationship to the theoretical aspects of mathematics allowed the dismissal of the idea that mathematics had some magical-symbolic associations. The discipline was finally raised to the level of *scientific knowledge*. For this cultural promotion a special mention is merited by Tartaglia's *General Trattato di Numeri et Misure* (1556-60), a mathematical yet practically minded encyclopedia on how to use the theory of numbers in merchant arts or the field of commerce. In the *Prolegomena* to the reader Tartaglia conveys the needs of the time: while until the end of the fifteenth century mathematics was related only to the practical disciplines, mathematics since the Renaissance has been taught for its speculative and theoretical aspects:

> The ancient wise men […] used to divide knowledge into two parts, the first of which [was] named by Ptolemy [as] *speculatione* [speculation], and the other of which was named *operatione* [operation]. These two parts are still commonly called *theoria*, or speculation, and *practica*, or active, or operative (knowledge) respectively. Between these two parts (in the way that Ptolemy

astronomy and astrology related to medicine. Padua is also the city in which Federico Commandino graduated in medicine.

[211] Gamba 2001, pp. 75-110.

says) there is a considerable difference due to the fact that they have different purposes. The aim of the science of speculation is (as Aristotle says in his second book of *Metaphysics*) nothing but the truth, and that of operation is the completed action […] and even though speculation (insofar as it aims to investigate through the proximate cause, and to argue through science) is far nobler than practice […], since the latter aims only at accomplishing what has already been discovered through speculation.

Out of these deliberations I have decided to compose a general treatise on numbers and measurements, on mathematics, according to the natural definition, and not only on practical arithmetic, and on geometry, proportions and proportionality, both irrational and commensurate. But also to investigate the 'arte magna', which in Arabic is called algebra, and the Almucabala, or the "rule of the thing".[212]

Tartaglia's encyclopedia is divided into six parts, which are dedicated to the three subfields of arithmetic, geometry and algebra. The first part deals with mercantile arithmetic, while the second part is more speculative and experimental and includes the paraphrase of Euclid's *Elements* II-X. The main purpose of the other parts is to present

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[212] *Gli antichi sapienti, […] dividerno la sapientia in due parti, la prima delle quali dal (detto) Ptolomeo è detta speculazione, e l'altra è chiamta operatione, le quali due parti communamente anchora l'una è detta theorica, over speculatione, e l'altra pratica, over attiva, over operativa, tra le quali due parti (come afferma esso Ptolomeo) non vi è poca differentia, la causa è che tendono a diverso fine, perché il fine della scientia speculativa (come dice Aristotile nel secondo della metafisica) non è altro che la verità, e della operazione […] l'opera compiuta, e abenche la speculazione (per esser invesigatrice delle propiunque cause, et argumentatrice della scientia) sia molto più nobile della operatione, […], la quale solamente attenda a saper con diligenza essequire […] tutte le cose già speculativamente ritrovate […]. Deliberai altresì nella mente mia di comporre a comun beneficio* (per l'arte speculativa-teorica e per quella pratica d'operazione) *un General Trattato di Numeri e Misure, si secondo la considerazion naturale, come Mathematica, e non solamente nella pratica Arithmetica, e di Geometria, e delle Proportioni, e proporzionalità, si irrazionali come razionali. Ma andar nella pratica speculativa dell'arte Magna detta in arabo Algebra, e Almucabala, over regola della cosa [...].* Tartaglia 1556-60, pp. 3-4. The English trans. is my own because no official translation is available.

Furthermore, let us not forget the most important and prominent pupil of Tartaglia, Giovan Battista Benedetti (1530-1590). As a court mathematician to the dukes of Savoy from 1567 to his death in 1590, Benedetti procured for himself the title of ducal philosopher. In his *De Philosophia Mathematica*, Benedetti regarded mathematics as philosophy, summoning Aristotle's epistemology in his defence. Moreover, like Aristotle, he makes mathematics the most honoured discipline of philosophy. Benedetti's actual mathematical method – which he called *speculatio mathematica* – is however more original than the method declared by Tartaglia in his *Nova Scientia*, especially because Benedetti takes for granted the recovery of ancient geometrical techniques, averring that there is no need to quote chapter and verse of Euclid, or any need to order mathematical propositions formally and sequentially. Instead of being part of a rigorous system, the propositions become intuitive *speculationes*. With this method of *philosophia* or *speculatio mathematica*, Benedetti achieved resounding success in purging science from the baneful influence of the pseudo-Aristotelian *Mechanica*. See further Benedetti 1585, p. 168, and Rose 1975, pp. 151-158.

a renewal to technical disciplines, with the addressees of the treatise being ostensibly geometricians, architects, engineers and machinery builders. In this respect, the *Trattato* represents the perfect synthesis of empirical investigation and speculative thought. In fact, Tartaglia's interests both in the speculative aspects of mathematics and in its practical and mechanical application can be noticed just by looking again at the quotation above and at the structure of the treatise. Tartaglia is among the first mathematicians to emphasise the importance of placing side-by-side empirical research and theoretical investigation, in order to find shorter expedients to those problems which are usually solved with a ruler and a compass, and eventually to reach theoretical solutions which can have a wide applicability. It seems, already in Tartaglia's time, that mathematics is the only means that can help to reach the goal of generality.

From the elements stated above, we can conclude that it is inconceivable to separate the classical basis of the mathematical Renaissance from a general classical revival undertaken by the Italian humanists.[213] Not only did the humanists collaborate with the mathematicians in the recovery of Greek mathematical manuscripts, but they also took an active role in the translation of the texts into Latin, and in particular, as we will see below, they played an exceptional role in the case of the works of Archimedes, which represent an authority in nearly all the practical disciplines. All these elements allowed for the elevation of mathematics to the status of an independent and self-reliant field of research and the appearance of a new way of doing and considering mechanics. Moreover, the arrival of Guidobaldo del Monte (1545-1607) sanctioned a new breakthrough: the second half of the sixteenth century was the context in which occurred discussion on how some sciences could use rigorous mathematical demonstrations to arrive at conclusions about physical objects. These theoretical considerations, which were themselves based on heuristic and empirical mechanical

[213] Besides the emergence of the culture of the middle class, which played an important part in accounting, geography, economics and finance, one should emphasise that the emergence of the humanist movement should be attributed to the role performed by the new social and economic conditions, offering new perspectives on the world, which on the one hand allowed the members of the middle class to be able to devote time to study, and on the other hand allowed the courts to play a more or less disinterested activity of patronage. The fifteenth century records a growth in the development of science and the publication of scientific papers. For a more general overview, see Kristeller 1956; *Id*. 1965 and *Id*. 1990.

procedures, gave rise to ontological and epistemological reflections on the nature of the centre of gravity, both as a purely geometrical entity and as a physical notion. We will return to this point in due course, but let us now continue with the account of the Renaissance rediscovery of the heritage of ancient books.

The fascination of rediscovering classical antiquity led to another phenomenon: the rapid formation and growth of the great humanist libraries of the fifteenth century. These Renaissance libraries differed greatly from their Medieval forebears with regard to both their size and their contents.

Florence was the focal point of the Greek revival, firstly because chairs of Greek were established in ca. 1396 at the University, and secondly because there sprung up a brilliant group of *Grecisti*[214] that included Palla Strozzi (1372-1462), Niccolò Nicoli (1365-1437), Fra Ambrogio Traversari (1386-1439), Ser Filippo di Ser Ugolino (XV century) and Antonio Corbinelli (1373-1425). The unanimous efforts of these scholars led to the composition of several great humanist collections that were assembled in the first half of the fifteenth century, collections that became public libraries after their acquisition by Cosimo de' Medici (1389-1464).

Two great libraries were inaugurated in Florence under Medici patronage. The first, known as the *Libreria Medicea Publica*, was opened to the public in 1444 and housed at the Dominican monastery of San Marco. The second, the *Libreria Medicea Privata*, was kept in the private household of the Medici and was usually accessible only to a group of scholars associated with the family. These family collections flourished particularly under Lorenzo de' Medici (1449-1492) and, since they were housed later in the cloister of San Lorenzo, became known as the *Laurenziana*. At present, most of the San Marco manuscripts are divided between the *Laurentiana* and the *Biblioteca Nazionale* in Florence.[215]

Many of the Greek codices were brought from Byzantium in the 1420s by the humanists Giovanni Aurispa (1376-1459) and Francesco Filelfo (1398-1481). During one of his journeys to the East in 1420-1423, Aurispa obtained an estimated 238

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[214] A *Grecista* was a term denoting a scholar of the language, literature and culture of Ancient Greece.
[215] Rose 1975, p. 33.

codices. Humanist sponsorship ensured the copying, translation and circulation of these texts besides their preservation. Their main interest was focused on mathematics thanks to the circulation of the Latin texts and translations of Ptolemy's *Almagest*, the *Liber Abaci*, *Pratica Geometriae Aritmeticus* and a *Prospectiva Generalis* of Leonardo Fibonacci. In addition, great interest was shown for the Greek codex of Ptolemy's *Geographia*, after it was brought to Florence by Emanuel Chrysoloras (1350-1415).

The intertwining of this humanist environment with the rediscovered scientific tradition led to a confluence of knowledge that allowed a huge improvement in conceptualising and using mathematics, geometry and scientific knowledge in general. In 1425, upon returning to Florence from the University of Padua, Paolo dal Pozzo Toscanelli (1397-1482) introduced the great architect Filippo Brunelleschi (1377-1446) to Euclid and geometry; thus it was due to this renewed discipline that Brunelleschi was acclaimed as the "second Archimedes". Doubtless this is a humanist exaggeration[216], but one which indicates the impression made upon contemporaries by the architect's discoveries in applied mathematics. Mathematical discoveries in perspective and engineering on the one side, and the restoration of classical techniques on the other, suggest immediate links between Renaissance engineering, as achieved by (among others) Brunelleschi and his friend Mariano Taccola (1381-1453), and Renaissance mathematics.[217] Moreover, because of his acquaintance with Leon Battista Alberti (1404-1472), Nicolaus Cusanus (1401-1464) and Regiomontanus, the mathematician and astronomer Toscanelli stands at the nexus of several important strands of the early mathematical renaissance. Among these figures the interest in Archimedean engineering had gradually increased, and it was thanks to these authors that the Renaissance tradition of arts and architecture spread all over the country and through Europe in the later centuries.

However, compared with Rome and Venice, Florence in the middle of the fifteenth century still had serious gaps in its possession of the texts of Greek mathematical authors. Only with the founding of the private library of Medici did most of the Greek

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[216] *Ibid.*, p. 29.
[217] *Ibid.*

manuscripts of Apollonius, Proclus, Diophantus and Archimedes appear in the Florentine catalogue.

In Rome, the Renaissance began with Nicholas V (1397-1455), who was acclaimed by Vespasiano as the ornament and light of letters and learned men. The symbol of the Roman Renaissance was the *Biblioteca Vaticana*, which can be essentially considered as the creation of popes, preeminently the papacies of Nicholas V and Sixtus IV (1414-1484). By 1484 the Vatican had excellent copies of the works of Apollonius, Diophantus and Euclid. In the next century, the Greek texts of Hero and Pappus were to be added. Nicholas' patronage, following the Medici's tradition, induced a great number of scholars to come to Rome in order to make free use of the Vatican assets and through the commissioning of translations from Greek to Latin. The most important translations made in this period were those made by Jacobus Cremonensis (1413-1453/54) of most of the works of Archimedes.

The translation of Archimedes commissioned by Nicholas V undoubtedly represents a significant layer in Renaissance mathematics, even though the Medieval Era Moerbeke version was ultimately to prove more important in the long run. In the early years of the fifteenth century, Jacobus' translations circulated among some of the central figures in that movement. Bessarion made a copy of it, and Regiomontanus corrected a copy for publication, which was also included in the *editio princeps* of Archimedes, having been published in Basel in 1544.[218]

Bessarion, the Greek monk who later became a Latin cardinal, attempted to join the two worlds of Byzantine and Renaissance Italian culture. The cardinal remained in Italy for most of the rest of his life, establishing himself in Rome as one of the dominant figures of this century. His house functioned as a humanist academy, being frequented by Poggio (1380-1459), Filelfo (1398-1481), Trapezuntius (1395-1472/73), Argyropulus (1416-1486), Cusanus and Regiomontanus. Bessarion's collection can be considered the richest and most helpful for the transmission of classical Greek culture into Europe from the fifteenth century onwards. According to the inventory of 1468, the

[218] Jacobus' version was superseded by Federico Commandino's edition in 1558.

collection numbered 182 Greek manuscripts and another 264 Latin texts. Between 1468 and 1472 several hundred more were added, thus making the library the largest collection in Europe before the time of Sixtus IV. After Bessarion's death the collection was bequeathed to the Venetian Republic, where the books were assembled to form the *Biblioteca Marciana*. From the inventory we can see that Bessarion's interests ranged widely, yet his main interest focused on Platonic philosophy, geometry and mathematics. The mathematical and astronomical sections alone run to 43 Greek and 11 Latin manuscripts.

At the end of the fifteenth century in Venice, there was besides Bessarion's collection another humanist collection, namely the library of the prominent humanist Giorgio Valla of Piacenza (1447-1500). Unfortunately, the detailed and precise catalogue of Valla's library has been lost, but a part of the books found their way to the *Biblioteca Estense* at Modena around the sixteenth century.

Valla's mathematical manuscripts became rather celebrated among the humanists of the late fifteenth century and attracted important visitors to his house in Venice, some of whom sought to procure copies of Archimedes' *Codex A*. Valla's interest in mathematics and the humanities is an example of the close relationship between the two fields of study in Renaissance Italy, a relationship shown directly by his *De Expetendis*[219], that is a great encyclopaedia which marks the state of humanist studies in Venice around 1500. Moreover, Valla used his position as public professor of humanities at Venice to further the cause of mathematics, by lecturing on astronomy, geometry and natural philosophy.

At that time in Venice there were three active mathematicians. The first was the Neapolitan Luca Gaurico (1475-1558), who published the *Tetragonismus id est de Circuli Quadratura*, comprehending the first printed Latin texts of Archimedes. The second, the Venetian Bartolomeo Zamberti, was far more deeply rooted in Valla's tradition than Gaurico. Zamberti was active as a lawyer and humanist and counted

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[219] Tartaglia's *Encyclopaedia* is made up of 49 books, and in it mathematics holds pride of place, for it enables one to move from natural philosophy to the higher realms of theology and methaphysics; Rose 1975, pp. 44-54.

among the first scholars who had a particular interest in Alexander of Aphrodisias, whose logical works were translated between 1511 and 1524. The third active in Venice in the early sixteenth century was Giovanni Battista Memmo (1466-1536), a patrician who held several government offices. His main work was the Latin translation of the Greek text of Apollonius, *Apollonii Perigei, philosophi, matematicique excellentissimi Opera* (Venice, 1537).

This extensive introduction allows us to state firstly the nature and extent of the influence that humanism had on the mathematical Renaissance. Certainly it was thanks to the determination and perseverance shown by Italian humanist manuscript-collectors of the *Quattrocento* that the assembly of an almost complete corpus of the Greek mathematical treatises was possible. In addition, all the humanists, mathematicians and other collectors presented above were directly involved in actively promoting the revival of ancient science, making the libraries the cultural centres of the mathematical Renaissance. Thanks to these scholars, the Greek heritage survived across the centuries and was restored from the vague and erratic Medieval style of translating. Finally, mathematics became a newly self-reliant discipline, which was not only useful for technical and engineering purposes, but which was also starting to be considered a theoretical discipline that needed to be axiomatized.

This goal was reached progressively in the following century to the detriment of Scholastic and Aristotelian philosophy. Until then, there had been a neglect of Archimedean mechanical works because almost none of the relevant treatises wrote by Archimedes were available in Arabo-Latin translation, with a lot of Euclidean and Aristotelian works instead being accessible in different languages. The right cultural environment for Archimedes' revival, and for explaining the meaning of the partition of Greek mechanics into the Aristotelian and Archimedean tradition, was provided by the development of Renaissance Scientific Humanism, whose main contribution derived from the School of Urbino.

However, before moving on to the next section, the purpose of which will be to display a detailed framework of the Urbino milieu, one more remark deserves our attention. At the beginning of the sixteenth century in Italy, a broad debate began to be had on two key issues: i) the role of mathematics among the natural sciences as a result of the increasing use of mathematics in applications, and as Tartaglia was trying to emphasise in his *General Trattato*; and ii) the fact that mathematicians were beginning to bestow a distinct status on those natural and technical disciplines. While almost no-one denied the fundamental role of mathematics in itself, not everyone agreed on the status of knowledge it might provide with regard to the physical world. The importance of the role of mathematics was certainly advocated by supporters of Platonist ideas, which, in addition to their diffusion through humanist circles, found their support from a professional mathematician, Luca Pacioli, whose *Summa* was read and appreciated by all the major mathematicians of the early sixteenth century, such as Tartaglia, Gerolamo Cardano (1501-1576), Giovan Battista Benedetti (1530-1590) and Commandino. Therefore, only in the second half of the sixteenth century can one truly see the dissemination of Archimedean mathematical and mechanical works, which deeply modified the approach to mechanics. Although Archimedes' work was influential everywhere, its stimulus was different in different regions. In the Northern school, which consisted of Benedetti, Tartaglia and Cardano, Archimedes' texts received less attention than those of Jordanus de Nemore or the *Problemata Mechanica*. The contrary held for the Centre school, which consisted of Commandino, del Monte, Bernardino Baldi (1553-1617) and the Southern school, who was made up of Maurolico, Nicola Antonio Stigliola (1546-1623) and Luca Valerio (1553-1618).[220]

Let us now turn to examine the role played by the Scientific Humanism in Urbino in the development of the new way of thinking about mathematics and the new way of considering mechanics as an independent and advanced field of study.

[220] Pisano and Capecchi 2016, pp. 163-164.

## 3.2 The rediscovery of Archimedes in the 'School of Urbino'

Perhaps the strongest evidence of the mathematical Renaissance is to be found in the writings of the Urbino School. The library of the Duke Federico da Montefeltro of Urbino (1422-1482) was considered by Vespasiano to be the most impressive humanist collection in Europe. Duke Federico was remembered as a "skilled geometer and arithmetician, and, with the great German philosopher and astrologer Master Paul of Middelburg (1446-1534), [he] before his death read books on mathematics, discoursing thereon like one learned in them."[221]

As for the other libraries described in the last section, the Urbino library was set up for humanist purposes, but came to be of the greatest use for the restorers of mathematics. The Duchy of Urbino assembled a significant circle of scholars, including the painter Piero della Francesca (1416/17-1492) and the mathematicians Luca Pacioli, Federico Commandino, Guidobaldo del Monte, Bernaldino Baldi and Muzio Oddi (1569-1639). Also the already well-known Leonardo da Vinci (1452-1519), a close friend of Pacioli, showed interest in both the assets of the Urbino library and in particular in the Archimedean codices preserved there.

As far as the cultural and scientific background of the Duchy's mathematicians is concerned, existing scholarly literature tends to place an exclusive emphasis on the importance of the lectures held by Commandino for the subsequent establishment of the Urbino School. These lessons were indeed surely influential for the education of Commandino's disciples, but unfortunately other important aspects of the technical and scientific environment have been sidelined in the scholarship as a result. Above all, the

[221] Vespasiano 1859, p. 213. Federico da Montefeltro was a military and civil leader as well as an intellectual humanist, and may be considered one of the most enlightened figures of the Italian Renaissance. He was the one to impose justice and stability on his state through the principles of his humanist education; he engaged the best copyists and editors in his private scriptorium to produce the most comprehensive library outside the Vatican; and he supported the development of fine artists, including the early training of the young painter Raphael. Finally, as mentioned above, he commissioned the constitution of one of the most complete and great libraries of the Renaissance. His humanistic court was built around the Ducal Palace of Urbino designed by Francesco di Giorgio Martini and Luciano Laurana. One of the most remarkable rooms of the Ducal Palace is the *studiolo,* commissioned by Federico and combining his personal vanity and interest in scientific knowledge.

cultural milieu which developed around the court of Urbino was particularly significant because of the rediscovery of Archimedes' scientific works. To single out but one figure, Commandino, who was notably influenced by Francesco Maria II della Rovere, published – among other works – Euclid's *Elements* in 1572 and Pappus' *Collectiones Mathematicae* in 1588. Furthermore, it was the Duke himself who entrusted both Commandino and his pupil Guidobaldo to supervise a series of construction and renovation projects in Urbino, Pesaro and the surrounding suburbs.[222] This exchange of knowledge and ideas, joined with the intense collaboration between the court and humanists, offered fruitful stimuli for these mathematicians. Indeed, as Martin Frank writes, "the cultural milieu at court was characterised by a profound interest in philosophical questions, with particular attention paid to the philosophy of Aristotle. Thus, in keeping with Guidobaldo, Baldi and Oddi's close ties to the court, some of their most important interlocutors were philosophers: Jacopo Mazzoni (1548-1598), Federico Bonaventura (1555-1602) and Cesare Benedetti. And in fact, certain arguments addressed in their writings seem to reflect discussions on natural philosophy they had had with them or appear to be developments of such exchanges of ideas."[223]

Aside from this general cultural trend, there were also specific concrete reasons that urged the Dukes of Urbino to stimulate the study of mathematics. The Dukes were traditionally military captains serving the Venetian Republic, the Pontifical State and the Spanish King. The activities that this role entailed were often fraught with mechanical challenges, such as the movement of heavy loads, the construction of walls, or the calculation of the trajectory of projectiles. These historical preoccupations remained prominent in the courtly milieu of the little duchy, even though the political situation in Italy increasingly calmed down over the course of the sixteenth century. It was in this context that the Prince Francesco Maria and Guidobaldo del Monte grew up together. Both of them were prepared for a life in the military. Guidobaldo was the Prince's page from the age of seven, and had the privilege of enjoying the same

[222] Those works were also supervised with the help of other mathematicians, architects, technicians and craftsmen, such as Muzio Oddi, the Count Giulio da Tiene, Girolamo Arduini, Francesco Paciotti and Simone Barocci.

[223] Frank 2013, p. 308.

education as Francesco Maria II della Rovere, which included the study of mechanics, and obviously a general background in mathematics. After all, preoccupations in war and certain mathematical disciplines were not the only traits that characterised the Urbino's environment: between the fifteenth and sixteenth centuries the duchy was an important cultural centre. Indeed, famous artists, architects and men of letters such as Raffaelo Sanzio (1483-1520), Piero della Francesca, Leon Battista Alberti, Luca Pacioli, Francesco di Giorgio Martini (1439-1501), the poets Pietro Aretino (1492-1556) and Torquato Tasso (1544-1595), were connected with the court.

In this scenario one also observes a decline in scholarly interest in Aristotle's natural philosophy counterbalanced by an increased attention in Archimedes. In fact, throughout the sixteenth century, all the scholars mentioned above became positively enthusiastic about Archimedes; in the field of *Discipline Mathematiche* a meeting of geometry and mechanics was expected. Renaissance scholars thought that Archimedes' work filled a gap that Aristotle was unable to fill because of a direct connection between Archimedes' and Euclid's studies. In particular Euclid's work represents the emblematic classical synthetic approach, proceeding from primitive propositions to deduce theorems and solve problems. On the other hand, although Archimedes kept the method of synthesis, he never rejected the laws of measures and the concrete application of mathematics. Archimedes' mathematics could be defined as a tool used to shape physical reality; according to him, building machines and the study of mechanical or hydrostatic phenomena were both jointly mathematical operations. He represents the perfect synthesis of the theoretical and practical qualities of the mathematician, in being able to combine theory and practice, knowledge and action.[224]

In order to understand the role played by Archimedes and to clarify how and why the idealized notion of the material point has its origins in the work of Reinassance mathematicians, craftsmans and machine-builders, we need to understand better the origins of mechanics itself. The theoretical physical notion of the point mass and the geometrical concept of the centre of gravity share some properties, since the description

[224] Gamba and Montebelli 1988, p. 69. In the section below on the science of weight (§ 3.6) I will supply more details on Greek problems relating to the conception of equilibrium.

and definition of the latter underwent a countless series of reshuffles over the centuries. Moreover, having mentioned at the beginning of the chapter the reasons that drive us to analyse the purposes of Renaissance mathematics and physics, we will show that it is exactly in this framework that the notion of the centre of gravity experienced a transformation, from being a simple geometrical and physical tool to becoming a theoretical and mathematical model used to perform both the idealization and the representation of physical reality. The idea is that pure mathematical objects have arisen on the basis of the idealization of quantitative properties from physical bodies, as the result of a theoretical operation that consisted in considering these properties as if they were not a part of sensible matter.

Before moving on to explaining the way in which this new branch of mechanics came to the fore, I will first discuss the ancient heritage of classical mechanics, which splits into two strands: the neglected Archimedean mechanics and the influential Aristotelian one.

In ancient Greece, mechanics was the science that dealt above all with the study of equipment or machines, transport, lifting, and weights, and that was employed as a response to other technological problems. *The search for equilibrium was not of practical interest*[225] – excluding the case of weighing by means of a balance – and mechanics, at least at the beginning, did not care for it. Rather, mechanics was above all preoccupied by problems of everyday life, being connected to manual work, and as such was considered insignificant by the intellectual aristocracy.

It is likely that mechanics began with Aristotle (384-322 BCE) or with the unknown author of the treatise called *Problems of Mechanics*.[226] This is a textbook of

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[225] This lies in striking contrast with the Renaissance tendency; in fact during the Renaissance, as we will see in this chapter, mechanics was associated with statics and thus with the science of equilibrium.

[226] During my research I have found that different opinions have been voiced concerning the author of this treatise. Some attribute it to Aristotle (e.g. Capecchi 2012, Duhem 1905 and Rose 1975), others (e.g. Dugas 1988) to an unknown author. Without entering into the debate pertaining to this issue, it is better to refer to it without authorship as the *Problems of Mechanics*. In short, the main purpose of this work concerns the mechanical problems relating to the shifting of heavy bodies. Nowhere in this text does either the concept or a word for *equilibrium* occur. The functioning of machines or devices such as the

practical mechanics devoted to the study of simple machines. According to its author, the power of the agency that sets a body in motion is defined as the product of the weight (or mass[227]) of the body and the velocity of the motion which the body acquires. By means of this law it is possible to formulate the condition of "equilibrium" of a straight lever with two unequal arms carrying unequal weights at their ends. Since nowhere in the *Problems of Mechanics* does the author use the word or refer to the concept of equilibrium, equilibrium can be seen at most in the dynamic key as the result of the cancellation of effects of opposing forces: when the lever rotates, the velocities of the weights will be proportional to the lengths of their supporting arms. So the author regards the efficacy of the lever as a consequence of a magical power of circular motion. This statement reduces all study of all simple machines to the same principles, or rather, the properties of the balance are related to those of the circle and the properties of the lever to those of the balance.[228]

In opposition to this magical interpretation of motion there is Aristotle, who thought that mechanics should be considered a mixed science defined as follows:

> These are not altogether identical with physical problems, nor are they entirely separate from them, but they have a share in both mathematical and physical speculations, for the method is demonstrated by mathematics, but the practical applications belong to physics.[229]

In both his treatises *On the Heavens* and the *Physics* the notion of movement includes both changes of position and changes of kind, of either a physical or chemical state; the Greek word *dunamis* denotes the force producing the movement. Its application to statics may be regarded as the origin of the "principle of virtual velocities" which will be encountered much later. Elsewhere Aristotle makes a distinction between *natural*

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wedge, pulley and winch is reduced to the lever. However, all authors agree in thinking that the validity of the law of the lever is suggested and may be the first in the history of mechanics. See further Capecchi 2012, pp. 19-32.

[227] The Greeks always confused these two concepts. In the next chapter (Ch. 4, especially § 4.2.2) we will see that the modern definition of the term 'mass' was given by Isaac Newton during the seventeenth century.

[228] Capecchi 2012 and Dugas 1988, pp. 19-31.

[229] Aristotle 1955, p. 331.

*motions*[230] and *violent motions*.[231] The fall of heavy bodies, for example, is a natural motion, while the motion of a projectile is a violent one. In particular, he writes: "to each thing corresponds a natural place. In this place its substantial form achieves perfection – it is disposed in such a way that it is subject as completely as possible to influences which are favourable, and so that it avoids those which are inimical. If something is moved from its natural place,[232] it tends to return there, for everything tends to perfection. If it already occupies its natural place it remains there at rest and can only be torn away by violence."[233]

In accordance with the idea that mechanics is a mixed science, one does not find any difference in the Aristotelian treatises between the two branches of dynamics and kinematics: all mechanical problems were held to accord with the same laws, and, as we will see below, it is only after the Renaissance period that things will change.

Instead, it is likely that Archimedes – as we have seen in the previous chapter – did not share the Aristotelian opinion and considered mechanics as a branch of pure

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[230] On the subject of the *natural* motion of falling bodies, Aristotle maintained in his treatise *On the Heavens* Book 1 that the "relation which weights have to each other is reproduced inversely in their durations of fall. If a weight falls from a certain height in so much time, a weight which is twice as great will fall from the same height in half the time." In *Physics* Book 5, he discusses the acceleration of falling heavy bodies: a body is attracted towards its natural place by means of its heaviness. The closer the body comes to the ground, the more that property increases. Additionally, if the natural place of heavy bodies is the centre of the world, the natural place of light bodies is the region contiguous with the sphere of the Moon. Inversely, heavenly bodies are not subjected to the same law as that which the terrestrial bodies are.

[231] All *violent* motion is essentially impermanent. Once a projectile is thrown, the motivating agency which ensures the continuity of the motion resides in the air which has been set in motion. Aristotle assumes that, in contrast to solid bodies, air spontaneously preserves the impulsion which it receives when the projectile is thrown, and that it can in consequence act as the motivating agency during the projectile's flight. This second Aristotelian thesis on motion, which is clearly stated in his *Physics*, concerns the motion *against* nature (commonly known as the "violent motion") of a heavy body: it occurs along a straight line, and the space covered in a given time is directly proportional to the 'force' applied to the body and inversely proportional to its weight. Over the centuries, the notions of force and work have always been superimposed and used synonymously, yet the precise differentiation between force and work will occur only in the eigtheenth century, and as late as the nineteenth century 'force' will ambiguously be used to mean both force and work. Directly connected with the causes and meaning of the "motion against nature of a heavy body", there is the problem of "resistance to a body's motion", but the causes of resistance are sometimes attributed to weight and sometimes attributed to the medium in which the body is embedded. See further Aristotle, *Physics* (trans. R. P. Hardie, R. K. Gaye). This trans. is available online: http://classics.mit.edu/Aristotle/physics.html.

[232] The natural place of the earth to which moving bodies tend is the concave surface, which defines the bottom of the sea, joined in part to the lower surface of the atmosphere, the natural place of air.

[233] Dugas 1988, p. 20.

mathematics. While Aristotelian mechanics is integrated into a theory of physics[234] which goes so far as to incorporate a system of the world, Archimedes made out of statics an autonomous theoretical science, which was based on postulates of experimental origin and supported by mathematical rigorous demonstrations. He focused mainly on the foundations of mechanics as a rational science, emphasising in particular the demonstration of the law of the lever. The lack of rigour and his indifference towards equilibrium were not central problems for the Aristotelian applications of mechanics; instead "it became a *central* theoretical problem for Archimedes (and of course for the Reinassance tradition that undertook a full-scale reevaluation of his science). He realized that once the problem of equilibrium was solved, the problem of lifting a weight was also solved. Indeed if a weight $p$ equilibrates a weight $q$ in a lever, a weight only slightly heavier than $p$ will lift $q$. But there is an advantage of this shifting of the theory from transport to equilibrium, because equilibrium is much easier to study in a rigorous way."[235]

Archimedes based his mechanical theory on a few *suppositiones* (both 'suppositions' and 'principles'), which were partly empirical in nature, and which certainly appear more convincing than the Aristotelian ones. His goal was to address the equilibrium of extended bodies, as we see in the examples in his treatise on hydrostatics: the equilibrium of a body, or set of bodies, was *reduced* to the determination of its centre of gravity and the ensurance that it was held in a stable position. Archimedes was certainly the first scientist to set rigorous deductive criteria for determining centres of gravity, and his theory was the first known physical theory formalised on a purely mathematical basis.

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[234] Physics according to Aristotle – considered as a subalternate science – can demonstrate *that* things are so (*demonstrationes quia*); while mathematics – the subalternating science – demonstrates *why* (*propter quid*) things are so. As a rule, the respective subject matter of the subalternating and subalternate science is not the same; if they were, one would have a single science and not two separate sciences. Apart from astronomy, the subalternate sciences which attracted the greater attention of mathematicians were geometrical optics and mechanics. They were structured on the basis of the Euclidean model, being based on definitions, suppositions (principles) and propositions (theorems). The main difference with respect to the Euclidean model was that some of the principles, rather than being purely geometric, are related to the physical world. See further De Pace 2009 and Kesten 1945.

[235] Capecchi 2012, p. 45.

So it was, and still is, an axiom of historiography that, since its origins, mechanics has followed two main routes, which are classified as Aristotelian and Archimedean, as we have shown above; it seems moreover that the two approaches differ only in the manner of their proofs. But studies from the second half of the nineteenth century[236] have proved that Archimedean mechanics represents simply the more formalized approach adopted by the mature work of Aristotle, avoiding, in the proofs, physical concepts such as e.g. force, whose meaning was difficult to grasp with certainty. The Greek concept of mechanics was thus revived in the Renaissance. It seems also that the Renaissance had a clear preference for Archimedean mechanics rather than its Aristotelian variant. In fact during this period, mechanics was considered a "new theoretical science" and it was mathematically well founded, although its object had a physical nature and it had social utility. However, for a full synthesis of the Archimedean and Aristotelian routes, we have to wait until Guidobaldo del Monte's and Bernardino Baldi's studies in the second half of the sixteenth century.[237]

What, then, exactly happened during the Middle Ages? How should this shift from one tradition to the other best be explained?

Texts in the Arabic Middle Ages usually divide mechanics into two parts. In particular Abu Nasr al-Farabi (870-950) established the epistemological status of mechanics by differentiating between the *science of weights* and *the science of devices*, both of which were considered parts of mathematics. Mathematics in turn was divided into seven disciplines: arithmetics, geometry, perspective, music, the science of weights and the science of devices.[238] The science of devices referred to the practical use and construction of machines. The science of weights, probably because of the fact that it centered on the balance, was a science not of motion but of equilibrium, as was Aristotelian mechanics. Arabic scholars had access to both Aristotelian and Archimedean works, as well as to Pappus, Hero of Alexandria and of course Euclid. However, among all the Archimedean treatises which are known today and

[236] See e.g. those by Jaouiche 1976 and Knorr 1982.
[237] We will return to this issue in § 3.7 and § 3.7.1 below.
[238] Abattouy 2006.

rediscovered only after the fifteenth century, Arabic scholars had access only to the treatise on floating bodies.

In the Latin milieu we have witnessed a process similar to that registered in the Arabic environment. A science of weight was founded and labelled *scientia de ponderibus*.[239] In addition, there was a branch of learning called mechanics, which was sometimes considered an activity of craftsmen, sometimes one of engineers (*scientia de ingeniis*).[240] What then was the legacy of those two traditions within the context of the School of Urbino? It was exactly in this milieu, due to Urbino's dynamic research and scholars, that what later became known as the "Equilibrium Controversy" took place, having become unfurled since the fourteenth century.[241] This dispute seeks to answer the question of whether a balance with equal arms, deflected from its horizontal equilibrium position, is able to return to its original position, or if it tilts to the vertical. Within the consideration of this problem, we find traces of the process of consolidation of the idealized concept of the material point, traces of idealization of a model nowadays widely used in mathematical physics. In fact, after geometrical analysis of the centre of gravity, the concept of the centre of gravity will achieve more and more autonomy and validity in the area of static equilibrium problems and in the demonstration of the equilibrium of the balance, above all for practical rather than theoretical problems.

In the above paragraph we have described briefly the duality of ancient mechanics in order to understand better its improvement and rejuvenation in the context of sixteenth century science. The next section (§ 3.3) is dedicated to the Renaissance of ancient science, above all to the chiefs of the Urbino School, in order to highlight the leap taken forwards both by mathematical physics towards a new theoretical level and by the new

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[239] As Danilo Capecchi shows, the expression *scientia de ponderibus* comes from the translation completed by Dominicus Gundissalinus of al-Farabi's works into Latin (whereas 'science of devices' was instead translated as s*cientia de ingenii*). For more details, see Abattouy 2006, p. 17 and Capecchi 2012, p. 65.

[240] For more information on the Greek, Arabic and Latin history of mechanics, see the following bibliography: Abattouy 2001; *Id*. 2002; *Id*. 2006 and *Id*. 2008; Capecchi 2012; Dugas 1988; Duhem 1905; *Id*. 1905-1906 and *Id*. 1911; Clagett 1959.

[241] We have discussed the legacy of the different Aristotelian and Archimedean viewpoints on mechanics during the Middle Ages, but in addition we should say that only in the late Renaissance does Aristotelic dynamics – based on the concept of *gravitas secundum situm* – meet with Archimedean statics. See further Nenci 2001 and section 3.7.1 below.

stage of idealization of the material point within the phenomenon of the objectification of procedure.

### 3.3 Mathematical Humanism in the Renaissance

The Renaissance was the period in which mathematics became related to matter, and in which mathematics and natural philosophy reached an extraordinarily high level of maturity, both of which were factors which, in combination, ended up opening the doors to rational mechanics. From a methodological point of view, demonstrations became more and more geometrically based, whereas from a subject-oriented point of view, we witness the development of the so-called *scientia de ponderibus*, the study of which involves both problems of statics and the notion of the centre of gravity.[242] The latter was totally unknown among those authors who had an Aristotelian education, because the Medieval philosophical tradition did not yet know Archimedes' treatises on statics and equilibrium. Some of these works, in particular the *Planes in Equilibrium* (Book 1), were translated for the first time into Latin around 1269 by William of Moerbeke. The Archimedean approach to mechanics was non-causal and non-dynamic, inasmuch as force and speed were not taken into account, but instead a rigorously mathematical method was applied to the study of equilibrium, with the aid of such physical postulates as the centre of gravity. This approach differed fundamentally from that of the *Problems of Mechanics* (attributed to Aristotle), which embodied vague dynamic ideas, above all that of the relationship between power and speed. When taken independently, as seen in the previous section, both traditions could be fruitful: Archimedes pioneered mathematical and rigorous proofs in mechanics, while Aristotle adumbrated the concept of virtual velocities. But in order to produce modern mechanics, it was essential to combine these two traditions in a complementary way. This step forward was unquestionably and authoritatively taken by Galileo, as we will see in the next chapter,

[242] For further study, see Pisano 2006.

but vaguely sketched by Guidobaldo del Monte and Bernardino Baldi from the School of Urbino, both of whom thought that the *Mechanica* merely represented a primitive and physico-causal account of mechanics.[243] Baldi's viewpoint flowered in sixteenth-century Italy, when the Aristotelian tradition was promoted by Niccolò Leonico Tomeo (1456-1531) and Alessandro Piccolomini (1508-1579), nor was it ignored by Tartaglia, Benedetti or Guidobaldo, all of whom attended Pietro Catena's (1501-1577) lectures in Padua. However, according to Baldi, the limitation of the pseudo-Aristelian *Problems of Mechanics* was the natural explanation of the lever and other machines. Baldi's *Mechanica Aristotelis Problemata Excercitationes*, printed albeit posthumously in Mainz in 1621, attempted to expose the "statistical obsolence of the *Mechanica* while retaining the Peripatetic doctrines of dynamics surrounding the tradition."[244]

Since the main goal of this study is to focus on the contribution of the Reniassance mathematicians to show the way in which a theoretical notion such as the point mass emerges, thanks to a process of idealization through analysing and observing the scientific practice which stands behind the activity of weighing, it is appropriate to analyse the rediscovery of Archimedean mathematics and the development of the Equilibrium Controversy by means of an analysis of the most prominent characters involved in this process. In the subsequent subchapters, I first analyse the way in which Federico Commandino reintroduced the Archimedean treatises to the Italian cultural milieu by circulating new translations of them, in order finally to clean them up from their earlier Medieval vagueness. Second I will scrutinise the work by Guidobaldo del Monte, who is first and foremost remembered for his most important Renaissance treatise on mechanics, but who is also estimated to be the first scientist ever to have tried to conjoin the two conflicting mechanical traditions.

[243] This led to the appreciation of the important work of other Renaissance mathematicians based in Italy, especially that of Tartaglia, who had tried to receive the medieval statics of Jordanus as an alternative critique of the pseudo-Aristotelian *Problems of Mechanics*. However, Jordanus' statics was in many ways dynamic in character and so impressed the Urbino School as being nothing more than a regression to an imprecise Aristotelian system long since superseded by Archimedes. See further Rose 1975, p. 249.

[244] Rose 1975, p. 249.

### 3.4 Federico Commandino, Head of the Urbino School[245]

During the sixteenth century, among the first thinkers to discuss the concept of Archimedes' centre of weight was Leonardo da Vinci, who used the centre of weight in his work on mechanical devices, as well as later in his studies on the equilibrium of the deflected balance.[246] But it is only in the Renaissance courts of Venice and Urbino, within a strongly humanistic background (which was at the same time interested in the practical application of mathematics and strongly connected with engineers and technicians), that we can observe the most important rediscovery of Archimedean science.

In Venice in 1503 Luca Gaurico edited a Latin edition of two Archimedean treatises, *De Mensura Circuli* and *De Quadratura Parabolae*.[247] Then in 1543, Tartaglia edited Gaurico's version (printed only posthumusly in 1565), adding the *Planes* in *Equilibrium* and *The Floating Bodies*.[248] In Basel in 1544 the first *editio princeps* of the Archimedean corpus was printed, including the Greek text and the Latin translation by Thomas Geschauff, better known by his Latin name Veratorius.[249]

At the court of the Duke of Urbino, the scholar who was considered the initiator of the Renaissance Mathematical Humanism, and who best represents the tradition of the

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[245] Despite the fact that, for my research purposes, the most important achievements towards the process of idealization and the objectification of the model of material point are reached by Guidobaldo dal Monte, his pupil Galileo Galilei, and Luca Valerio, I consider it to be significant at least to give a brief but expansive introduction on the "fathers" of this modern rational mathematical physics. Thus it seems impossible for me to outline such a process without taking into account the rule of Commandino and Bernardino Baldi as the foremost figures of the "Urbino School".

[246] Renn and Damerow 2012, p. 58.

[247] *Tetragonismus id est circuli quadratura, per Campanum, Archimedem Syracusanum atque Boetium mathematicae perspicacissimos adinventa*, Venezia, 1503.

[248] *Opera Archimedis Syracusani philosophi et mathematici ingeniosissimi, per Nico-laum Tartaleam Brixianum (mathematicarum scientiarum cultorem) multis erroribus emendata, expurgata ac in luce posita multisque necessariis additis... Appositisque manu propria guris quae graeco exemplari deformatae ac depravatae erant*, Venezia, 1543. In 1551 the same edtion was translated in vernacular and printed in 1565 with the title: *Ragionamenti sopra la sua Travagliata inventione, nelli quali se dechiara volgarmente quel libro di Archimede Siracusano intitolato De insidentibus aquae*, Venezia, 1551.

[249] *Archimedis Syracusani philosophi ac geometrae excellentissimi Opera, quae quidem extant, omnia multis iam seculis desiderata atque a quam paucissimis hactenus visa, nuncque primum et graece et latine in lucem edita. Adiecta quoque sunt Eutocii Ascalonitae in eosdem Archimedis libros Commentaria, item graece et latine, nunquam antea excusa*, Basel, 1544.

ancient mathematics, is Federico Commandino. This is certainly confirmed by his contemporaries and by Guidobaldo's praise, articulated in 1577:

> Yet in the midst of that darkness (though there were also some other famous names) Federico Commandino shone like the sun. He by his many learned studies not only restored the lost heritage of mathematics, but actually increased and enhanced it. For that great man was so well-endowed with mathematical talent that in him there seem to have lived again Archytas, Eudoxus, Hero, Euclid, Theon, Aristarchus, Diophantus, Theodosius, Ptolemy, Apollonius, Serenus, Pappus and even Archimedes himself, for his commentaries on Archimedes smell of the mathematician's own lamp. And so! Just as he had been suddenly thrust from the darkness and prison of the body (as we believe) into the light and liberty of mathematics, so at the most opportune time he left mathematics bereft of its fine and noble father and left us so prostrate that we scarcely seem able even by a long discourse to console ourseves for his loss.[250]

Commandino is defined as a humanist with a strong interest for the mathematical discipline, and, by the same token, as a mathematician with a strong competence in philology. In fact, between 1558[251] and 1575 he edited ten of the main ancient Greek treatises. But let us take one thing at a time.

Federico Commandino was born into a noble Urbinate family in 1509.[252] His grandfather had served the Duke Federico da Montefeltro, after whom the mathematician was named. His father was Battista Commandino and as a diligent student of architecture had supervised the fortification of Urbino's walls. Since

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[250] *Emicuit tamen inter istas tenebras (quamiis alij, quoquè nonnulli fuerint praedarissimi) Solis instar Federicus Commandinus, qui multis doctissimis elucubrationibus amissum mathematicarum patrimonium non modò restauravit, verùm etiam auctiùs, e kocupletiùs effecit erat enim summus iste vir omnibus adeò facultatibus mathematicus ornatus, ut in eo Architas, Eudoxus, Heron, Euclides, Theon, Aristarcus, Diophantus, Theodosius, Ptolemaeus, Apollonius, Serenus, Pappus, quin e ipsemet Archimedes (si quidem ipsius in Archimedem scripta Archimedis ilent lucernam) revixisse viderentur e ecce repentè è tenebris (ut confidimus) ac vinculis corporiis in lucem, liberatatemquè productus mathematicas alienissimo tempore optimo, e praestantissimo patre orbatas, nos veò ita conservatos reliquit, ut eius desideruum vix longo sermone mitigare posse dideamur. Ille tamen perpetuò in aliarum mathematicarum explicationem varsans, mechanicam facultatem, aut penitus praeter misit, aut modicè attigit.* del Monte 1577, *Prefatio*.
The English trans. is quoted in Renn and Damerow 2010, p. 57.

[251] Yet in 1558 Commandino was already 49 years old. What took him so long to get started?

[252] In this section, in order to recount the life and research carried out by Commandino, I have consulted the following works: Dennistoun 1851, III, p. 97; Grossi 1819, pp. 53-57; Mamiani 1828, pp. 4-42; Ugolino 1859, II, 271, 273f.

Commandino's father met figures as Jacob of Spira and Paul of Middelburg, his training and formative environment were both humanistic; in fact during his life and studies, Commandino remained in close contact with the Farnese's circle at Rome, which included Fulvio Orsini (1529-1600), Marcello Cervini (1501-1555) and the poet Torquato Tasso, who studied with Commandino and acquired enough mathematics to lecture at the University of Ferrara.[253]

Commandino's teachers were Giacomo Torelli of Fano, his tutor in liberal arts, Latin and Greek, and Giam-Pietro de' Grassi (both sixteenth century), who was skilled in both the humanities and mathematics. Thanks to both these teachers, when Commandino returned to Rome he entered the service of Cardinal Niccolò Ridolfi as a mathematics tutor. Ridolfi was a philologist and amassed over 600 Greek manuscripts. In this collection there was also a copy of the *Codex A* of Archimedes. It was surely thanks to his friendship with Ridolfi that Commandino resided in Rome in the 1530s and had unlimited access to Ridolfi's great collection. Moreover, owing to this acquaintance Commandino secured an introduction to Pope Clement VII, who appointed him as *cameriere secreto* and with whom he discussed mathematics and the scientific discipline in general.

After his years in Rome, he moved to the University of Padua where he attended medicine and philosophy under Marc'Antonio de' Passeri and Giovanni Battista Montano for ten years. Later he went to Ferrara where he completed his medical studies under the patronage of Antonio Brasavola. Meanwhile in 1537, he also met Tommaso Leonardi of Fano, an algebraist skilled in mathematics and astronomy, and it was likely that he was the major formative influence on Commandino.

In 1556 Commandino returned to Urbino, where he married Girolama Buonaventuri and had two daughters and a son. Given that both his wife and his son died prematurely, and since he also lost his father around the same time, Commandino harboured a sense of disillusionment towards medicine, labelling it as an uncertain branch of knowledge. It was in this context that Commandino abandoned the practice of

[253] For further details, see Guasti 1952.

medicine for the certainty and undeniability offered by mathematics.

In Urbino, Commandino was called to the service of Duke Guidobaldo della Rovere, under whose patronage Commandino met Ranuccio Farnese and the librarian Fulvio Orsini mentioned above. In the assets amassed by Orsini, Commandino had the chance to consult the entire Greek tradition including Aratus, Aristarchus' astronomical and musical works, and fragments of Archimedes' *On Floating Bodies*. Indeed owing to Farnese's network of acquaintances, Commandino met several scholars such as the Spanish humanist and mathematician Diego Huardo de Mendoza, who took lessons directly from Tartaglia in Venice.

Extremely important was also the friendship with the Cardinal Marcello Cervini, another collector of manuscripts and bibliophile who amassed among 440 codices now deposited in the *Biblioteca Vaticana*. This collection includes Leonardo Fibonacci's *Pratica Geometriae*, Pappus' *Problemata Geometrica*, Ptolemy's *Almagest* and *Geographia*, Hero's *De Geometria* and *Pneumatica*, and others. Through his friendship with Cardinal Cervini, Commandino acquired the Moerbeke translation of Archimedes' *On Floating Bodies*. Only after Marcello Cervini's death did Commandino return to Urbino definitively, where he accomplished the mathematical works which were published in 1556.

In the *Prolegomena* found at the beginning of the Archimedean edition, Commandino expresses the idea[254] that although all sciences have some elements of truth in common, it is only with the discipline of mathematics that uncertainty is dispelled. "The obscurity in nature and philosophy is exemplified in the disagreement between Plato and Aristotle. But Pythagoras, Plato, Aristotle and Galen all attest to the nobility and importance of mathematics", he says.[255] Commandino also remarks, according to the

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[254] This idea is also reintroduced in the *Prolegomena* at the beginning of the edition of the *Elements* published in Pesaro in 1575.

[255] Rose 1975, p. 195. As said above, Commandino in the *Prolegomena* begins by dividing philosophy into divine, natural and mathematical categories, of which the last provides the greatest epistemic certainty. After this, mathematics is distinguished from physics, because the former does not rely upon sense-phenomena and depends upon the intellect alone. At the same time mathematics is not merely an abstract discipline, for it has real connections with the physical world. This is obvious in relation to the mixed

current trend, how far mathematics was essential for military science and civil engineering works. The 1558 edition of Archimedes' works was not a complete corpus, but included only *De mensura Circuli*, *De Lineis Spiralibus*, *De Quadratura Parabolae*, *De Conoidibus et Spheroidibus* and the *Arenarius*. Despite the fact that he had also promised the publication of the *Aequeponderantibus*[256], Commandino never reached this goal, and the publication of this treatise was left to his pupil Guidobaldo in 1588. Hence, the climax of the restoration of the Archimedean corpus was reached with his publication of *On Floating Bodies* (Bologna 1565), whose delay was caused by the corruption of the ancient text. The excellence of Commandino's edition of *On Floating Bodies* may be observed in his elegant reconstruction of two lost proofs of Archimedes. In contrast to the Latin and vernacular editions by Tartaglia (published in Venice respectively in 1543 and 1565), Commandino supplied the missing proofs of Proposition 8 of Book 1 and Proposition 2 of Book 2. The difference between Tartaglia and Commandino can be found in the fact that these two proofs required a foreknowledge of the determination of the centre of gravity of a paraboloid segment (i.e. the centre of gravity of solid figures). On the one hand, Tartaglia never even tried to rework those proofs, and on the other hand, neither in any of the extant Archimedean works (the Archimedean *Method* was being rediscovered by Hilberg in 1906), nor in any other known Greek works, is this method described. Thus, Commandino merits to be called the first to achieve the task of reconstruction and restoration of the ancient Greek treatise *On Floating Bodies*.[257]

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sciences such as mechanics, astronomy, optics, music and geodesy. In addition to being a liberal art, mathematics is also utilitarian, as can also be seen from its application in medicine and military science.

[256] *Planes in Equilibrium*.

[257] Rose 1975, p. 201.

### 3.4.1 The new Centre of Gravity

Commandino's research on solids is embodied in his *Liber de Centro Gravitatis Solidorum*[258], published in 1565 and considered the apogee of his scientific production. This manuscript was principally written to clarify one of the most important propositions of the Archimedean *On Floating Bodies*, previously translated by Moerbeke into unclear and problematic Latin. Commandino was able not only to translate the *Floating Bodies* properly, but also, thanks to the additional work just mentioned above, to fill in the demonstrative and conceptual gaps about the law of the lever. Pier Napolitani writes: "As the only determinations of the centre of gravity that we have from antiquity only refer to two-dimensional figures, Commandino decided to provide scholars with an essay on centres of gravity of three-dimensional figures such as prisms, pyramids and cones, and also the parabola."[259] From Archimedes Commandino acquired not only the method for calculating the centre of gravity of each figure, but also the method of approximation. This method is useful for demonstrating that the centre of gravity of a figure lies on its axis or diameter (the segment that links the vertex with the centre of gravity of the base).[260] Starting from this point, Commandino was able to prove how to circumscribe (or inscribe) the studied figures with figures of increasing dimensions, in order to make the difference between the two of them arbitrarily small.

The definition of the *centrum gravitates*, which can be found in Commandino's treatise, arises from Pappus' *Mathematicae Collectiones* (third/fourth century BCE). Although in the previous chapter we have already given Pappus' definition, for the sake of convenience I reproduce it below:

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[258] The latest edition is recently published in 2015 and is edited and translated into Italian by E. Gamba and V. Montebelli: *Liber de Centro Gravitatis Solidorum*, Edizioni della Normale, Pisa, 2015.

[259] Napolitani 2001. During the Renaissance only the first book of the treatise *Planes in Equilibrium* was already known besides the treatise *On Floating Bodies*.

[260] The property that underlies the approximating figure technique – which was studied by Luca Valerio in 1604 – is the 'monotonicity' of the figure in question, that is the fact that the sections constantly decrease from the base to the vertex.

> The centre of gravity of any body is a point situated within, and such that, if the body is imagined to be suspended from it, the weight will be at rest as it hangs and will keep its original position.[261]

The idea is that a point is the centre of gravity of a body *if and only if* once we are hanging that body by its centre of gravity, it remains in that position, or in the equilibrium position, and it will never rotate around that point.

Given that definition and by combining it with Pappus' propositions, Commandino provides a new description, trying to be as accurate as he can, and additionally grounding his formulation on Archimedes' fragments. The result as he states it is the following:

> The centre of gravity of any solid shape is that point within it around which are disposed on all sides parts of equal moments [*aequalium momentorum*], so that if a plane be passed through this point cutting the said shape, it will always be divided into parts of equal weight.[262]

This definition contains two different statements: i) there exists in any solid a point in respect of which the two different parts making up the solid have *equal moments*; and ii) the two parts obtained from the operation by which we cut the whole solid will always be equal in weight.

This is absolutely the first work ever published which is completely dedicated to this topic, even though contemporaneously other mathematicians were working on the same subject, such as the Sicilian Francesco Maurolico, whose treatises were only published posthumously.[263]

It is not the aim of this section to compare or criticize the outcomes obtained by the Urbinate Commandino. Instead I wish to point out the importance of the introduction of

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[261] *Pappi Alexandrini Mathematicae Collectiones a Federico Commandino urbinate in latinum conversae et commentariis illustratae*, Pesaro 1588, f. 306*v*.

[262] *Centrum gravitatis uniuscuiusque solidae figurae est punctum illud intra positum, circa quod undique partes aequalium momentorum consistunt. Si enim per tale centrum ducatur planum figuram quomodocumque secans semper in partes aequeponderantes semper dividet*. Commandino 1565, pp. 1-2, with the English trans. taken from Frank, 2013.

[263] In the dedication of his book to Alessandro Ranuccio, Commandino explicitly states that in the previous years he was waiting for the publication of Maurolico's treatise, in order to be able to compare his own studies with Maurolico's outcomes.

the definition of centre of gravity or weight, to which both the newly introduced notion of *moment* and the meaning of the expression *equal in weight* are directly related.

The *centrum gravitatis* is a geometrical point equipped with specific static properties. *Gravitas* is not the force acting on a body, but the *gravezza* or weight of a body which is imagined to be concentrated in just one point. The centre of gravity is defined as the fulcrum of the virtual balance which is in turn made up of the elements of the solid figure; it is the point of equilibrium of all solids' inclinations and moments. Thus the moment represents the tendency to go downwards or upwards, and depends on the weight of the solid in itself and on its distance from the fulcrum. In this precise context the moment indicates the variable of the weight.

The meaning given to 'moment' by Commandino is not related to the concept of the moment of inertia[264], but rather is only a qualitative definition which tries to state the difference in the efficacy of weight in the balance and the lever. It will be Galileo who, between 1595 and 1597, will define the moment of a body – or of a particle – as the product of its mass multiplied by its velocity.[265]

Commandino's definition of the centre of gravity leads the general reader to misunderstand the term 'moment' and its relation to the expression 'equal in weight'. From the description given above it appears in fact that by cutting the solid figure in two parts, with a plane passing through the centre of gravity, the two parts must necessarily have the same weight in order to achieve the equilibrium position. But this cannot be the case, because, as we already know from Archimedes' law of the lever, once we hang two bodies at the end of the arm of a balance, that balance will be in equilibrium if and only if the bodies (hanging in their centre of gravity) lie at a distance which is indirectly proportional to the weight of the bodies. Thus, this does not endorse

[264] The moment of inertia, also known as the angular mass or rotational inertia, of a rigid body is a tensor that determines the torque needed for a desired angular acceleration about a rotational axis. In its modern physical conception, the moment of inertia $I$ is defined as the ratio of the angular momentum $L$ of a system to its angular velocity $\omega$ around a principal axis.

[265] For more on the development of the conception of 'moment', see Galluzzi 1979, pp. 41-62, which examines the occurrences of the word 'moment' in Commandino's *De Centro*, and which states that, besides the use made in the definition of the centre of gravity, in *no other* of his other works does it occur again.

the idea that the two parts of the same body must have the same weight. Only Guidobaldo in his *Mechanicorum Liber* will remedy this linguistic misunderstanding.

Now, let me introduce some examples to provide a basic overview of the evolution of the debate surrounding the notion of centre of gravity and its difference from the notion of moment. Francesco Guerrini, Guidobaldo's disciple, just few months after the death of his teacher, sent a letter to Clavius in which he reported a commentary given by Guidobaldo himself on the Archimedean propositions of the *Plane in Equilibrium:*

> After the death of the most Illustrious Guido del Monte, may God rest his soul, several gentlemen of Pesaro asked me to show them the practice of the *Mechanicorum Liber* of the aforesaid Sir, as I am doing. We already finished the first chapter *Della Libra* and at the beginning there has been a great controversy about the definition of the centre of gravity, about these words: "In fact, if a plane is drawn through this centre, intersecting the figure in an arbitrary way, so it always divides it into equiponderating parts." And if one wanted to insist on the wording "intersecting in an arbitrary way", it would seem that the two parts, after the section, would weigh equally, but in reality the contrary can be proven. [...] I beg You to tell me Your opinion which would be of great use for me [...].[266]

First of all, the term 'equiponderating' is a neologism used in translating the Latin word *aequeponderare*. Both the translations 'equal' and 'of equal moment' would distort the sense of the sentence.[267] Thus, Guerrini's letter (together with the discussions among the "gentlemen of Pesaro") concerns exactly the same problem already raised by Commandino, and which had only for a second time been reintroduced by Guidobaldo in the first book of the *Paraphrasis*. Formulated in terms of modern physics, the issue (in the case of three-dimensional geometry) is equivalent to the following question: does the intersection of a body by a plane passing through its barycentre create two separate parts of equal *weight*, or rather two separate parts of equal *moment*?

In the light of Commandino's activity later as a teacher to young engineers, the

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[266] Frank 2013, p. 322.
[267] For further in this regard, see the doctoral dissertation by M. Frank, *Guidobaldo Del Monte Mechanics in Context. Research on the Connections between his mechanical work and his life and environment*, 2011/2012, pp. 348-349.

double occurrence of this problem does not seem coincidental. Since the distinction between weight and a kind of (proto-)moment (*aequeponderare*) is one of the basic problems of Archimedes' barycentre theory, it is likely that Guidobaldo encountered the same quandary while he was teaching the *Mechanicorum Liber*'s content (and thus the very foundations of the Archimedean theory).[268] The remedy to this dilemma is later found thanks to the combination of the Archimedean tradition with its Aristotelian counterpart. Guidobaldo, and Baldi too, both give importance to machinery, for both the physical content and the physical outcomes of their research, while Commandino on his own was above all worried about the mathematical rigour of his reconstruction.

In fact also Guidobaldo himself, while praising Commandino as the greatest restorer of ancient mathematics, voiced that his teacher had failed to achieve a systematic treatment of mechanics:

> In his [sc. Commandino's] endless concern with the elucidation of other parts of mathematics, Commandino either left mechanics completely untreated or touched on it just casually. Therefore I began to devote myself more eagerly to this study, and in making my way through every branch of mathematics I never lost sight of my course to find whatever could be appropriated and derived from each of these branches, so that I might be better equipped to perfect and embellish mechanics.[269]

[268] In 1598 Guidobaldo wrote to the Jesuit Christoph Clavius, claiming that the *Mechanicorum*'s second supposition (i.e. that the centre of gravity of a body is always in the same location with respect to its body), asserts that when Archimedes, in the *De aequeponderantibus*, refers to centres of gravity, and to weights suspended in their centre of gravity, he intends that the weights must remain in any position whatever, and that in this sense they "have the same weight, or, are equally heavy [Lat: *aequeponderent*]". To this Guidobaldo also adds that Archimedes can only be interpreted in this way, since otherwise his conclusions would not be true. Thus, while Guidobaldo may or may not be right about his reading of Archimedes, it is clear that, by attributing to the nature of the centre of gravity the meaning of "having the same weight in all directions [that is, *aequeponderare per tutti i versi*]", he is *de facto* interpreting Archimedes according to Pappus' definition, and Pappus' definition in terms of moments, along the lines of Commandino's description. The main misinterpretation is, however, given by the fact that neither Commandino nor Guidobaldo have ever specified how the 'moment' is to be defined. Indeed Commandino throughout the *De Centro* never makes use of his own definition; rather following in Archimedes' footsteps, he proceeds by demonstrations that start from postulates. Moreover, neither Archimedes nor Pappus ever furnished criteria which are to be used for *directly* finding centres of gravity. In other words, from the descriptions and demonstrations given by those authors, we can only accrue knowledge about centres of gravity starting from postulates that describe simple configurations of weights, and only thereby progress to theorems about more complex configurations of weights. For further details, see Palmieri 2008, pp. 315-16.

[269] *Ille tamen perpetuò in aliarum mathematicarum ecplicationem versans, mechanicam facultatem, aut penitus praeter misit, aut modicè attigit. Qua propter in hoc studium ardentiùs ego incumbere caepi, nec*

Thus, Commandino's lack of theoretical knowledge did not allow him to enter actively into the equilibrium controversy, although numerous studies had been dedicated to the balance in practically all times and cultures, in particular in the Arabic and Latin Middle Ages, as well as in the early modern period. Specialist understanding of the question whether or not an equilibrated balance would return to its horizontal position or remain in whatever position it is brought was the paradigmatic topic of mechanical knowledge, the core of a science of weight. For centuries, it was a topic strictly related to physical concepts, from the law of the lever to the principle of conservation of energy. Therefore, what happened in this context and what was the gap that these scholars were able to fill in, by collaborating with each other and by allowing the overlap of different theoretical, empirical, geometrical and mathematical knowledge that they were in possession of?

Perhaps a more sophisticated knowledge was required to answer this apparently straightforward question. Physics was not enough. Although a series of sophisticated empirical experiments give us certain results, without a theoretical background we are not able to reply to the question in a proper way. In fact, the empirical context will yield nothing more than an accumulation of data based on single-case observations. The profound gap between these partial results and the theoretical framework used widely and confidently nowadays lies upon the lack of generalization or, better, on the absence of a systematic treatment of an ample series of cases, each of which may be included in a general law or in a general framework. Therefore, this *dilemma* was decided only before the turn to the early modern age, when – as stated by Guidobaldo himself – he was dealing with a systematic treatment of the equilibrium controversy.

Although Commandino did not make any particular contribution to the development of the notion of centre of gravity, or to its transformation from a purely geometrical notion relating to plane and solid figures into a theoretical entity used to represent all bodies detached from their physical features, a transformation which we

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*me unquam per omne mathematum genus vagantem ea solicitudo deservit; ecquid ex uno quoque decerpi, ac delibrari possit; quo ad mechanicam expoliendam, & exornandam accomodatior esse possem*. del Monte 1577, Preface.
The English trans. is taken from Drake and Drabkin 1969, pp. 239 ff, especially p. 245.

have already seen above, at least he had the merits to recover Archimedean studies in their entirety by giving back to them a measure of coherence.

Before switching to the core of the debate over the equilibrium of the balance, we will dedicate a section to the one of the most influential contemporaries of the Urbinate Commandino. Unfortunately, the indispensable role of Francesco Maurolico was only recognised later in the century, because during his lifetime he postponed the publication and circulation of the outcomes of his endeavors. According to today's scholarship, we do not find significant evidence of his influence on the theoretical development of the notion of the point mass; however, Maurolico deserves to be considered one of the main protagonists in the systematisation of the Archimedean mathematical corpus.

### 3.5 Francesco Maurolico

While Commandino is remembered for the philological accuracy of his translations, and for having illuminated for the first time the work of one of the greatest scientists of antiquity, his contemporary, the self-educated Francesco Maurolico from Messina,[270] also enters in a list of the greatest ancient "renovators" of the sixteenth century. He was the son of Antonio Maurolì, a Greek physician who had fled the sack of Constantinople to settle in Sicily, where he become master of the mint at Messina. Maurolico's father was attracted to mathematics and astronomy, acting as personal tutor to his son Francesco, who had other two teachers to train him in poetry, philology and history. As Rose says, "Maurolico's immersion in Greek culture – both through parentage and education – also gave him the technical skills in language necessary for his attempt to restore to life the mathematics of the Greeks."[271] Nothing is known of Maurolico's career until 1521. His first works on mathematics were completed between 1521 and 1523, and they represent a new departure in the history of optics. Then in 1528 he published

[270] The best source for Maurolico's biography can be found in the *Vita dell'Abate del Parto D. Francesco Maurolico*, Messina, 1613, written by his nephew Baron della Foresta.
[271] Rose 1975, p. 159.

the *Grammaticorum Rudimentorum Libelli Sex*. In this work it was his principal thesis that all five areas of natural philosophy (the *trivium*, metaphysics, physics, the *quadrivium* and ethics) were subject to epistemic doubt, while only mathematics contained certain demonstrations which keep it from falling into scepticism. Thus, he was directly involved in the renewal of the educational system, despite the fact that he was geographically isolated.[272]

Moreover, by 1534 he had finished a translation of a portion of Euclid's *Elements* based on Campanus' edition, and in the same year he turned his attention to Hero's *Pneumatica* and Archimedes. Maurolico's main attempt was now to restore and clean up the translated works of the previous authors thanks to his skills both in ancient Greek and in classical mathematics and geometry. Maurolico told his friend Pietro Bembo (1470-1547) that Euclid's original version of the *Elements* was so falsified by translators that he had become altogether altered, mutilated and full of new trifles. Despite this, his main interest was directed upon Archimedes. During his lifetime he enjoyed marginal success and his studies on Archimedes were published only posthumously. Fortunately, his intellectual heritage remained alive thanks to his Jesuit friend Christoph Clavius. Maurolico adopted his own editorial technique: while his predecessors, such as Commandino (as we have seen above) were only interested in the accuracy of their translations and adhered in this exercise to strict philological rules, Maurolico reworked the treatises entirely that he already had in his hands. He edited the books of Apollonius, Euclid and Archimedes, adding lemmas, demonstrating things which were omitted by the original authors, and treating their mathematical subjects with enormous expertise.

Maurolico's greatest work, and the most extraordinary example of his innovations, is the *De Momentis Aequalibus* (*Admirandi Archimedis Syracusani monumenta omnia mathematica*), an extensive rewriting of Archimedes' *Planes in Equilibrium* which was only published in 1685. This work is composed of four books, the first of which deals

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[272] As we read in Rose [1975 pp. 167-169], Maurolico during his lifetime received several offers from his contemporaries to stay in Rome, or to travel to other cities for research purposes or teaching. However, he always refused these offers, arguably because of his attachment to his Sicilian patron, Giovanni Ventimiglia.

with general statements relating to centres of gravity and equilibrium. The following three books concern the determination of barycentres in plane figures and solids (paraboloids and spheres, pyramids, prism and conoids). The *De Momentis* provides a new demonstration of the determination of the rotating parabola's centre of gravity. This demonstration is independent of the one given by Commandino (which he does not seem to be aware of at this time), and is based on the notion of moment. Maurolico's *De Momentis* cannot be considered either a simple edition of Archimedes' *Planes in Equilibrium* or an extension of this treatise. Rather, it is the result of deep reflection on Greek statics and Archimedean thought. It broke into the vanguard of Renaissance mathematical scholarship by presenting a new interpretation of i) the *scientia de ponderibus*, ii) the centre of gravity and iii) the relationship between the physical and mathematical interpretations of the equilibrium of balance.[273] Moreover, there is a conceptual transition from a qualitative notion of moment – whereby one is able to express the different efficacies of the weight in the scale and the lever – to a geometrical and quantitative notion of moment. Maurolico used his mathematical skills to integrate and redefine Archimedes' works, avoiding demonstrative gaps in his manuscripts.

At first glance, in comparing Commandino's and Maurolico's re-elaborated versions, it seems that their approach is basically the same, namely to elucidate the sequential development of the Archimedean proofs and to give alternative or additional demonstrations when required. Regarding this methodological consensus, there is a surviving amount of correspondence between them, which reports the following:

> All the mathematicians of our time ought to be grateful for the abundance of books written by you [Commandino is here referring to Maurolico]. With these you have opened the way to explaining and making intelligible those things which have lain, obstructed by many difficulties, in the greatest darkness for uncountable centuries [...]. I wish to see all your works printed as soon as possible, particularly the two unpublished books of Archimedes – *De operimetris Figuris* and *De Speculis Comburentibus* – and your books of *De Aequalibus Momentis*, *Photismi* and *Diaphana*.[274]

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[273] Napolitani 2001, p. 19.

[274] The surviving correspondence between Maurolico and Commandino can be found in the *Biblioteca Universitaria*, Urbino, MS, Busta Comune 120, fols. 185-188v, and may be dated to ca. 1557.

Although from the passage above Commandino and Maurolico's approaches appear to resemble each other, it is worthwhile highlighting the main difference between them. From the examination of Maurolico's texts we can assert that he seeks to establish the ancestry of algebra, mathematics and geometry by using the available Archimedean fragments, surpassing the translations made by Arabic and Latin *amanuenses* that were normally full of both grammatical and content-related errors and miscalculations. Out of scepticism towards these sources, he opts to change his expository method, and in fact was able to identify the *fil rouge* of the ancient mathematical-geometrical approach in order to make use of the common general features, and in so doing to yield a unitary method of enquiry and bring about the most consistent demonstrations. In this way the treatments of mechanical problems are now envisaged from two different perspectives: i) a purely physical analysis relating to the empirical experiments; and ii) a more general treatment involving the use of a systematic model to represent the events by means of this theoretical and idealized model. Thus in this respect, Maurolico's corpus denoted not only an inquiry into the science of mechanics with practical implications, but also a step towards the consolidation of a new general methodology and a new means of doing and conceiving mechanics. For example, in one of his works, the *Arithmetica* of Diophantus[275], the most striking aspect lies in the use of letters in place of numbers. The aim of this procedure is to try to convert the handling of numerical data to the same generality and abstraction as the letter-based handling of geometry.[276]

We have mentioned the steps forward taken by Commandino and Maurolico in the restoration of the main Archimedean treatises. Both authors provide two different approaches which were indispensable both for initiating the debate that led to the development of modern mechanics and, in particular, for kick-starting an epistemological debate about the nature of idealized geometrical entities. The process which the geometrical notion of the centre of gravity underwent by being transformed

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[275] Scinà, Bottazzini and Nastasi 1808, pp. 38-45; See further Fontana 1808, pp. 275-296 and Martines 1865, pp. 65-89.

[276] Some of the details used in this reconstruction are taken from Rose 1975, pp. 159-184.

into a mathematical notion, assuming that the physical meaning related to the definition of point mass, has its hints in the mechanical practice of weighing. However, the epistemological significance of idealizing and abstracting physical properties from its practical application or from real bodies are no more than outlined in Guidobaldo's research. So before entering into a detailed analysis of del Monte's mechanics, let us first consider the epistemological and historical meaning of the sixteenth century's Equilibrium Controversy.

### 3.6 The *Scientia de Ponderibus*

By understanding the epistemological and historical significance of the Equilibrium Controversy, we will better appreciate the difference between Guidobaldo del Monte, "the new Archimedes", and his contemporary Federico Commandino. Moreover, comprehension of this centuries-old debate will help us both to grasp the development of the theoretical field of modern mechanics, and to understand the essence of the process we have called "the second stage of objectification of procedure". Here it will be shown that, by looking at the practice of weighing and the behaviour of the balance, and by stating its mechanical principles, Guidobaldo and his contemporaries were able to abstract from a practical context an idealized concept such as the notion which we now call the point mass. Thus, in the context of the Equilibrium Controversy, the centre of gravity behaves as a geometrical point devoid of any physical content, i.e. any volume. This admittedly schematic representation is used mainly to simplify computations and to attempt to find a general mechanical law governing the behaviour of the balance. By means of these two items, the purely geometrical notion of the centre of gravity will approach – under a conceptual revision – the status of a (quasi-)idealized mathematical object. As we will see, even a simple problem such as that of the equilibrium of balance can in this light become a challenging issue. In this respect, the main aspect to be emphasised is the role played by historical contingencies[277] in conceptual development at the heart of mechanics. There is, first of all (as we have seen in the previous Chapter), the contingency of those aspects related to the centre of gravity, which had a physical, heuristic and mechanical meaning in aiming at pure geometrical and mathematical enquiry. Secondly, there is the contingency of the social and cultural conditions whereby knowledge is recorded, transmitted, and appropriated in the case of the Renaissance; this thorough re-examination of ancient sources occurred in order to determine the real nature of concepts in mechanics. Since science is by its nature a humanly guided enterprise, it is the purpose of this subchapter to display the key role played by mathematical humanism.

Furthermore, the analysis of the Equilibrium Controversy offers the chance to scrutinise the interaction of various components of mechanical knowledge and to investigate the consequences of the incomplete transmission of ancient scientific knowledge through the Arabic and Latin Middle Ages and into the early modern period. Scientific controversies have an unrestricted meaning, and can in general be defined as persistent antagonistic disagreements concerning a substantial scientific issue that cannot be resolved by standard means available to science in the given period.[278] In such a situation, the mathematicians of the time were trying to answer the question of the deflected balance by showing that their understanding agreed with widely accepted contemporary explanations. In this research they adapted their own interpretations to novel situations which had never been analysed before, hence widening the empirical range of their approach. They also attempted to show the inappropriateness and deficiencies of the opponents' accounts.

However, in this very context of methodological debate for the solving of unusual equilibrium problems, it appears that, in order to solve such a controversy, a demand for mature concepts and their generalization is required. Thus the attempt to find a

[277] With contingency we mean the circumstances in which the rediscovery of the ancient geometry took place, and the development of the related debate over the equilibrium.
[278] Freudental 2000, pp. 125-142; and *Id*. 2002, pp. 573-637.

universal "key", as it were, by which the effectiveness of a weight under varying mechanical circumstances, could be described, which could be applicable to *all* mechanical circumstances.

The need for generalization invades scientific controversies often triggered by what has recently been called "challenging objects".[279] As Renn and Damerow write, "these [sc. objects] are artefacts or other parts of the material culture that confront existing theoretical frameworks with explanatory tasks that cannot be accomplished with the available conceptual means, thus triggering their further development and ultimately their transformation. They typically embody other forms of knowledge, for instance, the practical knowledge of artisans to invent, produce, or make use of such objects. The development of the theoretical knowledge of mechanics in the early modern period can to a large extent be accounted for by the increasing attention scholars and engineer-scientists of the period paid to new objects of study which they investigated by means of the extant conceptual frameworks."[280]

A scientific controversy typically comes to an end through the development of a new conceptual framework, in which the original question either changes or loses its meaning. In this specific case, we witness the inception of a new theoretical object that will be extensively useful in a scientific context as a model or representational tool.

*Scientia de ponderibus* is the name given by Medieval scholars to the discipline that treats the equilibrium of heavy bodies, with particular reference to those hanging from a balance. This field was distinct from Greek mechanics both in scope – Greek mechanics aimed to interpret the transportation of weights, instead of their equilibrium – and in methodology – the science of weight was interested only in the *theoretical foundations* of equilibrium and not in its application.[281]

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[279] The notion of 'challenging objects' was first proposed by Renn in 2001. It became widely used in Bertoloni Meli 2006; Büttner, Damerow and Renn 2004; Büttner 2008 and *Id.* 2009, and Valleriani 2010. However, in this context, we only borrow the concept of 'challenging objects' without relating it explicitly to Renn's approach to the history of science.

[280] Renn and Damerow 2012, p. 13.

[281] As we will see in the next section, Guidobaldo del Monte, from Momabaroccio, represents an exception to this general rule among his contemporaries; he will be the first to instigate the reconciliation of mechanics as a scientific praxis and its theoretical aspects.

In the early third millennium BCE, balances with equal arms were introduced in Mesopotamia, Egypt, and China, and they became more widespread across the Mediterranean. From the extant papyri fragments, which report vignettes of the so-called 'judgment scenes', historians agree in maintaining that the balance had ritualistic origins, because in those scenes the main event consisted in the weighing of the heart of the deceased against the Father of Ma'at, a symbol of truth and rectitude. There are additional traces of ritual origins among the Incas of Peru and others of the pre-Columbian period. But any entertainment concerning the single or multiple origins of this device in the Old and New Worlds, or its original source, is not a matter for this reconstruction. However, the employment of these devices is well documented in literature, e.g. in Homer's *Iliad* and Aeschylus' fragmentary *Psychostasia*: in votive rituals, the balancing is seen as a special way of finding an equivalent to a sacrificed object, thus functioning as a prototype for the practical and physical use of the balance.[282] Gradually, these symbolic problems became associated with the practice of measuring with seeds, above all to solve mercantile disputes. Over time these arbitrary and geographically isolated methods underwent a process of standardization.[283] Their introduction to Europe was associated with the emergence of a quantitative concept of weight and a greater uniformity of weight measures. Despite its practical purposes, from the archaic era onwards, a theoretical field of knowledge can be reconstructed in which the balance and the law of the lever were central. This is further documented by texts in which the equilibrium of a balance is studied under various circumstances and fundamental statements about the balance and other mechanical devices are derived within a deductive structure from certain presuppositions.[284]

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[282] Seidenberg and Casey 1980, pp. 208-209.
[283] For further details, see Seidenberg and Casey 1980, pp. 179-226.
[284] Brentjes and Renn 2013. The authors establish that each of the extant ancient texts belongs to a different literary and intellectual tradition, thus for example the *Problems of Mechanics* – traditionally ascribed to Aristotle – explains the force-saving effect of a variety of mechanical devices with the help of a "balance-lever model", and makes use of the idea that differences of weight can be compensated by differences of length. This idea in turn is based on the practical experience of balances with unequal arms, a phenomenon which had been discovered in the late fifth century BCE. In this context, the sequel to Aristotelian natural philosophy deals with the challenge provided by mechanical devices to the correspondence between causes and effects fundamental to natural philosophy. The question concerned

However, the *scientia de ponderibus* in its original formulation saw its birth in the Arab world, with its status of a distinct *scientia* appearing in the *Book of Enumeration of the Sciences* by Al Farabi, who first definitively distinguished between a science of weights and a science of devices (or machines).[285] In the Arabic world, the new science of weights was characterised by a strong deductive system. The most common historical point of view is that the (Arabic) science of weights originated from the interplay of Aristotelian physics and physical-geometrical approaches to the equilibrium of bodies by Archimedes and (probably) Euclid. From a methodological viewpoint, the majority of treatises in the science of weights followed what is often called a dynamic (or more exactly a kinematic) approach, in which the equilibrium is seen as a balance of opposing forces, and movement – either virtual or real – plays an important role.[286] In addition, according to historians, the Arabic tradition was not merely a translation or reappropriation of the Greek science of weights, but constituted rather the emergence of a new science as a specific branch of mechanics, embodied in a large scientific and technical corpus.[287] The very expression *scientia de ponderibus* is only registered in the Latin context after the translation of al-Farabi into Latin, thanks both to Gerardo da Cremona and Dominicus Gundissalinus in the thirteen century. The latter reproduced al-Farabi's characterisation of the sciences of weights and devices, labelling them respectively *scientia de ponderibus* and *sciencia de ingeniis*.[288] So in the Middle Ages it

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was: how can a small force overcome a large weight? In his work *Planes in Equilibrium* Archimedes demonstrates the law of the lever with the help of the concept of the centre of gravity, and the observation that a redistribution of weights on a balance maintains its equilibrium, thus attaching it to a tradition of mathematical writing. Hero's *Mechanics* represents a later technical treatise belonging to the context of the *Museion* of Alexandria. And finally the *Liber de canonio* attributed to Euclid belongs to a tradition of deductive treatises in the style of Euclid's *Elements*.

[285] Some scholars, such as Ibn Sina (980-1037), divide the science of weights further into the *science of balances* and the *science of weight-lifting*. In contrast, other scholars, especially Al-Isfiza ¯r ¯ı (1048-1116) and al-Kha ¯zin ¯ı (1115-1130), distinguished the theory of centres of gravity from the science of weights.

[286] Capecchi 2012, pp. 115-116. The main sources consulted by these Arabic mathematians were Aristotle's *Physics* and *On the Heavens*, which had been available since the eleventh century; Pappus' and Heron's *Mechanics* were only available in Greek. With regard to Archimedes, only the treatises on hydrostatics were available at this time, but were seemingly not taken into account at all. See further Abattouy 2006, pp. 1-25.

[287] Capecchi 2012, p. 116.

[288] The main treatises of these two traditions are listed in Pisano and Capecchi 2016, pp. 120-121.

was possible to identify two distinct traditions of mechanics in Europe: i) the science of weights, in particular that pioneered by Jordanus Nemorarius; and ii) the philosophy of motion. Alongside these theoretical traditions there was the task of practical mechanics somehow continuing the tradition of the Roman period.

These two approaches both lack rigour and suffer from a structural weakness of the handling of *a priori* principles which lie at the very core of the *scientia de ponderibus*: that is, the principle of *gravitas secundum situm*, or, roughly speaking, the change in a body's gravity according to its position [*situs*], when the body moves along the circumference of a circle.[289] More precisely, this principle states that the positional gravity of a (punctiform) weight fixed at the end of one arm of a balance changes as the balance rotates around the fulcrum. The origin of this line of argument is given by two works, both attributed to Jordanus Nemorarius, but published only in the sixteenth century: the first, which appeared in 1533, is a short treatise entitled *Liber Iordani Nemorarii viri clarissimi De ponderibus propositiones XIII*, and the second, published in 1565 and titled *Iordani opusculum de ponderositate*, was printed in Venice on the basis of manuscript materials which had been collected and edited by Niccolò Tartaglia.[290]

It stemmed exactly from the practical question of whether or not a deflected balance will return to its horizontal position that initiated the Equilibrium Controversy. This in fact involves a set of causes and problems, and requires a set of concepts – such as centre of gravity, torque or moment, friction and bent levers – to be solved. In this context our twofold claim requires a clarification: the procedure of idealization and modelling which is required to represent physical bodies by means of punctiform mathematical objects also involves the balance, which in fact becomes an idealized model useful in order to build a theoretical set of mechanical principles. This model can

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[289] A further definition can also be provided: the heaviness of a body according to its location along the circumference described by the end of one of the arms of a balance rotating around the fulcrum.

[290] In this connection, E. Moody and M. Clagett have published a treatise by Jordanus Nemorarius, entitled *De ratione ponderis*, which is mostly reconstructed on the basis of a collation of medieval manuscripts, and whose contents overlaps only in part with the 1565 Tartaglia edition. See further Palmieri 2008, p. 309.

be used to interpret a whole series of single empirical cases, including a great variety of different experimental case-studies. Only by reaching this new theoretical layer could mechanics represent, through a broad and general model, a series of single cases.

From the beginning, the aim of mechanics was to explain the similarities between a set of mechanical devices and the ways it became possible to achieve a large effect with a small force.[291] The dispute over the deflected balance diminished only after the appearance of Guidobaldo del Monte's *Mechanicorum Liber*. Guidobaldo is considered by historians to have been the most influential writer on mechanics in the sixteenth century. He took a remarkable step forward towards the new classical mechanics. Despite the work of Maurolico, it was Guidobaldo del Monte, Commandino's pupil, who published in 1577 the most important text on mechanics. Guidobaldo was a soldier, an engineer and a scientist; his manuscript is treated as a continuation of the Greek tradition and represented the first work on mechanics to gain a prominent role in the equilibrium debate.

Up until Guidobaldo's time, the theorists of *scientia de ponderibus* had believed that an inclined balance does not remain in equilibrium but rather returns to the horizontal position. Only Guidobaldo ascertained *experimentally* that an inclined balance does in fact remain in equilibrium in the inclined position.[292] However, he argued that an inclined balance, far from remaining in equilibrium, and far from returning to the horizontal position, should rotate until it becomes perfectly vertical, with the initially lowest weight descending and the highest one ascending. This realisation must have been disconcerting to Guidobaldo, since it actually undermined the very foundation of his mechanics, which, as we shall see in the next two sections (§ 3.7 and § 3.7.1), rested mainly on Pappus' definition of centre of gravity. The theoretical step taken towards a new way of doing and conceiving mechanics amounted to showing the incoherence of Guidobaldo's interpretation of Pappus' definition of the centre of gravity, and ultimately called into question the physical existence of such a point in real

[291] Renn and Damerow 2012, p. 17.
[292] This is what is called *indifferent equilibrium*. A detailed analysis will be given in § 3.7.1 below.

bodies. Notwithstanding his vehement attack on Jordanus and his followers, Guidobaldo's discovery stems from his wholeheartedly embracing the *a priori* principles of *scientia de ponderibus*.[293]

### 3.7 Guidobaldo del Monte: the new Archimedes?

Guidobaldo del Monte was born in Pesaro on January 11th 1545. His father was Ranieri del Monte, created a *marchese* in 1543 by the Duke Guidobaldo II of Urbino. In 1564 Guidobaldo moved to Padua to attend lectures on the pseudo-Aristotelian *Problems of Mechanics* by Pietro Catena. However, as we already know, his main teacher was Federico Commandino. During his lifetime he remained close to the noble family and in particular to Prince Francesco Maria, whose half-sister married Guidobaldo. In 1587 he inherited the title of *marchese* and spent the rest of his life in Mombaroccio until his death on January 9th 1607.

Just as in the case of Commandino's education, Guidobaldo's education also involved intellectual exchange between literary men and mathematicians at the court of Urbino, which provided the perfect environment for him to grow both personally and professionally. For instance, Torquato Tasso and Jacopo Mazzoni (1548-1598) were both common friends of Commandino and Guidobaldo. One of the most prominent figures conversant with this milieu was the already-mentioned man of letters, Bernardino Baldi, who was also a skilled mathematician and historian of mathematics.

By analysing the correspondence between Guidobaldo and his contemporaries it emerges that almost all of Urbino's mathematicians were strongly connected with the engineers and technicians of the Duchy.[294] It is known that the Duchy of Urbino was an

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[293] Palmieri 2008, p. 302.

[294] This includes Francesco Maria della Rovere, Francesco di Giorgio Martini, Gerolamo Genga, Giovanni Battista Belluzzi, Francesco Paciotti, Giulio da Thiene, Muzio Oddi and Guidobaldo himself. On the figure of Guidobalso as manufacturer and inventor, see further the work of Muzio Oddi, another mathematician of Urbino (1569-1639), who was also Guidobaldo's pupil. Cf. e.g. *Degli orologi solari*, Milano, 1614, p. 19; *Fabrica et Uso del Compasso Polimetro*, Milano, 1633, "Proemio". The best modern literature on this topic is Rose 1968, pp. 53-69, together with corrections by E. Rosen 1968, pp.

important centre of military architecture. However, a particularly important technical branch was the fabrication of precision instruments, since both the distribution and renown of these devices was national, if not even international. The Venetian Giacomo Contarini in his *Problemata Astronomicorum* speaks of a geared and sophisticated instrument which could divide circular arcs into degrees, minutes, seconds, thirds and so on. Other instruments included a device for drawing ellipses and conic sections used to depict the planisphere, mechanical clocks and compasses (both reductional and proportional). Guidobaldo's attitude to mathematical instruments paralleled his attitude towards machines and mechanics. *Through these mechanical devices he felt that abstract mathematical truths could be made entirely visible*.[295] Mechanics had thus a special way of investigating the causes of things and demonstrating its propositions, which must be grasped before one can truly understand any of its claims. As Maarten van Dyck shows, "[i]n *Paraphrasis* (1588), Guidobaldo accordingly shows that the validity of the law of the lever is indeed grounded in a special method of demonstration – in the *argumentandi modus huius scentiae maximè proprius* – implying that whoever does not grasp this method of demonstration actually ignores the proper foundations for the mechanical science."[296] Thus, Guidobaldo's strategy consisted in observing the mechanical effects and their link to their being character *praeter naturam*[297] ('beyond nature'), in showcasing that mechanical science had both mathematical and physical characteristics. Furthermore, his skills in engineering were particularly evident due to the uninterrupted consultations with him and the confidence that Prince Francesco Maria displayed towards him.[298] As Frank says, "[t]he resulting engagement with

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377-400. In particular, for a detailed reconstruction on the mathematical and technical disciplines at the Urbino court, see Frank 2013; Laird 2013, pp. 35-52; Renn and Omodeo 2013, pp. 53-94.

[295] Rose 1975, p. 224.

[296] van Dyck 2013, p. 17.

[297] The same issue will be considered again in the next section (3.7.1). By way of indication here, in both *Physics* Book 2 and the *Problems of Mechanics*, Aristotle considers three ways in which art can operate: i) by imitating nature, ii) by completing what nature could not achieve and iii) by operating *praeter naturam* such as it does in mechanics. Guidobaldo inherits from Aristotle his own view on mechanics, allowing the rise of a science of mechanics that operates between mathematics and physics. See further van Dyck 2013, pp. 9-34.

[298] After consulting the correspondence that Guidobaldo kept in 1583, it appears that the ducal engineers worked with him to improve the water supply of Villa Mirafiore in Pesaro. However, after several months

questions of a practical nature and interaction with technical collaborators stimulated his scientific work. This is easily evidenced by an analysis of his manuscript *Meditatiunculae*.[299] Apart from using the manuscript to further his studies of perspective, astronomy and gnomonics, Guidobaldo also addressed practical topics: how to target with a cannon, the advantages and disadvantages of certain kinds of mechanical machines, the maximal and minimal inclination of roofs, and the water intake of a mill. In view of what is known about his practical activities, these reflections appear aimed at an elaboration of Guidobaldo's everyday observations and discoveries as a civil and military engineer and inventor of scientific instruments."[300]

With regard to his role among the translators of ancient Greek treatises, he translated the works of Pappus, Apollonius and Euclid and several other codices that were obtained from Contarini and Francesco Barozzi. His writings ranged widely across mathematics, mechanics, optics and astronomy over a 30-year period. In 1579 in Pesaro, Guidobaldo published an astronomical work entitled *Planisphaeriorum Universalium Theoretica*, which deals with the projection of the sphere onto a plane. It also contains theorems and transformations of figures, as well as sketches for building the compass to be used for drawing the projections of ellipses and conic sections. His interest in projection culminates in the work *Perspectivae Libri Sex* (Pesaro, 1600), in which he generalised and systematised the rules of perspective discovered in the previous 150 years. Until then, perspective was a theoretical instrument used by painters, but Guidobaldo was the first scholar to confer upon it a rigorous mathematical structure, with the treatise even containing a treatment of scenography. On the other hand, his interest in astronomy culminates in writings related to observational astronomy. Besides Guidobaldo's works published posthumously, there are different

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of vain endeavours to resolve the problem, Guidobaldo was summoned to perform an expert's inspection. The correspondence focuses on the discussion that Guidobaldo had with his collaborators, and in particular concerns the construction of the water reservoir and the water conduits. By these letters it also emerged that Guidobaldo oversaw the fabrication of clocks by craftsmen in the Duke's service. See further Frank 2013, pp. 315-319.

[299] Bibliothèque Nationale de France (Paris), ms. Latin 10246; for a transcription and analysis of the *Meditatiunculae*, see the doctoral dissertation by R. Tassora, *Le Meditatiunculae de rebus mathematicis di Guidobaldo del Monte*, 2001.

[300] Frank 2013, p. 323.

lost *opuscoli* mentioned by Orazio del Monte, namely *In Quintum Euclidis Elementorum Commentarius* and *De Proportione Composita,* which contained many comments related to the generalization of proofs.[301]

However, it is a cliché to consider the *Mechanicorum Liber* the apogee of Renaissance statics, and the culmination of the Equilibrium Controversy and mechanics. By analysing its rigorous mathematical proofs, we can maintain that the structure and content of the *Liber* trace back to the first personality of this contribution: Archimedes. After all, Guidobaldo himself was altogether aware of the contribution he gave to the restoration of the ancient tradition, as well as the Archimedean revival, as we see in his opening Dedication:

> After Archimedes, there were among the mechanicians Hero, Ctesibius and Pappus [...]. Although they did not reach the pinnacle of mechanics, as Archimedes did, still they had a remarkable understanding of mechanics and were excellent men. This is especially true of Pappus, so that no one could, I believe, blame me for following him as my leader. I have the more readily done so because Pappus does not depart even a nail's breadth from the principles of Archimedes. For I have always wished in this branch of science to follow in the footsteps of Archimedes. His thoughts, sought for some years now by many scholars, are to be found in his excellent book *On the Equilibrium of Planes*; in this, I believe, all the theories of mechanics are gathered as in an abundant store. Were the mathematicians of our present time to be better acquainted with this work, they would see repudiated many of the ideas they hold valid and correct. Let them see for themselves. Pappus, deeply devoted to a richer application of mathematics and so increasing the profits of the study, carried out a thorough and brilliant investigation of the five simple machines – the lever, pulley, wheel and axle, wedge and screw. [...] I wish that the ravages of time had not caused any loss in the writings of so great a man, for that earth would not have been covered almost entirely by such a thick mist of ignorance, nor would there be such ignorance of mechanics that men are esteemed leading mathematicians who by their inept distinctions remove some difficulties, but not those that are very arduous or obscure.[302]

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[301] These *opuscoli* are now held in the *Biblioteca Oliveriana* in Pesaro, nos. MSS 630-631 respectively.

[302] *Mechanici praeterea fuerunt Heron, Ctesibius & Pappus, qui licet ad mechanicae apicem, perinde atq; Archimedes, evecti fortasse minimè sint; mechanicam tamen facultatem egregiè percaluerunt; talesq; fuerunt, & praesertim Pappus, ut eum me ducem sequentem nemo (ut opinor) culpaverint. Quod & propterea libentinus feci, quòd nè latum quidem unguem ab Archimedeis principijs Pappus recedat. Ego enim in hac praesertim facultate Archimedis vertigijs haerere semper volui: & licet eius lucubrationes ad*

In addition, one more element proves his contribution and overtaking of the ancient works; by scrutinising Guidobaldo's relationships with expert craftsmen, we notice that this friendship had an impact on more than just his invention of new scientific instruments. In fact, in 1580 Guidobaldo exchanged several letters with Giacomo Contarini and Filippo Pigafetta, including a letter[303] addressed to the former, in which he explained the following:

> You should know that, before writing anything about the *Mechanicorum Liber*, I never wanted (so as not to make errors) to consider anything, irrelevant as it may have been, if first I had not seen that the experiment (*esperienza*) agreed exactly with the proof; and of very little thing I have made the experiment. [...] In any case, it is most sure that practice and theory always agree and do not differ from each other. And I tell you even this: the proofs have taught me much about how to make the experiments, regarding which many things have to be considered: firstly, the instruments should be small rather than big; as for example the pulleys with their wheels: if possible they should be made out of brass with very thin, iron axes; and the wheels should be well turned so that they do not waggle round the axes; but if possible, it would be very good if they turned around with just a blow of air. In fact, the big pulleys which are able to lift heavy weights, are not that adept at telling apart details, as the balances clearly show: in order to distinguish every little detail, one has to use those small ones for weighing coins, and not those big ones, with which large objects are weighed like meat or similar things, even if they are precise.

In this passage we first see the close relation between Guidobaldo's theoretical studies

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*mechanicá pertinentes multi sab hinc annis passim soleant doctis desiderari: eruditissimus tamen libellus de aequeponderantibus prae minibus hominú adhuc versatur, in quò tanquam in copiosissima poenu Omnia ferè mechanica dogmata reposita mihi videntur; quem sane labellum, si aetatis nostrae mathematici sibi magis familiarem adhibuissent; reperissent sanè sentétias multas, quos modò ipsi firmas,& ratasesse docent;subtilissimè, atque verissimè conuulsas, & labefactatas. Fed hoc viderint ipsi. Ego enim ad Pappus redeo, qui ad usum mathematicarum vberiorem, emulumentorumqué accessiones amplificandas penitus conversus, de quinque principibus machinis, Vecte nempè, Trochlea, Axe in peritrochio, Cuneo, & Cochlea, multa egregiè philosphatus est; demonstravitquè quicquid in machinis, aut cogitari peritè, aut acute definiri, aut certò statui potest, id omne quinquè illis infinita vi praeditis machinis referendum esse. Atquè utinam iniuria temporis nihil è tanti viri scriptis abrasisset: nec enim tam densa inscitiae caligo universum prope terrarium irbem obtexisset, neque tanta mechanicae facultatis esset ignoratio consecuta, ut mathematicarum proceres existimarentur illi, qui modo ineptissima quadam distinction, difficultates nonnullas, nec illastamen satis arduas, & obscuras è medio tollunt*. del Monte 1577, *Prefatio* (Guidobaldo's dedication to the Duke Francesco Maria).
The English trans. comes from Rose 1975, p. 231.

[303] Biblioteca Nazionale Marciana, Venezia, It. IV 63 (=Ven. 259); October 9th 1580; published in Favaro 1899-1900, pp. 307-310. The English transl. is taken from Frank 2013.

and the ‘experiences’ with his devices. Second, the instruments considered by Guidobaldo were not the everyday devices used in the marketplace (balances) or at building sites (pulleys): rather, he is referring to precision instruments, of small dimensions and made out of brass with very thin iron axes to keep friction to a minimum, so that a blow would suffice to make them turn around. Third, the role that Guidobaldo played in the late Renaissance was truly analogous to the methodological and theoretical function performed by Archimedes.[304] The main body of Guidobaldo’s treatise is concerned with the theoretical discussion of machines. This was achievable only by following the Archimedean model: his work started with definition and postulates on the centre of gravity, derived from both Pappus’ and Commandino’s previous works.

“The centre of gravity of any body is a certain point within it, from which, if it is imagined to be suspended and carried, it remains stable and maintains the position which it had at the beginning, and is not set to rotating by that motion”. This definition of the centre of gravity is taught by Pappus of Alexandria in the eighth book of his *Collections*. But, again, Guidobaldo’s teacher Commandino, in his *On Centres of Gravity of Solid Bodies*, explains the centre as follows: “the centre of gravity of any solid shape is that point within it around which is disposed on all sides of parts of equal moments, so that if a plane be passed through this point cutting the said shape, it will always be divided into parts of equal weight.”[305]

Since we have already discussed Commandino’s interpretation, it will be sufficient to add just a few more reflections which lead us directly onto the epistemological meaning of the scientific practice lying behind the second – ambiguous – step of the objectification of procedure. First of all, according to Guidobaldo, the uses that both Pappus and Commandino made of the notion of the centre of gravity should be regarded as “descriptions” rather than as definitions.[306] After this, Guidobaldo states that, on the one hand, bodies are taken individually and are subjected to gravity converging toward

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[304] For further information on Guidobaldo’s discoveries and handicraft, see Gamba 2001.
[305] Drake and Drabkin 1969.
[306] del Monte 1577, p. 9 and Palmieri 2008, pp. 301-346.

the centre of the world; and, on the other hand, gravity – the weight – is thought to be concentrated in the centre of gravity of the whole body, which is determined by the Archimedean rules.[307] Moreover, Pappus' description – as Guidobaldo would say – is purely experimental in nature, and as such leaves its mathematical determination completely open. To give such a precise determination is exactly the task of the first eight propositions of Archimedes' treatise. Guidobaldo's commentaries accordingly focus on precisely this problem: *How does this purely physical characterisation allow for precise mathematical determination?*[308] The issue can also be detailed as follows: as a point situated in a body, the centre of gravity is linked with some of the body's physical properties (i.e. tendency toward motion and equilibrium, dimensions and mass) but it can also be – as indeed it is – treated *as* a mathematical point. The question now becomes: *How is it possible to determine the mathematical features from this point, only by arguing from its physical nature and behaviour?*

By following the demonstrations Guidobaldo gives in the first few propositions of the *Mechanicorum Liber,* I argue that the context in which he developed his new way of doing and conceiving mechanics represents the perfect, albeit tantalizing, framework in which the centre of gravity, as an idealized model of the representation of physical reality, takes a new step forward in its process of objectification. More precisely, the operation by which the macroscopic bodies are considered abstract and idealized geometrical points (through which it is possible to study their motion and other physical features), will became more and more connected with geometrical investigation as it was applied to mechanics. In this light, let us examine first of all some of the fundamental propositions of the *Mechanicorum Liber* in order to upgrade and nuance our working hypothesis.

The propositions that support it are propositions 2, 3 and 4. The last of these – as

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[307] It should be noted that del Monte, along with many other mathematicians of his time, would unquestionably conclude that, from a practical point of view, to consider the lines of action of gravity parallel to each other or to consider them converging to the centre of the world did not matter much. Nevertheless Guidobaldo also believed that, to establish the 'reality' of things, one could not accept this approximation. See further Capecchi 2012, p. 109. For more clarification on this, see also my next section (§ 3.7.1).

[308] van Dyck 2013, pp. 12-15.

we shall see in the next section in more detail (§ 3.7.1) – also includes the attempt to combine the Archimedean and Aristotelian mechanical traditions. In particular, Guidobaldo saw himself as pursuing a mechanical approach that could be traced directly to Archimedes; however, at the same time he was also following the Aristotelian tradition, giving great importance to the concept of the centre of the world. Guidobaldo is thus seen by both his contemporaries and historians as the reconciler of two divergent and parallel traditions in deriving the mechanical properties of his machines from the mutual relation of three centres: the fulcrum, the centres of gravity and the centre of the world. Now, with regard to this last item, we will examine in the next section the lively polemical debate concerning indifferent equilibrium, which turns out to be one more piece of evidence in favour of our methodological approach – the objectification of procedure.

With this idea in mind, let us now turn to the first two propositions mentioned above. In Proposition 2 we can observe (below, Figure 3.1) a balance whose point of suspension *C* is placed above the centre of gravity *D* of the system of weights. If the balance is deflected from its horizontal equilibrium position, it will return to its original position because the barycentre, raised from the position *D* to the position *G*, will tend to occupy the lowest position, and then return to *D*.

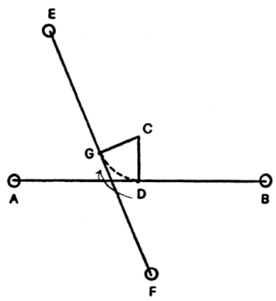


Figure 3.1

*Proposition 2* states: A balance parallel to the horizon, with its centre above and having equal weights at its extremities which are equidistant from the perpendicular *CD*, when moved from this position and released, will return and rest in it.

> Given the balance *AB* which is placed parallel to the horizon, and whose centre of suspension *C* is placed above the scale, the line *CD* is perpendicular to *AB* and *DA* is equal to *DB*. The two weights, *A* and *B*, located at the ends of the bars, are equal to each other. From this position of stable equilibrium the scale is moving from *AB* to the position *EF*, and it is left. From here the balance returns to the original position *AB*. But how is that possible? Since the point *C* is stationary during the counterclockwise movement, the point *D* describes the *CD* circle diameter. And since *CD* is always perpendicular to the arm *AB* of the scale, whether the balance is in the position *EF* or not, the line *CD* is located in *CG*. In *D* there is the centre of gravity of the system of weights *AB*, while *G* is the centre of gravity of the *AB* system of weights when the balance is in *EF* position. Since then *CG* is *not* perpendicular to the horizon, *EF does not* remain in this site, but returns to the original position *AB*.[309]

This demonstration discusses the *stable equilibrium* of the balance by means of a deductive procedure. Later on, in Proposition 3, Guidobaldo discusses the *unstable equilibrium* of the balance, in the case in which the fulcrum is under the barycentre of the system of weights *AB* (see below, Figure 3.2).

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[309] *Libra horizionti aequidistans, cuius centrum sit supra libram, aequalia in extremitatibus, aequaliterq; à perpendiculo distantia habens pondera, si ab eiusmodi moveatur situ, in eundem rursus relicta, redibit, ibìq; manebit.*

*Sit libra AB recta lineahorizonti aequidistans, cuius centrum C sit supral ibram; sitq CD perpendiculù, quod horizonti perpendiculare erit: atq; distantia AD sit distantia DB aequalis; sitq; in AB pondera aequalia euorù gravitatis centra sint in AB libra ab hoc situ, putà in EF, deinde relinquatur. Dico libram EF in AB horizonti aequidistantem redire, ibìq; manere. Quoniam autem punctum C est immobile, dum libra movetur, punctum D circuli circumferentiam describet, cuiud semidiameter erit CD. Quare centro C, spatio verò CD, circulus describatur DGH. Quoniam enim CD ipsi libra semper est perpendicularis, dum libra erit in EF, linea CD erit in CG, ita ut CG siti psi EF perpendicularis. Cùm autem AB bisariam à puncto D dividatur, et pondera in AB sint aequalia; erit magnitudinis ex utrisq; EF compositae centrum gravitatis in medio, hoc est in D et quando libra unà cum ponderibus erit in EF; erit magnitudinis ex utrisq; EF compositae centrum gravitatis G et quoniam CG horizonti non est perpendicularis; magnitudo ex ponderibus EF composita in hoc situ minimé persistet, sed deorsum secundum eius centrum gravitatis G per circumferentiam GD movebitur; donce CG horizonti fiat perpendicularis, scilicet donec CG in CD redeat. Quando autem CG erit in CD, linea EF, cùm ipsi CG semper ad rectos sit angulos, erit in AB; in quo situ quoq; manebit libra ergo EF in AB horizonti aequidistantem redibit, ibìq; manebit. Quod demonstrare oportebat.* del Monte 1577, pp. 4-4*r*.
The English trans. is my own.

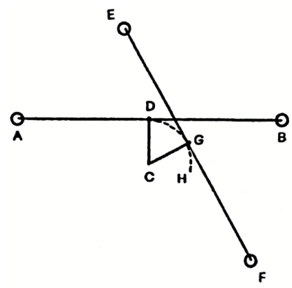


Figure 3.2

*Proposition 3* states: A balance parallel to the horizon, with its centre below and having equal weights at its extremities which are equidistant from the perpendicular *CD*, will be at rest; but if moved and left tilted, it will move towards the lower side.

> Given the balance *AB* placed parallel to the horizon, whose centre of suspension is placed under the balance itself, the line *CD* is perpendicular to the horizon and to the arm *AB*. The distance *AD* is equal to *DB*. The two weights placed at the ends *A* and *B* are equal to each other. At the beginning, the balance is at rest in this site *AB*, and the arms *AB* is divided in half by the point *D*. *D* is therefore the centre of gravity of the system of weights *AB*. Let us move the balance *AB* to *EF* and then leave it. The system of weights *EF* will move downwards, i.e. towards the weight *F*. But how is that possible? Since the line *CD* is perpendicular to the horizon, when the balance reaches the position *EF*, *CD* is in *CG*. Since then *CG* is *not* perpendicular to the horizon, *EF will descend* towards *F*.[310]

In order to solve the *unstable equilibrium* problem, Guidobaldo argued that, if a balance is deflected from its horizontal position, whether the centre of suspension remains fixed in *C* or not, the centre of gravity of the system of weight *EF* does not remain fixed.

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[310]*Libra horizonti aequidistans, aequalia extremitatibus, aequaliterq; à perpendiculo distantia habens pondera, centro infernè, collocato, in hoc situ manebit. Si verò inde moveatur deorsum relicta, secundùm partem decliviorem movebitur.*

*Sit libra AB recta linea horizonti aequidistans cuius centrum C sit infra libram; perpendiculumq; sit in CD, quod horizonti perpendiculare erit; et distantia AD sit distantiae DB aequalis; sintq; in AB pondera aequalia, quorum gravitatis centra sint in punctis AB. Dico primum libram AB in hoc situ manere. Quoniam enim AB bisariam dividitur à puncto D, et pondera in AB sunt aequalia; erit punctum D centrum gravitatis magnitudinis ex utrisq; AB ponderibus compositae. Et CD libram sustinens horizonti est perpendicularis, libra ergo AB in hoc situ manebit moveatur autem libra AB ab hoc situ, putà in EF, deinde relinquatur. Dico libram EF parte F moveri. Quoniam igitur CD ipsi librae semper est perpendicularis, dum libra erit in EF, erit CD in CG ipsi EF perpendicularis. Et puncto G magnitudinis ex EF compositae centrum gravitatis erit; quod dum movetur, circuli circimferentiam describet DGH, cuius semidiameter CD, et centrum C. Quoniam autem CG horizonti no est perpendicularis, magnitudo ex EF ponderibus composita in hoc situ minimè manebit; sed secundum eius gravitatis centrum G deorsum per circunferentiam GH movebitur. Libvra ergo EF ex parte F deorum movebitur, quod demonstrare oportebat*. del Monte 1577, pp. 4*r*-5.
The English trans. is my own.

Thus additionally, the whole system of weights will not remain in its original place and will move downwards towards *F*.[311]

In order to solve the problem of the unstable equilibrium, Guidobaldo needed to imagine the balance as a punctiform system, having position in the centre of weight of the system *EF*. In this respect, we can maintain that he used, in a most confident way, a theoretical model to solve a practical problem, and that he thereby counts among the scholars who prepared the correct context for developing the idealized concept of the point mass. The idea is that at the base of the *Mechanicorum Liber* there is the following theoretical approach at play: *the bodies placed in equilibrium at the extremities of the balance's arm are considered as bodies devoid of size and mass and other physical features, as if they were just punctiform objects subject to any possible movement*. The endorsement of this interpretation can be traced back to the demonstrations that we have just seen, and precisely in these words: "the system (of weights) made up of the *EF* weights, will not stay in its place, but, *according to its centre of gravity*, it will descend along the circumference *GH*". Guidobaldo postulates this idealization, which is very akin to the modern concept of the *material point*. He imagines that the macroscopic balance no longer exists, but that there is instead its centre of gravity. In other words, he thinks that, in order to understand the unstable equilibrium, the whole system of weight – i.e. the balance in itself – could be substituted with a geometrical point to which we can ascribe certain physical properties. Such an operation frees the material bodies of any dimensional properties, and they can finally be considered as if they were moving material points.

Guidobaldo's understanding of the stability of the balance, which took an additional step forward in Proposition 4 of the treatise at hand, allows a wider conception of the three-fold organisation of his mechanics to emerge, one which is based not only on the notion of centre of gravity but also on the Aristotelian centre of the world – which has appeared in the thinking of almost all previous scholars – and the fulcrum – which had hitherto appeared as a only weak element in the development of

[311] Gamba and Montebelli 1988, p. 231.

the Equilibrium Controversy. In fact Guidobaldo's conceptualization of mechanical phenomena involves, on the one hand, what he found in Aristotle and his followers, and, on the other hand, what he learned from Archimedes. Its basic conceptual feature, the barycentre, is of Archimedean origin, but the methodological approach and the way it functions is co-determined by an Aristotelian cosmological framework and a particularly Aristotelian understanding of the balance.

This new framework leads us to endorse the idea that it was exactly by means of Guidobaldo's mechanics that a theoretical and idealized notion of centre of gravity took a new step towards its process of the objectification of procedure. Guidobaldo's mechanics is characterised by a series of mathematical considerations. In fact he exploited mechanical and practical procedures in order to formalize mathematically the natural properties observed in nature. Following in Archimedes' footsteps, mechanical science has a special heuristic way of demonstrating its propositions, which must be grasped before one can truly understand any of its theoretical claims. From the next section, it will emerge that the concept of the centre of gravity grounds this new approach to the science of mechanics in two ways: i) by providing the artificial effects with a well-defined ontological place in the Aristotelian cosmos; and ii) by allowing the epistemological grounding of the law of the lever. With regard to i), the concept of the centre of gravity can be adduced as a further proof that Guidobaldo considered important to situate his mathematical science of mechanics within a properly Aristotelian framework. In fact, he had a sophisticated understanding of abstraction as a mental operation (e.g. as is evident in his solution to Proposition 3, on which see above). The main idea advocated by del Monte – following Aristotle – was that pure mathematics arose on the basis of the abstraction of quantitative properties as if they were not a part of sensible matter, which in reality they are. For a second time the analysis of these abstractions stimulated renewed attention to problems regarding the ontological status of the objects of Archimedes' science. Furthermore, with regard to ii), given that a purely empirical approach would not suffice to lay the mathematical foundations of the law of the lever, it is important to bring out the essential properties of a body's centre of gravity, including the very fact of its existence ontologically. It is exactly in the scenario arising from the polemical discussion on indifferent equilibrium – which is the subject of § 3.7.1 – that it becomes possible to recognise the indifferent equilibrium in itself as the essential property for the proof of the law of the lever.[312]

### 3.7.1 Guidobaldo the Natural Philosopher

In the fourth proposition Guidobaldo goes on to demonstrate that a balance – defined as a weightless beam with two extended weights fixed to the beam's ends in their centres of gravity – will always remain in the position where it is initially placed, be it horizontal or inclined, contrary to the belief of all the previous theories of *scientia de ponderibus*.[313] Guidobaldo's demonstration of the indifferent equilibrium constitutes a direct application to the balance of Pappus' definition of the centre of gravity. In what follows, I will argue that, not only by solving the stable and unstable equilibrium, but also by defining and explaining the problem of indifferent equilibrium, Guidobaldo gave rise to a twofold theoretical discovery. First, by discussing the conjunction of the two divergent Hellenistic mechanical traditions, he legitimates the idea that mathematics makes free use of theoretical entities, such as the punctiform entity corresponding to the geometrical and physical centre of gravity of rigid bodies. Second, he significantly does not neglect the fundamentals of the heuristic and mechanical procedures – as Archimedes did – in discovering the mathematical foundations of any other science.

To this end, we will in this section mainly be interested in stressing two considerations: i) the main difference between Guidobaldo's mechanics with respect to that of his contemporaries, and ii) his pronounced interest in natural – or Aristotelian – philosophy. Laird summarizes the first of these issues as follows: "first, it [sc.

[312] See further Laird 1986, pp. 43-68 and van Dyck 2013, pp. 28-32.
[313] I have sketched a brief review of the *scientia de ponderibus* in § 3.6. For further analysis, see Palmieri 2008 and van Dyck 2006, pp. 373-407.

mechanics] was a theoretical science rather than a manual art; second, it was mathematical, although its subject was natural; third, it concerned motion and effects outside of or even against nature; and fourth, it produced them for human ends."[314] As pointed out above, the arc of sixteenth-century views on the status of mechanics as a scientific discipline derived from the rediscovery of the pseudo-Aristotelian *Problems of Mechanics*. Due to the presuppositions set down in this treatise, mechanics was seen as a *subalternate* science[315], above all because both Tartaglia and Baldi, following the Aristotelian preface, claim that the 'How?' of mechanical problems is known through mathematics, and the 'About What?' through physics. Maurolico noted that mechanics represented a *scientia media* between the mathematical and the natural (physics). However, as maintained by Van Dyck, "a potential complication arises when we try to see how this can be squared with the third aspect singled out by Laird: that mechanics treats effects *praeter naturam*."[316] It is well known that Aristotle more than once singled out the difference between "what is according to nature" and "what is beyond nature", promoting the distinction between the natural and the artificial. Thus, the claim that art imitates nature means that the former does not simply overrule nature, but that it profits from the natural constitution of things to bring about these effects by imitating nature.

[314] Laird 1986, pp. 45-46.
[315] In his *Posterior Analytics* Aristotle alludes to some sciences which are "under" other sciences; on this basis, commentators on the Stagirite elaborated and analysed the category of the *subalternate* sciences. The context in which Aristotle had introduced the germs of this concept stemmed from his discussion of how some sciences could use mathematical demonstrations to arrive at conclusions about physical things, apparently by violating the essential Aristotelian requirement of homogeneity, which states that all terms in valid scientific demonstrations must belong to the same genus. Nevertheless, according to Aristotle, sciences such as astronomy or optics *can* use mathematical principles because they are related to mathematics as "one [sc. science] under the other". Furthermore in the *Physics*, Aristotle also called astronomy, optics and harmonics as the "more physical of the mathematical sciences". While in geometry one treats physical lines as mathematical rather than physical, in optics one treats mathematical lines as physical rather than mathematical. Finally, in the *Metaphysics*, Aristotle makes the seemingly contrary claim that optics treats visual rays (i.e. physical lines) but only as mathematical lines. Debate concerning the use of purely mathematical and abstract entities in understanding how the world functioned naturally was already around in earlier philosophy, but the scientific community only became fully aware of its capacity close to the end of the Renaissance, when the conjunction of the Aristotelian-mathematical and Archimedean-mechanical traditions eventually took place.
[316] van Dyck, 2013.

In this regard Guidobaldo, first in the *Mechanicorum Liber*,[317] expressed the idea that mathematics was *not* sufficient to describe mechanics. Thus, mechanics would need to include elements of natural philosophy alongside mathematics, for instance in the context of the "true motion" of weights:

> [...] [I]n fact, there are some keen mathematicians of our time who assert that mechanics can only be considered either mathematically or physically; as if mechanics could sometimes be considered either without geometrical demonstrations or without the true motion.[318]

Later, Guidobaldo opens his *Paraphrasis* with a broad discussion of art's imitation of nature. There he asserts that art is able to bring about effects which are *praeter naturam* because it imitates nature. Accordingly mechanical demonstrations are not wholly mathematical, and, at least on an Aristotelian view, nature certainly does not operate according to mathematical principles. Del Monte seeks the interrelation of aspects relating to mathematics and natural philosophy in the discipline of mechanics, but without referring to the idea of subalternate or mixed science.[319] To be precise, he describes mechanics in his 1577 treatise as the Aristotelian natural field, and as a discipline which refers to phenomena apparent in nature, whilst at the same time holding that mechanics possesses intrinsically a mathematical tendency, insofar as it makes recourse to notions such as *distance* or *ratio* which belong to mathematics and are represented by the Archimedean conception. Guidobaldo's view of the relation between Archimedes and Aristotle goes further, attributing to them an equally influential status, as Frank states: "[h]e [sc. Guidobaldo] presents a kind of concordism,

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[317] Guidobaldo too opened his *Mechanicorum Liber* with a discussion of mechanics operating against nature, in which he claimed that mechanics comes from the union of geometry and physics.

[318] *Reperiuntur enim aliqui nostraque aetate emunctae naris mathematici, qui mechanicam tum mathematice seorsum, tum phisice considerari posse affirmant; ac si aliquando vel sine demonstrationibus geometricis, vel sine vero motu res mechanicae considerari possint.* del Monte 1577, *Prefatio*, p. 8.

A further reference to the 'need' for a connection between mathematics and natural philosophy can be found in *Cochlea* (Venice, 1615), where Guidobaldo claims that mathematics and mechanics both shed light on those phenomena which at first glance seem to be in contradiction with common sense, such as the fact that heavy loads can be moved by exiguous forces.

[319] *Ibid.*, p. 16.

claiming that in his axioms Archimedes followed what Aristotle had shown and that they agree also in their perception of mechanics as subdivided in the two fields. Consequently, although Guidobaldo was undoubtedly a follower of Archimedes' mechanical theory, he did not consider Aristotle's approach as inferior:

> At the beginning of the *Quaestiones Mechanicae*, Aristotle gave many extraordinary clues for discerning the *causes* of mechanical phenomena. In his writings, Archimedes followed him and brought to light the principles of mechanics more clearly, making them even more intelligible. But Aristotle is not diminished in stature for this reason: in fact, he masterfully explained the *causes* behind the problems that he had presented and discussed. [...] [F]or example, Aristotle asks why we move heavy weights with a lever. And he replies that the *cause* is the greater length of the law on the side of the force: and he is indeed right.[320]

It seems that Guidobaldo interpreted Aristotle's approach as an explanation of the *causes* of the mechanical phenomena which – in Aristotelian terminology – corresponds to the teleological task of philosophy in its search of the causes. Archimedes in turn dealt with the mathematical description of these phenomena, and reached a more complete formalisation in this regard."[321] Guidobaldo focused on the rational understanding of mechanical phenomena and *vice versa*. The exercise in which Archimedes seems to have been engaged was not so much a mathematization of physics, but rather a physicization of mechanics.

Therefore, the examination of the demonstration of indifferent equilibrium addressed in the fourth proposition of the *Mechanicorum Liber* can help us to understand the following two points: i) how Guidobaldo interpreted and recovered the writings of his predecessors, trying to make them cohere and complement each other; and ii) why this new conceptual and theoretical approach also has to be seen as an

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[320] *Aristoteles enim in principio Quaestionum Mechanicarum multa, eaque praecipua ad causas rei mechanicae dignoscendas aperuit. Quem secutus Archimedes in his libris mechanica principia explicatius patefecit eaque planiora reddidit. Nec propterea Aristoteles diminutus exstitit: etenim eorum quae ab ipso proposita et explicata fuere, problematum causas egregie patefecit. [...] Aristoteles enim (gratia exempli) quaerens cur vecte magna movemus pondera. Causam esse ait longitudinem vectis maiorem ad partem potentiae: et recte quidem*. del Monte, *Paraphrasis*, p. 4.

[321] Frank 2013, pp. 326-327.

influential example which Galileo – and earlier, Luca Valerio – at some point tried to emulate in their own endeavours in the field of mechanics.

Guidobaldo opens his treatise with both Pappus' and Commandino's definitions of centre of gravity followed by a few obvious axioms about weight as a magnitude, together with three *suppositions,* which read as follows:

1. Every body has but a single centre of gravity.
2. The centre of gravity of any body is always in the same place with respect to that body.
3. A heavy body descends according to its centre of gravity.[322]

After the proof of the stable and unstable equilibrium in Propositions 2 and 3 respectively, Guidobaldo in Proposition 4 deals with the *indifferent equilibrium* demonstrated as follows:

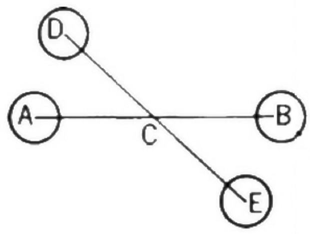


Figure 3.3

*Proposition 4* states: A balance parallel to the horizon, having its centre within the balance and with equal weights at its extremities, equally distant from the centre of the balance, will remain stable in any position to which it is moved.

Let the balance be the straight line *AB*, parallel to the horizon, with its centre *C* in the line *AB* and the distance *CA* equal to the distance *CB*; and let the weights *A* and *B* be equal, and have their

[322] I reproduce the Latin text here:
1. *Unius corporis unum tantùm est centrum gravitates.*
2. *Unius corporis centrum gravitatis semper in eodem est situ respect sui corporis.*
3. *Secundùm gravitatis centrum pondera deorsum feruntur.*

del Monte, 1577, p. 1*r*.
The English trans. is taken from Drake and Drabkin 1969, p. 256.

centres of gravity in the points *A* and *B*. Let the balance be moved to *DE* and left there. I say, first, that the balance *DE* will not move and will remain in that position. Now, since the weights *A* and *B* are equal, the centre of gravity of the combination of the two weights *A* and *B* will be at *C*. Hence the same point *C* will be the centre of gravity of the balance and of the whole weight. And since the centre of gravity of the balance *C* remains motionless while the balance *AB*, together with the weights, moves to *DE*, the centre of gravity is not moved. Therefore the balance *DE*, being hung on this, will not move, by the definition of the centre of gravity. The same likewise happens with the balance *AB* parallel to the horizon or in any other position. Hence the balance will remain where it is left; which was to be demonstrated.[323]

Both in propositions 4, and, in particular, in the propositions (2-3) studied in the previous section, Guidobaldo maintains that the centre of gravity of the system of weight – which coincides with the point of suspension of the balance – is raised if the balance is deflected from its horizontal position. In the case of indifferent equilibrium (Figure 3.3), if one moves the balance from the stationary horizontal position, the centre of gravity of the system of weight will remain stationary while the arm of the balance keeps on moving.

At first glance, it may appear to the reader that the aim of the proposition is rather ambiguous, because del Monte is disagreeing with the idea upheld by Jordanus and Tartaglia which had argued *against* indifferent equilibrium and *in favour of* the idea of building a science of mechanics that was completely dependent on the notion of positional gravity.

According to these authors, a balance would never be in a state of indifferent equilibrium, since the weight on a depressed arm is always "positionally lighter" (as they said) than the weight on the other arm. Hence, a balance with equal weights, suspended in its centre, *always* returns to its horizontal position. At the very beginning

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[323] *Libra horizionti aequidistans aequalia in extremitatibus, aequaliterq; à centro in ipsa libra collocato, distantia habens pondera; sive inde moveatur, sive minus, ubicunq; relicta manebit.*

*Sit libra recta linea AB horizonti aequidistand, cuius centrum C in eadem sit linea AB; distantia verò CA sit distantiae CB aequalis: sintq; pondera in AB aequalia, quorum centra gravitatis sint in puntis AB. Moveatur libra ut in DE, ibiquè relinquatur. Dico primùm libram DE non moveri, in eoquè situ manere. Quoniam enim pondera AB sunt aequalia; erit magnitudinis ex utroq; pondere, videlicet A, & B compositae centrum gravitatisC. Quare idem punctum C, & centrum librae, & centrù gravitatis totius ponderis erit. Quoniam autem centrum librae BC, dum libra AB unà cum ponderibus in DE movetur, immobile remanet,centrum quoq; gravitatis, quod est idem C,non mevebitur. Nec igitur libra DE movebitur, per definitionem centri gravitatis, cum in ipso suspendatur. Idipsum quoq; contingit libra ABhorizonti aequidistante, vel in quocunq; alio situ existente. Manebit ergo libra, ebi relinquetur. Quod demonstrare oportebat.* del Monte 1577, pp. 5-5*r*.

The English trans. is taken from Drake and Drabkin 1969, p. 261.

of his discussion, Guidobaldo simply reiterates his proof of Proposition 4 without supplying a direct proof of the existence of indifferent equilibrium, but employs a *reductio ad absurdum* argument to refute the claim that an equal arm balance sustained in its centre would have stable equilibrium, by showing that this would imply that the centre of gravity of a given body *would not be unique*, contrary to the first postulate.

In this refutation Guidobaldo aimed at undermining the arguments of Tartaglia and Jordanus through two main ideas: i) by showing a mathematical error in their argument concerning the supposedly smallest ratio of angles; and ii) by criticizing the argument of the parallel nature of the descending lines from the weights at both ends of the balance, by introducing the convergence of the line of descent towards the centre of the world.[324]

The main point of the first argument consists in showing that, despite the fact that the weight on the elevated arm is positionally heavier than the weight on the depressed arm, the difference in heaviness is always infinitesimally small and consequently cannot be offset by adding a small weight to the positionally lighter weight. The relevance of this argument for Guidobaldo's point lay in the fact that it could be used to argue that although the one weight would be positionally heavier than the other, the centre of gravity of both weights would not change and as a result it would *still be unique*.

Guidobaldo's second aim is to show that Tartaglia was also wrong in maintaining that the lines of descent of the weights at both ends of the balance are parallel. To this end, he first re-introduced the convergence of the lines of descent towards the centre of the world into the argument. Therefrom he deduced that, as a consequence of this convergence, the weight on the depressed arm should always be positionally heavier, and that even stable equilibrium is inconsistent with the theory of positional gravity. (See Figure 3.4)

[324] As we shall see in a moment, the argument of the parallel nature of the descending lines is used not only to criticize Tartaglia, but also to maintain the possibility of indifferent equilibrium. Although this seems to be contradictory, or a self-contradiction for Guidobaldo's argument in itself, I explain the meaning of its choice below.

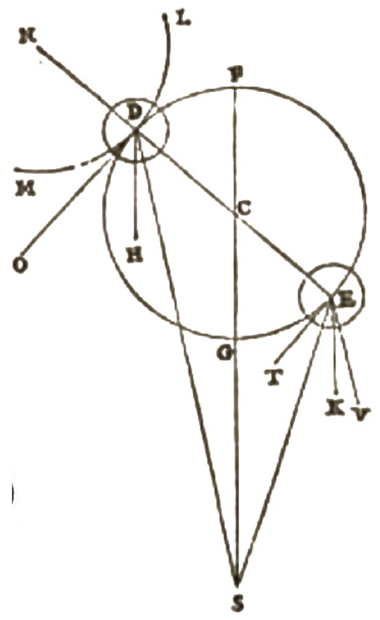


Figure 3.4

Since the lines of descent of the bodies at *D* and *E* converge in S, the centre of the world, the body at the lower position *E* will always have to be positionally heavier, according to the views of Tartaglia and Jordanus, because the angle *SEG* is smaller than *SDG*. It follows that even stable equilibrium would be impossible on these authors' own assumptions. *(Mechanicorum Liber,* 8r.)

In the second part of his polemical debate in favour of indifferent equilibrium, Guidobaldo first refers back the parallel nature of the lines of descent of the weights suspended on the opposing arms of a balance. Secondly, he stresses that the different types of stability (stable, unstable and indifferent) are governed by the *duality* between centre of suspension and centre of gravity. Immediately after having criticized Jordanus and Tartaglia for having neglected the effect of the conjunction of weights moving on the opposite arms of a balance in assessing stability, Guidobaldo aims to show that as a further effect of this conjunction the lines of descent will become parallel.[325] Why does Guidobaldo return here to parallel lines? If he did not do this, he would be confronted with the same problem as he had uncovered for the proponents of positional gravity, i.e. Tartaglia and Baldi. In fact he says:

[325] van Dyck 2006.

> But if the weights *E* and *D* are joined together and we consider them with respect to their conjunction, the natural inclination of the weight placed at *E* will be along the line *MEK,* because the weighing down of the other weight at D has the effect that the weight placed at *E* must weigh down not along the line *ES,* but along the line *EK*.[326]

The line *ES* is the line connecting the weight *E* with the centre of the world *S*, whereas the line *EK* is a line through *E* but parallel with the line connecting the centre of gravity of *E* and *D* with the centre of world. Now, van Dyck maintains that the lines of descent are posited not just to be parallel to each other, but also to be parallel to the line connecting their centre of gravity with the centre of the world. This is immediately relevant, because if Guidobaldo has a means to justify this, he also has resources which are unavailable to the proponents of a theory based on the notion of positional gravity. Hence, he could at the same time criticize them for neglecting the convergence of the lines of descent and retain the parallel lines in his own conception. Moreover, if we remember Guidobaldo's understanding of the notion of the centre of gravity as it was evinced in his comments in the *Planes in Equilibrium,* it becomes clear that he is not just positing an arbitrary stipulation. In fact, one of the main properties of the centre of gravity is that it is connected with the cosmological structure of the Aristotelian cosmos. We saw that it is the centre of gravity which truly wants to unite itself with the centre of the universe (a fact which is also expressed in the third supposition of the *Mechanicorum Liber,* quoted above).[327] Once again, following Pappus, Guidobaldo links the existence of such a point – as the centre of gravity – within any body which possesses the natural propensity that all bodies have to move towards the centre of the world.[328] The present argument for the parallel nature of the lines of descent can be understood as a straightforward extension of this reasoning. The balance is also shown

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[326] *Ac propterea non est inconveniens idem pondus modò in E, modò in D, gravius esse in E, quàm in D. Si verò pondera in ED sibi inuicem connexa, quatenusq; sunt connexa consideraverimusù; erit ponderis in e naturalis propensio per lineam MEK: gravitas enim alterius ponderis in D efficit, nè pondus in e per lineam ES gravitet, sed per EK.* del Monte 1577, 20 *r*.
The English trans. is taken from Drake and Drabkin 1969, p. 282.
[327] van Dyck 2006, pp. 390-391.
[328] van Dyck 2013, p. 20.

with its centre of gravity in the centre of the world, its arms parallel to the original position. If we now draw lines from the weights in their original position to the same weights in this latter position, we have their paths of descent as their centre of gravity descending towards the centre of the world: lines which are parallel with each other, and with the line of descent of the centre of gravity.[329]

From the extensive passage that follows Proposition 4 – more or less fifty pages of polemical reasoning against Tartaglia and Jordanus! – it emerges that Guidobaldo is interested not so much in attributing a theory of barycentric physics to Aristotle, but rather in taking over an Aristotelian focus on the physical effects of the stationary character of the point around which the weights move, and in integrating this within a barycentric theory. Guidobaldo's conceptualization of mechanical phenomena essentially involves both what he had found in Aristotle and his followers, and what he had learned from Archimedes. Its essential conceptual element, the centre of gravity, is of Archimedean origin, but the way it functions is co-determined by an Aristotelian cosmological framework and the particular Aristotelian understanding of the balance. This straightforward connection between the possibility of indifferent equilibrium and the existence and uniqueness of the centre of gravity brings to light what is really at stake for Guidobaldo in his polemic against the proponents of the notion of positional gravity. In his view, *by denying indifferent stability*, *they take away the well-foundedness of the whole concept of centre of gravity* (hence also Guidobaldo's confidence in claiming that Archimedes seems to have been of the same opinion as himself concerning the stability of balances, a topic never explicitly discussed by Archimedes).

The very basic assumption of the demonstration makes recourse to the Aristotelian cosmos, namely that a heavy body is at rest in the centre of the world. Consequently, all the world's parts must have equal moments with respect to the point that coincides with

[329] What happens if a body is hypothetically placed at the centre of the world? It will be absolutely at rest, and there must additionally be a point in the body around which all parts of the body have an equal moment (or equal weight). This clearly resembles Archimedes' hydrostatic arguments, in particular Propositions 2-7 of *On Floating Bodies* which are used to prove the sphericity of the Earth, and which have been discussed in the previous chapter (§ 2.5).

the centre of the world; otherwise, one part would outbalance the other and produce movement, thus contradicting the hypothesis of the body's rest at the centre of the world. But according to the definition given above, this point is also the centre of gravity itself. Therefore, to say that a body moves to the centre of the world *naturali propensione* – or by its own *positional gravity* – is to say that it is the centre of gravity which truly wants to unite itself with the centre of the universe. Since it is the gravity, or the weight, of the body itself which generates the natural propensity of a body towards the centre of the world, and since both the centre of the world and the centre of gravity coincide, we can assert, using Guidobaldo's own words, the following conclusion:

> Although we have considered in the foregoing only the weights of the bodies which are at the ends of the balance, without that of the balance itself, nevertheless since the arms of the balance are equal, the balance will behave the same whether we consider its weight together with those of the bodies or without them, for the same centre of gravity without weights will be that of the balance alone. Likewise if the weights are attached to the ends of the balance, in the usual manner, it will be the same, provided that the lines drawn from where the weights are attached toward the centre of heavy things (the balance being moved in any manner) go to meet in the centre of the world, since, when the weights are attached in this manner, they bear down as if they had their centres of gravity in those same points. Whence we may consider the results in just the same way. But with regard to this last conclusion, many things are said by men who believe otherwise. Hence it will be well to dwell further on this; and according to my ability I shall endeavour to defend not only my own opinion but Archimedes too, who seems to have been of the same opinion.[330]

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[330] *Cum vero in iis, quae dictasunt, gravitatis tantùm magnitudinum, quae in extremitatibus librae prositae sunt aequales, absq; librae gravitate consideraverimus; quoniam tamen adhuc librae brachia sunt aequalia, idcirco idem librae, eius gravitate considerata, unà cum ponderibus, vel fine ponderibus eveniet. Idem enim centrum gravitatis fine ponderibus librae tantùm gravitatis centrum erit. Similiter si pondera in librae etrmitatibus appendantur, ut fieri solet, idem eveniet; dummodo ex suspensiorum punctis ad centra gravitatum ponderum ductae lineae (quocunq; modo moveatur libra) si protrahantur, in centrum mundi concurrant. Ubi enim pondera hoc modo sunt appensa, ibi gravescunt, acsi in iisdem punctis centra gravitatum haberent. Praeterea, quae sequuntur, eodem prorsus modo considerare poterimus. Quoniam autem huic determination ultimate multa à nonnullis aliter sentientibus dicta officere videntur; idcirco n hac parte aliquantulum immrari oportebit; et pro viribus, non solum propriam sententiam, sed Archmedem ipsum, qui in hac eadem esse sentential videtur, defencere conabor*. del Monte 1577, p. 5*r*.
The English trans. is taken from Drake and Drabkin 1969, p. 262.

Thus, all bodies behave as if they possess weight exclusively in their centre of gravity. Does this mean that the weight of a body, or what is nowadays simply called mass, is fully concentrated in the body's centre of gravity? Could it be the case that this idea conceals some insights related to the process of idealization that will only become an ordinary process in the following centuries? The *common* centre of two equal bodies is located *in the middle* of the line connecting their centres of gravity. It is absolutely crucial that since this is one magnitude – given by the union of the weights posed at the extremities of the balance's arm – it also has one unique centre of gravity, and this must be *completely independent* from the shape of the composing magnitude. On the same subject, Guidobaldo had already argued (in Propositions 2 and 3) that the definition of a body's centre of gravity is given by stating that it is exactly in this point that the tendency towards the motion of the body is concentrated. We can thus validly assume that the equilibrium which subsists between two bodies will not be disturbed if we *replace* one of these bodies with two equal bodies, both of which are half its weight, and, placed in such a way that their centres of gravity are equally far from its centre of gravity, which follows from the fourth proposition (quoted above) together with the definition of centre of gravity, that both situations are completely equivalent with respect to the physical causes determining the system's equilibrium.[331]

By investigating the physical behaviour of the centre of gravity, we can conceptualize mechanical phenomena and give them a mathematical meaning. It is by means of an odd wordplay – somehow already adopted by Commandino – that Guidobaldo expressed the relation between the physical and the mathematical approaches, when he speaks about the fact that two bodies, which are suspended at their common centre of gravity, are *aequipollent*. This term expresses first that the bodies have equal power, and second it constitutes a term with "a well-engrained technical meaning within medieval logic, where it expresses something like truth-valued equivalence because of the syntactic features of language."[332]

Whilst being careful not to make too much out of Guidobaldo's reasoning, we can

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[331] van Dyck 2013, pp. 25-26.
[332] *Ibid.*, p. 31.

say that his insights show favour towards a new emerging tradition, a new mechanics or new mathematical science of nature, which would become more developed from Galileo onwards. This new mechanics can be seen as an attempt to systematise the practice of introducing abstract concepts to denote physical objects. "These can be *transformed into each other without altering the effect*, rather allowing the construction of a logic that is supposed to do justice to the syntax of the world."[333]

Despite the fact that there are no traces of indifferent equilibrium in either Aristotle or Archimedes, Guidobaldo makes recourse to the Aristotelian conception of the cosmos in order to justify his integration of the Pappian and Archimedean definitions of the centre of gravity into his own mechanical theory. Since the basic assumption of his reasoning is that a heavy body is at rest in the centre of the world, thus both parts of the body must have equal moments with respect to the point that coincides with the centre of the world. This case, in which the centre of the system of weight is in the balance itself – so coinciding with the fulcrum – is exactly the most crucial example that is missing in the Greek treatises. What is interesting is that Guidobaldo takes over an Aristotelian focus on the physical effects of the stationary character of the point around which the weights move, and integrates this within a barycentric theory. In this manner, to simplify the case, we can just imagine that instead of considering the equilibrium of the whole balance, thus imagining the physical effects of the stationary character of the mechanical and material device, we can replace the balance itself with the point around which the weights are suspended.

Due to this interplay between theoretical and empirical considerations, i.e. between Aristotelian and Archimedean mechanics, Guidobaldo opens the doors to modern mechanics and to a more strongly theoretical input. The core of the whole argument lies in the idea that the complete weight of each body can be replaced by its centre of gravity, when considered as a point without any physical content. This replacement is justified, on the one hand, by the definition of centre of gravity, which, on the basis of Pappus' and Commandino's viewpoints, clearly imply that a body suspended in its own

[333] *Ibid*.

centre of gravity will always be in what has been called indifferent equilibrium – that is roughly, the idea according to which no matter what the orientation with respect to that point, the body will remain in equilibrium. On the other hand, the replacement of the physical body with the barycentre is endorsed by the procedure through which Guidobaldo raises his worry whether two bodies merely connected by a line can be considered to be natural constituents of the physical universe. In other words, as he writes in this passage: "the balance will behave the same whether we consider its weight together with those of the bodies or without them, for the same centre of gravity without weights will be that of the balance alone." It turns out that we can ascribe a centre of gravity to any combination of physical bodies; we can consider them to be unified in just one point, or we can also neglect their magnitude altogether. Guidobaldo conceptualises mechanical phenomena in a new way. In his *Mechanicorum Liber* he says that to have control over physical things does not come about by the application of geometrical arguments to physical matter, but by employing those physical properties in a cunning and opportunistic way.

Following the same line of argument, we will now pursue the theoretical forward step subsequently taken by Luca Valerio.

### 3.8 Luca Valerio: the turn towards the mathematical analysis of physical states of affairs

The mathematician Luca Valerio[334] was a pupil to Christoph Clavius in Rome and was

[334] Luca Valerio was born in Naples in 1553, the son of Giovanni Valeri from Ferrara (a chef) and Vincenza Rodomano from Corfù, Greece. Unfortunately we know very little about the family's origins and Valerio's childhood. We know that he lived in Corfù, where he received a training in liberal arts. His humble origins forced him to pursue a religious career in order to continue his study. In 1570 he became a Jesuit, and his tutor Christoph Clavius helped him to familiarize himself with various scientific disciplines, mathematics in particular; in the following years he also established contact with F. Maurolico and his works. From 1580s he abandoned the Jesuit order and from 1580 to 1600 we have no more than fragmentary testimonies and information about his life and interests. However, it was in these years that his first mathematical treatises can be safely dated. The first two works – which are now lost –

probably influenced by the work of Francesco Maurolico, because of their mutual friendship with Clavius the Jesuit. Valerio represents an exception in the Renaissance era, because he was able to build a huge corpus of theorems which was useful for finding the centre of gravity of any figure, and which worked efficiently for a large group of figures. This corpus was divided into two categories: *circa axim* ('around the axis') and *circa diametrum* ('around the diameter'). This division allowed him to distance himself from the *ad hoc*[335] system which was already used by Commandino as part of the Archimedean legacy. Already an entire set of theorems was developed with Maurolico which was useful to calculate the centres of gravity for each class of figures, but only following Valerio's *De Centro Gravitatis Solidorum Libri Tres* (1604) did it become possible to achieve a *general* demonstrative technique[336] and thereby bring into

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are the *Phylogeometricus Tetragonismus* and the *Subtilium Indagationum Liber*, both of which were dedicated to mechanics.

In the last twenty years of the sixteenth century, he gave private lessons to Cardinal Ippolito Aldobrandini and simultaneously worked in the *Biblioteca Vaticana* as teacher of Greek, while his governor was Cardinal Marco Antonio Colonna. In the same time, he met G. Galilei in Pisa and the Ducal family of Zagarolo.

Finally, it was only in 1600 that he held his first classes at the Sapienza University in Rome, one in ancient Greek and the other in mathematics. During the teaching activities he had time to carry out further research, reshuffling the problems related to the centres of gravity. In 1604 he published the *De Centro Gravitatis Solidorum Libri Tres*, and almost all his life and career from this point was dedicated to the refinement of the methodology introduced in the *De Centro*. His outcomes are also mentioned by Galileo in the appendix of the *Two New Sciences*, where Galileo says loudly that Valerio has to be considered the Archimedes of the seventeenth century, who gave a large contribution to the birth of modern mechanics. For more detailed biographical information, see Baldini and Napolitani 1991.

[335] The objects of classic geometry were always particular (i.e. *ad hoc*) objects given by a more or less axiomatized construction procedure. If no general object exists, then a general method could not exist either. Therefore, the classic method was always to deal with particular objects on a case-by-case basis, introducing an *ad hoc* technique for each plane or solid figure under scrutiny.

But Luca Valerio, in his *De Centro Gravitatis Solidorum*, followed a different method. He constructed an enormous edifice of theorems valid for a whole class of figures, namely *circa axim* and *circa diametrum*. Roughly speaking, the former denotes the solid figure generated by rotation around an axis, while the latter denotes the plane figure with an axis of symmetry. Thanks to his methodological innovations, the way was opened to the development of modern rational geometry. Napolitani and Saito 2004, pp. 67-124.

[336] As Napolitani and Saito state in their paper, and as we see in the previous footnote, the objects of classical Greek geometry were particular objects and if no general object exists, then a general method could not exist either. For us, an ellipse is the *locus* of the zeros of a particular quadratic equation, or in more elementary terms, the *locus* of the points, such that the sum of the distances from two fixed points is constant. The property precedes the object, which, *a priori*, might not even exist. For the Greeks, the ellipse was the object determined by a plane which cuts a cone, and which meets all of its generatrices; this curve therefore existed, but all its properties were unknown and needed to be investigated. Now, it was arguably thanks to Valerio that the transition from the classical to the modern perspective was

production a corpus of *general* theorems on centres of gravity applicable to a variety of different figures.

Although the *De Centro* represents an *unicum* in the overview of the history of science, the most important and fascinating results regarding debate around the notion of the centre of gravity are not found in its demonstrations, but in a treatise apparently less important and influential, the *Quadratura Parabolae per Simplex Falsum*. This work, published in Rome in 1606 and dedicated to Marzio Colonna, Duke of Zagarolo, can be considered a manifesto or public declaration of the methodology which was more widely and explicitly used in the *De Centro*, and which aimed at explaining the geometrical outcomes presented in it. Not only that, this work also aimed at answering a single much-discussed epistemological question of the time. This question can be restated in the following form: *are we committed to considering plane and solid figures such as they are*, i.e. *just as mathematical abstractions without any physical or material properties* (e.g. mass, dimension, extension and volume)? *Or, are we committed to attributing weight* – as a physical property – *to purely geometrical entitie*s*?*

The cultural environment in which Guidobaldo and Valerio worked was characterised by the clear attempt of attributing methodological certainty and certainty relating to the outcomes obtained in mathematics to other associated fields of study. In particular, while Guidobaldo was attempting to understand the ambit of machines and technical constructions, Valerio's aim was to broaden the results of mathematics to physics and thereby discover the epistemological meaning of this revolutionary overlapping of disciplines.

Let us now turn to the most interesting proposition of the treatise at hand, the statement of the first theorem, which proceeds as follows: *Si qualibet figura plana gravitatem acciperet, tanta esset, quanta si nullam gravitatem accepisset*. In other words, if we add weight to a geometrical object, its extension (i.e. area or volume) will not change. Thus, the core of the proposition is the following:

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possible. Valerio thus took a significant step forward in departing from classical mathematics, in that he found a general method and theorems to discover the centres of gravity of the two classes of figures he dealt with. His new and general method of demonstration, as we further note above (fn. 260), is based on the property of 'monotonicity'.

Proposition I: If to any plane figure we add a certain weight, this figure will continue to have the same dimension as (its dimension) before we have added some weight.

Let us take a square of two feet (*bipedale*) and denote it as *A*. Now, if we confer gravity (weight) upon that square *A*, it will be the same square as before, or rather (it will be) a surface surrounded between four straight lines and four angles, each of 90° degrees.
The heavy square *A* will be the same with respect to the square without that attributed weight, namely of two feet (*bipedale*).
In the same way, if any man would become white and that man is also an animal, he will be an animal in the same way as he was before acquiring his whiteness.
Another proposition is evident in itself: the assumption that *A*, from not being white, will become white is evident for itself. Whether *A* has already become white or is becoming white, it is still the same *A* as (it was at) the beginning. This is true because of a[n Aristotelian] definition that a thing can be either true or false but it cannot be both at the same time[337].
If Caesar would be declared orator – for the same Aristotelian definition expressed in his *Physics* – he would remain to be alike, because something that persists is something that is for itself and it cannot be lost. In the same way, if a square figure acquires some weight, its geometrical extension would not be greater than before, and indeed it would be the same as before acquiring the weight. This is not subject to demonstration, but it is *materiam falsitatis* that we are not using for our purpose. When we say that the squared figure *A* has acquired a certain weight can be true or false, it will be the same as saying that the squared *bipedale A* is heavy. Having stated that, the demonstration is easy to do.[338]

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[337] Cf. Aristotle, *Physics*, VIII, 263b, 10-25: "Let us suppose a time *ABG* and a thing *D*, *D* being white in the time *A* and not-white in the time *B*. Then *D* is at the moment *G* white and not-white: for if we are right in saying that it is white during the whole time *A*, it is true to call it white at any moment of *A* and not-white in *B*, and *G* is in both *A* and *B*. We must not allow, therefore, that it is white in the whole of *A*, but must say that it is so in all of it except the last moment *G*. *G* belongs already to the later period, and if in the whole of *A* not-white was in process of becoming and white in the process of perishing, at *G* the process is complete. And so *G* is the first moment at which it is true to call the thing white or not-white respectively. Otherwise a thing may be non-existent at the moment when it has become, and existent at the moment when it has perished: or else it must be possible for a thing at the same time to be white and not-white, and in fact to be existent and not-existent."

[338] *Sumamus hic exempli gratia quadratum bipedale, quod voco A. Quonim igitur si quadratu hoc A, pondus acciperet; idem esset quadtratum, quod nunc est: at nunc quadratum A, nihil est aliud quam superficies plana, quatuor restis lineis ad rectos inter se angulos comprehesa certae mensurae; grave igitur quadratum A, tantundem esset, quantum nunc est omnis ponderis expers, nimirum bipedale. Quemadmodum si quis homo animo albesceret, is pereque esset animal, atque ante fuisset, cum animo non esset albus homo enim non aliud est, quam animal rationis particeps. Altera etiam propositio est per se nota: si enim verum ilud consequitur; A iam factum album, vel albescens idem esse A, quod ante erat manifestum hoc autem ex generali definitione eius quod est certam quampiam rem fieri, vel factam esse aliquid: velut si Caesar fiat, vel factus sit orator: illa enim definitiom quam Aristoteles in physicic optimè ecpressit, significat permanere id, quod sit aliquid, non autem interire. Quoòd autem, si figura quadrata gravis fierer, ea corpus esser non amplius superficies, nec ergo tanta quanta fuisset antequam gravis fierit: genere enim differentium magnitudinum, genere differentes sunt mensurae: hoc non facessit negotium demonstrationi, sed material arguit falsitatis, qua non utimur, nulla siquidem his verbis. Si quadratum A sit ponderosum, veritas affirmatur, aut negatur: quemadmodum neque his. Sit quadratum A grave, dicimus, sed facimus quadratum A esse grave. His iactis fundamentis facilis est demonstratio istius alterius dicti.* Valerio 1606, pp. 8-9.
The English trans. is my own because no official translation is available.

The example suggested by Valerio takes into account a square measuring two feet (*quadratum bipedale*) which possesses certain geometrical properties. If we add weight (*pondus*) to it, its extension (i.e. area or volume) will remain unchanged because the addition of weight – as a physical feature – refers to a different category of size or magnitude. However, he also says: Let us imagine that we can confer gravity upon that plane figure, in which case its geometrical properties will not change. The theoretical concept and properties related to this geometrical figure (i.e. it has square sides and four right angles) does not change; in fact its substance or category, understood in an Aristotelian sense, does not change, whether we talk about plane or solid figures.

The outstanding crux to the issue is this: Can a geometrical size have a physical meaning? The addition of weight to a geometrical plane figure does not change its own geometrical nature, and we can freely and arbitrarily attribute physical content to geometrical simple objects, such as point, lines, planes and solids, or else add weight to geometrical entities without changing their extension. To this end, it is legitimate to suppose that a simple unextended point has some idealized physical features, such as weight or mass, if it is understood in its modern Newtonian sense; or otherwise velocity, charge, and so on.

However, I think that the contrary also holds, or rather that given an extended body – no matter how it is shaped – we can abstract, omit or neglect its physical content and make free use of our imagination in order to associate that body with a geometrical entity. In this respect, these kinds of mathematical entities can be used to represent any physical contingencies.

Therefore, the quantity of mass should be considered a mere attribute, or a simple feature which is tied to the idealized notion that we want to express. Valerio thereby imagines theoretical and explicative situations which have a particular connection with empirical data.

Valerio's main aim thus seems to account for an investigation into a theoretical procedure which has a pre-eminently epistemic function, namely the investigation of a process of attributing physical properties (e.g. weight/*pondus*) to geometrical figures (planes and solids). Up until this time, the theorisation stressed by Luca Valerio had aimed at conferring weight upon simple plane figures, and at explaining the movement of these bodies by the action of an external force (*vis*) which brings the bodies to their natural place (i.e. the centre of the world). Moreover, it was common in Valerio's day to use the terms '*pondus*', 'gravity' and weight' in a virtually synonymous fashion. In Valerio's proposition as seen above, we can recognise a *difference* in both the meaning and the use of these words, since when Valerio uses the term 'gravity', he is referring to the natural 'propension' that a body possesses to move towards its natural place (i.e. the centre of the world), whereas when he uses the term *pondus*, he is referring to the physical property of extended bodies such as their weight. These terms were used in the same way by Galileo, all of which conveyed the same idea of burden, or heaviness as measured by weighing, describing a mechanical procedure whose principles can be understood through purely mathematical analysis, by representing the physical bodies as a geometrical entity.

Perhaps the most interesting way of interpreting Valerio's argument, which follows not only in Guidobaldo's footsteps but also a general trend widespread during this time, is to suggest that he is attempting an investigation into the physical meaning that one should attribute to purely mathematical principles. His work can thus be considered as an epistemological analysis of the conditions under which mathematical principles can be considered to be true of physical things.

Valerio's breakthrough seems to suggest that a purely empirical approach does not suffice to understand the way in which the physical world behaves or how physical contents are related to each other. This is perfectly understandable, since the mathematical principles state precise relations which can only be approximated in reality. Even though a purely heuristic approach will not suffice to lay the foundations of a mathematical science, it is still important to stress that these foundations *do* crucially involve an empirical input.

In Chapter 2 we have seen that in antiquity, above all in the case of the Archimedean law of the lever, the empirical approach allowed the essential properties of a body’s centre of gravity to be emphasised. Thereafter, from Guidobaldo’s proofs of indifferent equilibrium, there arose the idea that the centre of gravity should be considered as a point of ‘indifferent equilibrium’, because otherwise the form of a body would matter, and the crucial transformations could not be affected in the proof. The polemical discussion that followed Guidobaldo’s fourth proposition constituted not only a way to discredit the followers of Jordanus, but also the perfect framework in which to introduce the ill-fated idea that the natural tendency of bodies to move downwards must converge at the centre of the earth. Such a centripetal convergence reflects the physical situation on which Guidobaldo agreed with his opponents, but he also showed in detail that it invalidates their explanatory scheme, whereas he can accommodate the physical facts. The difference marked from his opponents lies in the requirement of absolute mathematical rigour. However, to this rigour an empirical item needed to be added: in a passage included in the Italian translation of his *Mechanicorum Liber*, Guidobaldo claims to have been able to construct a balance that exhibits indifferent equilibrium. This proves empirically that bodies do indeed have a point situated within them that shows the required property, contrary to Jordanus’ misguided arguments. The rigour that Guidobaldo seeks is not absolute mathematical rigour, which would describe the empirical world in detail, but the rigour of any well-founded applied mathematical science. And in order to ground such a science, one has to select – and, where possible, support experimentally (e.g. by building a balance) – those properties of the empirical world that can be linked with fruitful mathematical demonstrations.[339]

What was not explicitly asserted by Guidobaldo has now been added by Luca Valerio. Valerio seems to ask: How can we claim to offer a treatment of the equilibrium of plane figures, since such a predicate – the equilibrium – is completely alien to the nature of plane figures? Insofar as they are plane figures and do not have weight at all, we can still “mentally conceive” of plane figures to be equilibrating, and thus still showing the effects of gravity, or in other words their tendency to move downwards towards the centre of the earth. How could science make use of mathematical

[339] van Dyck 2013, pp. 29-31.

demonstrations in order to achieve conclusions about physical reality? Or rather, how could science make use of physical experiments in order to endorse mathematical demonstrations or concepts? Is one science subordinate to another? The discussion of this exemplary scenario brings out the essentially dual nature of a body's centre of gravity. It is a notion that can be ascribed to every physical body which has a natural tendency for motion, but which is *at the same time* connected with some of the mathematical accidents of this body, such as its geometrical form and position (i.e. properties which are not mathematical). It is this double aspect that lies behind the revised Renaissance centre of gravity. While Archimedes considered mathematical things, such as distances and proportions, through geometrical demonstrations, giving rise to a new heuristic and mechanical approach to geometry, Guidobaldo considered natural things through natural considerations, such as those relating to the nature of the centre of gravity and upward or downward motion. Thus, the Renaissance notion of the centre of gravity essentially binds together both kinds of considerations, the mathematical joined with the physical; it connects itself with physical properties, such as the equilibrium effects of weight, but at the same time it is to be considered as a mathematical point which can be easily introduced into physical demonstrations. Furthermore, later at the very beginning of the seventeenth century, Valerio explicitly declares that pure and theoretical mathematics have arisen on the basis of the abstraction of quantitative properties from physical bodies, an abstraction that consisted in considering these properties as if they did not belong to sensible matter. This abstraction is nothing more than the approximation by means of which scientific theory represents *how the physical reality appears to our eyes*. Conversely, this opportunistic procedure allowed Valerio to apply the abstracted properties *back* to natural things, an application which was understood as the predication of a sensible condition.

These mathematical entities, such as the material point, rely on the idea that it is possible to ascribe a *sufficiently rich* mathematical structure to empirical reality, which can in turn truly ground a fruitful science. But on the other hand, the contrary also holds, since these kinds of abstract or idealized mathematical entities allow us to represent physical reality by means of simple geometrical entities devoid of their physical properties. The centre of gravity is thus hybrid in method, both i) in being geometrical in origin, due to its belonging to geometrical figures; and ii) in representing on the one hand that point in which a figure is cut into two halves equal in extension, but on the other hand a point endowed with physical properties, such as shape, weight and velocity.

To this end, the centre of gravity becomes an issue of crucial importance because it fulfils a fundamental and irreplaceable role, due to the fact that it allowed scientists to explain physical and material configurations in mathematical terms. In fact, the centre of gravity represents a hybrid case, since it is considered – by definition – a 0-dimensional geometrical point to which one can nevertheless attribute an idealized physical content. Yet on the other hand, if we have a physical body one can omit its physical content and represent it as a geometrical entity. In this respect, the weight becomes a quality, not a quantity, and as evidence of this, we can consider the fact that the weight is neglected.

# Chapter 4

## From Galileo to the Modern Turns

The reconstruction undertaken in the previous chapter showed the way in which the mathematics of the Renaissance had managed to create a suitable set of meaningful materials, techniques and methods that could be used to create a modern rational mathematical physics. These Renaissance scholars were able to work – with mathematical accuracy and methodological opportunism – towards the reinforcement of a particular branch of physics, namely that which consists of important theoretical notions. They worked towards the foundational programme of constructing an epistemological debate over the nature of abstract and idealized mathematical entities. This was the context in which a purely mathematical treatment of physical reality now came to the fore. The notion of the point mass interpreted as the centre of gravity still relates, on the one hand, to heuristic and practical issues, but now on the other hand it started to be used as a representational model for a great variety of physical (specifically dynamic, kinematic and static) phenomena. Seventeenth-century mathematical physics represents the framework for the completion of the third stage of the objectification of procedure, insofar as it was at this time that the centre of gravity became an independent object of study as the point mass, an idealized entity used to represent physical objects and to which we can ascribe natural properties such as volumetric extension, mass and forces. The point mass is still a representational geometrical point whose features are idealized, but it now assumes the role of being an independent object of research useful for building the axiomatic foundational principles of rational mechanics.

Historically, Galileo Galilei dates to the last phase of the Renaissance period, when in the 1580s he led the renaissance of mathematics at its high tide. In fact, he is usually considered the author who initiated the tradition of modern mathematical physics, either because in his early years of research he focused on problems directly related to the

Hellenistic tradition[340], and because, by the turn of the new century, his interest had shifted to different topics and purposes, or better, because his approach was driven by the desire to give a mathematical interpretation to natural and physical phenomena. By analysing some of the Galilean treatises we can isolate the reasons that he can be considered the author who placed himself in a half-way position between the earlier mechanics of the Renaissance and the prototypical conception of modern rational mechanics. Having completed our analysis of Galileo's mathematical approach to mechanics, above all to the branch of kinematics, we will now turn to consider the theoretical elements which characterise the last stage of the objectification of procedure, the stage in which the point mass enters the scene as an intelligible entity, a primitive term, and a model for scientific representation. In particular, this last phase is characterised by the fact that the algebraic approach replaced the geometrical one, and allowed for the abstract theoretical reasoning of mathematical entities such as ideal numbers or simple equations. In order to carry out this examination, we will scrutinise the contributions and achievements of three more vital thinkers, Thomas Hobbes, Isaac Newton and Leonard Euler. In the findings of these authors we can locate the answers to our twofold purpose. First, we will note the way in which the geometrical centre of gravity assumed its status as an object of mathematical idealization, that is as a model used to represent any physical heavy body, not only at rest but also in motion, either uniformly, in acceleration or parabolically. Between the seventeenth and eighteenth centuries the point mass turned into an independent mathematical entity detached from any physical features, which was applicable to a series of different cases due to its general nature and its power to serve as a representational model. It became a sort of simple and malleable centaur which can be modified on the basis of the analysis of the cases and purposes at hand. This achievement is directly related to the second purpose

[340] It is also often said that Galileo's research represents the revival and continuation of the Archimedean tradition, while he had constantly taken a position against the influence of Aristotle. However, even if this is true in many areas of dispute such as in hydrostatics, it is nevertheless quite erroneous in the domain covered by mechanics, which is the one in which we are most interested. Indeed, almost the reverse holds here: Galileo operates as his mentor Guidobaldo did in dealing with an attempt to reconcile himself with both the Archimedean and the Aristotelian traditions. For further details, see Galileo 1960, p. 141.

of our study, namely to ascertain how Galileo assisted in the development of the first kernel of the theoretical procedure involving idealization and abstraction in mathematical physics. In this connection, we will see how Galileo used the practical and artisanal achievements of the Urbino School to win mechanics a more general level of applicability. Thus, the first item on the agenda of this chapter focuses on Galileo's contributions (§ 4.1) to the science of mechanics. Here we shall point out how Galileo was able to shift the investigative methodology used in the field of mechanics to a more theoretical and physical perspective, by showing how the purely practical operations carried out through machinery were now, in Galileo's view, able to be used to interpret the working principles behind the phenomena of the natural world. Within the second part of this chapter, § 4.2, we will first examine (§ 4.2.1) the more metaphysical contributions given by Thomas Hobbes, which relate to the rule of human imagination in building our representational model of physical phenomena and the point mass. Then in §§ 4.2.2-4.2.3, we will establish the way in which the model of the point mass from a purely mathematical point of view is used to represent, with purely algebraic language, a series of states of affairs which were not only made up of simple rigid bodies, but which were also typified by a higher level of complexity. This represents the climax of the procedure of objectification of our centre of gravity. Now it is no longer a geometrical tool shifted along the arms of a real mechanical balance, but instead a mathematical entity, an investigative tool or a model of intelligibility – explicitly introduced for the first time in Newton's and Euler's research – that has two main characteristics: i) it is the centre of mass of any rigid body, no matter how it is shaped, and ii) it is the point of applicability of all the forces – gravity, works, pressure, and so on – acting on a stationary or a moving body, be it travelling in uniform motion, in a state of acceleration or parabolically.

Before entering into the details, let us examine some historical features of the period under consideration. The debate over the equilibrium, which was widely discussed in the previous chapter (esp. § 3.6), continues until the turn into the eighteenth century, and although Galileo took part in it – especially in the first years of his career, with his research presented in the *De Motu* (*ca*. 1590) and in *Le Mecaniche* (*ca*. 1600) – his approach, as we shall see, was completely different by comparison with the one of his tutors Guidobaldo. The relationship of the Equilibrium Controversy to the branches of mechanics became increasingly close: while study on the equilibrium was previously connected to statics, or otherwise confined to the field of kinematics, now with Galileo's works, dynamics and statics are considered two aspects of the same field of research. My interest in this section will be directed to the causes[341] of the loss of equilibrium of bodies on a balance, thus shifting the attention towards the forces acting in the equation for a falling body and towards the nature of gravity of the energy involved in the motion of bodies. The Aristotelian dynamics, which remained prevalent until the late Middle Ages, erroneously believed that to keep a moving body in constant velocity did not require the application of an external force, and that the motion of bodies was rather caused by a power that possessed intrinsic self-motion. In other words, the conventional opinion asserted that the cause of the falling of a body was inherent to its heaviness and that motion was a natural property of any body in itself. The idea of the existence of an external and independent cause of motion was not yet conceptualised. The resurgent interest in the causes of motion, which found its application in the field of the science of equilibrium, led in turn to the science of the impetus. This field was probably first identified by Jean Buridan (1295-1358), who theorised that the application of a force to a body leads to the generation of an impetus, which gives the body the power to continue moving at a constant velocity. The velocity does not change if the body is not stopped by obstacles or if there is no resistance, i.e. from friction on the surface; otherwise in the case of resistance the speed will gradually decrease. The existence of this impetus seems to be the cause due to which the natural fall of bodies acquired an undefined acceleration. Buridan also applied this theory not only to earthly bodies but also to the celestial bodies.

Therefore between the sixteenth and seventeenth centuries the combined studies of static and dynamic phenomena allowed above all the achievement of the most important

[341] Galileo might be considered one of the first investigators to inspire this new approach.

results in astronomy, in particular thanks to the work of Johannes Kepler (1571-1630) and Isaac Newton. Moreover, there appears to be a direct connection between Buridan's studies and the astronomy of Johannes Kepler. This connection involved the theoretical innovations into astronomy by Buridan's pupil, Nicholas Oresme (1323-1382), Nicholas of Cusa (1401-1464), Nicolaus Copernicus (1473-1543) and Giambattista Benedetti (1530-1590). From the doctrine of impetus Kepler established the notion of 'inertia' – which had already been theorised by Benedetti – and defined it as the tendency of a body to maintain its static condition or its linear motion, unless an external force changes its state.[342]

At the same time, studies were being carried out into the nature of space. The law of inertia in seventeenth-century physics was a theory of absolute space, a continuum of utterly featureless and indistinguishable parts against which motion can be comprehended. Even more importantly, space was understood not as deriving from the relations between objects, but in and of itself as a "pseudo-substance" that is not dependent on anything else for its existence.[343] The chief figures of this new mathematical science of space presumed that motion can be understood to occur in relation to space itself without reference to any other bodies around it, and being accounted for only by appeal to the law of inertia. Thus, for example, a single isolated body alone in space can be said either to move or to remain stationary. The difference between the two states are explained by the difference of the surrounding environment. These two concepts – absolute space and inertia – were inseparable and together were absolutely crucial for the development of modern mathematical physics.[344] In this scenario, bodies moving in the space are conceived as point masses, hence as models of intelligibility. This latter notion represents the tradition according to which a great

[342] For further bibliography, see Dugas 1988; Jammer 1957 and Jammer 1961.
[343] These two different definitions of space (and time) date back to the well-known substantivalist-relationalist debate. The ontological contents related to the nature of space originate from Newton's mechanics and Leibniz's philosophy. The former, as a substantivalist, argues that space is a somewhat particular substance that does not depend upon the objects which are located in it, whereas the latter, as a relationist, states that space is nothing but some sort of collection of spatial relations between more fundamental physical objects, which thus thoroughly depends on them for its existence.
[344] For further details, see Herbert 1987, pp. 709-717 and Brandt 1927.

variety of phenomena can be represented and investigated in a simplified way. Originally, Peter Machamer introduced the notion of 'model of intelligibility' to capture the multiple functions the balance plays within Galileo's science, but I would suggest that it applies not only in Galilean mechanics, but across the whole dispute on the equilibrium. Machamer states that:

> Its physical concreteness, mathematical describability, and physical manipulability leading to experimental possibilities gave intelligibility and structure to the abstract concepts of the mechanical world picture.[345]

According to this research here, the appearance of the theoretical notion of balance has to be considered fundamental to sketch the development of the notion we are trying to investigate. Already in ancient Greek mechanics, especially in Archimedean science, the virtual balance represented not only a novelty but also a mechanical and heuristic insight which could be used to simplify and represent much more complicated situations occurring in (among others) artisanal contexts. The virtual balance model can be considered as a parallel to the centre of gravity as a point mass model. In this connection, we can borrow the notion of the model of intelligibility in order to describe the role which is fulfilled by the material point in this last step of the objectification of procedure. Just as the virtual balance represents an investigative tool, according to van Dyck[346] and Machamer, the material point is a mechanical and heuristic tool used in investigating natural principles. In this new step taken during the modern turn, the idea of using mechanical tools in investigating natural principles loses much of its paradoxical character. Rather it becomes an ordinary practice, whereby it is almost natural to investigate these everyday physical events *exactly through manipulations*. It is through our way of interacting with nature that nature can now truly show itself for the first time. This allows mathematical instruments which had been primarily practical

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[345] Machamer 1998b, p. 71.
[346] van Dyck 2006.

problem-solving tools to function now additionally as investigative tools and models of intelligibility.

To sum up, it will be pointed out from the analysis at hand that the behaviour of any rigid body, be it standing at the extremities of a balance or in motion along a flat or inclined surface, or again in motion following a parabolic path, may be represented with a model of intelligibility which has the structure and features of a point mass. The geometrical centre of gravity will leave the stage to be replaced by the notion of the centre of mass. The centre of mass or centre of inertia is a point in any body, around which the mass or inertia[347] is equally distributed in some manner according to the equality of the moments. This point simultaneously represents the point of application of the forces acting on a body, which is a force equally distributed on the entire surface of the body, but it is imagined to be concentrated on the centre of gravity of the body in question.

It was, however, only in the eighteenth century that this notion widened in terms of its heuristic applicability. Before then until the end of the first half of the nineteenth century, there were several attempts to extend and systematise mechanics as it had been developed in the works of Galileo, Christiaan Huygens (1629-1695), Leibniz, Jakob Bernoulli (1654-1705) and Newton. What will evidently change in the approach suggested and adopted, first by Leonard Euler, and later by his pupil Lagrange, was the search for principles that allowed the study of systems more complex than those formed by simple point masses that had been studied in the previous century. It was a period in which scientists debated by publishing their research in scientific journals, a period in which the spread of knowledge became faster and more immediate than before. The eighteenth century generally traces its influences back to the presentation of the infinitesimal calculus – this having been achieved by both Leibniz and Newton,

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[347] As Drake [1970, p. 4] recalls in his *Galileo Studies,* all historians agree that the chief importance of the concept of inertia lay in its application to projectile motion, which is usually traced back to Galileo's mature work. It is likewise agreed that Galileo had to do with the treatment of the inertial concept, but not in connection with the treatment of the projectile motion. In fact, the origins of this latter concept lay in the logical refutation of Aristotle's classification of motion in general, which was carried out at a time when Galileo was already satisfied with the anti-Aristotelian impetus theory of projectiles, a theory which he himself adapted to the explanation of acceleration in fall at that same time.

apparently independently of each other – that was widely used for solving complex problems.

Along with the search for new principles, their foundations were examined as well as the nature of mechanics and its relationship with mathematics. It was a period dominated by empiricist philosophy, in which mechanics was seen by some of the greatest scientists as a purely mathematical discipline, shedding the characterisation of a mixed science that had accompanied its inception, first as statics in the ancient world and later as dynamics in the modern era. The new emphasis laid upon the causes of motion, instead of upon the causes and the conditions which guaranteed the equilibrium.[348]

## 4.1 Galileo Galilei: a Bridge to the Modern Mathematical Physics

Even though Guidobaldo del Monte's science of mechanics was still confined to statics and the study of simple machines, through the use of a series of central concepts, he nevertheless had the meritable aim not only to give coherence to mechanics, but also to trigger the process of representing facts about physical phenomena through mathematics. Just after the publication and distribution of his *Mechanicorum Liber*, which became the most respected treatise on mechanics from the second half of the sixteenth century, one of his pupils, Galileo Galilei, published a treatise containing the most outstanding physical and mathematical results related to this issue, *Discourses and Mathematical Demonstrations concerning Two New Sciences* (1638). Galileo in this treatise attempted to develop a mathematically grounded natural philosophy as a new field of knowledge; moreover, due to the values embedded in late sixteenth-century society, he was capable of structuring this field in a different way. However, whilst this masterpiece represents the highest point of Galileo's scientific research, it marked the result of a long series of practical and thought experiments combined with attempted

[348] See further Dugas 1988 and Capecchi 2014.

theoretical treatments.

First, let us provide some details on Galileo's personal life and an overview of his *oeuvre*. Galileo was born in Pisa on February 15th 1564, and moved with his family to Florence in 1572, where he started to study for the priesthood. But this he soon left and signed up for a medical degree at the University of Pisa. However, he never completed this degree, but instead in 1583-84 came in contact with Ostilio Ricci (1540-1603), a mathematician of the Tuscan court and a friend of his father's, who introduced him to Euclid, Archimedes and all the mathematicians of his time, including the Urbino School. Later, he visited the mathematician Christopher Clavius in Rome and began a correspondence with Guidobaldo del Monte. He applied and was turned down for a position in Bologna, but a few years later in 1589, with the help of Clavius and del Monte, he was appointed to the chair of mathematics in Pisa. In 1592 he moved to Padua for a position as mathematician at the University; during this time he also became married and from two spouses in the period 1600-1606 had two daughters and a son. In Padua he worked on mechanics and his telescope, and in 1610 he published *The Starry Messenger*. After this period he accepted a teaching position in Pisa, where he carried out observations on the moon of Jupiter, which were certified by Clavius and the *Collegio Romano* in 1610.

In 1612 Galileo published his *Discourses on Floating Bodies* and in 1613 his *Letters on the Sunspots*. In 1623 he published *The Assayer,* an essay dealing with the comets which posited that they were sublunary phenomena. At the same time, he began work on his *Dialogues concerning the Two Great World Systems*, which was first published in Florence in 1632. This is the book that caused the trial promoted by the Church, with the book being immediately banned from sale, and in 1633 Galileo himself was condemned by the Inquisition. Upon his arrest he began work on his last masterpiece, the *Discourses and Mathematical Demonstrations concerning Two New Sciences*.[349]

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[349] For a brief entry on Galileo's life and works, see P. Machamer, "Galileo Galilei", The Stanford Encyclopedia of Philosophy (Summer 2017 Edition);
On-line: https://plato.stanford.edu/archives/sum2017/entries/galileo/.

The central influence of Galileo's scientific life was Archimedean mechanics as it was laid down by Commandino and Guidobaldo. In each of the Galilean works can be recognised influences from his contemporary scholars. For example, in the *Theoremata circa centrum gravitatis solidorum* written between 1585 and 1587, but published only as an appendix to the *Two New Sciences* in 1638, Galileo's affinity with Luca Valerio's treatise is an evident trait. In the *De Motu* composed in the late 1580s at Pisa, the influence of Filippo Fantoni is discernible, who was among the first mathematicians to give lectures on the application of mathematics to physical problems, treating the subject of mathematical certainty and addressing problems of motion that were usually neglected in the Archimedean context but discussed in the Aristotelian domain. By the time of his departure from Pisa to Padua, Galileo was in possession of "an effective Archimedean method for the interpretation of physical subjects".[350] At the University of Padua, Galileo encountered an established mathematical and philosophical tradition: in the 1350s Giovanni de' Dondi (1330-1388), designer of the *planetarium*, had lectured mathematics, Biagio Pelacani (1355-1416) and Vittorino da Feltre on Euclid, and in 1463-1464 Regiomontanus gave a course of lectures on astronomy and his famous *Oratio in Omnias Scientias Mathematicas*. Moreover, sixteenth-century debate now centered on two main problems: i) the relationship between mathematics and physics, or more exactly the methodological applicability of mathematics to mechanics as the mixed science; and ii) the claim of the superiority of mathematics – which posits epistemological certainty in virtue of its reliance on theoretical abstractions – over Aristotelian syllogistic reasoning, in which certainty is supposed to derive from the intrinsic power of its proofs. Again in his youthful contributions, it appears clear that obvious and decisive influences derive from Greek geometers, and in fact in 1584 when he was a student at Pisa, Galileo encountered Euclid's treatises for the first time, while in 1585 at Florence he approached Archimedes' works. Therefore, it is likely that it was the study of the treatises on statics and hydrostatics that settled Galileo's true vocation. In 1586, the young Galileo constructed a hydrostatic balance in order to reproduce the

For further details, see Drake 1978.
[350] Rose 1975, p. 283.

experiments by which Archimedes had established the density of various substances. But while these young contributions failed to introduce any novel concepts, they nevertheless make it clear that as early as 1587/1588 Galileo was already familiar with the essential contributions of the Greek mathematicians.[351] On the same line of thought, the second source of inspiration was his acquaintance and correspondence with the Urbinate, Guidobaldo, who was in all likelihood appointed as Galileo's mentor and intellectual patron, and whose own scholarship gave rise to the reunion of the two Hellenistic traditions, as I have argued in the previous chapter (esp. § 3.7 and 3.7.1). These may be considered the preparatory years that provided Galileo with the instruments for the transformation of the old approach and the gradual elaboration of a new and original method of analysis. This method represents the outcome of his research into the so-called 'problem of nature': he uncovered the way in which nature functions *discursively* as a normative instance that regulates the kind of claims that can be scientifically made about any individual objects under study.

It is suggestive to consider Galileo's achievements from two distinct points of view. First, we shall consider the theatres of machines and the epistemological and theoretical meaning which Galileo gave to the use of the virtual balance as an artisanal practice that however still retains its heuristic value. In this context it is worthwhile to analyse the youthful work *Le Mecaniche*, together with some passages deriving from Galileo's last work, *Discourses and Mathematical Demonstrations concerning Two New Sciences*. From this it emerges that, in order to represent natural phenomena by means of a virtual balance model, bodies – no matter how they are shaped or how big they are – have to be represented as *volumeless* point masses. Secondly after this, we shall see how Galileo took part in, and provided a significant contribution to, the ongoing dispute over the truth-value attributed to idealizations, and to the idealized model of the point mass – something that Galileo tends to label as 'moving particle'. Here it will be pointed out that at the very core of the use that Galileo made of the model of balance, the inclined plane model and even the pendulum model, there stands the idea of bodies conceived as

[351] Clavelin 1974, p. 118.

simple idealized point masses. These conclusions are drawn from the dispute that is staged between Simplicio, Sagredo and Salviati[352] in Galileo's last masterpiece concerning the nature of motion, in which he suggests a free play of human imagination that is only constrained by the intentions that must be put into practice. Galileo hereby regiments the "role played by the imagination" through his abstract analysis[353], or through what we have called in Chapter 1, 'Galilean idealizations'. It will appear that for Galileo human imagination has no limit, neither does the range of applicability that we can confer upon our idealized models, to which we give the label of investigative tools and which derive from abstracting the purely practical purposes for which we have built machines and the abstraction of the natural properties of our physical bodies. The result will be the extrapolation of a general and simple model applicable to a large variety of issues. According to Galileo's regimentation of model-building practice, there is one limit that we cannot overcome as users of a new mathematical discursive practice: this is not the limit of practical feasibility or efficiency, but rather the limit of what we can do with machines and what is imposed on us by nature itself.

What will therefore emerge from this section is a conclusion concerning how the material point will achieve a new step forward. Galileo will appear as the last Renaissance mathematician, but also as the one who opens a window onto the accomplishment of the third stage of the objectification of procedure. In other words, Galileo gave some clues as to how we should conceive the point mass as a mathematical idealization represented by an unextended geometrical point, imagined as devoid of weight, and as the idealized point in which all forces acting on a moving body – be it suspended at the extremity of a balance and moving downwards, or in motion along an inclined plane – are imagined as exerting their power *uniformly on the centre of gravity of the body*; this point has since taken the name 'centre of pressure'. Galileo was the

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[352] Salvaiti is Galileo's spokesman, Sagredo is a Venetian senator and friend of Galileo, and Simplicio represents the spokesman of the Aristotelian tradition. Those three characters are used in the two main last works written by Galileo: *Dialogues concerning the Two Great World Systems,* Florence, 1632; and *Discourses and Mathematical Demonstrations concerning Two New Sciences,* Leiden, 1638.

[353] Galilei, *Opere* II, pp. 156-157. See also Galilei 1960a-b, pp. 46 and 148. Throughout this chapter, when I refer to "Galilei, *Opere* n.", I refer to the corresponding volume of Favaro's National edition in the bibliography.

author who delivered the biggest contribution for reaching the level of theorisation shown progressively by authors such as Newton and Euler, who both also contributed to the legitimation of the representational relation linking mathematical structure to concrete physical events.

It has already been mentioned that Guidobaldo was one of the earliest and most important patrons of Galileo: both men had the same interest in scientific matters, and also carried out some experiments together. Thus it may not be surprising for the contemporary reader to find common elements between Guidobaldo's mechanics and Galileo's mechanical-theoretical writings. Yet despite these general similarities between them, Galileo was consciously trying to do something else in his conceptualization of mechanical phenomena: he was trying to achieve a new layer of generality and simplicity but, above all, instead of paying attention to conditions that preserve the equilibrium of a balance, he was focussing on the *causes* behind the loss of this equilibrium, with his efforts being specifically oriented towards dynamics as a theory concerning heaviness and lightness. However, in Galileo's science of motion the centre of gravity still plays an important role.

In the early 1590s Galileo published the treatise entitled *Le Mecaniche*, which was intended as a clear and coherent account of statics, and only in its last few pages did he apply the results of his analysis to the revision of some crucial concepts of dynamics. More precisely, he underlined there the practical usefulness that can be abstracted from the theoretical analysis of the mechanical machine – or instruments of precision – which were ordinarily used by artisans in the sixteenth and seventeenth centuries. Galileo opened this treatise with an explanation of a mechanical phenomenon by introducing a proof of the long-established Archimedean law of the lever. Not only this proof, but indeed the entire work relies on this demonstration already given by Guidobaldo, or rather the explanations which made use of the interplay between the centre of gravity and the point of suspension on the one hand, and the relation between the centre of gravity and the tendency towards the centre of the world on the other hand. However, there are some methodological differences between the two authors. Let us first look at the proof through a summary borrowed from Marteen van Dyck:

"A uniform solid is suspended at its endpoints from a line *AB* which at its turn is suspended at the point *G* exactly in the middle (see Figure 4.1 below). It will be in equilibrium. Now divide the solid in two unequal parts, and add an extra string at the point of division. It remains in equilibrium, as it will also if we now hang it from two other strings right above the parts' respective centres of gravity and cut the other strings. At this point follows a geometrical proof of the fact that the ratio of the weights of the two unequal parts equals the ratio between the distances from which they are respectively suspended. Galileo then comments as follows: And from what has been said it seems to me clearly understood not only how the two unequal bodies *CS* and *SD* weigh equally when hanging from distances inversely proportional to their weights, but moreover how, in the nature of things, this is the same effect as if equal weights were suspended at equal distances, since in a certain sense the heaviness of the weight *CS* virtually spreads out beyond the support at *G*, and that of the weight *SD* shrinks back from it, as any speculative mind can understand by examining closely what has been said about the present diagram."[354]

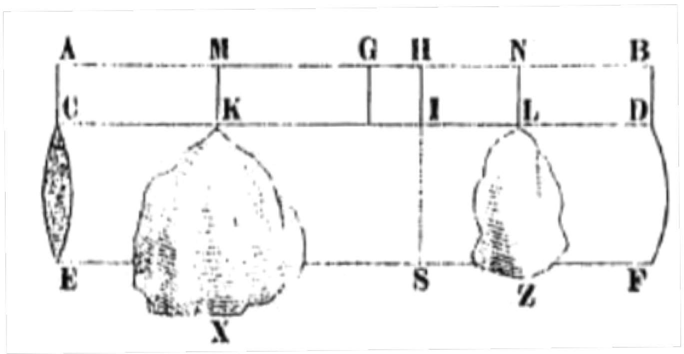


Figure 4.1

[354] van Dyck 2006, p. 140. For the complete demonstration, see the English trans. in Galilei 1960b, p. 155, and the Italian verion in Galilei 2005a, pp. 150-152.

The uniform solid *CF* is suspended at its endpoints from a line *AB* which at its turn is suspended at the point *G* exactly in the middle. It will be in equilibrium. Now divide the solid in two unequal parts *CS* and *DS*, and add an extra string at the point *I*. It remains in equilibrium, as it also will if we now hang it from two other strings right above the parts' respective centres of gravity at *K* and *L* and cut the other strings. It can easily be geometrically proven that the ratio between the distances *MG* and *GN* equals the ratio of the weights of the respective unequal parts.[355]

Our attention is immediately drawn to Galileo's use of the notion of *moment,* or *downward impetus*.[356] This notion plays an important role in Galileo's scientific writings and has subsequently been inherited by today's mathematical physics:

> Moment is the tendency to move downward [sc. with the downward impetus] caused not so much by the heaviness of the moveable body as by the arrangement which different heavy bodies have among themselves.[357]

From the demonstration of the law of the lever and this new definition, two aspects need to be underlined, respectively: i) the conclusion drawn by Galileo from the law of the lever, and ii) the causes that stand behind the loss of equilibrium. Galileo is offering his interlocutors a visual reasoning, teaching to the reader to see for himself what makes for an equilibrium in mechanical situations:

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[355] Galilei, *Opere* IV, p. 161.

[356] In Galileo' research there is no space for the notion of acceleration. He thinks that this is a sort of accident attributed to the moving bodies; however, there is the notion of impetus which represent a *vis motiva* exerting its tendency in an opposite manner to the force of gravity. More precisely, in the youthful treatise *De Motu*, Galileo established that the time taken by a body to descend along an inclined plane is equal to the square of the plane's length (*i tempi di discesa lungo i piani inclinati di uguale elevazione stanno tra loro come i quadrati delle lunghezze di tali piani*). However, Galileo also observed that this conclusion did not comply with empirical observations. The theory makes a prediction of the velocity of a body moving uniformly or downwards along an inclined plane; but on the other hand, when we introduce this ratio to the treatment of kinematic problems, the theoretical outcomes stand in contrast with the empirical observations. This discrepancy is due – in Galileo's conception – to the presence of the acceleration as a disturbing element, because i) it does not allow the user to grasp the "real" nature or behaviour of the bodies under consideration, and ii) it precludes any possibility of a purely mathematical explanation of the motion of heavy bodies. The purely mechanical and heuristic approach adopted in the youthful treatise drives Galileo to look at the acceleration as the core concept of a new kinematics, especially thanks to the analysis of the pendulum motion (in the *Discourses*), which is accelerated since it begins to swing, and only after a series of oscillation does its velocity decrease. Thus, it emerges that the acceleration plays a dominant role in the motion of bodies and it is no longer a simple accident.

[357] Galilei 1960b, p. 151.
*[…] Momento è la propensione di andare al basso, cagionata non tanto dalla gravità del mobile, quanto dalla disposizione che abbino tra di loro diversi corpi gravi […]*. Galilei 2005a, p. 147.

> One can see how the relative positions of the respective centres of gravity are responsible for the fact that the effect of the separate bodies' weights is distributed over space in such a way that they are conceptually reducible to a situation where a single body is hanging from its two end points. In this way one can see through the apparent marvelousness of this kind of situation and perceive the underlying and inherently stable configuration.[358]

As in Guidobaldo's proof IV of the first book of the *Mechanicorum Liber*, Galileo used the idea that the centres of gravity of the two different bodies hanging at the extremities of the balance may be imagined as being collected only in one point – *the centre of suspension of the balance*. Moreover, in that proof Galileo moved from a consideration of the conditions that ensure stability to the balance to multiple considerations on the causes of motion, asking 'What would happen if the two bodies *A* and *B*, situated at different distances on a balance, would start to move?' During the Renaissance, considerations about the equilibrium of a body that has been displaced from its original position by means of a force – or power – applied to a machine were justified as soon as the equilibrium of the body was explained. Indeed, if a balance is in equilibrium with a force *f*, a force *F* greater than *f* causes the displacement. However, the question 'By how much should *F* be greater than *f*?' was left open. In reply several interpretations can be given: for engineers *F* represented the force necessary to overcome frictions, while for mathematicians *F* stood substantially for a vanishing value (i.e. such that the value is negligible). As Danilo Capecchi writes, "[t]he equilibrium was seen from two points of view that clashed for a long time in the history of mechanics. On the one hand the equilibrium was considered between two actions that in 'potentiality' would have been able to produce contrasting displacements; on the other hand between two actions that in 'actuality' produce opposite displacements. They are however thought as imaginary or virtual, resulting thus from a power only partially in act."[359]

Moreover, there are other similarities and differences between Guidobaldo's and Galileo's explanations. Besides the dynamic interplay between the three centres,

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[358] van Dyck 2006, p. 140.
[359] Capecchi 2014, p. 3.

Guidobaldo's attempt to join the Archimedean tradition with the Aristotelian one was due on the one hand to a large humanistic project aimed at restoring the ancient science of mechanics, whilst on the other hand it relied on the dispute over the possibility of indifferent equilibrium in which he was already engaged. Thus, from the analysis above, it should appear clear to the reader that in Guidobaldo's mind the centre of gravity was of crucial importance for a rational organisation of his mechanics. Similarly, in the first version of Galileo's treatise *Le Mecaniche,* Guidobaldo's statement appeared again by means of a corollary in the following way:

> But it must be remarked that so much as we make it easier on ourselves using a lever, that much more time will we have to take; and that so much as the force will be less than the weight, that much larger will be the distance over which the force travels than the distance over which the weight travels.[360]

However, this passage also contains the point through which Galileo was attempting to take a new further step towards a discursive mechanical practice. But it was just while he was preparing the second version of the treatise *Le Mecaniche* (in the late 1590s) that he realized that the instrument of the virtual balance[361] could have been used for a different purpose, namely as a *general mechanical principle* and an *investigative* instrument:

> The utility which is drawn from this *instrumentis* not that of which common mechanics persuade themselves; that is, that nature comes to be overpowered and in a sense cheated.[362]

So, rather than thinking of a machine as an instrument whose only use is to shift the centre of gravity of bodies, Galileo now starts to think of it as an instrument by which to

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[360] Galilei 2002, p. 7. For critical analysis, see van Dyck 2006, pp. 143-145.
[361] As will also be noted by Galileo about all the other mechanical instruments.
[362] Galilei 1960b, p. 158.
*E qui si deve notare […] che la utilità, che si trae da tale strumento, non è quell ache i volgari meccanici si persuadono, ciò è che si venga a superare, ed inun certo modo ingannare, la natura […].* Galilei 2005a, pp. 157-158.

understand *what the causes are that move the bodies* at the extremities of the balance, how the force exerts this power, and how the balance could be used to *redistribute the moment*. Additionally in some passages of his works, he presents an analogy between the mechanical model of the balance as a visual representation of a practical instrument and the physical situations of bodies moving downwards along a rigid inclined plane. Given that the definition of moment as the tendency of a body to move downward is caused not so much by the heaviness of the moveable body as by the arrangement which different bodies have among themselves, in order to understand why and how a body will move at the extremity of the balance, we have to investigate what that tendency is. This tendency in the same work is defined as that force with which the mover moves the mobile and with which the mobile resists this motion, which force depends not only upon simple weight, but also upon the velocity of motion and the various inclinations of the spaces in which the motion occurs.[363] Therefore, the moment was conceived as the motive force that is exerted on the weight, and it is frequently used by Galileo to denote the action of a force.

The main problem deriving from this term 'moment', however, is to be found in the fact that it was originally a static concept (as is especially the case in Giovanni Battista Benedetti's interpretation, which – according to Mach – was an inspiration to Galileo on this occasion), which is now applied in a dynamic context. Indeed in *Le Mecaniche*, Galileo used the term as a purely dynamic concept, recalling that etymologically 'moment' derives from the Latin *movimentum* ('movement'). And yet in the *Dialogues*, the term *momento* (or *momento della potenza*) denotes the force in the static sense. This represents the turning point in the conception of force, because – as we have already said – Galileo shifts the attention from "why" to "how" the free falling of bodies could be conceived and explained, by looking for a more general principle from which the laws of free fall could be deduced.[364]

A step forward was reached in Galileo's mature work, the *Two New Sciences*,

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[363] Galilei 1960b, pp. 151-52.
[364] For more thoroughgoing investigation on the evolution of the notion of *momentum*, see Galluzzi 1979 and Jammer 1957.

which contains everything that Galileo had to say on the subject of physics. As Stillman Drake has written, “[i]n Galileo’s day there was no such profession as that of physicist. The role of the theoretical physicist was played by the philosopher. By temperament and tradition, the philosopher liked to generalise and was not unduly perturbed by apparent anomalies; indeed, he welcomed them as things to explain, or at any rate to explain away. The role of the experimental physicist, to the extent that it was played at all, fell to craftsmen, artisans, and mechanics. But philosopher and mechanics did not work on the same team, nor was there any apparent reason why they should. Consequently, there was a highly developed technology, which was generally not even noticed by philosophers, let alone integrated with their physics. Philosophers knew how physical objects ought to behave, and cared relatively little if they didn’t always seem to behave that way; craftsmen knew how objects behaved and cared relatively little for theoretical explanations. Though both were deeply concerned (each in his own way) with precision, neither habitually associated that with mathematics.”[365] However, Galileo represented the exception in his time, since his disposition was almost perfectly balanced between the two extremes described above; in fact he liked particularly to carry out observation on natural phenomena and generalise about them, since he saw mathematics as a *common ground* of the two demands for precision.[366] This attitude was certainly helpful, if not absolutely necessary, to the birth of modern physical science, namely that someone should formulate mathematical laws without waiting for their precise experimental confirmation. The last work published by Galileo in 1638, which contains much of his scientific results on physics over the preceding thirty years, may thus be considered a manifesto of this emerging mathematical tendency. The *Two New Sciences* is divided into Four “Days”, each of which addresses a different area of physics. In the First Day Galileo focuses on topics deriving from Aristotle’s *Physics* and

[365] Drake 1970, p. 68.

[366] Drake mentions that this kind of temperament was quite exceptional in that century. For example, Marin Mersenne represents the extreme of the critical temperament and was much more involved in proving that his experiments with falling bodies departed from the mathematical laws confidently announced by Galileo. In contrast, René Descartes privileged the route of generalization over concern for precise experimental observations and went in the wrong direction in formulating the general laws of physics, such as the laws of impact.

Hellenistic mechanics, and of course also provides an overview on the "two new sciences". The topic of the resistance of bodies to separation leads into a wide discussion on *infinites* and the *continuum*, the composition of matter, atoms and the existence of void. The Second Day's discussion deals with the strength of materials, and the cause of cohesion. Both these arguments rely on the well-known Archimedean law of the lever, and the mechanical instrument of the balance is used to support Galileo's experimental and theoretical speculations. Almost all the discussion of the Second Day contains the main outcomes stated in the early treatise *Le Mecaniche*. After this, both the Third and the Fourth Days focussed on dynamics and the science of motion. In particular, the Third Day discusses uniform and naturally accelerated motion and is shaped on the basis of the early treatise *De motu locali*. It is also divided into three parts: i) *De motu aequabili*, ii) *De motu Naturaliter accelerato* and iii) *De motu violento*. The Fourth Day calls attention to the projectile motion, and the treatise entitled *De motu violento seu de proiectis*. In the first of these, uniform steady motion is defined as "the one in which the distances traversed by the moving particle during any equal intervals of time, are themselves equal"[367], while the uniformly accelerated motion is motion in which the speed increases by the same amount in increments of time. In the latter day, the motion of projectiles is defined as a combination of uniform horizontal motion and a naturally accelerated vertical motion which produces a parabolic curve.

This mathematically rigorous method prevails chiefly in the exposition of the last two days, and the whole treatise aims at defining the two new sciences, the strength of materials and the motion of objects. In this way mechanics is no longer only the science either of equilibrium or of statics, but also refers to the motion and the causes of bodies' motion.

In the first place we can now examine a demonstration which is considered an added fragment of the treatise at hand, but which cannot be found in any printed edition of it because, according to Raffaele Caverni, there are no suggestions from Galileo's successors and scholars to dating it and reorganising it within the entire exposition of

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[367] Galilei 1914, p. 154.

the *Discourses*. But because the methodology adopted in the demonstration sounds very similar to the one found in the youthful treatise *Le Mecaniche*, I have decided to present it first:

> *Sagredo*: Let a balance having unequal arms be suspended at the point *C*, with *AC* greater than *CB*, and let us look for the reason why, although we have two equal weights *A* and *B* at the ends of the balance, the balance will not remain in equilibrium once it is deflected from its horizontal position, but rather moves downwards towards the greater arm reaching the *EF* position (see Figure 4.2).
> The reason that is commonly attributed to this reaction is because the weight of velocity *A*, when it comes down, would be greater than the speed of weight *B*, because *CA*'s distance is greater than the distance *CB*. Given that the movable *A* is equal to the *B* in weight, it will exceeds it at the moment of speed, so *A* prevails *B*, and the former (*A*) goes down by lifting the latter (*B*).
> We doubt the value of that reason, which seems to have no strong conclusion: it is true that the moment of a weight is increasing, together with the speed of the movable, with respect to the moment and the velocity of another weight that is made in quiet; however, given that both of the weights are at rest, we are not able to see how this kind of behaviour is even possible; And I really feel a noticeable difficulty.
>
> *Salviati*: Your Lordship has very good reason to doubt; and I still am not satisfied with this kind of argument. I found myself silent for another very simple and expedient direction, without the support of anything other than the very first and common notion, that heavy things go down in all the ways that they are allowed. When you put two equal weights in the balance *AB*, if you let it go freely, it will fall to the centre of the heavy things, always maintaining the centre of its gravity (which is the middle point *D*) in the line that goes to the centre of the world; but if you run into a bump under the centre *D*, the motion of the balance will be stopped. So let the balance in equilibrium in its position. But if the obstruction gets out of centre *D*, as it would be in *C*, such an obstruction will not stop the balance, but will divert the centre *D* from the perpendicular towards which it was walking and make it down by the *DO* arc.
> In this way the balance with the two weights is one and only one heavy body, whose centre of gravity is the point *D*, and this only body will fall as far as it can, and its fall is governed by its own centre of gravity: the centre *D* falls in *O*, and so what goes down is the whole system of weight composed of the two initial weight *A* and *B* and the balance too. The answer to the question why the inclined balance moves downwards to the side of the heaviest body is that an entire weight comes

down and approaches what can be the common centre of all the graves (or centre of the world).[368]

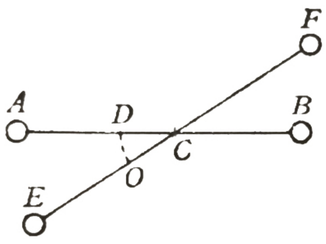


Figure 4.2

What should be noticed from this passage is that, first of all, it resembles the arguments used at the beginning of *Le Mecaniche,* which in turn show some affinities with Guidobaldo's abstraction. In particular, the passage maintaining "[…] the centre *D* falls in *O*, and so what goes down is the whole system of weight composed of the two initial

[368] As stated above, this fragment cannot be found in the printed version of the 1638 treatise, so there is no official English trans. of it. Galilei 1831, p. 154. The Italian text runs:

Sagr. *Sia sostenuta nel punto C la libra di bracci diseguali, AC maggiore, CB minor; cercasi la cagione onde avvenga che, posti nell'estremità due pesi eguali, A, B, la libra non resti in quiete ed equilibrio, ma inclini dalla parte del braccio maggiore, trasferendosi come in EF. La ragione che comunemente se ne assegna è perché la velocità del peso A, nello scendere, sarebbe maggiore della velocità del peso B, per essere la distanza CA maggiore della CB; ode il mobile A, quanto al peso, eguale al B, lo supera quanto al momento della velocità, e però gli prevale, e scende sollevando l'altro. Dubitasi circa il valore di tal ragione, la quale pare che non abbi forza di concludere: perché è ben vero che il momento di un grave si accresce, congiunto con velocità, sopra il momento di un grave eguale che sia costituito in quiete; ma che, posti amendue in quiete, cioè dove non sia pur moto, non che velocità maggiore di un'altra, quella maggioranza che non è, ma ancora ha da essere, possa produrre un effetto presente, ha qualche durezza nel potersi apprendere; ed io veramente ci sento difficultà notabile.*

Salv. *V. S. ha molto ben ragione di dubitare; ed io ancora non restando ben soddisfatto di simile discorso, trovai di quietarmi per un altro verso molto semplice e speditivo, senza suppor niente altro che la prima e comunissima nozione, cioè che le cose gravi vanno all'in giù in tutte le maniere che gli viene permesso. Quando nella libbra AB voi ponete due pesi eguali, se voi la lascerete andare liberamente, ella se ne calerà al centro delle ose gravi, mantenendo sempre il centro della sua gravità (che è il punto di mezzo D) nella retta che da esso va al centro universale; ma se voi a cotal moto opporrete un intoppo sotto il centro D, il moto si fermerà, restando la libra con i suoi pesi in equilibrio; ma se l'intoppo si metterà fuor del centro D, come sarebbe in C, tale intoppo non fermerà la bilancia, ma devierà il centro D dalla perpendicolare per la quale camminava, e lo farà scendere per l'arco DO. Insomma la libbra con i due pesi è un corpo ed un grave solo, il cui centro della gravità è il punto D, e questo solo corpo grave scenderà quanto potrà, e la sua scesa è regolata dal centro di gravità suo proprio: quando voi gli supponete il sostegno, il centro D cala in O, e così quel che scende è tutto il corpo aggregato e composto della libra e i suoi pesi. La risposta, dunque, propria alla interrogazione perché inclina la libbra etc. è perchè, come quella che è una mole solo, scende e si avvicina quanto può al centro comune di tutti i gravi.*

weight *A* and *B* and the balance too […]"[369] makes an explicit use of the point mass model as a representation for the entire system of the balance. This argument supports exactly our main hypothesis, inasmuch it might be used to maintain that in this period of transition (between the Renaissance and the modern period) the abstraction method, which stands behind the notion of the material point, is a customary and explicit procedure. This passage represents one more proof in support of the controversy triggered by Luca Valerio in his *Quadratura Parabolae per simplex falsum*. The mechanical procedures commonly used by Archimedes in the Greek period and usually referred to in a geometrical context now bring to bear in the field of the science *de ponderibus*, with the result that the mechanical instruments are used to represent not only artisanal and practical evidence, but also natural phenomena that can be simplified by using representations which have a mechanical structure.

At the beginning of the second day of the *Discourses,* after having established the condition of equilibrium, Galileo lets his characters deal with the following issue:

> *Salviati*: […] [I]f what preceded is clear [sc. referring to the condition of equilibrium of two weights suspended at a certain equal distance from the centre of suspension of the balance], you will not hesitate, I think, to admit that two prisms *AD* and *DB* are in equilibrium about the point *C*, since one half of the whole body *AB* lies on the right of the suspension *C* and the other half on the left; in other words, this arrangement is equivalent to two equal weights disposed at equal distances. I do not see how anyone can doubt, if the two prisms *D* and *DB* were transformed into cubes, spheres, or into any other figure whatever, and if *G* and *F* were retained as points of suspension, that they would remain in equilibrium about the point *C*, for it is only too evident that a change of figure does not produce a change of weight so long as the mass [quantity of mass] does not vary. From this we may derive the general conclusion that any two heavy bodies are in equilibrium at distances which are inversely proportional to their weights.
>
> This principle established, I desire, before passing to any other subject, to call your attention to the fact that these forces, resistance, moments, figures, etc., may be considered either in the abstract, dissociated from matter, or in the concrete, associated with matter. Hence the properties which belong to the figures that are merely geometrical and non-material must be modified when we fill

[369] *"[…] il centro D cala in O, e così quel che scende è tutto il corpo aggregato e composto della libra e i suoi pesi […]*".

these figures with matter and therefore give them weight. Take, for example, the lever *BA* which, resting upon the support *E*, is used to lift a heavy stone *D*. The principle just demonstrated makes it clear that a force applied at the extremity *B* will just suffice to equilibrate the resistance offered by the heavy body *D*, provided this force [*momento*] bears to the force [*momento*] at *D* the same ratio as the distance *AC* bears to the distance *CB*; and this is true so long as we consider only the moments of the single force at *B* and the resistance at *D*, treating the lever as an immaterial body devoid of weight. But if we take into account the weight of the lever itself – an instrument which may be made either of wood or of iron – it is manifest that, when this weight has been added to the force at *B*, the ratio will be changed and must therefore be expressed in different terms. Hence before going further, let us agree to distinguish between these two points of view; when we consider an instrument in the abstract, i.e. apart from the weight of its own material, we shall speak of "taking it in an absolute sense" [*prendere assolutamente*]; but if we fill one of these simple and absolute figures with matter and thus give it weight, we shall refer to such a material figure as a "moment" or compound force! [*momento o forza composta*]

*Sagredo:* I must break my resolution about not leading you off into a digression; for I cannot concentrate my attention upon what is to follow until a certain doubt is removed from my mind, namely, you seem to compare the force at *B* with the total weight of the stone *D*, a part of which – possibly the greater part – rests upon the horizontal plane: so that … –

*Salviati*: I understand perfectly: you need to go on further. However please observe that I have not mentioned the total weight of the stone, and [this] varies with its shape and elevation.

*Sagredo*: Good: but there occurs to me another question about which I am curious. For a complete understanding of this matter, I should like you to show me, if possible, how we can determine what part of the total weight is supposed by the underlying plane and what part by the end A of the lever.

*Salviati*: The explanation will not delay request. In the accompanying figure, let us understand that the weight having its centre of gravity at *A* rests with the end *B* upon the horizontal plane and with the other end upon the lever *CG*. Let *N* be the fulcrum of a lever to which the force [*potenza*] is applied at *G*. Let the perpendiculars, *AO* and *CF*, fall from the centre *A* and the end *C*. Then, I say, the magnitude [*momento*] of the entire weight bears to the magnitude of the force [*momento della potenza*] at *G* a ratio compounded of the ratio between the two distances *GN* and *NC* and the ratio between *FB* and *BO*. Lay off a distance *X* such that its ratio to *NC* is the same as that of *BO* to *FB*; then, since the total weight *A* is counterbalanced by the two forces at *B* and at *C*, it follows that the force at *B* is to that at *C* as the distance *FO* is to the distance *OB*. Hence [*componendo*] the sum of the force *C* at band *C*, that is, the total weight *A* [*momento di tutto il peso A*], is to the force at *C* as the line *FB* is to the line *BO*, that is, as *NC* is to *X*: but the force [*momento della potenza*] applied at *C* is to the force applied at *G* as the distance *GN* is to the distance *NC*; hence it follows [*ex aequali in proportione perturbata*[370]] that the entire weight *A* is to the force applied at *G* as the distance *GN* is to *X*. But the ratio of *GN* to *X* is compounded of the ratio of *GN* to *NC* and of *NC* to *X*, that is, of *FB* to *BO*; hence, the weight *A* bears to the equilibrating force at *G* a ratio compounded of that of *GN* to *NC* and of *F* to *BO*: which was to be proved.[371]

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[370] Having been affected (or disrupted) in equal proportion.

[371] Galilei 1914, pp. 112-115.

Salv. *Inteso sin qui, non credo che voi porrete difficoltà in ammettere che i due prismi AD, DB facciano l'equilibrio dal punto C, perchè la metà di tutto 'l solido AB è alla destra della sospensione C, e l'altra metà dalla sinistra, e che così si vengon a rappresentar due pesi eguali disposti e distesi in due distanze eguali. Che poi li due prismi AD, DB ridotti in due dadi, o in due palle, o in due qual'altre si siano figure (purchè si conservino le sospensioni medesime G, F), seguitino di far l'equilibrio dal punto C, non credo che sia alcuno che ne possa dubitare, perchè troppo manifesta cosa è che le figure non mutano peso, dove si ritenga la medesima quantità di materia. Dal che possiamo raccor la general conclusione, che due pesi, qualunque si siano, fanno l'equilibrio da distanze permutatamente respondenti alle lor gravità. Stabilito dunque tal principio, Avanti che passiamo più oltre devo metter in cosiderazione come queste forze, resistenze, momenti, figure, etc., si posson considerar in astratto e separate dalla materia, ed anco in concreto e congiunte con la materia; ed in questo modo quelli accidenti che converranno alle figure considerate come immateriali, riceveranno alcune modificazioni mentre li aggiugneremo la materia, ed in consequenza la gravità. Come, per esempio, se noi intenderemo una leva, qual sarebbe questa BA, la quale, posando su 'l sostegno E, sia applicata per sollevare il grave sasso D, è manifesto, per il dimostrato principio, che la forza posta nell'estrmità B basterà per adequare la resistenza del grave D, se il suo momento al momento di esso D abbia la medesima proporzione che ha la distanza AC dalla distanza CB; e questo è vero, non mettendo in considerazione altri momenti che quelli della semplice forza in B e della resistenza in D, quasi che l'istessa leva fusse immateriale e senza gravità: ma se noi metteremmo in conto la gravità ancora dello strumento stesso della leva, la quale sarà talor di legno e talvolta anco di ferro, è manifesto che, alla forza in B aggiunto il peso della leva, altererà la proporzione, la quale converrà pronunziare sotto altri termini. E però, prima che passar più oltre, è necessario che noi convenghiamo in por distinzione tra queste due maniere di considerare, chiamando un prendere assolutamente quello quando intenderemo lo strumento preso in astratto, cioè separato dalla gravità della propria materia; ma congiugnendo con le figure semplici ed assolute la materia, con la gravità ancora, nomineremo le figure congiunte con la materia momento o forza composta.*

Sagr. *È forza ch'io rompa il proposito che vevo di non dar occasione di digredire; ma non potrei con attenzione applicarmi al rimanente, se non mi fusse rimosso certo scrupolo che mi nasce; ed è questo: che mi pare che V. S. faccia comparazione della forza posta in B con la total gravità del sasso D, della qual gravità mi pare che una parte, e forse forse la maggiore, si appoggi sopra 'l piano dell'orizonte; sì che …*

Salv. *Ho inteso benissimo, V. S. non soggiunga altro ma solamente avverta che io non ho nominata la gravità totale del sasso, ma ho parlato del momento che egli tiene ed esercita sopra 'l punto A, estremo termine della leva BA; il quale è sempre minore dell'intero peso del sasso, ed è, variabile secondo la figura della pietra e secondo che ella vien più o meno sollevata.*

Sagr. *Resto appagato; ma mi nasce un altro desiderio, che è, che per intera cognizione mi fusse dimostrato il modo, se vi è, di poter investigare qual parte sia del peso totale quella che vien sostenuta dal soggetto piano, e quale quella che grava su 'l vette nell'estremità A.*

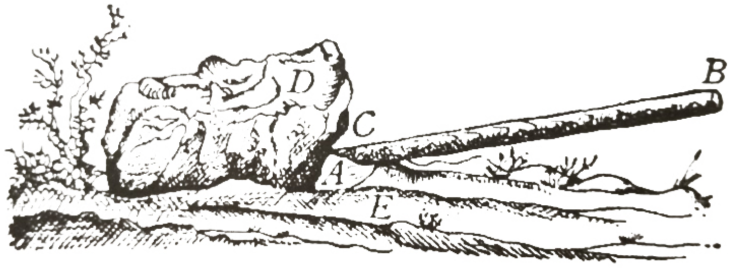


Figure 4.3

This passage has, first of all, something in common with the previous one, namely that it deals with the new notion of moment, which refers to the action of a body on a mechanical system, or more precisely, to the tendency of moving downwards composed of heaviness, position and everything else by which this tendency may be caused. Apropos of this aspect, two features need to be emphasised: i) each body in the demonstrations is replaced with a material point detached of its natural weight or its volume, so that it becomes an heavy point mass – the weight is imagined to be concentrated in the centre of mass; and ii) on each body moving downwards a specific force exerts its tendency, and it is equally distributed over the entire surface of the body; however, the force is imagined to be directed only on the centre of gravity – or the centre of pressure – of the moving body. Therefore, it is clear that Galileo's thought is perfectly in harmony with the entirety of sixteenth-century mechanics whose background already hinged on the causes of motion. However, the precise definition of

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Salv. *Perchè posso con poche parole dargli sodisfazzione, non voglio lasciar di servirla. Però, facendone un poco di figura, intenda V. S. il peso il cui centro di gravità sia A, appoggiato sopra l'orizonte co 'l termine B, e nell'altro sia sostenuto col vette CG, sopra 'l sostegno N, da una potenza posta in G; e dal centro A e dal termine C caschino, perpendicolari all'orizzonte, AO, CF: dico, il momento di tutto il peso al momento della potenza in G aver la proporzion composta della distanza GN alla distanza NC e della FB alla BO. Facciasi, come la linea FB alla BO, così la NC alla X: ed essendo tutto il peso A sostenuto dalle due potenze poste in B e C, la potenza B alla C è come la distanza FO alla OB; e componendo, le due potenze B, C insieme, cioè il total momento di tutto 'l peso A, alla potenza in C è come la linea FB alla BO, cioè come la NC alla X: ma il momento della potenza in C al momento della potenza in G è come la distanza GN alla NC: adunque, per la perturbata, il total peso A al momento della potenza in G è come la GN alla X. Ma la poroporzione di GN ad X è composta della proporzione di GN ad NC e di quella di NC ad X, cioè di FB a BO; adunque il peso A alla potenza che lo sostiene in G ha la proporzione composta della GN ad NC e di quella di FB a BO: ch'è quello che si doveva dimostrare*. Galilei 2005b, pp. 684-687.

*force* as a gravitational force that causes the falling of bodies cannot be attributed to Galileo himself. He in fact studied the kinematic aspects of motion, a motion originated mainly in a constant force, without delving into the nature of force in itself.

Within this historical framework Galileo showcases some anticipation of the development of what we define as force and what we define as work. In fact, a differentiation between the two concepts had already appeared clearly in the introductory part of Galileo's *Le Mecaniche*[372], but it was with Descartes and his reflection on the operations of simple machines that this distinction was made fully explicit.[373]

A second aspect that emerges from the last passage relates to the idea that machines are no longer characterised by the fact that they conserve an abstract quantity, or that they should be used practically by artisans or engineers – as it was said in the introduction of *Le Mecaniche* – but that they are built and interpreted as idealized and representative devices that could be used to *cheat nature*. In the statement: "[…] forces, resistance, moments, figures, etc., may be considered either in the abstract, dissociated from matter, or in the concrete, associated with matter. Hence the properties which belong to the figures that are merely geometrical and non-material must be modified when we fill these figures with matter and therefore give them weight", we can find the

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[372] Galilei 1890-1909b, pp. 147-190.

[373] As far as the classical concept of force is concerned, Galileo's contribution to the development of the notion of force is related to the one given by Kepler. While the latter arrived at his concept as a result of his astronomical investigations, Galileo studied the kinematic aspects of motion originated by a constant force without examining properly the nature of force itself. Moreover, he rejected any conjecture about the true essence of force, so he avoided any purely metaphysical considerations. Rather Galileo declares in his *Discourses* (1638) that force is first and foremost a physical concept. In the Third Day of the *Dialogue* he furthermore grappled with an intuitive notion of force and sought an exact formulation: he gave 'force' by definition an "agency that caused unnatural motion and is a sort of intruder in the otherwise harmonious system of natural process." In sixteenth-century mechanics *forza* (force) was used as a synonym, and in his writing we can find more than just two terms, such as *forza*, *potenze*, *virtù*, *possanza*, *momento della potenza*, etc. In his works *impeto* is widely used to express the instantaneous action of a force and is taken for what we call "impulse"; however, his favourite term is *momento* (a notion which is already analysed in the current section). From the analysis of Galileo's whole corpus, it emerges that he was one of the first thinkers to compare and distinguish two different concepts: i) muscular force (the impelling force or the impetus) and ii) the force of gravity. "In fact, when the stone, thrown upwards, attains its highest position, its momentary state of rest is for Galileo an indication that these two forces, the 'impressed impetus' and the 'weight of a body', are in equilibrium". In his *Dialogues*, where this twofold definition could be found, Galileo comes very near to the modern notion of force. For further details, see Jammer 1957, pp. 94-115.

kernel of model-building practice which is not limited to the model of balance, but is extended even to the simplest bodies in nature. It is likely that this transition, and this new way of observing natural phenomena, was suggested to Galileo by his reformulation of the treatment of mechanical phenomena by means of the concept of moment.[374] Writers like Guidobaldo had previously maintained that mechanical phenomena stood *outside nature*, or that technicians and craftsmen worked in opposition to the laws of their disciplines, even though they did not claim that they were able to overstep the boundaries of what was possible. Rather they merely indicated their awareness of the Aristotelian way of identifying objects by the origin of their principles of coming into existence and being organised. Yet, on the contrary, when Galileo states that it is impossible to achieve any effect that is "outside the constitution of nature", he is trying to ascertain the boundaries between the possible and the impossible, between what is natural and what is not.

The new forward step that Galileo took by following this approach consisted in the idea that machines – above all the classical balance – were conceived as instruments for building discursive practices over artisanal ones. These instruments, the model of balance and the point mass model are investigative tools used to confer a discursive structure onto the phenomenal world. This process takes the name of model-building practice, suggested by a free play of the imagination, and thus machines were only constrained by the intentions that must be put into practice. Galileo hereby regiments this play of the imagination through his abstract analyses; he is effectively stressing that certain things are "absolutely impossible to accomplish with any machine imagined *or imaginable*".[375]

In Galileo's thought we can find two different steps of the idealization procedure. The first relates to single bodies, in which he has his mouthpiece Sagredo says, that the change of shape of the body hanging at the extremity of the balance does not have any

[374] van Dyck 2006, p. 144.
[375] Galilei *Opere* II, pp. 156-157. See also Galilei 2002, p. 46.

influence on the loss of equilibrium.[376] Thus, the representation given by the theory stated by Galileo does not change whether the bodies are transformed into cubes, spheres, or any other figures; this means that any figure can also be represented by a simple geometrical point made devoid of its weight intended as volume. The second step deserving of mention relates to the idea that the balance or lever in itself henceforth became an instrument devoid of natural properties, and was used to represent any other phenomena that resemble, for some properties, the balance in itself. To be more clear, we should stress this point again: whether in the second half of the sixteenth century the centre of gravity represents an idealized instrument that stands for the bodies hanging at the extremities of the balance, or whether the idealization procedure was restricted to rigid bodies only, at the turn of the new century, "the *identity* of a machine no longer lies in its functional organisation of material to a specific end, but in the fact that it is a closed system that conserves the amount of moment that is put into it. It is the unity of nature rather than the intention of men that constitutes their ontological character."[377]

The machines being no longer useful tools for attaining specific practical purposes, they enter now into the new century as investigative tools, and in a parallel development, the centre of gravity becomes a general and independent intelligible entity used for analysing and solving *purposes extrinsic to their specific functioning*. The way in which they behave becomes a general method used to represent a large variety of natural phenomena, and by understanding the pragmatic functioning of the balance – as Renaissance mechanics was perfectly able to do – we can understand a greater variety of phenomena that follow almost the same physical principles by the weighing practice. The balance in turn becomes a means for investigating the working process of nature, as a representational model of what is going on in the phenomenal world. In this sense, Galileo has to be considered the one who gave to those practical tools – indeed to machines in general – the role of investigative tools, rather than merely a means of

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[376] Here there ought to be pointed out the difference in meaning between force and weight. As Jammer maintains in his book on the concept of force, in Galileo's early writing the concept of force is nearly equivalent to weight. This is an inheritance from the Archimedean conception, according to which force and weight can be seen as a natural inclination of a body to move nearer to the centre of the world.
[377] van Dyck 2006, p. 148.

representation. However, as he himself says in the Fourth Day of his *Discourses,* they (i.e. machines) should be considered as tools that not only represent the physical reality, but even stand for it, or rather that they do not depart from the physical reality in itself, no matter how idealized they are. We can always add back the properties that have undergone idealization in order to bring those models closer to the particular phenomenon at hand.

> What happens in the concrete […] happens the same way in the abstract. It would indeed be surprising if computations made in abstract number did not thereafter correspond to actual gold and silver coins and merchandise. Did you know what does happen, Simplicio? Just as the accountant who wants his calculations to deal with sugar, silk and wool must subtract the boxes, bales, and other packings, so the mathematical physicist, when he wants to recognize in the concrete the effects which he has proven in the abstract, must deduct the material hindrances; and if he is able to do that, I assure you that matters are in no less agreement than for arithmetical computations. The sources of errors, then, lie not in the abstractness or concreteness, not in geometry or physics, but in a calculator [sc. as the user of this method] who does not know how to make a true accounting.[378]

For Galileo the question of how much an abstract model, designed on the basis of the mechanistic operations of craftsmen, departs from reality does not make any sense, because he was completely confident in the idea that nature in itself is written in a mathematical language and codes.

For these reasons, by understanding how the balance acquired representational power, we can also understand how the geometrical centre of gravity could be conceived as a point mass model that stands for natural bodies, no matter how they are shaped and no matter whether they are at rest or in motion. The point mass could thus be considered as a sort of preliminary notion compared with the balance, because, before entering into the analysis of complex phenomena, or a state of affairs involving a system of moving or accelerated rigid bodies, we should understand the behaviour of single simple bodies.

The balance as a machine is, in Galileo's time, thought of as exemplifying

[378] Galilei 1914, pp. 207-208.

inviolable invariance, and human intentions have become extrinsic to their ontological identity, so that the idea of using mechanical tools in investigating natural principles also loses much of its paradoxical character. While in Luca Valerio's treatises there are worries about the legitimacy of the attribution of purely mathematical notions to geometrical entities, now with Galileo, when it is admitted that the basic principles of this new mechanical science express the limits of our manipulative capabilities, it becomes natural to investigate these exactly through manipulations. It is through our way of interacting with nature that nature now can truly show itself for the first time. This allows mathematical instruments that had been primarily practical problem-solving tools now also to function as investigative tools.[379] Galileo thus regiments the model-building practice, especially in the *Two New Sciences,* in which Simplicio, the spokesman for the Hellenistic and Scholastic tradition, and Salviati, Galileo's mediator, debate the techniques of idealization that underlie the proposed 'new science' of mechanics. The former strongly objects to these techniques, because they tend to falsify the real world which is not regular, as the idealized laws would make it seem, but rather complicated and messy. The latter, on the contrary, claims that by choosing the right set of conditions we can reach the right way of representing the complexity of the empirical world. By choosing the invariances that underlie the phenomena taken into account, we can create isolated subsystems which have the role of representing the empirical phenomena at hand. There is, of course, a set of conditions that need to be followed: crucially, "one has to choose the right level of abstraction."[380]

Galileo was able to achieve a level of isolation of real properties and thereby take them under the same approximated closed systems (closed in the sense that the representational system which we are building up is constituted by a series of similar and stable properties behaving analogously). This is the preliminary commitment to the development of full-fledged mathematical theories[381], and it represents the second key issue of his research:

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[379] Bennett 1986, p. 2.
[380] van Dyck 2006, p. 153.
[381] For a detailed analysis see van Dyck 2006, chapters 6 and 7, and McMullin 1985, pp. 247-73.

> This way of proceeding actually installs the specific interplay between universality and locality that has become so specific for modern physical sciences. If one has been able, in a very specific and local situation, to isolate a sufficiently closed system that shows some stable behaviour, one can transfer the lessons learned from this behaviour to all similar situations. *And one can do this exactly because this stability expresses what lies outside our manipulative capabilities and hence must be ascribed to nature*.[382] [my italics]

According to Galileo, the model has to represent the invariances observed in natural phenomena. In his works, the virtual balance plays this very role, and the same has to be attributed to the pendulum system or the inclined plane. We can use these models as a system for helping scientists to build up the laws that stand behind nature. This led Galileo to make *abstract* mathematical representations of *concrete* physical events.

Irrespective of whether or not we can overcome ideally true propositions by purely material hindrances or by machines for Guidobaldo, for Galileo this natural limit cannot be violated, and machines cannot be used to violate nature. Guidobaldo is thus offering a picture of theoretical mathematics that is somewhat different from the one expounded by Galileo. The former is still too much related to practical and material considerations. Galileo, by contrast, introduced a frictionless fulcrum of a balance, a frictionless, inclined plane, or a pendulum which oscillates with the same frequency, without considering the fact that a real pendulum reduces its frequency of oscillations over time. When we move to Galileo's thought, the conditions under which something can count as a deviation from true principles apparently becomes changed. I propose to interpret this change as follows: *what for Guidobaldo was an invalid abstraction becomes an innocuous idealization for Galileo*. Guidobaldo was still too heavily anchored to mechanics, because his theory relies on the purpose of giving practical scope to the investigation of the phenomenon of equilibrium. He belonged to the century in which the main interest lay in practical physics, in making suppositions in order to build real artisanal machines, using the outcomes of his theory in order to build instruments of

[382] van Dyck 2006, p. 154.

precision. Galileo, and previously Luca Valerio, albeit in a less precise and mature way, both represent a change in the way of approaching mechanics and looking at the practice, because they wanted to build a purely mathematical form of physics and mechanics. They want to make sure that the laws can be applied to general phenomena. As van Dyck sees it, "Galileo sees a continuum from a fulcrum with friction towards an ideal fulcrum which only alters the way in which the precise relations show up in empirical reality."[383] Here, Galileo shows his awareness of the difference between a mechanical and a geometrical approach – linked to the name of his tutor – with an approach that is designed to offer the foundations of future mathematical physics:

> Experience bore Guido out in a sense, as some power *is* lost in actual simple machines; [...] [y]et Guido was in the habit of showing side by side material machines and schematic figures of them, and as a mathematician he should have been able to see the idealized truth. The fact that he did not is strong evidence that it is simpler for us to see this than it was for Galileo, who was the first to do so. Nor is this surprising; it was he who made it simpler for us.[384]

Regardless of whether Guidobaldo was able to see an idealized truth in the moving point along the downward motion of the balance's arms, Galileo interpreted this truth as a truth impacting on bodies and nature in itself. In fact he says that the truth expressed by this representation does not depart from reality, and has to be interpreted as the truth about reality in itself. This idea derives from the fact that throughout his career, Galileo was aware that nature was written in the abstract language of mathematics, and that any practical mechanism could be used in order to understand how nature works and behaves. Again, whilst Guidobaldo was using the moving point mass as an abstract body detached from its purely physical content, and whereas he was not confident in using this as a vehicle for truth concerning real-world phenomena, Galileo, on the other

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[383] *Ibid.*, p. 164.
[384] *Ibid.*, p. 165. See further the translator's footnote in Galilei 1960b, pp. 166-167, n. 24.

hand, was the advocate of the idea that idealization truly expressed the course of nature. Nature should hence be investigated by applying these general models; machines should assume a theoretical layer in order to be used as investigative tools for a general framework by which to study a series of phenomena through common instances.

Although the question of the extent to which the model departed from the real situation has already been introduced, we should focus more on the necessary answer for it. This is so especially because our claim already stands in favour of the idea that the conclusions which we as users draw from models are not suitable for truth, but rather that they are surrogative inferences which need to be adapted to the observations. By adopting the Galilean point of view, let us now consider the following comparison. The most peculiar property of a pendulum is its isochrony, i.e. whatever the amplitude given to a swing, the time it takes to execute that swing remain unchanged. As a consequence, the pendulum seems to be a particularly interesting closed system, *comparable with* but at the same time *interestingly different from* a balance. Most importantly, since any swing always starts from zero speed, isochrony is only intelligible if we take into account that any downward motion is accelerated, and that this acceleration moreover obeys precise proportions which make the overall time always come out as equal. No matter what was the precise historical chronology between his empirical discovery of isochrony and his mathematical derivation of the law of chords, it is clear that once Galileo had realized this connection, he was determined to see what could be learned from it concerning the proportions characterising all natural accelerations.[385]

The issue we attempt here to argue can be rephrased in the form of the following question: is the idealization something that allows us to observe the phenomenon under study in its ideal circumstances, where precise ratios can be discerned, or are we rather dealing with an illegitimate abstraction where we illicitly alter the scope of the theory (from natural to fictitious situations)? Why would the idealized situation teach us

[385] For more details, see van Dyck 2006, pp. 172-173; Machamer and Hepburn 2004, pp. 333-347 and Wisan 1974, p. 175.

something valuable about the real freefall of bodies, or about the real instrument of the pendulum, to enable us to claim that in a void they would fall with equal speeds, when we see arising clear differences in their speeds in all actual instances? Aren't we just dealing with a different kind of phenomenon? For example, in the First Day of the treatise, Galileo takes as a model the situation in which all the bodies fall freely in the void, since in this environment all bodies exhibit the same behaviour independently from any other factors. Here Galileo is only constructing a theoretical model with a certain grade of simplicity and generality; but in fact, there will never be the case of a body moving in the void. The achievement reached by Galileo is the following: it has been shown that the proper domain to model freefall mathematically is fall within a void, since in this environment *all* bodies will exhibit the same behaviour, independently from any other factors. Notice that he has not yet established the exact relations constituting such models; this will only be done in the Third Day of the *Discourses*, where the 'times squared' relation will find its place in an elaborated deductive structure built on the supposition of uniform acceleration. *That the models thus constructed will still be relevant for all actual occurrences of freefall is secured by his particular experimental procedure, which guarantees that the case of fall in a void is not merely the simplest case, but the most general*. There can thus be no question of an invalid abstraction. By isolating all the proper dissimilarities between different kind of bodies with respect to the phenomenon of freefall, it becomes possible for Galileo to attribute the presence of the "pure phenomenon" to *actually occurring* instances of freefall, even if these might show considerable deviations from the theoretical models. The model represents the pure phenomenon by means of the target system. He maintains that:

> No firm science can be given of such events [*accidenti*] of heaviness, speed, and shape, which are variable in infinitely many ways. Hence to deal with such matters scientifically, *it is necessary to abstract from them*. We must find and demonstrate conclusions abstracted from the impediments, in order to make use of them in practice under those limitations that experience will teach us.[386]

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[386] Galilei 1974, p. 225.

To this end he then tries to estimate the effect of air friction on different kind of bodies and under different conditions (again using the pendulum as an investigative tool).

In this last work Galileo is doing something strikingly new: he is learning something about the ideal case (the model) in which bodies exhibit their behaviour *from* the way in which they are disturbed by the presence of a medium (an interactive medium, i.e. matter). In order to do "science" Galileo maintains that we need to rely upon our conclusions on abstraction. Although he is still not giving a scientific treatment of the disturbances themselves, *he shows how to exploit their presence to epistemic ends*. Thus, Galileo shows how to retract a meaningful signal from the noisy real-world behaviour by looking at how signal and disturbances interact with each other.

In Galileo, we find a precursor of what in Bachelard and in van Dyck is called 'phenomenotechnique'; and the transition from a phenomenon to a phenomenotechnique is what makes possible a mathematical science of nature. His idealizations are instantiations of practice deriving from establishing or regimenting the use of imagination for building up rules for engaging with material objects. In Galileo's eyes, we can start to discern stable relationships (if they are to be found – this is of course never guaranteed) which can be modelled mathematically as constant ratios. The high level of applicability and management which every model shows allow us to fit them to a cluster of similar states of affairs. Moreover, this is the language of mathematics which is seen as an abstract layer that overlaps with nature in itself. The material point, from our point of view, is a kind of primary model which is used in order to build more complex models such as the model of the balance, and so on. These models' representative power relies on the fact that they can account for the exemplification of principles of *natural* philosophy and thus can be used to generate evidence for our physical theories. The ground for this representative power must be sought in the discursive function of nature, but crucially a further question arises from this issue: why can these concrete material objects be represented in turn on an abstract

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*De i quali accidenti di gravità, di velocità, ed anco di figura, come variabili in modi infiniti, non si può dar ferma scienza: e però, per poter scientificamente trattar cotal materia, bisogna astrar da essi, e ritrovate e dimostrate le conclusioni astratte da gl'impedimenti, servircene, nel praticarle, con quelle limitazioni che l'esperienza ci verrà insegnando*. Galilei 2005b, p. 779.

level through mathematical structures exemplified in geometrical diagrams or figures? Galileo has no methodological instruments to attempt to reply to this question, since the purely algebraic method is still far off, at least in its modern formulations.

### 4.2 The Modern Turns

In the second half of the seventeenth century Italy was no longer the main centre in Europe for the development of mathematics and physics as independent disciplines; this role was left mainly to France, the Netherlands and England. It was, however, the century labelled as the 'Golden Age'. The great variety of catalogues and archives collected during the Renaissance period were no longer of exclusive access to the Italian scholars; now the development of the printing press system, joined with the growth of the means of communication and transport, allowed for the rapid spread of knowledge, gaining in turn to an increased exchange of knowledge between scholars who had the same scientific interests. This was also a period of great and significant transformation in the fields of politics and religion, both of which underwent intense cultural and economic development.

The Netherlands, for example, promoted the dissemination of scientific knowledge through the creation of new schools at a local level. Higher studies were also enhanced and the University of Leiden[387], founded in 1575, became one of the most popular in Northern Europe. A special role for the development of mathematics was played by the surveyors, who faced complex problems for the preparation of reliable nautical and land charts required for the trade policies of the new state. This fervor of scientific activity set to root an important cultural tradition.

In England too the seventeenth century saw a revival of the sciences in general and these exact sciences in particular. It was not that there had not been great British scientists before this time, but, excepting William Harvey (1578–1657) and William

[387] In Leiden in 1638 the first *editio* of the *Discourses* was also published, which however failed to found any promoter in Italy, due to Galileo's excommunication from the Church.

Gilbert (1544–1603), these were sporadic cases. According to some historians, the real flowering of British science started around the 1640s, in particular in connection with the beginning of the Puritan Revolution, before the restoration of the British monarchy in 1658. Although there is no unanimity on this point, it is likely that a close connection could be attributed between the Puritan movement and scientific improvement.

The main attempt of this century was thus directed towards the solution of all the problems that afflicted Galileo. The mathematics of this century saw the birth and early development of the infinitesimal calculus, known as *Calculus* as the alternative to the existing mathematics. The very necessity of its development was certainly determined by the problems posed by physics, in particular by mechanics, which could be solved only in part by Hellenistic geometry or unsophisticated Renaissance algebra. For example, many much-discussed problems were related to the kinematics of accelerated motion, the determination of the maximum range of a cannon, the direction of reflection of the light incident on non-planar surfaces, and the determination of the centroids of solid figures having complex forms.

Still in the same century the foundation of the method of indivisibles was also generally attributed to Bonaventura Cavalieri (1598–1647), who posited a *fully geometric method* that was very fertile for calculating areas and volumes. He considered lines, surfaces and volumes as generated respectively by points, lines and surfaces that were added up together. Almost over the same years, Gilles Personne de Roberval (1602-1675) developed similar methods. Later on, Evangelista Torricelli (1608-1647), following in Cavalieri's footsteps, arrived upon some extremely interesting and apparently paradoxical results, including the demonstration that if one rotates a branch of a hyperboloid around an axis, despite the area subtended by the hyperboloid being infinite, the volume of the solid of rotation stays as finite. At this time, Galileo and Cavalieri had had systematic scientific relations through intense correspondence, since the latter was trying to obtain Galileo's approval for his method of indivisibles, but Galileo was always sceptical towards it. The fundamental idea in the *Geometria indivisibilibus continuorum nova quadam ratione promota* (1653) was that it is possible to compare two continua by comparing their indivisibles. This method allowed one to measure the area or volume of a plane or solid figure respectively as a continua of lines – or plane – parallel with each other. While Cavalieri adopted a geometrical conception and was sceptical towards a description of real-world phenomena in purely mathematical language, for Galileo the converse was true.

Unfortunately, the history of the developments of the method of indivisibles is too complex to be covered fully here, and it would also entail a dealing with both the history of seventeenth-century calculus and the definition of the notion of the infinitesimal. But given that this is not the right place to attempt such a reconstruction, we will instead choose in the next section (§ 4.2.1) to reconstruct Hobbes' attempt to regiment the role and process of abstraction and idealization from a more philosophical – and metaphysical – point of view.

### 4.2.1 Hobbes: Abstraction and Idealization as Acts of Mind

In the seventeenth and the first half of the eighteenth centuries an intense debate had been registered concerning the interrelationship between mathematics and philosophy. It involved figures such as the English philosophers John Wallis (1616-1703), Thomas Hobbes[388], Isaac Barrow (1630-1677) and George Berkeley (1685-1753). This debate

[388] Thomas Hobbes was born on April 5th 1588 in Malmesbury, and left the city in *ca*. 1602-1603 in order to study at Magdalen Hall, Oxford. After graduating from Oxford in February 1608, Hobbes went to work for the Cavendish family, initially only as a tutor to William Cavendish (1590-1628); here, he would work for most of the rest of his life, and his work as a philosopher was strongly influenced by the Cavendish family's background and acquaintance with other philosophers and scientists. Hobbes had also interacted with various prominent intellectual figures. On a trip around Europe in the mid-1630s, Hobbes met Marin Mersenne in Paris; there is here some evidence that he also met Galileo Galilei and that he worked as a secretary to Francis Bacon. Hobbes first made a notable impact in his philosophical writings in the early 1640s. These included his *Elements of Law* and *De Cive*. The *Elements of Law* is the first work in which Hobbes follows his typical systematic pattern of *starting with the workings of the mind and language,* and in turn, *developing the discussion towards political matters*. *De Cive* was meanwhile conceived as part of a larger work, the *Elements of Philosophy*, the first part of which turned out to be *De Corpore,* which deals with logic, language, method, metaphysics, mathematics, and physics. About this I will have more to say later on in this section. At this time Hobbes also had a series of interactions with

concerns the alleged merits of algebra – a blooming field of the new mathematical physics – and its struggle with the ranks of geometry. One main point of contention concerned one of the larger mathematical questions of the days: *our ability to accrue reasoning about signs without consideration of their meaning*. It was a debate that runs parallel to the problem of imaginary numbers, which similarly raised the question of reasoning about signs for which there were absolutely no corresponding ideas.

Accordingly, three main questions surrounding this period of transition were: i) Does the use of symbols really shorten mathematics? ii) Is the human mind able to reason about symbols, or do the mathematicians need to translate them back into appropriate prose expressions or into the ideas for which they originally stood? and iii) Can the mathematician draw accurate conclusions – those that we have previously labelled surrogative inferences – just by grounding them in symbolic expressions, as for instance $x^2 + y^2 = 9$, without recalling that the curve under consideration is a circle with all its common geometrical properties?

This later critical attitude was strongly advocated by Hobbes, according to whom the algebraic symbolism effectively wasted the mathematician's time by forcing him to translate his ideas into mathematical abstract symbolism and then to translate it back again in the form of prose.[389] Let us first clarify one point: from our perspective, Hobbes was not in opposition to the so-called process of abstraction or idealization, but he was doubtful about the idea of substituting the geometrical perspective and reasoning for

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Descartes: they probably met in 1648 but apparently did not get along very well. Descartes suggested that Hobbes was more accomplished in moral philosophy than in other disciplines, but also that he had wicked views there [Descartes 1643, 3.230-1], and Hobbes equally thought that Descartes would have been better off sticking to geometry. Hobbes spent the next decade in exile in Paris, leaving England late in 1640, and not returning until 1651. His exile was related to the civil wars of the time. During his time in France, Hobbes continued to associate with Mersenne and his circle, including Pierre Gassendi, who seems to have been a particular friend of Hobbes. Before moving back to England he wrote the *Leviathan*, which was published in 1651. The *De Corpore* was only published in 1655 and provides Hobbes's main statements on several topics, such as method and the workings of language. In his later years he took part in discussions with the mathematician John Wallis on the merits of geometry and algebra, and with Robert Boyle about the experimental physics of the Royal Society. Hobbes died on December 4th 1679 at Hardwick Hall.

For further biographical information, see D. Stewart, "Thomas Hobbes", The Stanford Encyclopedia of Philosophy (Summer 2017 Edition), Edward N. Zalta (ed.), forthcoming;
On-line: https://plato.stanford.edu/archives/sum2017/entries/hobbes/.

[389] Pycior 1987, pp. 265-286.

physical reality with the algebraic interpretation. In other words, according to Hobbes, geometrical symbols have representational power, and they are still sensible tools, with the difference that they should be considered by neglecting physical properties. Due to the powers of our imagination, we can abstract and idealize, with a simple act of omission, all the natural properties of the physical phenomena.

In fact, I suggest that the debate promoted by Hobbes now comes into direct contact with the one promoted by Valerio and Galileo, because the claim is now the following: how legitimate is it to draw conclusions about purely symbolic and theoretical representations, rather than about the objects they stand for? This issue took the name of the 'problem of symbolic reasoning', and it focused not only – as it had done for Valerio and Galileo – on the legitimacy of having symbolic representations of physical phenomena, but even on the epistemological meaning of the conclusions we draw about those symbols. Thus on some occasions, it also supplied hints that touched on the realistic/anti-realistic debate, and in fact in this respect Hobbes followed Galileo's footsteps in supporting the idea that the abstractions we use in geometry are *not* mind-dependent, so they do not depart from reality, but rather than it was only a matter of omission and simplification in order to make easier computations and find a form for representing the physical reality in itself.

Although his main interest lay with the analysis of human behaviour and social aspects of our society, Hobbes' ambition was to attempt an investigation following the scientific method. More precisely he aimed to apply the rigour and the foundational principles of geometry in his political and sociological investigations. On this basis, individuals were considered as inanimate objects subjected to motion, force and external power. For the sake of simplicity and in order to reach valid general conclusions on how we as human beings should behave in our society, it was in his eyes simpler to adduce reasoning about general statements and to build general laws on objects considered in their simplest form, being detached from their purely natural content.

It appears clear that, from Hobbes' perspective, bodies come in two varieties, the natural and the political, in the same way that philosophy is divided into the natural and the political. A single methodological approach with geometrical foundations can be used for both kinds of investigations.

At that time, Hobbes was also the author who went upstream with respect to the use of the *inertia principle*. In fact, whereas on the one hand he made only little use of the doctrine of inertia in his natural philosophy, on the other hand he had a lot to say about the foundations of geometry and mathematical physics, the differences between the two and the processes of abstraction and idealization that were flowering in that century. In particular, it seems that his doctrine of inertia helped him considerably in favour of the aim of this overall contribution.

Taking into account his entire corpus, we can see that Hobbes was throughout still anchored to a geometrical representation of physical reality. After all, he was also aware that his speculations were of a metaphysical nature instead of a physical, mathematical or a strictly mechanical one, like that of Galileo or Wallis. Yet the difference between Hobbes and Galileo lies in the fact that whereas the Galilean theory on idealizations originates from a technical interest and in part from the theoretical and epistemological leftovers of the Equilibrium Controversy, the Hobbesian thought was mainly driven by the desire to gain a certain ground for his political speculations. The intellectual ferment of the seventeenth and early eighteenth centuries provided Hobbes not only with elements of his mathematical philosophy but also with problems and insights which dominated his general philosophy. Hobbes and his contemporaries were involved in what we can characterise as the mathematical phase of the battle of the ancients against the moderns; his thought, and in particular the defence of the ancient thought expressed in the *Elementorum Philosophiae Sectio Prima de Corpore* published in 1655, made of him – within this framework – a maverick in geometry, insofar as he favoured geometry over arithmetic and synthesis over analysis. In summary, his inclination was in favour of a nominalistic position.

We will argue that whereas on the one hand, the Hobbesian conception has been considered a materialistic one because of the rejection of pure abstract mathematical reasoning, on the other hand his consideration of the line as a body whose length is considered without its breadth constitutes from our point of view an act of omission.

In 1655 Thomas Hobbes published the *Elementorum Philosophiae Sectio Prima de Corpore*, in which he sets out the principles, methods, and ends of his philosophy, together with his doctrines of the first part of that philosophy[390], which concerns bodies. Once he had established that the most fundamental and primary doctrines on bodies is geometry, which also illuminates Hobbes's views on metaphysics, logic, and philosophy of science, he stated that the foundations of science were to be found in the most universal treatment of the simplest bodies[391], of which others are merely component parts. This foundation of science is represented by geometry. As he frequently maintains throughout the treatise, *philosophia prima*, i.e. mathematics and physics, may be gathered together under the same *a priori* science, namely geometry.

To understand better the connection between our purpose and Hobbes' thought, let us first summarize his conception on bodies and motion. According to Hobbes, bodies are imagined as "that, which, having no dependence on our thought, is coincident or coextended with some parts of space."[392] Whether or not bodies are mind-independent, space is certainly not, because the latter is an imaginary entity; it is "the phantasm of a thing existing without the mind simply."[393] More precisely, besides imaginary space, there is a complementary conception of space labelled 'real space', which is roughly considered as the magnitude of some object just as it are measured in itself.

[390] As we can see in the *De Corpore*, Hobbes, following his contemporaries, used the term "philosophy" in a broad sense. Thus in this case philosophy is, roughly speaking, the working-out of the doctrines of logic and metaphysics to give knowledge of a particular sort. This knowledge may be divided up according to the categories of body, man, and commonwealth. One of the most interesting aspects of Hobbes' natural philosophy, which unfortunately cannot be analysed thoroughly in this section, is the "extension" of the geometrical, synthetic and deductive methods from the context of the strict scientific knowledge to the field of politics. Thus, Hobbes' idea is that, since we can use the rigid and coherent geometrical and synthetic methods in order to interpret and represent physical reality, the same should be done in other realms. In other words, the model given by scientific knowledge can be transferred to the foundation of political science. Geometry thus became a central discipline in two quite distinct ways: i) as a methodological guide and example, and ii) as the most basic of all branches of knowledge, from which "synthesis" might deduce, step by step, the immutable laws of social justice.

[391] One question remains open: are we committed to consider these "simplest objects" as geometrical points detached from any of their physical content?

[392] Hobbes 1839-45, Vol. IV, part II, ch. VII, §1.

[393] *Ibid.,* Vol. IV, part II, ch. VII, §2.

Thus, first we have the property that makes a body real, or rather its being mind-independent: a body is not simply an idea or a theoretical conception, so it is still a sensible tool. Secondly, we have the idea that what makes space imaginary is the converse; it is simply a conception and as such is dependent on some thought – i.e. it is the conception of extension, specifically the extension[394] of some body. The extension of a body is an abstraction and so, considered in itself, it is imaginary; since it is appraised as the extension of some body, it is real. Given that the extension of a body is not mind-dependent, the notions of real and imaginary space are not two sorts of space, but they rather overlap as if they were the same space. In other words, the extension of a body can be considered in two different ways: either i) as the magnitude of a (real) body, so that geometrical objects are instances of magnitude; or ii) as the abstracted extension (not necessarily with any body), which is thus existent only in thought (and which is hence imaginary).[395] Since space, whether it is real or imaginary, is given by the extension of bodies, the latter overlaps at least with a part of that space.

Having defined bodies, Hobbes is now in a position also to define the elements of geometry. A *point*, he says, is a body considered without its magnitude. The path of a body through space considered without its breadth is a geometrical *line*. A *superficies* (surface) is the space made by the motion of a body considered as a line (i.e. a one-dimensional body considered without its depth).[396] On the one hand, the Euclidean notion is considered, from Hobbes' point of view, as nothing more than meaningless symbols and not names. On the other hand, the Hobbesian points, lines and surfaces are considered as abstract entities with their own magnitude. It is simply that bodies as points are considered without their magnitude. They are real instances of their own properties; those real bodies are just now represented *by means of points*. Again,

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[394] *Ibid.,* Vol. IV, part II, ch. VIII, § 44, p. 105: "The *extension* of a body, is the same thing with the *magnitude* of it, or that which some call *real space*. But this *magnitude* does not depend upon our cognition, as imaginary space does; for this is an effect of our imagination, but *magnitude* is the cause of it; this is an accident of the mind that of a body existing out of the mind".

[395] Bird 1996, p. 224.

[396] This is clearly in contrast with the well-known Euclidean definition of point and line given at the very beginning of the *Elements*. Euclid says there that "a line is a breadth-less length", and "a surface is that which has length and breadth only". For further study, see also Sacksteder 1981.

Euclidean notions of point and line are just empty symbols without any reference to physical reality; Hobbesian notions, and indeed our material point, are on the other hand names or models, with a direct correlation to the physical reality; thus they are instances of magnitude.[397] A point, being a body considered without its magnitude, is imaginary in the sense that the omission of its natural properties derives solely from the competence of the user's own imagination:

> Though there be no body which has not some magnitude, yet if, when any body is moved, the magnitude of it be not at all considered, the way it makes is called a *line*, or one single dimension; and the space, through which it passeth, is called *length*; and the body a *point*.[398]

These mathematical objects – point, line and surface, as Hobbes argues – perform an act of representation, since they do not arise from observation but from a descriptive or *representational mental act*. Magnitude is omitted solely in order to find a means for calculating it; they can thus be considered as a meta-geometrical model.

On this interpretation, our point mass accomplishes the rule of a model or meta-geometrical tool which we might call an act of imaginary abstraction, since it leaves out of consideration any magnitude belonging to an actual and real body.

Hobbes took part in what we have called the theoretical definition of the model-building practice – or simply the idealization and abstraction process – and became one of the philosophers who tried to confer relevance and legitimacy on metaphysical speculations about abstract, idealized and theoretical mathematical tools or meta-geometrical tools. He clearly explained how, from an act of the mind, we can create purely mathematical representational objects of physical reality:

> Mathematical objects do not derive from observation of a motion presented. They depend, instead, on our election to attend to such and such consequences resulting from causal act or from a motion which is proposed or undertaken by ourselves or in our imagination. Dimensions of magnitude are not given to the geometer. Rather they are made by his own descriptive motions or taken up

[397] See fn. 394 of this chapter above on the difference between magnitude and extension.
[398] Hobbes 1839-45, part II, ch. VIII, §12, p. 111.

according to his mental discriminations.[399]

Bodies in themselves, and even their actual magnitudes, are left out in any account and reasoning. Geometry, and science in general for Hobbes, both make calculations from the geometrical representation of the physical reality. Moreover, the *philosophia prima* studies the accidents common to all bodies:

> Though there be no body which has not some magnitude, yet if, when any body is moved, the magnitude of it be not at all considered, the way it makes is called a line, or one single dimension; and the space, through which is passeth is called length; and the body itself, a point; in which sense the earth is called a point, and the way of its yearly revolution, the ecliptic line. But if a body, which is moved, be considered as long, and be supposed to be so moved, as that all the several parts of it be understood to make several lines, then the way of every part that is called breadth, and the space which is made is called superficies, consisting of two dimensions, one whereof to every several part of the other is applied whole. Again, if a body be considered as having *superficies*, and be understood to be so moved, that all several parts of it describe several lines, then the way of every part of that body is called thickness or depth, and the space which is made is called solid, consisting of three dimensions, any two whereof are applied whole to every several parts of the third.[400]

From our point of view, the seventeenth-century philosophical and mathematical tradition houses the conjunction of two different lines of thought, which together led to the fulfillment of the theoretical notion of the point mass. The first of these two traditions derives from a more technical framework and is disposed to analyse the functioning of artisanal and mechanical practices; the second line of thought conversely gives rise to the abstraction and idealization procedure, or rather to the practice of abstracting some natural properties from any particular set of phenomena at hand in order to simplify computations. Thus in the seventeenth century, we envision a process in which any mechanical analysis was abstracted from its original context and used in a broader sense, in order to understand the physical world in which we are embedded. The basic idea is that from the observation of a practical context – such as the context of

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[399] Sacksteder 1981, p. 578.
[400] Hobbes 1839-45, Vol. IV, part II, ch. VIII, §12.

Renaissance mechanics – a kernel of idealization emerges and leads to the objectification of a purely practical procedure. The process of measuring the weight of heavy bodies hanging at the extremity of a balance is shifted to a physical context, in which any heavy bodies are conceived as reduced to their centre of gravity, devoid of its own area or volume.

### 4.2.2 Newton

The reconstruction undertaken so far in §§ 4.1 and 4.2.1 to showcase the model building practice and the role that imagination played in conferring legitimacy upon a theoretical procedure used for building mathematical entities in a representational way represent an important part of the whole story. Whether from a philosophical and metaphysical slight of speculation or not, the debate over the nature of mathematical entities kept going on in the way that we have just delineated, and both from the side of mechanics and from a more technical perspective, things were also improving differently. While Hobbes, from his *non-technical* perspective, was enquiring into the foundations of geometry and mathematics, Newton and some of his other contemporaries were investigating the field of mechanics, stimulating the development of the so-called rational mechanics, and those branches of physics which had mathematical foundations. On the one hand, del Monte and his pupils pointed out the utility of mechanics, maintaining that the role of mechanics and the philosopher of mechanics were strictly separate, and that the latter was considered superior to the former. On the other hand, with Newton, the term 'mechanics' acquired its modern prevalent meaning and began to be defined as "a theoretical discipline founded on mathematics, often referred to as the science *par excellence*".[401] We have already said that mechanics in the past was considered a branch of mathematics together with astronomy, music and optics, whose principles derived from empirical observations about nature and as such could no longer be considered as

[401] Capecchi 2014, p. 7.

irrefutable and unquestionable but simply highly probable. In Galileo's conception, experiments still occupied a marginal role, and were not considered as a useful means of discarding theoretical principles or giving confirmations to them. However, on the advent of Newton's *Principia*, a new tradition emerged, since this treatise was "for one half a treatise on theoretical mechanics, for the other half a treatise of applied mechanics. Regarding the theoretical parts, the experiments that Newton presented were rather simple and had a predominantly educational value. The matter on applied mechanics, especially celestial astronomy, is different. It was based on the law of universal attraction and this was not so immediate to be accepted. Here empiric observations were important and could not be of qualitative nature."[402]

From this perspective, Newton maintained the following:

> In mathematics we are to investigate the quantities of forces with their proportions consequent upon any conditions supposed; then, when we enter upon physics, we compare those proportions with the phenomena of Nature, that we may know what conditions of those forces answer to the several kinds of attractive bodies.[403]

More precisely, in the second half of the seventeenth century, mechanics had a different background from the background of earlier periods, and almost all scientific knowledge could now be framed under the following four schemes: i) the validity of the principle of inertia, ii) uniform acceleration for falling bodies, iii) the introduction of the infinitesimals in mathematics, and iv) the indifference to presence or absence of a vacuum.[404] Besides the *cause of motion and its variation* – an aspect that was already part of the Galilean research – a new interest now came to the fore, namely that variation in motion was no longer associated with *impetus* – this sort of inner power

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[402] *Ibid*., p. 8.
[403] Newton (1687) 1726, p. 218.
*In mathesi investigandae sunt virium quantitates et rationes illae, quae ex conditionibus quibuscunque positis consequentur: deinde, ubi in physicam descenditur conferendae sunt hac rationes cum phaenomenis; ut innotescat quaenam virium conditiones singulis corporum attractivorum generis competant*.
[404] These four different approaches are widely analysed in Capecchi 2014.

which was attributed to bodies – but to something labelled *force* (flanked with *power*), which now took on various important connotations. As Daniel Capecchi has written:

> It [sc. force] might indicate the muscle force, the elastic force of a spring, the pressure, corresponding more or less to the modern force; the effort or the fatigue to move bodies, corresponding more or less to the modern work; the force of bodies in motion which modifies the state of rest or motion of the collided body, corresponding more or less to the modern kinetic energy.[405]

These different connotations of force began to evolve from confused to more or less clear conceptions. Moreover, we also notice an extension of the new discoveries from astronomy to mechanics. Newton was also the first to introduce the notion of mass in mechanics, and his laws of motion represented the summit of mathematical physics between the sixteenth and seventeenth centuries. In these centuries, mathematics proceeded towards the preparatory phase of the birth of modern rational mechanics. This new way of conceiving mechanics dealt with the study of the movement of the bodies, *excluding* other phenomena such as thermal, chemical or electromagnetic ones.[406] In order to describe a great variety of natural bodies such as fluids, three-dimensional or deformable bodies, physics used conceptual abstractions and idealizations that derived from the strict criterion of approximation and simplification which had been mainly determined in the works of Galileo. The application of the notion of mass in the field of dynamics – as I have already said – represents a part of the process of conceptual abstraction, because the dynamic of three-dimensional bodies now involved the possibility of analysing their natural behaviour without considering their volumetric dimensions. The idea to reduce rigid bodies to simple point masses is already explicit in Newton's research, but we need to wait until the turn of the eighteenth century and Euler's writings in order to see these theoretical entities applied in a purely algebraic context. Moreover, the term 'point mass' was put forward only in Euler's *Mechanica* of

[405] Capecchi 2014, p. 228.
[406] D'Anna and Renno 1995, pp. 1-2.

1736.

Let us first analyse the structure of Newton's[407] *Philosophiae naturalis principia mathematica*. This work is made up of four different parts: (i) the introduction containing eight definitions, the famous *scholium* on absolute space and time, and the axioms or laws of motion; and (ii) three books, the first two of which are both entitled *The Motion of Bodies*, and the third of which is entitled *The System of the World*. Together they embody a powerful and coherent research program for linking mathematical representations with real world structures, which has been dubbed "the Newtonian style" by Bernard Cohen.[408] A crucial passage where Newton himself expresses clearly what he is up to in his *Principia* occurs in a *scholium* to section XI of the first book. This section concerns the motion of bodies with centripetal forces which mutually attract each other, and it maintains the following:

> These propositions naturally lead us to the analogy there is between centripetal force, and the central bodies to which those forces used to be directed; for it is reasonable to suppose that forces which are directed to bodies should depend upon the nature and quantity of those bodies, as we see they do in magnetical (*sic*) experiments. And when such cases occur, we are to compute the attractions of the bodies by assigning to each of their particles its proper force, and then collect the sum of them all. I here use the word attraction in general for any endeavour of what kind so ever, made by bodies to approach to each other; whether that endeavour arises from the action of the bodies themselves, as tending mutually to or agitating each other by spirits emitted, or whether it arises from the action of

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[407] Isaac Newton was born into a Puritan family in Woolsthorpe, a small village in Lincolnshire near Grantham, on December 25th 1642, just after Galileo's death. His father, a farmer, died two months before Isaac was born. Isaac learned to read and write from his maternal grandmother and mother, both of whom were literate. In 1656 he went to boarding school in Grantham, returning full time to manage the farm, not very successfully, in 1659. After further schooling at Grantham, he entered Trinity College, Cambridge in 1661. These years of Newton's youth coincided with the most turbulent in the history of England. The English Civil War had begun in 1642, King Charles was beheaded in 1649, Oliver Cromwell ruled as lord protector from 1653 until his death in 1658, to be followed by his son Richard in 1658-1659, leading to the restoration of the monarchy under Charles II in 1660. How much the political turmoil of these years affected Newton and his family is unclear, but it has been noticed that the intellectual world of England at the time when Newton matriculated in Cambridge was thus very different from its previous state when he was born.

For further information, see S. George, "Isaac Newton", The Stanford Encyclopedia of Philosophy (Fall 2008 Edition), Edward N. Zalta (ed.);
On-line: https://plato.stanford.edu/archives/fall2008/entries/newton/.

[408] Cohen 1980.

> the aether or of the air, or of any medium whatsoever whether corporeal or incorporeal, any how impelling bodies placed therein towards each other in the same general sense I use the word impulse, not defining in this treatise the species or physical qualities of force, but investigating the quantities and mathematical propositions of them; as I observe before in the Definitions. In mathematics we are to investigate the quantities of forces with their proportions consequent upon any conditions supposed; then, when we enter upon physics, we compare those proportions with the phenomena of Nature, that we may know what conditions of those forces answer to the several kinds of attractive bodies. And this preparation being made, we argue more safely concerning the physical species, causes, and proportions of forces. Let us see then with what forces spherical bodies consisting of particles endued with attractive powers in the manner above spoken of must act mutually upon one another: and what kind of motions will follow from thence[409].

Here, we encounter not only the idea that natural state of affairs could be represented throughout a mathematical language, but, as Newton explicitly says, that spherical bodies only consist of particles, an issue which is deeply developed in section XII of the same first book. In particular, in expounding the theorem known as "the shell theorem", we can find the reasons which drives Newton to posit the *Calculus*, or rather the idea of showing that in applying his Law of Universal Gravitation to spherically symmetrical massive bodies (such as planets, stars, and the like), one can regard these bodies as point masses with all of their mass concentrated at a point. The key ingredient in showing this is to show that for a thin mass shell, the gravitational force at a point outside this shell is the same as if all the mass of this shell is concentrated at its centre.

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[409] Newton (1687) 1726, pp 217-218.
*His propositionibus manuducimur ad analogiam inter vires centripetas, et corpora centralia, ad quae vires illae dirigi solent. Rationi enim consentaneum est, ut vires, quae ad corpora diriguntu, pendeant ab eorundem naturâ et quantitate, ut fit in magnetics. Et quoties hujusmodi casus incidunt, aestimandae erunt corporum attractiones, assignando singulis eorum particulis vires proprias, et colligendo summas virium. Vocem attractionis hic generaliter usurpo pro corporum conato quocunque accedendi ad invicem: sive conatus iste fiat ab actione corporum vel se mutuo petentium, vel per spiritus emissos se invicem agitantium; sive is ab actione aetheris, aut aëris, mediive cujuscunque seu corporei seu incorporei oriatur corpora innatantia in se invicem utcunque impellentis. Eodem sensu generali usurpo vocem impulsus, non specie virium et qualitates physicas, sed quantitates et proportiones mathematicasin hoc tractatu expendens ut in definitionibus explicui. In mathesi investigandae sunt virium quantitates et rationes illae, quae ex conditionibus quibuscunque positis consequentur: deinde, ubi in physicam descenditur conferendae sunt hac rationes cum phaenomenis; ut innotescat quaenam virium conditiones singulis corporum attractivorum generis competant. Et tum demum de virium speciebus, causis et rationibus physicistutius disputare licebit. Videamus igitur quibus viribus corpora spherica, ex particulis modo jam exposito attractivis constantia, debeant in se mutuò agere; et quales motus inde consequantur.*

Before looking at this theorem in detail, we should focus on the way in which the concept of force and mass are used, modified, and applied to this new theoretical layer added to mathematical physics.

The force in this case needs to be considered and understood as a purely instrumental concept that could also be removed. Indeed Newton avoided giving force any ontological content. This idea that the force can be sometimes removed or neglected gave rise to some metaphysical questions on the model-building theory, something I already considered when we defined the material point with its theoretical modern definition. In order to understand *how* a force acts on a rigid body, let us consider Newton's following definition:

> I have hitherto been treating of the attractions of bodies towards an immovable centre; though very probably there is no such thing existent in nature. For attractions are made towards bodies, and the actions of the bodies attracted and attracting are always reciprocal and equal, by Law III; so that if there are two bodies, neither the attracted nor the attracting body is truly at rest, but both (by Cor. 4, of the Laws of Motion), being as it were mutually attracted, revolve about a common centre of gravity, And if there be more bodies, which are either attracted by one single one which is attracted by them again, or which all of them, attract each other mutually, these bodies will be so moved among themselves, as that their common centre of gravity will either be at rest, or move uniformly forward in a right line. I shall therefore at present go on to treat of the motion of bodies mutually attracting each other; considering the centripetal forces as attractions; though perhaps in a physical strictness they may more truly be called impulses. But these propositions are to be considered as purely mathematical; and therefore, laying aside all physical considerations, I make use of a familiar way of speaking, to make myself the more easily understood by a mathematical reader.[410]

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[410] Newton (1687) 1726, p. 194.
*De motu corpurum viribus centripetis se motuo petentium.*
*Hactenus exposui motus corporum attractorum ad centrum immobile, quale tamen vix extat in rerum naturâ. Attractiones enim fieri solent ad corpora; et corporum trahentium et attractorum actiones semper mutuae sunt et aequales; per Legem tertiam: adeo ut neq; attrahens quiescere neq; attractum, si duo sint corpora, sed ambo (per Legum Corollarium quartum) quasi attractione mutuâ, circum gravitatis centrum commune revolvantur: et si plura sint corpora , quae vel ab unico attrahantur, et idem attrahant vel omnia se mutuo attrahant; haec ita inter se moveri moveatur in directum. Quâ de causâ jam pergo motum exponere corporum se mutuo trahentium, considerando vires centripetas tanquam attractiones, quamvis fortasse, si physicè loquamur, verius dicantur impulsus. In mathematicis enim jam versarur; et propterea, missis disputationibus physicis, familiari utimur sermone, quo possimus a lectoribus mathematicis facilius intelligi.*

That "immovable centre" is something which schematically represents the body on which the force we are taking into account exerts its power. This case should be considered as one in which an external force applied upon a body is equally distributed on its whole surface with the same strength. Using a representational picture, we should imagine the force as exerting its power only on the (material) centre of gravity of the body itself, the centre in which the mass is not only *imagined* to be fully concentrated, but the force is also acting as a unique power, or considered to be exerting its strength in only one point, which is labelled as the point of application of a force.

Then, as we have already said, the technical meaning of *mass* as the *quantity of matter* can also be traced back to Newton, although its use was limited. In fact, as Max Jammer and Danilo Capecchi remind us, in the *Principia* 'mass' appears not more than ten times with its "new technical meaning", and usually it seems that Newton preferred the use of the expression 'quantity of matter' instead.[411] More precisely, mass entered

[411] Jammer claimed that even in the nineteen century, "despite the decisive role of this notion there was no formal definition of mass." [Jammer 1961, Ch. VIII] In the introductory part of this chapter, Jammer mentioned that the conservation of mass was implicitly assumed in Newton's *Principia*, and the basic assumption on the conservation of mass had been introduced by Kant [Kant, *Meta Anfangsgrunde*] and Lavoisier [Lavoisier, *Traite*]. Jammer mentioned further authors who do not refer to Euler: e.g. M. Brisson [Brisson, *Dictionnaire*], Sigaud de la Fond [de la Fond, *Dictionnaire*], despite Euler's treatise was published in 1736. Jammer stated: "All the textbooks in [the] 18th century do not provide a better definition of mass (as "the mass of the body is the quantity of matter which it contains"). The only exception is Euler's *Mechanica* which had been written aiming a construction of rational mechanics as a science based on axioms, definitions and logical deduction. [...] In the history of the notion of mass Euler's *Mechanica* is of exceptional importance, because it demonstrates the logical transition from the Newtonian axiomatics, which is based on the notion of the force of inertia to a widely contemporary notion of mass as a numerical coefficient, which characterises the special physical body and is determinate by the relation of the force to the acceleration. [...] Euler followed Newton's *Principia*. However, in the demonstration of Proposition 17 (§ 142) appeared a new idea. Euler described the force of inertia [...] as determined by the force which is necessary to force the body to change its state of rest or motion. Different bodies need different forces which are proportional to their quantity of matter (Euler E015/016, § 142). Therefore, the quantity of matter or the mass is determined using moving forces, an idea, which is of great importance in the following chapters of *Mechanica* as well as in the *Theoria* (1765) [Euler E289, § 154-156]. Euler stated that the matter (mass) of a body is not measured by its volume, but by the force which is necessary to bring the body in the given motion (acceleration). Here, we find, consequently, the expression of the well-known formula, 'Force is equal to mass multiplied by acceleration' which expression is used for the precise definition of mass." Jammer continued with the analysis of the development in France in the nineteenth century, due to Duhamel, Resal, Paul Appell, and other countries, due to Saint-Venant, Reech, Andrade, Kirchhoff, Mach, Hertz and Poincare, who all used the same definition of mass as Euler (in most cases without reference to Euler). Jammer also mentioned that the new concept of mass also stimulated investigation of the concept of force (compare [Jammer, 1957]). For further details, see Dieter 2009, p. 160, n. 143.

into Newton's mechanical theory in two ways, firstly as a measure of the resistance of a body to be accelerated, and secondly as a measure of the force exerted on a body located in a gravitational field. The former meaning occurred in the laws of motion, for which the mass is the constant of proportionality between force and acceleration ('inertial mass'), while the latter one is mainly used in the development of the theory of universal gravitation and is linked to the idea of attraction, where mass is the constant of proportionality between the centripetal force and the gravitational field ('gravitational mass').

The definition no. VIII of the introduction represents only an anticipation of what Newton intended to develop and prove in the course of his masterpiece. In this definition, he defined the motive quantity of a centripetal force as the measure of the same, which is proportional to the motion which it generates in a given time.[412] Thus he explains as follows:

> Thus the weight is greater in a greater body less in a less body; and, in the same body it is greater near to the earth, and less to remote distances. This sort of quantity is the centripetency, or propension to the whole body towards the centre, or, as I may say, its weight; and it is always known by the quantity of an equal and contrary force just sufficient to hinder descent of the body.
>
> This quantity of force, we may, for the sake of brevity, call by the name of motive, accelerative, and absolute force; and, for the sake of distinction, consider them with respect to the bodies that tend to the centre, to the place of those bodies, and to the centre of force towards which they tend; that is to say, I refer the motive force, to the body as an endeavor and propensity of the whole towards a centre, arising from the propensities of the several parts taken together; the accelerative force to the plane of the body, as a certain power diffused from the centre to all places around to move the bodies that are in them; and the absolute force to the centre, as endued with some cause, without which those motive forces would not be propagated through the space round about; whether that cause some natural body (such as is the magnet in the centre of the magnetic force, or the earth in the centre of the gravitational force), or anything else that does not yet appear. For I here design only to give a mathematical notion of those forces, without considering their physical uses and seats.

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[412] Definitio VIII: *Vis centripetae quantitas motrix est ipsius mensura proportionalis motui, quem dato tempore generat.*

Wherefore the accelerative force will stand in the same relation to the motive, as celerity does to motion. For the quantity of motion arises from the celerity multiplied by the quantity of matter; and the motive force arises from the accelerative force multiplied by the same quantity of matter. For the sum of the actions of the accelerative force, upon the several particles of the body, is the motive force of the whole. Hence it is, that near the surface of the earth, where the accelerative gravity, or force productive of gravity, in all bodies is the same, the motive gravity or the wright is as the body; but if we should ascend to higher regions, where the accelerative gravity is less, the weight would be equally diminished, and would always be as the product of the body by the accelerative gravity. In those regions, where the accelerative gravity is diminished into one-half, the weight of a body two or three times less, will be four or six times less.

I likewise call attractions and impulses, in the same sense, accelerative and motive; and use the word attraction, impulse, or propensity of any sort towards a centre, promiscuously, and indifferently, one for another; considering those forces not physically, but mathematically: wherefore the reader is not to imagine that by those words I anywhere take upon me to define the kind, or the manner of any action, that causes or the physical reason thereof or that I attribute forces, in a true and physical sense, to certain centres (which are only mathematical points); when at any time I happen to speak of centres as attracting, or as endued with attractive powers.[413]

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[413] Newton (1687) 1726, pp. 76-77.
*Uti pondus majus in majori corpore, minus in minore; inq; corpore eodem majus prope terram, minus in caelis. Haec vis est corporis totius centripetentia seu propensio in centrum & (ut ita dicam) pondus, & innotescit semper per vim ipsi contrariam & aequalem, qua descensus corporis impediri potest.*

*Hasce virium quantitates brevitatis gratia nominare licet vires absolutas, acceleratrices & motrices, & distinctionis gratia referre ad corpora, ad corporum loca, & ad centrum virium: Nimirum vim motricem ad corpus, tanquam conatum & propensionem totius in centrum, ex propensionibus omnium partium compositum; & vim acceleratricem ad locum corporis, tanquam efficaciam quandam, de centro per loca singula in circuitu diffusam, ad movenda corpora quæ in ipsis sunt; vim autem absolutam ad centrum, tanquam causa aliqua præditum, sine qua vires motrices non propagantur per regiones in circuitu; sive causa illa sit corpus aliquod centrale (quale est Magnes in centro vis Magneticæ vel Terra in centro vis gravitantis) sive alia aliqua qua non ap paret. Mathematicus saltem est hic conceptus.*

*Nam virium causas & sedes physicas jam non expendo. Est igitur vis acceleratrix ad vim motricem ut celeritas ad motum. Oritur enim quantitas motus ex celeritate ducta in quantitatem Materiæ, & vis motrix ex vi acceleratrice ducta in quantitatem ejusdem materiae. Nam summa actionum vis acceleratricis in singulas corporis particulas est vis motrix totius. Unde juxta Superficiem Terrae, ubi gravitas acceleratrix seu vis gravitans in corporibus universis eadem est, gravitas motrix seu pondus est ut corpus: at si in regiones ascendatur ubi gravitas acceleratrix fit minor, pondus pariter minuetur, eritq; semper ut corpus in gravitatem acceleratricem ductum. Sic in regionibus ubi gravitas acceleratrix duplo minor est, pondus corporis duplo vel triplo minoris erit quadruplo vel sextuplo minus.*

*Porro attractiones et impulsus eodem sensu acceleratrices & motrices nomino. Voces autem attractionis, impulsus vel propensionis cujuscunq; in centrum, indifferenter et pro se mutuo promiscue usurpo, has vires non physice sed Mathematice tantum considerando. Unde caveat lector ne per hujusmodi voces cogitet me speciem vel modum actionis causamve aut rationem physicam alicubi definire, vel centris (quae sunt puncta Mathematica) vires vere et physice tribuere, si forte aut centra trahere, aut vires centrorum esse dixero.*

In the last paragraph of the quotation above, Newton is referring to the difference between mathematical and physical concepts. Until this time, mathematics and physics were still considered independent disciplines, because the former develops hypotheses, whereas the latter inspects principles and causes. Mathematics is still considered as a tool which has no relationship with the actual world, and does not give any description on the actual world. This is, by far, the clearest point in opposition with the Galilean reasoning.

However, it is only between section XI and XII of Book I that the features characterising the entity labelled the point mass are made explicit. The theorem XXX says that: "[I]f the individual points of the surface of some sphere may be drawn to the centre by equal forces inversely proportional to the square of the distance from the points, I say that a corpuscle within the surface agreed upon is not attracted by these forces in any direction." Its demonstration is only preliminary to theorem XXXI, which states that *a corpuscle put in place outside the surface of the sphere is attracted to the centre of the sphere, by a force inversely proportional to the square of its distance from the same centre*. From the demonstration Newton states essentially two things, both of which have a very important consequence. First, he says that the gravitational field outside a spherical shell having total mass $M$ is the same as if the entire mass $M$ is concentrated at its centre (i.e. its centre of mass). Secondly, it says that for the same sphere, the gravitational field inside the spherical shell is identical to 0.

Newton is here regimenting the use of the point mass as an entity which can be used to *represent* natural phenomena. He makes explicit in this theorem the two main features attributed to the point mass in everyday mathematical physics. The only difference between this approach and the one promoted by Euler is that Newton's approach is in some way still traditional; in other words, Newton's approach is mostly related to the geometrical approach inherited from the Euclidean tradition and method, which was still much alive within his own scientific context. This Newtonian style is in some way related to the previous century; he was in fact extremely convinced of the superiority of ancient geometry, and in the *Principia* he always adapted it to the study of figures whose sides were infinitesimal distances, movements and velocities using the procedures of the *Calculus*.

### 4.2.3 Euler, Rational Mechanics and the Algebraic Representation of Physics

The last achievement was reached by Euler, perhaps the most inventive mathematician of his period, who wrote the *Mechanica sive motus scientia analytice exposita*[414], a work on mechanics or the science of motion demonstrated by means of either analytical methods or the application of the calculus.[415] He aims to create – through providing axioms, definitions and logical deductions – a rational mechanical science that can be used to show the unquestionable character of Newtonian mechanics. Euler's solution functioned correctly for *all* those problems in which the bodies' dimensions and the motion of their parts are negligible in respect of more important issues. In fact, he clearly says in Scholium 98, which will be discussed further below, that his program for mechanics relies on the concept of bodies of infinitesimal magnitude.

For the purpose at hand, we shall not give an extensive overview of the role fulfilled by Euler or the other scientists of his time, because we are already well aware of the remarkable role they played, mostly because there are numerous works and publications on the subject. In fact, as I have already stated at the beginning of this chapter, our main interest is to give an account of the historical moment in which the material point comes to stand as a model that is used for representing physical phenomena. This breakthrough was mainly due to the work of Luca Valerio and Galileo

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[414] L. Euler, *Mechanica sive motus scientia analytice exposita*. Auctore Leonhardo Eulero academiae imper. scientiarum membro et matheseos sublimioris professore. Tomus I. Instar supplementi ad commentar. acad. scient. imper. Petropoli: Ex typographia academiae scientiarum. A. 1736.

The whole corpus of Euler's works can be found via the online digital resource: http://eulerarchive.maa.org/.

[415] The father of this type of foundational method is of course Leibniz, who, in his *Nova Methodus*, invented the operation of the calculus, defining it as the operational rules that applied to quantities. See Leibniz 1684.

Galilei. Thanks to the circulation of the Galilean works, this issue became one of great epistemological and metaphysical relevance for new generations of mathematicians and within the new field of mechanics. On the one hand, Galileo was responsible for having proposed again the way in which Guidobaldo del Monte used the centre of gravity as a moving material point, and on the other hand he was also one of the first who gave an epistemological and metaphysical meaning to the representational power fulfilled by those abstract or idealized objects, which, according to him, did not depart from physical reality and truth, because these abstract mathematical entities already constituted the very language of nature.

Thus, this last part of the earlier Galilean research aims at completeness, and performs the role of giving a complete account of the methodological approach of the objectification of procedure. In both Newtonian and Eulerian mechanics, the material point is *already conceived* as a model and is already used in this explicit way, namely as a simple tool that gives to the mathematician the chance to simplify computations and represent in a mathematical way a large variety of phenomena that have common elements and properties. The program of the *Mechanica* focuses mainly on the basic concepts of i) rest and motion;[416] ii) the model of an infinitely small body; and iii) the basic distinction between internal principles, such as the property of inertia and the impenetrability of bodies, and external principles.[417] Every rigid body is treated as if it were a point mass, and in the Scholium 98 the *material point* stands finally as an explicit concept, and is deliberately used exactly for what it stands for. The bodies of infinitesimal magnitude are used, detached from their purely physical content:

> These laws of motion, which a body observes that is left to itself, either at rest or in continued motion, are seen particularly for these indefinitely small bodies, which can be consider as points. For in bodies of finite magnitudes, of which the individual parts have their own motions, the body will exert itself to observe these laws, but which will not always be possible to happen on account of the state of the body. The body therefore will follow a motion which is composed from the

[416] A basic distinction is made between i) rest, ii) uniform motion and iii) non-uniform motion, according to which even the smallest element of the path is able to be conceived as equably transferred.

[417] Dieter 2009, p. 118.

individual exertions of the parts of the body, and this hitherto on account of the insufficiency of the principles has not been possible to be defined, but this discussion is to be differed to the following. The different kinds of bodies will therefore supply the needs for the primary division of our work. For in the first place, we will consider very small bodies or which can be considered as points. In the next case we will approach these bodies of finite magnitudes which are rigid and are not allowed to change their shape. In the third case we will consider flexible bodies. Fourthly, we are concerned with these which allow extension and contraction. Fifthly, we put under our examination the solution of the motion of many bodies, that are impeded by others, that their own motion may be completed as they exert themselves. Truly in the sixth case the motion of fluids will be the agenda. For these bodies we will not only see, how the remainder of the motion is to be continued [p. 38]; but in addition we will inquire, how these are affected by the external causes or forces. Finally from all these inquiries the large scale variation of the whole body can be inferred, whether it is free or not. For a non-free state, I understand this: when bodies are impeded, by which they are unable to progress in that direction, and which they try to overcome; the motion of pendular bodies is of this kind which, since they are unable to descent directly, as they try, and so make oscillations. For the free state is to be understood: when bodies are progressing and which come upon no impediments to their motion anywhere, not only on account of their own force, or from disturbing forces pulling on them. Therefore it appears, from the things Mechanics will have as its agenda, and that there are many which have not even been touched upon. For besides the motion of points, which have been dealt with hitherto, nevertheless there are so few that it will be necessary to derive nearly all from first principles. I begin therefore with the motion of free points with any kinds of disturbing forces, because these left to themselves will follow the motion shown in this chapter. Hence on account of this I have resolved for the First Volume to be concerned with the motion of free points, and the following Volume truly set up to explore the motion of points which are not in free motion; in both of which, and which will occur, as with these already dealt with, so likewise for these that follow, I shall derive the motions from first principles using the analytical method.[418]

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[418] *Istae motus leges, quas corpus sibi relictum vel quietem vel motum continuando observat, spectant proprie ad corpora infinite parva, quae ut puncta possunt considerari. In corporibus enim finitae magnitudinis, quorum singulae partes alios habent motus insitos, quaelibet pars quidem has leges observare conibitur, quod autem non semper propter corporis statum fieri potest. Corpus igitur ipsum eum sequetur motum, qui ex singularum partium conatibus componitur, isque adhuc ob insufficientiam principiorum non potest definiri, sed haec tractatio ad sequentia est differenda. Diversitas igitur corporum suppeditabit nobis operis divisionem primariam. Primo enim contemplabimur corpora infinite parva seu quae tanquam puncta spectari possunt. Deinde corpora finitae magnitudinis aggrediemur ea, quae sunt rigida neque figuram suam mutari patiuntur. Tertio agemus de corporibus flexibilibus. Quarto de iis, quae extensionem ex contractionem admittunt. Quinto plurium corporum solutorum motus examine subiiciemus, quorum alia impediunt, quin motus suos possint, ut conantur, absolvere. Sexto vero de motu fluidorum erit agendum. De his vero corporibus non solum videbimus, quomodo sibi relicta motus continuent; sed praeterea inquiremus, quomodo ea a causis externis scilicet potentiis afficiantur.*

By way of this approach, Euler deserves to be considered as the successor of a tradition which can be traced back to Descartes, Newton, Leibniz, Huygens, Galileo and even – for some aspects – to Archimedes. In fact, he clearly refers to Archimedes' considerations and the Archimedean approach on the equilibrium between bodies in order to explain the conservation of state.[419] However, Euler, in the same manner as his forerunners, develops simultaneously mathematics and mechanics with a preference for the mathematically established algorithms. The main difference between him and Newton thus rather lies in the fact that, although the latter in his *Principia* conceived a mechanics relying on an original and new arithmetical algorithm – which is known as the 'Method of Fluxions'[420] – he still preferred geometrical methods for the confirmation of the results that had been obtained analytically. Moreover, during the same years, an alternative foundation based on arithmetic had been discussed by Leibniz, in his *Historia et origo calculi differentialis*[421], who based his method on the correlated operations formed by differences and sums (as were already stated in his *Elementa*[422]),

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*Denique in his omnibus disquisitionibus magnam inferet varietatem status corporum vel liber vel non liber. Per statum non liberum hic intelligo, quando corpora impediuntur, quo minus in ea directione progrediantur, qua conantur; cuiusmodi est motus corporum pendulorum, quae, quia non possunt directe, uti conantur, descendere, oscillationes efficiunt. Ex quo intelligitur statum liberum esse, quando corpora nullum inveniung impedimentum in quamvis plagam progrediendi, in quam tum ex propria vi, tum a potentiis sollicitata tendunt. Apparet igitur, quibus de rebus in Mechanica sit agendum, et quam sint multa, quae etiam nunc nequidem sunt libata. Nam praeter motum punctorum, quae adhuc sunt tractata, tam pauca sunt, ut fere omnia demum invenire et ex principiis derivare necesse sit. Incipio igitur a motu punctorum liberorum a potentiis quibuscunque sollicitatorum, quia, quos sibi ipsa relicta sequantur motus, hoc capite iam est ostensum. Hanc ob rem primum istum Tomum motui punctorum libero destinavi, in sequente vero punctorum motum non liberum pertractare constitui; in quorum utroque, quae occurrent, cum ex his iam traditis, tum ex sequentibus principiis methodo analytica sum derivaturus.*

[419] Euler, E015/016 § 56.

[420] John Wallis too developed a method of fluxions. However, he was strongly influenced by the method promoted by Leibniz, so much so that it can be said that Wallis developed his own method independently from Newton. However some key differences can be explained as follows: while Newton was affected by physical reasons, Leibniz was not. The latter placed a great deal of attention upon the symbolism and the general rules of differentiation and integration. The Newtonian approach relies on an arithmetical algorithm, but he still was inclined towards a geometrical method for the confirmation of the results he had obtained analytically.

[421] Cf. G. W. Leibniz, "Historia et origo calculi differentialis", in *Mathematische Schriften, herausgegeben von C. I. Gerhardt*, Georg Olms Verlag, Hildesheim New York, 1971, p. 392.

[422] G. W. Leibniz, *Elementa calculi novi pro differentiis et summis, tangentibus et quadraturis, maximis et minimis, dimensionibus linearum, superficierum, solidorum, aliisque communem calclulum transcendentibus*. 1669-71.

where the first operation resulted in a differentiation, whereas the second one resulted in the inverse operation, which he called 'integration'[423].

Now, to go back to Euler, he introduced an alternative approach where the relation to geometry is not assumed. Geometrical truths – such as the truth deriving from the Euclidean approach – are still considered as necessary and obviously stand in contrast to the laws of mechanics or other sciences which relate to experience and contingent truths. As a direct consequence it follows that, although mechanics is related to experience, *the theory has to be based on principles of the same reliability (or necessity) as mathematics*. "The program for mechanics has to be supposed by transfer of mathematical principles to mechanics. Then, the mathematical principles are not softened or violated, but form a constructive part of the theory. As a consequence mathematical and mechanical principles have been not only applied or transferred into the other discipline, but preferentially confirmed and mutually tested in their reliability an applicability."[424]

Whereas on the one hand Newton proved this program for the transfer of geometrical principles to mechanics, Euler on the other hand stated that a core defect of the geometrical method was the lack of an *algorithm* which could be used for the *modelling and calculation of problems which deviated only slightly in some detail from the standard formulation and solution*. A new basis for modelling a contingent truth and finding the solution to a problem now arises from the application of the calculus to mechanics; or as Dieter writes, "the motion of bodies where the finite and infinitesimal quantities are interpreted not only geometrically, i.e. related to distances, but also temporally, i.e. to bodies travelling a certain distance in a certain time."[425] Indeed in the *Scholium,* Euler asserts:

> These laws of motion, which a body observes when left to itself in continuing rest or motion pertain properly to infinitely small bodies, which can be considered as points. [...] The diversity of bodies therefore will supply the primary division of our work. First indeed we shall consider infinitely

[423] Dieter 2009, xi.
[424] *Ibid.*, ix.
[425] *Ibid.*, xiv.

small bodies [...]. Then we shall attack bodies of finite magnitude which are rigid.

Euler's program may be interpreted as the unification of statics and dynamics, since rest and motion are always treated as correlated notions.

Euler's pupil, Joseph Louis de Lagrange (1736-1813), took another step forward to achieving a programme of the reorganisation of mechanics. In his *Mècaniquie analytique* published in 1788, an essay considered as a compendium of all branches of mechanics from statics to hydrostatics and from hydrodynamics to dynamics, the same methodology of the algebraic approach is used to treat all these different but correlated issues. In this work Lagrange, beyond providing a detailed historical introduction, managed to produce a work that systematised and organised the science of mechanics. Additionally, he improved upon the analytical-mathematical language of his predecessors.[426] The combined work of Lagrange and his tutor Euler allowed mechanics to reach its current structure.

The brightest insight reached by Euler was the way of his conceiving the material point as an explicit idealized tool and of representing its trajectory by means of equations with an algebraic structure instead of following a conventional geometrical treatment. Of course, a geometrical structure cannot be avoided, since the properties of surface, lines and solids were *essential components* of mechanics, but they were necessary only at the very beginning of the treatise in order to write down the equation of motion, and at the very end in order to interpret the results. In this way the intermediate phase became completely algebraic in nature. The motion of any rigid body follows the same treatment and rules applied to the point mass model, but it becomes a mechanics dominated by a phase of de-geometrization, independent so far as possible from a geometrical account.

The 'old centre of gravity' became the 'new centre of mass' or 'centre of inertia'. This is now a point in any body around which the mass or the inertia is equally distributed according to the equality of the moments. This notion is completely

[426] Dugas 1988, pp. 332-349.

equivalent to what we have called 'centre of gravity'. But it is likely that the main conceptual difference relies on the fact that the previous notion was used without having in mind the precise epistemological meaning of the idealization process and its epistemological and theoretical consequences. In fact, the achievements reached during the Renaissance owed themselves most of all to their practical target; their simplifications were driven only by the aim to solve purely mechanical problems linked to practical and artisanal purposes. As Dieter writes of Euler's discoveries,

> Euler introduced an alternative approach where the relation to geometry is not assumed. Therefore, the questions which previously emerged, if geometrical curves were analyzed, does not persist any longer. Euler based the calculus on the transfer of the rules, i.e. addition, subtraction, multiplication and division, valid for finite quantities to the operation with infinitesimal quantities. As a consequence, Euler considered infinitesimal quantities of different magnitude which are related to infinite quantities of different magnitude [Euler E387, § 84]. All these quantities are to be treated as numbers [Euler E101, § 1-10] and are allowed to appear in expressions composed of infinitesimal, finite and infinite quantities [Euler E212].[427]

Euler replaced the geometrical representation of mechanics with the analytical one, using the calculus in the Leibnizian arrangement. The basic facts observed for all kinds of bodies – no matter how they are shaped and how large they are – were the preservation of the states of rest and motion due to the inertia of bodies.

Both the approaches seen above, that followed by Newton and that followed by Euler, are representational, exactly because they are using the point mass model as a representational vehicle, or better, because their model of point mass, be it geometrical or algebraic, is in any case a representation which can help every competent user of the model to make computations and deduce some surrogative inferences from it, accruing knowledge that is relative to this 'surrogate' piece of physical reality. This new approach required the distinction to be made between a necessary and a contingent

[427] Dieter 2009, p. 13.

statement[428], according to which mechanics should rely on necessary mathematical principles, even though the theory aims to give a representation of experiential events, so both the principles should have the same reliability. In other word, as Dieter says, "[a]s a consequence mathematical and mechanical principles have been not only applied or transferred into the other discipline, but preferentially confirmed and mutually tested in their reliability an applicability."[429]

This new program, which aimed towards a wider application than the mechanical discipline that had been developed previously, related not only to mechanical problems strictly speaking, but also to natural phenomena in general. Euler says that "the shortcoming of the geometrical method is the lack of an *algorithm* which can be used for the modelling and calculation of problems, which deviated only slightly in some detail from the standard formulation and solution."[430]

The point mass – or the material point, as I prefer to call it – is not used as the elementary unit without extension, out of which all bodies are composed, but rather as an *appropriate geometrical model* – because it can be defined as a geometrical point – for the relation between numbers which have been assigned to geometrical objects, such as point, lines, surface and solids.

> Every material body has not only one extent, but is always extended in the so-called three dimensions of length, breadth and depth, and it must consequently have a definite shape in all directions. Whatever has extent in only one direction is called a line, and what has extent in two directions is called a surface; both have extent, but nevertheless they are not bodies, and we have here an example showing that not all things with extent can be considered to be bodies. A body must have threefold extent, in length, in breadth and in depth. There are no further kinds of extent, and consequently a body must possess all possible kinds of extent, that is to say it must have extent in all directions. A body is also bounded all round, and the outer circumference of its extent is called its

[428] This distinction is specified in the *Specimen* I, 11 and the *Monadology* §31-36, both of which were written by Leibniz, who based his program for mechanics on the assumption that geometry has to be completed by principles which explain the action and being acted upon of bodies. The geometrical truths are necessary, whereas the laws of mechanics are usually connected to the physical experience, and so they can be classified as contingent truths.

[429] Dieter 2009, ix.

[430] *Ibid.*, x.

shape, of which there are infinitely many types, as is shown in geometry. In each individual body this shape is fully determined in all its parts. But when we speak of a body in general, we can ascribe to it no more than the property of adopting a shape, and one must regard this property as indeterminate. But of the parts of a body one cannot say that they have a definite shape, not even if one states the result of how many subdivisions the part is. This is so because one can in infinitely many ways cut from a given body a piece that represents half of the whole, and this piece can therefore have infinitely many shapes. The question as to the shape of the parts of a body is ill-put, and can only be raised by those who maintain that bodies have ultimate parts. But since these parts have no size, the concept of shape becomes inapplicable. This appears sufficient to give us a complete understanding of extent and its properties, so that we will now proceed to the examination of other general properties of material bodies.[431]

The novelty introduced by Euler consisted of having attributed a *purely numerical parameter* to material bodies detached from any physical properties such as length, breadth and depth. On this basis, not only solids may have a weight – which is usually the volume of the body – but also simple plane figure or lines. Euler was able to generalise this procedure by that assuming the *same modus of assignment* was true for all other mechanical quantities.[432]

Thanks to this manoeuvre we have the reply to what, already in the *Quadratura Parabolae per Simplex Falsum,* Luca Valerio had considered the legitimacy of abstracting the *pondus* from the bodies, or conversely, the legitimacy of the procedure of assigning a weight to idealization without physical extension, such as points and lines or plane figures. This attribution of physical properties to geometrical entities, which are generally conceived as not having the classical three-dimensional properties – the "threefold extent", as Euler called them –, was all part of this new practice of abstraction and idealization, which at this point became consolidated, deeply rooted and epistemologically relevant in order to develop a new science of motion, as it were to "give a new face" to mechanics.

[431] Euler, E842, §15. The original title of this work is "Anleitung zur Naturlehre", published in *Opera Postuma 2*, 1862, pp. 449-560. The classic English trans. was produced by Ernest Hirsch. The text is also available online: http://eulerarchive.maa.org.

[432] Dieter 2009, p. 182.

As Danilo Capecchi has written, "[t]he efforts of the 18th century for the development of a unified theory of motion of bodies, which ideally can be considered found their end with Lagrange's 1788 *Méchanique analitique*, had given classical mechanics a modern aspect. Discussions upon principles passed now from their kind to their justification and their utility. The issue of Lagrange's treatise and its great success, due mainly to the use of the principle of virtual work, was the occasion in the 19th century for a deep discussion on the logical status of the principles of mechanics, mainly statics, probably still more heated than that of half of the 18th century on the principles of dynamics. However, after this debate, the science of motion found itself at a turning point."[433] The models of point mass used by the mechanicians of the eighteenth century had finally exhausted their outcomes, or, as Capecchi again writes:

> The problems which could be solved with them are either too difficult, such as the problem of *n* bodies, or of little importance.[434]

This turning point was reached from two different directions. On the one hand, there has been an enhancement in theoretical study, by examining phenomena which were not yet explored in classical mechanics, as those observed in moving frames, and by adopting a more powerful formal language for this theory. On the other hand, the point mass is no longer a representative tool for rigid bodies only, but assisted in the opening of new perspectives. The new era allowed for the introduction of the deformation of bodies, by adopting either a molecular or a continuum model for matter and by introducing the concepts of energy and dissipation of thermodynamics.

> But the science of force, or of power acting by law in space and time, has undergone already another revolution, and has become already more dynamic, by having almost dismissed the conceptions of solidity and cohesion, and those other material ties, or geometrically imaginably conditions, which Lagrange so happily reasoned on, and by tending more and more to resolve all connexions and actions of bodies into attractions and repulsions of points: and while the science is advancing thus in

[433] Capecchi 2014, pp. 368-369.
[434] *Ibid*. p. 369.

one direction by the improvement of physical views, it may advance in another direction also by the invention of mathematical methods.[435]

It seems taken for granted that, although scientists from the eighteenth and nineteenth centuries were trying to find a way to represent more complex systems than that represented by a single body, this did not mean that for a series of simplified and straightforward cases, the model of the point mass cannot be used anymore. The classic dynamic laws of motion are still valid in the construction of inferential conclusions with regard to macroscopic objects and events.

This allows us to give to the point mass as a mathematical entity an appropriate conventional meaning. Now we have all the elements to define the essence of a physical theory: i) an abstract calculus which makes sense of undefined or theoretical terms, definitions, principles and rules of inference; and ii) a conceptual model which more or less gives a traditional representation of the interested part of the world. For example, a mechanical theory of the solar system took as its primitive terms the point mass. Its principles are the Newtonian laws of motion and its rules of inference are those offered by the differential calculus. The conceptual model may be the set of planets, thought of as spheres. Then, there is a certain set of rules adopted by competent users in order to translate the conclusions drawn on the model into the target system's knowledge.

The debate over the applicability of idealized and abstract notions arose from a cluster of practical problems, all of which related to the Equilibrium Controversy and were solved with the same methodological approach, with the aid of the virtual balance. Only for a second time did these issues acquire additional philosophical content. We have seen how these authors who have appeared in our discussion tried to establish their right to speak on empirical matters by representing aspects of reality in purely mathematical terms.

Guidobaldo, Valerio and Galileo all played the role of important mediators, who translated the outcomes of purely practical investigations back into a respectable philosophical idiom. They provided a philosophical discourse which would be used at a

[435] Hamilton 1834, p. 247.

later time to grant philosophical legitimacy to mathematical entities within a mathematical and physical context.

## Conclusions

In this research, I have worked on the boundaries between the so-called problem of scientific representation and the model-building practice. In particular, I have forwarded a hypothesis pertaining to the possible development of the idealized model of the point mass considered as a tool for epistemic representation, which, I have argued, is rooted in the formalisation of scientific practice. When we bump against some practical or technical difficulty in dealing with the understanding of how the world works and in giving an exact description of any actual situation – which is the art of doing physics – we can find an adequate approximate or idealized description to replace the actual surrounding variety of phenomena in order to simplify this representation and carry out computations to do with the prediction and the explanation of them.

The present dissertation has consisted in the main of two attempted enquiries. On the one hand, it has aimed at showing that the model of the point mass can be considered as an idealized entity, having mainly an epistemic representational role. On the other hand, using the methodological approach – previously introduced by Giusti in 1999 – which has been called the 'objectification of procedure', this research has forwarded a more historical claim in providing a possible reconstruction of the development of such a controversial mathematico-physical tool, as indeed the point mass is. This reconstruction has adopted the point mass as a new case study for supplying evidence in favour of the objectification of procedure, attempting to augment its consistency as a modelling technique that is mainly based on the objectification of scientific practice. Therefore, our case study is nothing other than a further confirmation of the fact that models are not always and foremost abstractions of qualitative features of physical objects deriving from a purely mental activity, but rather they can also be objectifications of pragmatic procedures having an epistemic role.

Let me retrace the main outcomes of this research. According to the account presented in the first chapter – which followed in the footsteps of Giere and Contessa – a model, in our case the model of the point mass, was shown to merit being called an epistemic representation of a certain target for a certain user, if and only if the user takes that vehicle to denote the target, and adopts an interpretation – on the basis of a certain set of rules arbitrarily laid down within the scientific community – of the vehicle (in terms of the target). These two processes, i.e. denotation and interpretation, jointly account for what is known as scientific representation. Therefore *any* competent user can adopt an interpretation of the vehicle, in virtue of the fact that i) a vehicle is an epistemic representation of a target for her; and ii) she can perform valid inferences from the vehicle to the target (i.e. as surrogative inferences). In this respect, the inferential conclusions made of the vehicle are no more than partial, because they cannot account for an epistemically faithful representation of the physical phenomena under observation.

For this purpose I used the point mass as a case study to exemplify the idea that an idealized representational model could be formalized by observing the scientific practice. From this there followed an attempt to give a detailed account of the theoretical development of this mathematical entity. The basic idea behind this approach was to argue that the point mass is not necessarily a mind-independent Platonic entity deriving from an act of abstraction from physical objects, which is used to represent only the underlying peculiarities common to a large variety of the most disparate physical objects observable in our surroundings. On the contrary, the perspective in which the development of my case study has been placed aims to show that it is likely that some models theoretically represent both practical enquiries and heuristic or mechanical demonstrative procedures carried out by the scientific community. In this respect, we have seen through three stages that the geometrical centre of gravity and the point mass model share some properties, a fact which becomes evident especially if we look at the use that ancient mathematicians – above all Archimedes – and the Renaissance community – above all the School of Urbino, which was also responsible for the restoration of a considerable part of the ancient scientific corpus – made of the barycentre for heuristic, mechanical and practical enquiries.

More precisely, in the first stage, we have seen that Archimedes operates a physicization of mechanics in the sense that he made use of mechanical elements (e.g. the balance) and principles (e.g. the law of the lever) to understand the geometrical proportionality among planes and solids. His demonstrative method is based on mechanical principles, because the geometrical features belonging to planes and solids whose area and volume one wants to measure are used for building up a virtual balance, which follows the law of the lever. Therefore, from Archimedes' perspective, the centre of gravity is used as a point in which the volume (considered as weight) of the figure is fully concentrated, and is hung at one of the extremities of the virtual balance. This mechanical illustration lays the ground for an ultimately geometric proof, in helping the untrained audience to visualize the empirical correctness of the geometrical theorem which one wants to prove.

The Renaissance scholars pointed out the practical use we can make of geometrical elements, and specifically they used the centre of gravity as a zero-dimensional geometrical point as an investigative tool for building up the foundations of modern statics. In fact, it was in the field of the *scienza de ponderibus* that the role of the barycentre became of central importance for a purely mathematical treatment of physical reality. The Urbinate Guidobaldo del Monte is held as the author who was able to imagine that macroscopic bodies at the extremities of the balance no longer exist, but that there are, instead, their centres of gravity. In other words, he thought that at the extremities of the balance existed only moving points, instead of physical objects. Such an operation releases the materials bodies of any dimensional and physical properties, and they can thereafter be considered as if they were moving (material) points, to which one can ascribe physical properties.

Working on the cusp of the sixteenth and seventeenth centuries, Luca Valerio was the first scholar who forwarded an epistemological query concerning the use of mathematical entities in a physical context, or, in other words, was the first to challenge the legitimacy of assigning weight – i.e. a physical feature – to unextended idealizations, such as points. The attribution of physical properties to geometrical entities, which are generally conceived as not having the classical three-dimensional properties, was part of this new practice of abstraction and idealization, which at this point became epistemologically relevant in order to develop a new science of motion, or, as it were, to "give a new face" to mechanics.

Within the same epistemological debate, Bernardino Baldi's edition of the pseudo-Aristotelian *Problems of Mechanics* (1621) made use of the centre of gravity for a slightly different epistemic purpose. Specifically he used the geometrical concept of the centre of gravity and the conclusions derived from the mechanical treatment of his tutor Guidobaldo, in order to augment the foundational principles of architecture, which was his main field of study and research. He used the notion of centre of gravity, the Archimedean law of the lever and Euclidean mathematics to interpret geometrically any architectural structure, i.e. buildings, arches and architraves.

Following in Guidobaldo's footsteps with this geometrical approach, Bernardino Baldi was among the first Renaissance scholars to argue in favour of the idea that, besides a purely empirical approach, a theoretical treatment of the construction techniques was required. In this revolution the centre of gravity became now an epistemic tool that was useful for understanding the limitations of practical Renaissance techniques, and so mathematics became in turn the field of study which can help the user to analyse the advantages and disadvantages of our construction techniques. The centre of gravity, in this respect, allows for a purely mathematical treatment of any material configuration.

This epistemological line of thought deserves to be investigated in greater depth in future research.

The aim of developing a mathematical representation of nature by means of idealized and abstract models was thereafter pursued systematically by Galileo. It was with his *Two New Sciences* (1638) that mathematical physics began to be considered as the framework for the completion of the third stage of objectification of procedure, whereby the point mass assumes the role of being an independent object of research, which is useful for building the axiomatic principles of rational mechanics. By using the foundational principles of machines – as established in Guidobaldo's *Mechanicorum*

*Liber* – Galileo shifted his attention to the working principles behind nature: the book of nature is now written in a mathematical form. An idealized entity is used to represent physical objects, and it is thanks to the power of our imagination [Hobbes 1655] that we can ascribe natural properties, such as volumetric extension, mass and forces, to entities such as points. It was thereafter only with Newton and Euler that the centre of gravity entered rational mechanics as a model of intelligibility, or rather as a purely theoretical model used to investigate natural phenomena and principles. Now, it is the centre of mass of any rigid body – differently shaped and moving uniformly, accelerating or moving in parabolic motion – that represents a series of states of affairs, which were not only made up of simple rigid bodies, but which were also typified by a higher level of complexity, in an algebraic language.

On the basis of this historical evidence, it seems coherent to maintain that some mathematical entities of today's mathematical physics do not necessarily derive from a simple act of abstraction from the material objects' main features. Rather it is likely that some of them have their own foundation in the objectification of a series of practical enquires (i.e. when we understand this process of objectification as turning a scientific practice into a mathematical entity) which are deeply interconnected, because they all laid down the same principles such as the mechanical ones.

It is only when one attempts to investigate the scientific practice that the development of some mathematical entities can be understood as playing a key epistemic role in order to understand how the world works and that therefore they need to be interpreted by the user on the basis of the context she is working within. Moreover, since models are – from the perspective I have attempted to show – generators of hypothesis upon the physical system under observation, their correctness, together with the truth-value of the inferences derived from them, can be tested only *a posteriori* to an empirical investigation.

From this analysis it emerges that the point mass – and perhaps some other mathematico-physical models which should be further investigated – is created and developed by a dialectical series of revisions in the methodological and theoretical approach to geometry and mechanics. When we study the history of scientific practice we find a repeated refinement of old concepts with a gradually increasing standard of rigor; in each of the historical periods surveyed, mathematicians have used the centre of gravity opportunistically for their own specific purpose at hand, be it geometrical, mechanical or even theoretical and epistemic. In this, each generation has discarded what was faddish and recast what was still fertile into new and sharper form.[436]

[436] Goodman 1979, p. 545.

## References


M. Abattouy, "Greek mechanics in Arabic context, Thabit ibn Qurra, al-Isfizari and the Arabic Traditions of Aristotelian and Euclidean Mechanics", *Science in Context,* 14, (1/2), 2001, pp. 179-247.

M. Abattouy, "The Aristotelian foundations of Arabic mechanics. From the ninth to the twelfth century", in C. Leijenhorst, C. Lüthy, and H. Thijssen (eds.), *The dynamics of Aristotelian natural philosophy: from antiquity to the seventeenth century,* Brill, Leiden, 2002, pp. 109-140.

M. Abattouy, "The Arabic transformation of mechanics: the birth of the science of weight", *Foundation for Science, Technology and Civilization*, FSTC Limited, Manchester, 615, 2006, pp. 1-25.

M. Abattouy, "The Arabic science of weights, Textual tradition and significance in the history of mechanics", in E. Calvo, M. Comes, R. Puig and M. Rius (eds.), *A shared legacy. Islamic science East and West*, Publications Edition, Barcelona, 2008, pp. 83-114.

E. A. Abbott, *Flatland. A Romance of Many Dimensions*, 1882.

F. Acerbi, *Tutte le Opere di Euclide*, Bompiani, Milano, 2007.

F. Acerbi, *Il silenzio delle sirene. La matematica greca antica*, Carocci, Roma, 2010.

F. Acerbi, C. Fontanari and M. Guardini, *Archimede. Metodo. Nel laboratorio del genio*, Bollati-Boringhieri, Torino, 2013.

F. Acerbi and B. Vitrac, *Héron d'Alexandrie. Metrica*, Serra Edizioni, Pisa, 2014.

Archimedes, "De Centris gravium vel de planis equerepentibus", in N. Tartaglia (ed.), *Opera Archimedis Syracusani philosophi et mathematici ingeniosissimi per Nicolaum Tartaleam Brixianum multis erroribus emendata, expurtaga, ac in luce posita*, Rufinelli, Venice, 1543a, pp. 2v-19r.

Archimedes, "Liber Archimedis de insidentibus aquae", in N. Tartaglia (ed.), *Opera Archimedis Syracusani philosophi et mathematici ingeniosissimi per Nicolaum Tartaleam Brixianum multis erroribus emendata, expurtaga, ac in luce posita*, Rufinelli, Venice, 1543b, pp. 31v-35v.

Archimedes, "On the Equilibrium of Planes", in T. L. Heath (ed.), *The Works of Archimedes Edited in Modern Notation with and Introductory Chapter by T. L. Heath*, Dover, New York, 1953, pp. 189-220.

Aristotle, "Questiones Mechanicae", in N. L. Thomaeo (ed.), *Problematum: sectiones duae de quadraginta; Questiones Mechanicae; De miraculis naturae; Physiognomica; De Lineis insecabilibus, haec & alia sequens pagina indicabit*, Bruniolo, Venice, 1585.

Aristotle, "Mechanical Problems", in W. S. Hett (ed.) *Aristotle minor works*, William Heinemann, Cambridge, 1955.

Aristotle, *Opere III. Fisica, Del Cielo*, Laterza, Roma-Bari, 1973.

Aristotle, "Mechanical problems", in *Minor Works*, Vol. 14 of *Aristotle in Twenty-three Volumes*, Harvard University Press, Cambridge, Mass, 1980, pp. 329-414.

Aristotle, "On Generation and Corruption", in J. Barnes (ed.), *The Complete Works of*

*Aristotle*, (English trans. by A. A. Joachim), Princeton University Press, Princeton, 1984a.

Aristotle, "Physics", in J. Barnes (ed.), *The Complete Works of Aristotle*, (English trans. by A. A. Joachim), Princeton University Press, Princeton, 1984b.

Aristotle, *Metaphysica,* in The internet classics Archive, (English trans. by W. D. Ross); On-line: http://classics.mit.edu/Aristotle/metaphysics.html .

Aristotle, *Physica*, in The internet classics Archive, (English trans. by R. P. Hardie, and R. K. Gaye);
On-line: http://classics.mit.edu/Aristotle/physics.html.

Aristotle, *Posterior Analytics,* in The internet classics Archive, (English trans. by G. R. G. Mure);
On-line: http://classics.mit.edu/Aristotle/posterior.html.

B. Baldi, *Cronica de matematici overo epitome dell'istoria delle vite loro*, Monticelli, Urbino, 1707.

U. Baldini, G. Zanier, P. Farina and F. Trevisani, *Ricerche sull'atomismo del Seicento*, Atto del Convegno di studio di Santa Margherita Ligure, 14-16 ottobre 1976, La Nuova Italia Editrice, Firenze, 1977.

U. Baldini and P. D. Napolitani, "Per Una biografia di Luca Valerio. Fonti edite e inedite per una ricostruzione della sua carriera scientifica", *Bollettino di Storia delle Scienze Matematiche*, XI, 1, 1991.

R. W. Batterman, "Critical Phenomena and Breaking Drops: Infinite Idealizations in

Physics”, *Studies in History and Philosophy of Modern Physics*, 36b, 2, 2005, pp. 225-244.

A. Becchi, B. D. Meli and E. Gamba, (eds.), *Guidobaldo del Monte (1545-1607). Theory and Practice of the Mathematical Disciplines from Urbino to Europe*, Max Planck Research Library for the History and Development of Knowledge, Berlin, 2012.

G. B. Benedetti, *Resolutio omnium Euclidis problematum aliorumque ad hoc necessatio inventorum una tantummodo circini data apertura*, Bartolomeo Cesano, Venice, 1553.

G. B. Benedetti, *Demostratio proportionum motuum localium contra Aristotelem et omnes philosophos* (1st ed.), Bartolomeo Cesano, Venice, 1554.

G. B. Benedetti, *Demostratio proportionum motuum localium contra Aristotelem et omnes philosophos* (2st ed.), Bartolomeo Cesano, Venice, 1555.

G. B. Benedetti, *Diversarum Speculationum mathematicorum et physicarum liber*, Nicolò Bevilacqua, Turin, 1585.

G. B. Benedetti, *Diversarum Speculationum mathematicorum et physicarum liber*, Francesco Ziletti, Venice, 1586.

J. A. Bennett, “The mechanics’ philosophy and the mechanical philosophy”, *History of science,* 24, 1986, pp. 1-28.

S. Berryman, “Ancient Atomism”, The Stanford Encyclopedia of Philosophy (Winter 2016 Edition), E. N. Zalta (ed.), forthcoming;
On-line: <https://plato.stanford.edu/archives/win2016/entries/atomism-ancient/>.

D. Bertoloni Meli, *Thinking with object: The trasformation of mechanics in the Seventheen century*, Johns Hopkins Univeristy Press, Baltimore, 2006.

A. Bird, "Squaring the Circle: Hobbes on Philosophy and Geometry ", *Journal of the History of Ideas*, 57, 2, 1996, pp. 217-231.

R. B. Braithwaite, *Scientific Explanation. A Study of the function of theory, Probability and Law in Science*, Cambridge University Press, Cambridge, 1968.

F. Brandt, *Thomas Hobbes' Mechanical Conception of Nature*, Levin and Munksgaard, Copenhagen, 1928.

S. Brentjes, "Additions to Book I in the Arabic traditions of Euclid's Elements", *Studies in History of Medicine and Science*, 15, 1997–1998, pp. 55–117.

S. Brentjes and J. Renn, *The Arabic Transmission of Knowledge on the Balance*, Max Planck Research Library for the History and Development of Knowledge, Berlin, 2013.

J. E. Brown, *The* Scientia the Ponderibus *in the Later Middle Ages*, UMI Dissertation Service, Ann Arbor, 1967.

S. G. Brush, *The Kinetic Theory of Gases*, Imperial College Press, London, 2003.

J. Burckhardt, *La civiltà del Rinascimento in Italia*, Sansoni, Firenze, 1876.

J. Büttner, *Galileo's Challenges: The origin and early conceptual Development of Galileo's theory of Naturally Accelerated Motion on Inclined Plane*, Ph.D. thesis, Humbolt-Universität su Berlin, 2004.

J. Büttner, “Big Wheel Keep on Turning”, *Galilaeana*, 33-62, 2008.

J. Büttner, “Galilei, Galileo: Discorsi e dimostrazioni matematiche intorno a due nuove scienze attenenti alla mecanica & i movimenti locali”, in H. L. Arnold (ed.), *Kindlers Literatur Lexikon,* Vol. 6, Stuttgart, Metzler, 2009.

J. Büttner, P. Damerow and J. Renn, “Galileo’s unpublished treatises. A case study on the role of shared knowledge in the emergence and dissemination of an early modern ‘New science’”, in C. R. Palmerino (ed.), *The Reception of the Galilean Science of Motion in Seventeenth-century Europe*, J.M.M.H. Thijssen, Dordrecht, Kluwer, 2004, pp. 99-117.

C. Callender and J. Cohen, “There Is No Special Problem About Scientific Representation”, *Theoria*, 2006, pp. 67-85.

N. R. Campbell, *Physics. The Element,* Cambridge Unversity Press, Cambridge, 1920.

D. Capecchi, *History of virtual work laws*, Birkhaüser, Basel, 2012.

D. Capecchi, *The Problem of the Motion of bodies. A historical View of the Development of the Classical Mechanics*, Springer, Cham, Heidelberg, New York, Dordrecht, London, 2014.

R. Carnap, “Uber die Aufgabe der Physic und die Anwendung des Grundsatzes der Einfachstheit”, *Kant-Studien*, 1923, pp. 99-107.

R. Carnap, *Der Logische Aufbau der Welt*, Weltkeis-Verlag, Berlin-Schlachtensee, 1928.

R. Carnap, "Über Protokollsätze", *Erkenntnis*, 3, 1932, pp. 215–228; English trans. by R. Creath and R. Nollan, "On Protocol Sentences", *Noûs*, 21, 4, 1987, pp. 457–470.

R. Carnap, *The Logical Syntax of the Language*, Routledge & Kegan Paul, London, 1937.

R. Carnap, "Foundations of Logic and Mathematics", *International Encyclopedia of Unified Science*, Vol. 1, No. 3, University of Chicago Press, Chicago, 1939.

R. Carnap, "The methodological character of theoretical concepts", *Feigl and Scriven*, 1956, pp. 38-76.

R. Carnap, *Philosophical Foundations of Science*, Basic Books, New York, 1966; reprint in 1972 as *An Introduction to the Philosophy of Science*.

N. Cartwright, *How the Laws of Physics Lie*, Oxford University Press, New York, 1983.

N. Cartwright, *Nature's Capacities and Their Measurement*, Oxford University Press, New York, 1989.

B. Cavalieri, *Geometria indivisibilibus continuorum nova quadam ratione promota*, ex typographia de Ducijs, Bononiae, 1653.

R. Caverni, (1972), *Storia del metodo sperimentale in Italia,* 6 Vols., Stabilimento G. Civelli, Florence, 1891-1900.

B. Centrone, *Introduzione a i Pitagorici*, Laterza, Roma-Bari, 1996.

M. Clagett, *The science of Mechanics in the Middle Ages*, The University of Wisconsin

Press, Madison, 1959.

M. Clagett, *Archimedes in the Middle Ages*, The University of Wisconsin Press, Madison, 9 Vols., 1964-1984.

M. Clagett, "The works of Francesco Maurolico", *Physis,* 16, 1974, pp. 149-198.

M. Clavelin, *La philosophie naturelle de Galilée. Essai sur les origines et la formation de la mécanique classique*, Armand Colin, Paris, 1968; English trans. by A. J. Pomerans, *The Natural Philosophy of Galileo. Essay on the Origins and Formation of Classical Mechanics*, The Massachusetts Institute of Technology, Cambridge, 1974.

C. Clavio, *Euclidis Elementorum Libri XV*, Romae, 1574.

I. B. Cohen, *The Newtonian Revolution*, Cambridge University Press, Cambridge, 1980.

F. Commandino, *Federici Commandini Urbinatis liber de centro gravitatis solidorum*, Alexander Benacius, Bologna, 1565.

F. Commandino, *Euclidis Elementorum Libri XV*, Pesaro, 1572.

F. Commandino, *De gli Elementi d'Euclide libri quindici con gli scholii antichi tradotti prima in lingia latina da M. Federico Commandino da Urbino con commentarij illustrati, et hora d'ordine dell'istesso trasportati nella nostra vulgare e da lui riveduti con provilegio*, Urbino, 1575.

F. Commandino, *Pappi Alexandrini Mathematicae Collectiones a Federico Commandino urbinate in latinum conversae et commentariis illustratae*, Pesaro, 1588.

G. Contessa, *Representing Reality: The Ontology of Scientific Models and their Representational Function*, Ph.D. dissertation, 2007.

G. Contessa, “The Ontology of Scientific Models”, *Synthese*, 172, 2, 2010.

C. F. Craver, “Structures of Scientific Theories,” in P. K. Machamer and M. Silberstein (eds.), *Blackwell Guide to the Philosophy of Science*, Blackwell, Oxford, 2002, pp. 55-79.

R. Creath, “Logical Empiricism”, The Stanford Encyclopedia of Philosophy (Spring 2014 Edition), E. N. Zalta (ed.);
On-line: <https://plato.stanford.edu/archives/spr2014/entries/logical-empiricism/>.

P. Damerow and J. Renn, “The transformation of Ancient Mechanics into a Mechanistic World View”, in G. Toepfer and H. Böhme (eds.), *Transformation antiker Wissenschaften*, De Gruyter, Berlin, 2010, pp. 243-267.

A. D’Anna and P. Renno, *Elementi di Meccanica Razionale*, CUEN, Napoli, 1995.

C. da Novara, *Elementa geometriae*, Venice, 1482.

B. della Foresta, *Vita dell’Abate del Parto D. Francesco Maurolico*, Messina, 1613.

G. del Monte, *Guidi Ubaldi e marchionibus montis mechanicorum liber*, Hieronymus Concordia, Pesaro, 1577.

G. del Monte, *Le mechaniche dell’illustriss. Sig. Guido Ubaldo de’ Marchesi Del Monte: Tradotte in volgare dal sig. Filippo Pigafetta*, Francesco di Franceschi, Venice, 1581.

G. del Monte, *Meditatiunculae Guidi Ubaldi e Marchionibus Montis Santae Mariae de rebus mathematicis (cs. 1587-1592),* Bilbiothèque Nationale de Paris, manuscript, catalogue no. Lat. 10246, 1587.

G. del Monte, *Guidi Ubaldi e marchionibus montis in duos Archimedis aequeponderantium libros paraphrasis scholiis illustrata*, Hieronymus Concordia, Pesaro, 1588.

G. del Monte, *Giudo Ubaldi e marchionibus montis perspectivae libri sex*, Hieronymus Concordia, Pesaro, 1600.

J. Dennistoun, *Memories of the Dukes of Urbino illustrationg the Arms, Arts and Literature of Italy, 1440-1630*, 3 vols, London, 1851.

A. De Pace, *Niccolò Copernico e la fondazione del cosmo eliocentrico*, Mondadori, Milan, 2009.

H. de Ridder-Symoens, *A History of the University in Europe: Volume 1, Univerisities in the Middle Ages*, Cambridge University Press, Cambridge, 2003a.

H. de Ridder-Symoens, *A History of the University in Europe: Volume 2, Univerisities in the Early Modern Europe (1500-1800)*, Cambridge University Press, Cambridge, 2003b.

V. De Risi, "The development of Euclidean axiomatics The systems of principles and the foundations of mathematics in editions of the Elements in the Early Modern Age", *Archive for History of Exact Sciences*, 70, 2016, pp. 591- 676.

S. Dieter, *Euler as a Physicis*, Springer, Berlin, 2009.

E. J. Dijksterhuis, *Archimedes*, Munksgaard, Copenhagen, 1956.

E. J. Dijksterhuis, *Archimedes,* Pinceton University Press, Princeton, 1987; English trans. By c. Dikshoorn.

A. G. Drachmann, "Fragments from Archimedes in Heron's mechanics", *Centaurus*, 8, 1963, pp. 91-146.

S. Drake, *Galileo Studies. Personality, Tradition, and Revolution*, The University of Michigan Press, Ann Arbor, 1970.

S. Drake, *Galileo at work. His scientific Biography*, Dover Publication Inc., Mineola, New York, 1978.

S. Drake and I. E. Drabkin (eds.), *Mechanics in Sixteenth-Century Italy*, *Selections from Tartaglia, Benedetti, Guido Ubaldo, and Galileo*, The University of Wisconsin Press, Madison, 1969.

R. Dugas, *Historie de la mécanique*, Griffon, Neuchatel, 1950.

R. Dugas, *A History of Mechanics*, Dover Publications, New York, 1988; English trans. by J. R. Maddox.

P. Duhem, *L'evolution de la mécanique*, Hermann, Paris, 1905.

P. Duhem, *Les origines de la statique* (2 Vols.), Hermann, Paris, 1905-1906.

P. Duhem, *Traité d'énergétique ou de thermodynamique générale*, Gauthier-Villars, Paris, 1911.

P. Duhem, *La Théorie Physique. Son object, sa structure,* 2nd ed., Librarie Marcel Rivière, Paris, 1914.

P. Duhem, *The Aim and Structure of Physical Problems*, Academic Press, New York, 1977.

L. Euler, *Mechanica sive motus scientia analytice exposita. Auctore Leonhardo Eulero academiae imper. scientiarum membro et matheseos sublimioris professore. Tomus I. Instar supplementi ad commentar. acad. scient. Imper*, Ex typographia Academiae Scientiarum, Petropoli, 1736;
On-line: "The Euler Archive", http://eulerarchive.maa.org/.

L. Euler, "Anleitung Zur Naturlehre", in *Opera Postuma 2*, 1862, pp. 449-560.

V. Fano, *Comprendere la Scienza. Un'introduzione all'epistemologia delle scienze naturali*, Liguori Editori, Napoli, 2005.

V. Fano, *I paradosi di Zenone*, Bussole Carocci, Roma, 2012.

A. Favaro, "Due lettere inedite di Guidobaldo del Monte a Giacomo Contarini", *Atti del Reale Istituto Veneto di scienze, lettere ed arti*, LIX 2, 1899-1900, pp. 307-310.

A. Favaro, *Archimede*, Formiggini Editore, Roma, 1923.

A. Favaro (ed.), *Le opere di Galileo Galilei, nuova ristampa della edizione nazionale 1890-1900,* 20 Vols., Barbera, Florence, 1968.

H. Feigl, "The 'Orthodox' View of Theories: Remarks in Defense as Well as Critique," in M. Radner and S. Winokur (eds.), *Analyses of Theories and Methods of Physics and*

*Psychology,* Minnesota Studies in the Philosophy of Science, Vol. 4, University of Minnesota Press, Minneapolis, 1970, pp. 3-16.

M. Fontana, "Osservazioni Storiche sopra l'Aritmentica di Francesco Maurolico", *Memorie dell'Istituto Nazionale Italaino*, *Cl. Di Scienze Fisiche e Matematiche*, 2, i, 1808, pp. 275-296.

A. Frajese, *Galileo Matematico*, Editrice Studium, Roma, 1964.

A. Frajese, *Opere di Archimede*, Utet, Torino, 1974.

A. Frajese and L. Maccioni (eds.), *Euclide. Gli Elementi*, Utet, Torino, 1970.

M. Frank, *Guidobaldo del Monte Mechanics in Context. Research on the Connections between his mechanical work and his life and environment*, Ph.D. dissertation, 2011/2012.

M. Frank, *Mathematics, technics, and courtly life in Late Renaissace Urbino*, Springer, Berlin, 2013, pp. 306-330.

S. French and J. Ladyman, "Superconductivity and Structures: Revisiting the London Account", *Studies in History and Philosophy of Modern Physics*, 28, 3, 1997, pp. 363-393.

S. French and J. Ladyman, "Reinflating the Semantic Approach", *International Studies in the Philosophy of Science*, 13, 2, 1999, pp. 103-121.

S. French and J. Ladyman, "Remodelling Structural Realism: Quantum Physics and the Metaphysics of Structure," *Synthese*, 136, 1, 2003, pp. 31-56.

G. Freudental, “A rational Controversy over Compounding forces”, in P. Machamer, M. Pera and A. Baltas (eds.), *Scientific Controversies. Philosophical and Historical Perspectives*, Oxford University Press, Oxford, 2000.

G. Freudental, “Perpetuum Mobile: The Leibniz-Papin Controversy”, *Stud. Fil. Hist. Sci. 33*, 2002, pp. 573-637.

R. Frigg, “Models and representation: why structures are not enough”, in P. Dietsch (ed.), *Measurement in physics and economics discussion papers*, 25/02, The London School of Economics and Political Science, London, 2002.

R. Frigg, “Fictions and Scientific Representation”, in R. Frigg and M. C. Hunter (eds.), *Beyond Mimesis and Convention, Boston Studies in the Phylosophy of Science,* 262, Springer, Berlin, 2010.

R. Frigg and S. Hartmann, “Models in Science”, The Stanford Encyclopedia of Philosophy (Spring 2017 Edition), E. N. Zalta (ed.);
On-line: <https://plato.stanford.edu/archives/spr2017/entries/models-science/>.

D. J. Furley, *Two Studies in the Greek Atomists*, Princeton University Press, Princeton, 1967.

G. Galilei, *Discorsi e Dimostrazioni matematiche intorno a due nuove scienze attenti alla mecanica e i movimenti locali*, Elsevier, Leiden, 1638.

G. Galilei, *Discorsi e Dimostrazioni matematiche intorno a due nuove scienze attenti alla mecanica e i movimenti locali*, Dozza, Bologna, 1666.

G. Galilei, “De motu”, in A. Favaro (ed.), *Le opere di Galileo Galileo*, Vol. 1, Barbèra, Florence, 1890-1909a.

G. Galilei, “Le mechaniche”, in A. Favaro (ed.), *Le opere di Galileo Galileo*, Vol. 2, Barbèra, Florence, 1890-1909b.

G. Galilei, *Dialogues Concerning Two New Sciences*, Leyden, 1638; English trans. by H. Crew and A. de Salvio, with an Introduction by A. Favaro, MacMillan Company, Dover Publication, Inc. New York, 1914.

G. Galilei, *Dialogue concerning the two chief world systems*, translated and with revised notes by S. Drake, The Modern Library, New York, 1953.

G. Galilei, “On motion”, in S. Drake and I. E. Drabkin (eds.), *On motion and on Mechanics*, The University of Wisconsin Press, Madison, 1960a, pp. 1-131.

G. Galilei, “On mechanics”, in S. Drake in I. E. Drabkin (eds.), *On motion and on Mechanics*, The University of Wisconsin Press, Madison, 1960b, pp. 133-186.

G, Galilei, *Two New Sciences, Including Centers of Gravity and Force of Percussion*. Wall & Thompson, Toronto, 1989.

G. Galilei, *Le Mecaniche. Edizione critica e saggio introduttivo di Romano Gatto*, Olschki, Florence, 2002.

G. Galilei, *Opere 1,* F. Brunetti (ed.), Utet, Turin, 2005a.

G. Galilei, *Opere 2,* F. Brunetti (ed.), Utet, Turin, 2005b.

P. Galluzzi, *Momento. Studi Galileiani*, Ed. dell’Ateneo & Bizzarri, Rome, 1979.

E. Gamba, “Le scienze fisiche e matematiche dal Quattrocento al Seicento”, *Pesaro nell’età dei Della Rovere*, Vol. III, 2, Marsilio ed., 2001, pp. 75-110.

E. Gamba and V. Montebelli, *Le scienze a Urbino nel tardo Rinascimento*, Quattroventi, Urbino, 1988.

E. Gamba and V. Montebelli, *Liber de Centro Gravitatis Solidorum*, Edizioni della Normale, Pisa, 2015.

L. Gaurico, *Tetragonismus id est circuli quadratura, per Campanum, Archimedem Syracusanum atque Boetium mathematicae perspicacissimos adinventa*, Venice, 1503.

L. Gaurico, *Opera Archimedis Syracusani philosophi et mathematici ingeniosissimi, per Nicolaum Tartaleam Brixianum (mathematicarum scientiarum cultorem) multis erroribus emendata, expurgata ac in luce posita multisque necessariis additis... Appositisque manu propria guris quae graeco exemplari deformatae ac depravatae erant*, Venice, 1543.

V. Gavagna, “Euclide a Venezia”, in E. Giusti and M. Martelli (ed.), *Pacioli 500 anni dopo*. Celebrazioni cinquecentenario *Divinae Proportione*. *1509-2009*, Atti del Convegno di studi Sansepolcro, 22-23 maggio 2009, pp. 97-124.

V. Gavagna, *La storia della matematica in classe: dalle materne alle superiori*. *Testi classici: Euclide gli Elementi*, Relazione secondo convegno nazionale Ivrea, 2013.

S. George, “Isaac Newton”, The Stanford Encyclopedia of Philosophy (Fall 2008 Edition), E. N. Zalta (ed.);
On-line: <https://plato.stanford.edu/archives/fall2008/entries/newton/>.

T. Geschauff, *Archimedis Syracusani philosophi ac geometrae excellentissimi Opera, quae qui- dem extant, omnia multis iam seculis desiderata atque a quam paucissimis hactenus visa, nuncque primum et graece et latine in lucem edita. Adiecta quoque sunt Eutocii Ascalonitae in eosdem Archimedis libros Commentaria, item graece et latine, nunquam antea excusa*, Basel, 1544.

R. N. Giere, *Explaining Science: A Cognitive Approach*, Chicago, University of Chicago Press, 1988.

R. N. Giere, "Using Models to Represent Reality", in L. Magnani, N. J. Nersessian and P. Thagard (eds.), *Model-Based Reasoning in Scientific Discovery*, Kluwer Academic, Plenum Publisher, New York, 1999, pp. 41-57.

R. N. Giere, "How Models Are Used to Represent Reality", *Philosophy of Science,* 71, Supplement, 2004, pp. 742-752.

R. N. Giere, "Models, Metaphyisics, and Methodology", in S. Hartman, C. Hoefer and L. Bovens (eds.), *Nancy Cartwright's Philosophy of Science*, Routleg, New York, 2008, pp. 123-133.

R. N. Giere, "Why Scientific Models should not be regarded as Works of fiction", in M. Suárez (ed.), *Fictions in Science. Philosophical Essays on Modelling and Idealization,* Routledge, London, 2009, pp. 248-258.

R. N. Giere, "An Agent-based Conception of Models and Scientific Representation", *Synthese*, 172, 2, 2010, pp. 269-281.

R. N. Giere, "Representing with Physical Models", in P. Humphreys and C. Imbert (eds.), *Model, Simulation and Representation*, Routleg, New York, 2011, pp. 209-215.

R. N. Giere, B. Bickle, and R. Mauldin, *Understanding Scientific Reasoning*, Thomson/Wadsworth, Belmont CA, 5th edition, 2006.

M. Giunti, A. Ledda and G. Sergioli, *I modelli nelle teorie scientifiche*, Carocci, Roma, 2016.

E. Giusti, *Ipotesi sulla natura degli oggetti matematici*, Bollati Boringhieri, Torino, 1999.

E. Giusti, *Piccola introduzione alla storia del calcolo inifinitesimale dall'antichità al Novecento*, Istituti editoriali e poligrafici internazionali, Pisa-Roma, 2007.

N. D. Goodman, "Mathematics as an Objective Science", *The American Mathematical Montly*, 86, 7, 1979, pp. 540-551.

I. Grattan-Guinnes, "Numbers, Magnitudes, Ratios, and Proportions in Euclid's Elements: How Did He Handle Them?", *Historia Mathematica, 23,* 1996, pp. 355-375.

W. R. Grønfeldt, "The Structure of Scientific Theories", The Stanford Encyclopedia of Philosophy (Winter 2016 Edition), E. N. Zalta (ed.);
On-line: <https://plato.stanford.edu/archives/win2016/entries/structure-scientific-theories/>.

C. Grossi, *Degli Uomini Illustri di Urbino*, Urbino, 1819.

C. Guasti, *Le lettere di Torquato Tasso*, Le Monnier, Firenze, 1952.

W. R. Hamilton, "On a general method in dynamics; by which the study of the motions of all free systems of attracting or repelling points is reduced to the search and differentiation of one central relation, or characteristic function", *Philos. Trans. R. Soc. Lond., 124*, 1834, pp. 247-308.

T. L. Heath, *The Works of Archimedes,* The Macmillan Company, New York, 1897.

T. L. Heath, *The Thirten books of Euclid's Element*, translated from the text of Heiberg, 3 Vols., Introduction and book I, II, Cambridge Univesity Press, Cambridge, 1908.

T. L. Heath, *The Method of Archimedes recentely discovered by Heiberg*, a supplement to *the Works of Archimedes 1897*, Cambridge University Press, Cambridge, 1912.

C. G. Hempel, "The Theoretician's dilemma", in H. Feigl, M. Scriven, and G. Maxwell (eds.), *Minnesota Studies in the Philosophy of Science* (Vol. 2), University of Minnesota Press, Minneapolis,1958, pp. 142-163.

C. G. Hempel, *Aspects of Scientific Explanation and other Essays in the Philosophy of Science,* Free Press, New York, 1965.

C. G. Hempel, "On the 'Standard Conception' of Scientific Theories," in M. Radner and S. Winokur (eds.), *Minnesota Studies in the Philosophy of Science* (Vol. 4), University of Minnesota Press, Minneapolis, 1970, pp. 142-163.

C. G. Hempel, "Scientific Rationality: Analytic vs. Pragmatic Perspectives", in T. F. Geraets (ed.), *Rationality To-Day*, The University of Ottowa Press, Ottawa, 1979, pp. 46–58.

H. R. Hertz, *The Principles of Mechanics presented in a new form*; English trans. by D. E. Jones and J. T. Walley, Macmillan and Co., London, 1899.

H. R. Hertz, *I principi della meccanica delineati in una nuova forma*, Bibliopolis, Naples, 2010.

T. Holden, *The Architecture of Matter. Galileo to Kant*, Oxford University Press, Oxford, 2004.

J. L. Heiberg, *Quaestiones Archimedeae*, Copenhagen, 1879.

J. L. Heiberg, *Archimedis opera omnia cum commentariis Eutocii*, 3 Vols., B. G. Teubner, Leipzig, 1880-1881.

J. L. Heiberg, "Eine Neue Archimedesschrift", *Hermes,* 42, 1907, pp. 234-303.

J. L. Heiberg, *Archimedes/Opera* (3 Vols.) (2nd ed.), Leipzig, 1910-15.

J. L. Heiberg and H. Menghe, *Euclidis opera omnia*, ediderunt, in aedibus G. B. Teubneri, Lipsiae, 1883-1895.

G. B. Herbert , "Hobbes's Phenomenology of Space ", *Journal of the History of Ideas* , 48, 4, 1987, pp. 709-717.

M. Hesse, *Models and Analogies in Science*, Sheed and Ward, London, 1963.

T. Hobbes, *Elementorum Philosophae Sectio Prima de Corpore*, London, 1655.

T. Hobbes, *On the Body*, London, 1656.

T. Hobbes, *The English Works of Thomas Hobbes of Malmesbury*, Sir William Molesworth (ed.), 11 Vols., London, 1839-45.

N. Huggett, “Zeno’s Paradoxes”, The Stanford Encyclopedia of Philosophy (Winter 2010 Edition), E. N. Zalta (ed.);
On-line: <https://plato.stanford.edu/archives/win2010/entries/paradox-zeno/>.

R. Hughes, “Models and Representation”, *Philosophy of Science*, 64, 1997, pp. 325-336.

F. Hultsch (ed.), *Heronis Alexandrini geoetricorum et stereometricorum reliquiae*, Berlin, 1864.

G. Israel, “Lo strano concetto di punto materiale”, in M. Emmer (ed.), *Matematica e Cultura*, Springer-Verlag, Italia, 2007, pp. 11-27,

M. Jammer, *Concepts of Force: a study in the foundations of dynamics*, Cambridge University Press, Cambridge, 1957.

M. Jammer, *Concepts of Mass in Contemporary Physics and Philosophy*, Cambridge University Press, Cambridge, 1961.

K. Jaouiche, *Le livre du Qarastun de Thabit Ibn Qurra*, Brill, Leiden, 1976.

M. R. Jones, “Idealization and Abstraction: a Framework”, in M. R. Jones and N. Cartwright (eds.), *Idealization XII: Correcting the Model. Idealization and Abstraction in the Sciences* (*Poznan Studies in the Philosophy of the Sciences and the Humanities)*, Vol. 86, pp. 173-217, Amsterdam/New York, Rodopi, 2005.

A. Jones (ed.), *Book 7 of the Collection/Pappus,* Springer-Verlag, New York, 1986.

H. Kesten, *Copernicus and his world*, Secker & Warburg, London , 1945.

P. Kitcher, “Explanatory Unification and the Causal Structure of the World”, in P. Kitcher and W. E. Salmon (eds.), *Scientific Explanation,* University of Minnesota Press, Minneapolis, 1989, pp. 410-505.

W. R. Knorr, *Ancient sources of the medieval traditn of mechanic*s*: Greek, Arabic and Latin studies of the balance*, Istituto e museo di storia della scienza, Florence, 1982.

E. Knox, “Abstraction and its Limits: Finding Space for Novel Explanation”, *Noûs,* 50, 1, 2012, pp. 41-60.

P. O. Kristeller, *Studies in the Renaissance Thought and Letters*, Edizioni di Storia e Letteratura, Rome, 1956.

P. O. Kristeller, *La tradizione classica nel Rinascimento*, La Nuova Italia, Firenze, 1965.

P. O. Kristeller, *Reinassance Thought and the Arts*. Collected Essays, Princeton University Press, Princeton, 1990.

T. S. Kuhn, *The structure of Scientific Revolutions*, University of Chicago Press, Chicago, 1962.

S. Kusukawa, *Picturing the book of nature: image, text and argument in sixteenth-century human anatomy and medical botany*, The Chicago University Press, Chicago , 2012.

J. A. C. Ladyman and S. French,”Reinflating the Semantic Approach”, *International Studies in Philosphy of Science*, 13, 1999, p. 99-117.

J. A. C. Ladyman, “Review: Unifying Scientific Theories: Physical Concepts and Mathematical Structures”, *The Journal of Economic Methodology*, 10, 2002, pp. 91-96.

W. R. Laird, “The Scope of Renaissance Mechanics”, *Osiris,* 2(II), 1986, pp. 43-68.

W. R. Laird, “Guidobaldo del Monte and Renaissance”, in A. Becchi, D. Bertoloni Meli and E. Gamba (eds.), *Guidobaldo Del Monte (1545-1607). Theory and Practice of the Mathematical Disciplines from Urbino to Europe*, Max Planck Research Library for the History and Development of Knowledge, Berlin, 2013, pp. 35-52.

G. W. Leibniz, *Elementa calculi novi pro differentiis et summis, tangentibus et quadraturis, maximis et minimis, dimensionibus linearum, superficierum, solidorum, aliisque communem calclulum transcendentibus*. 1669-71a.

G. W. Leibniz, *Elementa Juris Naturalis*, 1669-71b.

G. W. Leibniz, “Historia et origo calculi differentialis”, in *Mathematische Schriften, herausgegeben von C. I. Gerhardt*, Georg Olms-Verlag, Hildesheim New York, 1971.

G. W. Leibniz*, Nova Methodus pro maximis et minimis,* 1684; English trans. in D. J. Struik, *A Source Book in Mathematics, 1200-1800*, Harvard University Press, Harvard, 1969, pp. 271-281.

T. Levi-Civita and U. Amaldi, *Lezioni di meccanica razionale*, Zanichelli, Bologna, 1929.

S. Lutz, “The Semantic of Scientific Theories”, in A. Brożek and J. Jadacki (eds.), *Księga pamiątkowa Marianowi Przełęckiemu w darze na* 90-*lecie urodzin,* Norbertinum, Lublin, 2014, pp. 33-67.

P. Machamer, (ed.), *The Cambridge Companion to Galileo*, Cambridge University Press, Cambridge, 1998a.

P. Machamer, *Galileo's machines, his experiments, and his mathematics*, 1998b, pp. 53-79.

P. Machamer and B. Hepburn, "Galileo and the pendulum: Latching on to time", *Science & Education,* 13, 2004, pp. 333-347.

P. Machamer, "Galileo Galilei", The Stanford Encyclopedia of Philosophy (Summer 2017 Edition), E. N. Zalta (ed.), forthcoming;
On-line: <https://plato.stanford.edu/archives/sum2017/entries/galileo/>.

U. Mäki, "Isolation, Idealization and Truth in Economics", in B. Hamminga and N. B. De Marchi (eds.), *Idealization VI: Idealization in Economics*, (*Poznan Studies in the Philosophy of the Sciences and the Humanities)*,, Vol. 38, Rodopi, Amsterdam, 1994 pp. 147-168.

G. Mamiani, *Elogi Storici di Federico Commandino, G. Ubaldo del Monte, Giulio Carlo Fagnani*, letti all'Accademia Pesarese, Pesaro, 1828, pp. 4-42.

D. Martines, *Origine e Progressi dell'Aritmentica*: *Sunto Istorico*, Messina, 1865, pp. 65-89.

F. Maurolico, (1548), "Archimedis de momentis aequalibus, ex traditione Francisci Maurolico", in *Monumenta omnia mathematica quae extant quorumque catalogum inversa pagina demonstrat, ex traditione Francisci Maurolici*, Panomi, Hesperius, 1685a, pp. 86-180.

F. Maurolico, *Monumenta omnia mathematica quae extant quorumque catalogum inversa pagina demonstrat, ex traditione Francisci Maurolici*, Panomi, Hesperius, 1685b.

E. McMullin, "Galilean Idealization", *Studies in the History and Philosophy of Science,* 16, 1985, pp. 247-73.

H. Meyer, "On the Heuristic value of Scientific models", *Philosophy of Science*, 18, 2, 1951, pp. 111-123.

R. Migliorato, *Archimede. Alle radici della modernità tra storia scienza e mito*, Dipartimento di Matematica e Informatica Università di Messina, 2013.

E. A. Moody, and M. Clagett (eds.), *The medieval Science of Weight*, University of Wisconsin Press, Madison, 1960.

M. Morgan and M. Morrison (eds.), *Models as Mediators. Perspectives on Natural and Social Science*, Cambridge University Press, Cambridge, 1999.

M. Morgan and M. Morrison, "Models as Mediating Instruments" in M. Morgan and M. Morrison (eds.), *Models as Mediators: Perspectives on Natural and Social Science*, Cambridge University Press, Cambridge, 1999, pp. 10-37.

C. Moulines, "Approximate Application of Empirical Theories: A General Explication", *Erkenntnis*, 10, 2, 1976, pp. 201-227.

C. Moulines, "Introduction: Structuralism as a Program for Modelling Theoretical Science", *Synthese*, 130, 1, 2002, pp. 1-11.

C. Mugler, *Dictionnaire historique de la terminologie géométrique des Grecs,* Paris, 1958.

A. Musgrave, "'Unreal Assumptions' in Economic Theory: The F-Twist Untwisted", *Kyklos,* 34, 1981, pp. 377–387.

P. Musso, *Rom Harré e il problema del Realismo Scientifico*, Franco Angeli, Milan, 1993.

P. D. Napolitani, "La tradizione Archimedea", in E. Giusti (ed.), *Luca Pacioli e la matematica nel Rinasciemnto*, Petruzzi, Sansepolcro, 1998, pp. 77-102.

P. D. Napolitani, "Archimede alle radici della scienza moderna", *I grandi della scienza* (Le Scienze), IV, 22, 2001.

P. D. Napolitani and K. Saito, "Royal Road or Labirinth? Luca Valerio's De Centro Gravitatis Solidorum and the beginnings of modern mathematics", *Bollettino di Storia delle Scienze Matematiche,* 24, 2004, pp. 68-69.

E. Nenci (ed.), *Le vite de' matematici. Edizione annotata e commentata della parte medievale e rinasciemntale*, Franco Angeli, Milan, 1998.

E. Nenci, (ed), *Bernardino Baldi, In Mechanica Aristotelis problemata exercitationes*, Max Planck Research Library for the History and Development of Knowledge, Source 3, Berlin, 2001.

R. Netz, *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History*, Cambridge University Press, Cambridge, 1999.

R. Netz, *The works of Archimedes, Translated into English, together with Eutocius' commentaries, with commentary, and critical edition of the diagrams*; Vol. I, The Two Books On the Sphere and the Cylinder, Cambridge University press, Cambridge, 2004.

R. Netz and W. Noel, *Il codice perduto di Archimede*, Rizzoli, Milano, 2007.

O. Neurath, (1932), "Protocol Statements", in R. S. Cohen and M. Neurath (eds.), *Philosophical Papers 1913-1946*, Dordrecht, Reidel, 1983, pp. 91-99.

I. Newton, *The Principia: The Authoritative Translation: Mathematical Principles of Natural Philosophy*, 1687; English trans. by I. B. Cohen and A. Whitman, University of Califonia Press, California, 2016 (based on the 1726 (A. Motte) edition).

P. Nolan, *Fondamenti di Fisica. Meccanica, Onde, Termodinamica, Elettromagnetismo, Ottica*, Zanichelli, Bologna, 1996.

J. D. Norton, "Approximation and Idealization: Why the difference matters", *Philosophy of Science,* 79, 2012, pp. 207-232.

P. Palmieri, "Breaking the circle: the emergence of Archimedean mechanics in the late Renaissance", *Archive for History of Exact Sciences*, 62, 2008, pp. 301-346.

M. Panza and A. Sereni, *Il Problema di Platone. Un'introduzione storica alla filosofia della matematica*, Carocci, Roma, 2010.

Pappus of Alexandria, *Pappi Alexandri mathematicae collections a Federico Commandino Urbinate in Latinum conversae, et commentariis illustrate*, Hieronymus Concordia, Pesaro, 1588.

Pappus of Alexandria, *Pappi Alexandri mathematicae collections a Federico Commandino Urbinate in Latinum conversae, et commentariis illustrate. In hae nostra edition ab innumeris, quibus scatebant mendis, et praecipue in Graeco contextu diligenter vindicatae,* Ducius, Bologna, 1660.

Pappus of Alexandria, *Collectio*, *instruxit*, in F. Hultsh (ed.), 3 Vols., Berolini, Weidmannos, 1876-78.

R. Pisano, *Brief historical notes on the theory of centres of gravity*, The Global and The Local: The History of Science and the Cultural Integration of Europe, proceedings of the 2nd ICHEHS, (Cracow, Poland, September 6-9, 2006), M. Kokowski (ed.), pp. 934-941.

R. Pisano and P. Bussotti, "On Popularization of Scientific Education in Italy Between 12nd And 16th Centuries", *Problems of Education in the 21st Century,* 57, 2013, pp. 90-101.

R. Pisano and P. Bussotti, "Fibonacci and the Reception of the Abacus Schools in Italy. Mathematical Conceptual Streams and their Changing Relationship with society", *Almagest*, 2015, pp. 126-164.

R. Pisano and D. Capecchi, *Tartaglia's Science of Weights and Mechanics in the Sixteenth Century*, Springer, Dordrecht, 2016.

Platone, *Timeo*, G. Reale (ed.), Bompiani, Milano, 2000.

M. Proclus, *The Philosophical and Mathematical commentaries of Proclus on the first Book of Euclid's Elements,* Oxford University Press, Oxford, 1788.

M. Proclus, *In Primum Euclidis Elementorum librum commentarii*, Friedlein ed., Teubner, Leipzig, 1873.

M. Proclus, *A commentary of the First Book of Euclid's Elements*, by G. R. Morrow (ed.), Princeton University Press, Princeton, 1970.

M. Proclus, *Commento al I libro degli Elementi di Euclide*, M. Timpanaro Cardini (ed.), Giardini editori e stampatori, Pisa, 1978.

H. Putnam, "What Theories Are Not", in E. Nagel, P. Suppes, and A. Tarski (eds.), *Logic, Methodology and Philosophy of Science*, Stanford University Press, Stanford, 1962, pp. 240-51; (Reprinted in *Mathematics, Matter and Method,* 1975, pp. 215-27).

H. M. Pycior, "Mathematics and Philosophy: Wallis, Hobbes, Barrow, and Berkeley", *Journal of the History of Ideas*, 48, 2, 1987, pp. 265-286 .

M. Redhead, "Models in Physics", *British Journal for the Philosophy of Science,* 31, 1980, pp. 145-163.

Regiomontanus, *Tabulae Primi Mobilis*, Rupprich, Frühzeit, 1514.

H. Reichenbach, *Experience and Prediction: An Analysis of the Foundations and the Structure of Knowledge*, University of Chicago Press, Chicago, 1938.

J. Renn, and P. Damerow, *Guidobaldo del Monte's Mechanicorum Liber*, Max Planck Research Library for the History and Development of Knowledge, Source 1, Berlin, 2010.

J. Renn, and P. Damerow, *The Equilibrium Controversy. Guidobaldo del Monte's Critical Notes on the Mechanics of Jordanus and Benedetti and their Historical and Conceptual Background,* Max Planck Research Library for the History and Development of Knowledge. Source 2, Berlin, 2012.

J. Renn and P. D. Omodeo, "Guidobaldo del Monte's Controversy with Giovan Battista Benedetti on Positional Heaviness", in A. Becchi, D. Bertoloni Meli and E. Gamba (eds.), *Guidobaldo Del Monte (1545-1607). Theory and Practice of the Mathematical Disciplines from Urbino to Europe*, Max Plank Research Library for the History and Development of Knowledge, Berlin, 2013, pp. 53-94.

P. L. Rose, "The origins of the proportional Compass from Mordente to Galileo", *Physis*, 10, 1968, pp. 53-69; together with corrections by Edward Rosen, "The invention of the Reduction Compass", *Physis*, 10, 1968, pp. 377-400.

P. L. Rose, *The Italian Renaissance of Mathematics: Studies on Humanists and Mathematicians from Petrarch to Galileo,* Droz, Geneve, 1975.

W. Rüegg, "Universities in the nineteenth and early twentieth centuries", in S. de Ridder and W. Rüegg (eds.) (1992-2011), *A history of the University in Europe* 4 Vols., Cambridge University Press, Cambridge, 2004

L. Russo, *La rivoluzione dimenticata*, VII edizione, Feltrinelli, Milano, 2013.

W. Sacksteder , "Hobbes: Geometrical Objects" , *Philosophy of Science*, 48, 4, 1981, pp. 573-590 .

W. Salmon, *Scientific Explanation and the Causal Structure of the World,* Princeton University Press, Princeton, 1984.

D. Scinà, U. Bottazzini and P. Nastasi, *Elogio di Francesco Maurolico*, Dalla Reale Stamperia, Palermo, 1808.

A. Seidenberg and J. Casey, “The ritual origin of Balance”, *Archive of History of Science*, 23, 1980.

L. Sklar, *Theory and Truth: Philosophical Critique within Foundational Scinece*, Oxford University Press, New York, 2000.

W. Stegmüller, *The Structure and Dynamics of Theories*, Springer, New York, 1976.

W. Stein, “Der Begriff des Schwerpunktes bei Archimedes”, *Quellen und Studien zur Geschichte der Mathematik, Physik und Astronomie*, Abt. B: Quellen, I, 1930, pp. 221-244.

D. Stewart, “Thomas Hobbes”, The Stanford Encyclopedia of Philosophy (Summer 2017 Edition), E. N. Zalta (ed.), forthcoming;
On-line: <https://plato.stanford.edu/archives/sum2017/entries/hobbes/>.

M. Suárez, “The Role of Models in the Application of Scientific Theories: Epistemological Implications,” in M. S. Morgan and M. Morrison (eds), *Models as Mediators. Perspectives on Natural and Social Science*, Cambridge University Press, Cambridge, 1999, pp. 168-196.

M. Suárez, *The pragmatics of scientific representation,* Discussion paper, DP 66/02, Centre for Philosophy of Natural and Social Science, London, 2002.

M. Suárez, “Scientific Representation: Against Similarity and Isomorphism”, *International Studies in the Philosophy of Science,* 17, 2003, pp. 225-244.

M. Suárez, “Representation in Science”, in P. Humphreys (ed.), *The Oxford Handbook of Philosophy of Science,* 2016, pp. 440-459.

F. Suppe, *The Structure of Scientific Theories*, University of Illinois Press, Urbana, 1977.

F. Suppe, *The Semantic Conception of Theories and Scientific Realism*, University of Illinois Press, Chicago, 1989.

F. Suppe, “Understanding Scientific Theories: An Assessment of Developments”, 1969-1998, *Philosophy of Science*, 67, Supplement, 2000, pp. 102-115.

P. Suppes, “Models of Data”, in E. Nagel, P. Suppes and A. Tarski (eds.), *Logic, Methodology and Philosophy of Science: Proceedings of the 1960 International Congress*, Stanford University Press, Stanford, 1962, pp. 252-261; (Reprinted in P. Suppes, *Studies in the Methodology and Foundations of Science. Selected Papers from 1951 to 1969*, Reidel, Dordrecht, 1969, pp. 24-35).

P. Suppes, “A Comparison of the Meaning and Uses of Models in Mathematics and the Empirical Sciences”, *Synthèse,* 12, 1969, pp. 287-301; (Reprinted in Freudenthal, 1961, pp. 163-177, and in P. Suppes, *Studies in the Methodology and Foundations of Science. Selected Papers from 1951 to 1969*, Reidel, Dordrecht, 1969, pp. 10-23.

C. Swoyer, “Structural Representation and Surrogative Reasoning”, *Synthese,* 87, 1991, pp. 449-508.

N. Tartaglia, *Euclide megarense philosopho: solo introduttore delle scientie mathematice: diligentemente reassettato et alla integrita ridotto per il degno orifessore di tal scientie Nicolo Tartarea Brisciano*, Venice, 1543.

N. Tartaglia, *La prima [-sesta] parte del general trattato di numeri, et misure di Nicolo Tartaglia, nella quale in diecisette libri si dichiara tutti gli atti operativi, pratiche, et regole necessarie non solamente in tutta l'arte negotiaria, & mercantile, ma anchorin ogni altra arte, scientia, over disciplina, dove interuenghi il calculo*. In Vinegia per Curtio Troiano de i Navò. MDLVI [MDLX], Curzio Troiano, Venice, 1556-60.

R. Tassora, *Le Meditatiunculae de rebus mathematicis di Guidobaldo del Monte*, Ph.D. dissertation, 2001.

P. Teller, "Quantum Mechanics and the Nature of Continuous Physical Quantities", *The Journal of Philosophy*, 76, 9, 1979, pp. 345-361.

F. Ugolino, *Storia dei conti e Duchi d'Urbino*, Florence, 1859, II, 271, 273 f.

E. Ulivi, "Scuole e maestri d'abaco in Italia tra Medioevo e Rinascimento", in E. Giusti et al. (eds.), *Un ponte sul Mediterraneo. Leonardo Pisano, la scienza araba e la rinascita della matematica in occidente*, Polistampa, Florence, 2002, pp 121-159.

G. Vailati, *Del concetto di centro di gravità nella Statica d'Archimede*, Atti della R. Accad. d. Scienze di Torino, 32, 1996-97, pp. 742-758.

L. Valerio, *De Centro Gravitatis Solidorum Libri Tres*, Bonfadini, Rome, 1604.

L. Valerio, *Quadratura parabolae per simplex falsum*, Rome, 1606.

G. Valla, *De expetendis et fugiendis rebus opus,* Aldo Manuzio ed., Venice, 1501.

M. Valleriani, *Galileo engineer, Volume 269 of Boston Studies in the Philosophy of Science*, Springer, Dordrecht, 2010.

M. van Dyck, *An Archeology of Galileo's Science of Motion*, Ph.D. dissertation, 2006a.

M. van Dyck, "Gravitating towards stability: Guidobaldo's Aristotelian-Archimedean Synthesis", *History of Science,* 44/3, 146, 2006b, pp. 373-407.

M. van Dyck, "Argumentandi modus huius scientiae maxime proprius." Guidobaldo's Mechanics and the Question of Mathematical Principles", in A. Becchi, D. Bertoloni Meli and E. Gamba (eds.), *Guidobaldo Del Monte (1545-1607). Theory and Practice of the Mathematical Disciplines from Urbino to Europe*, Max Planck Research Library for the History and Development of Knowledge, Berlin, 2013, pp. 7-33.

B. van Fraassen "On the Extension of Beth's Semantics of Physical Theories", *Philosophy of Science*, 37, 3, 1970, pp. 325-339.

B. van Fraassen, *The Scientific Image*, Oxford University Press, Oxford, 1980.

B. van Fraassen, *Laws and Symmetry*, Oxford university Press, New York, 1989.

B. van Fraassen, "Science as Representation: Flouting the Criteria", *Philosophy of Science*, 71, Supplement, 2004, pp. 794-804.

Vespasiano da Bisticci, *Vite di uomini illustri del secolo XV*, Barbèra, Florence, 1859.

V. Vita, "Il punto nella terminologia matematica greca", *Archive for the History of Exact Sciences,* 27, 1982, pp. 101-14.

G. Volpe, *Urbino e i Luoghi della scienza,* Vol. 1, I Quaderni del Centro Internazionale di Studi Urbino e la Prospettiva.

A. Wayne, “Expanding the Scope of Explanatory Idealization”, *Philosophy of Science*, 78, 5, 2011, pp. 830-841.

M. Weisberg, *Simulation and Similarity: Using Models to Understand the World*, Oxford University Press, New York, 2013.

L. Wessels, “Laws and Meaning Postulates in van Fraassen’s View of Theories”, in *PSA: Proceedings of the Biennial Meeting of the Philosophy of Science Association 1974*, 1976, pp. 215-234.

W. L. Wisan, “The new science of motion: A study of Galileo’s De motu locali”, *Archive for the History of Exact Sciences,* 13, 1974, pp. 103-306.

B. Zamberti, *Euclidis Megarensis mathematici clarissimi Elementorum Geometricorum* Lib. XV, Basileae, 1537.

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