A new characterization of P_6-free graphs

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Abstract. We study P_6-free graphs, i.e., graphs that do not contain an induced path on six vertices. Our main result is a new characterization of this graph class: a graph G is P_6-free if and only if each connected induced subgraph of G on more than one vertex contains a dominating induced cycle on six vertices or a dominating (not necessarily induced) complete bipartite subgraph. This characterization is minimal in the sense that there exists an infinite family of P_6-free graphs for which a smallest connected dominating subgraph is a (not induced) complete bipartite graph. Our characterization of P_6-free graphs strengthens results of Liu and Zhou, and of Liu, Peng and Zhao. Our proof has the extra advantage of being constructive: we present an algorithm that finds such a dominating subgraph of a connected P_6-free graph in polynomial time. This enables us to solve the Hypergraph 2-Colorability problem in polynomial time for the class of hypergraphs with P_6-free incidence graphs.

1 Introduction

All graphs in this paper are undirected, finite, and simple, i.e., without loops and multiple edges. Furthermore, unless specifically stated otherwise, all graphs are non-trivial, i.e., contain at least two vertices. For undefined terminology we refer to [8]. Let $G = (V, E)$ be a graph. For a subset $U \subseteq V$ we denote by $G[U]$ the subgraph of G induced by U. A subset $S \subseteq V$ is called a *clique* if $G[S]$ is a complete graph. A set $U \subseteq V$ *dominates* a set $U' \subseteq V$ if any vertex $v \in U'$ either lies in U or has a neighbor in U. We also say that U *dominates* $G[U']$. A subgraph H of G is a *dominating subgraph* of G if $V(H)$ dominates G. We write P_k, C_k, K_k to denote the path, cycle and complete graph on k vertices, respectively.

A graph G is called H*-free* for some graph H if G does not contain an induced subgraph isomorphic to H. For any family $\mathcal{F}$ of graphs, let $Forb(\mathcal{F})$ denote the class of graphs that are F-free for every $F \in \mathcal{F}$. We consider the class $Forb(\{P_t\})$ of graphs that do not contain an induced path on t vertices. Note that $Forb(\{P_2\})$ is the class of graphs without any edge and $Forb(\{P_3\})$ is the class of graphs all components of which are complete graphs.

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The class of P_4-free graphs (or cographs) has been studied extensively (cf. [5]). The following characterization of $Forb(\{P_4, C_4\})$, i.e., the class of C_4-free cographs, is due to Wolk [19, 20] (see also Theorem 11.3.4 in [5]).

Theorem 1 ([19, 20]). *A graph G is P_4-free and C_4-free if and only if each connected induced subgraph of G contains a dominating vertex.*

We can slightly modify this theorem to obtain a characterization of P_4-free graphs.

Theorem 2. *A graph G is P_4-free if and only if each connected induced subgraph of G contains a dominating induced C_4 or a dominating vertex.*

Since this theorem can be proven using similar (but much easier) arguments as in the proof of our main result, its proof is omitted here.

The following characterization of P_5-free graphs is due to Liu and Zhou [14].

Theorem 3 ([14]). *A graph G is P_5-free if and only if each connected induced subgraph of G contains a dominating induced C_5 or a dominating clique.*

A graph G is called *triangle extended complete bipartite (TECB)* if it is a complete bipartite graph or if it can be obtained from a complete bipartite graph F by adding some extra vertices $w_1, \dots, w_r$ and edges w_iu, w_iv for $1 \leq i \leq r$ to exactly one edge uv of F (see Figure 1 for an example).

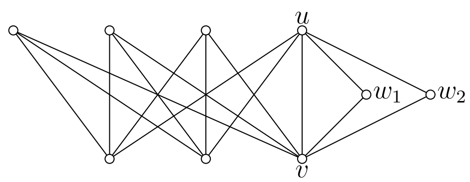

Fig. 1. An example of a TECB graph.

The following characterization of P_6-free graphs is due to Liu, Peng and Zhao [15].

Theorem 4 ([15]). *A graph G is P_6-free if and only if each connected induced subgraph of G contains a dominating induced C_6 or a dominating (not necessarily induced) TECB graph.*

If we consider graphs that are not only P_6-free but also triangle-free, then we have one of the main results in [14].

Theorem 5 ([14]). *A triangle-free graph G is P_6-free if and only if each connected induced subgraph of G contains a dominating induced C_6 or a dominating (not necessarily induced) complete bipartite graph.*

A characterization of $Forb(\{P_t\})$ for $t \geq 7$ is given in [1]: $Forb(\{P_t\})$ is the class of graphs for which each connected induced subgraph has a dominating subgraph of diameter at most $t - 4$.

Our results

Section 3 contains our main result.

Theorem 6. *A graph G is P_6-free if and only if each connected induced subgraph of G contains a dominating induced C_6 or a dominating (not necessarily induced) complete bipartite graph. Moreover, we can find such a dominating subgraph in polynomial time.*

This theorem strengthens Theorem 4 and Theorem 5 in two different ways. Firstly, Theorem 6 shows that we may omit the restriction "triangle-free" in Theorem 5 and that we may replace the class of TECB graphs by its proper subclass of complete bipartite graphs in in Theorem 4. Secondly, in contrast to the proofs of Theorem 4 and Theorem 5, the proof of Theorem 6 is constructive: we provide a (polynomial time) algorithm for finding the desired dominating subgraph. Note that we cannot use some brute force approach to obtain such a polynomial time algorithm, since a dominating complete bipartite graph might have arbitrarily large size.

In Section 3, we also show that the characterization in Theorem 6 is minimal in the sense that there exists an infinite family of P_6-free graphs for which a smallest connected dominating subgraph is a (not induced) complete bipartite graph. We would like to mention that the algorithm used to prove Theorem 6 also works for an arbitrary (not necessarily P_6-free) graph G: in that case the algorithm either finds a dominating subgraph as described in Theorem 6 or finds an induced P_6 in G. Furthermore, we can easily modify our algorithm so that it finds a dominating induced C_5 or a dominating clique of a P_5-free graph in polynomial time. This yields a constructive proof of Theorem 3 and generalizes the algorithm by Cozzens and Kelleher [7] that finds a dominating clique of a connected graph without an induced P_5 or C_5. We end Section 3 by characterizing the class of graphs for which each connected induced subgraph has a dominating induced C_6 or a dominating *induced* complete bipartite subgraph (again by giving a constructive proof). This class consists of graphs that, apart from P_6, have exactly one more forbidden induced subgraph. This generalizes a result in [2].

As an application of our main result, we consider the Hypergraph 2-Colorability problem in Section 4. It is well-known that this problem is NP-complete in general (cf. [10]). We prove that for the class of hypergraphs with P_6-free incidence graphs the problem becomes polynomially solvable. Moreover, we show that for any 2-colorable hypergraph H with a P_6-free incidence graph, we can find a 2-coloring of H in polynomial time.

Section 5 contains the conclusions, discusses a number of related results in the literature and mentions open problems.

2 Preliminaries

We use the following terminology throughout the paper for a graph $G = (V, E)$. We say that an order $\pi = x_1, \ldots, x_{|V|}$ of V is *connected* if $G_i := G[\{x_1, \ldots, x_i\}]$

is connected for $i = 1, \ldots, |V|$. Let $w \in V$ and $D \subseteq V$. Then $N_G(w)$ denotes the set of neighbors of w in G. We write $N_D(w) := N_G(w) \cap D$ and $N_G(D) := \cup_{u \in D} N_G(u) \backslash D$. If no confusion is possible, we write $N(w)$ (respectively $N(D)$) instead of $N_G(w)$ (respectively $N_G(D)$). A vertex $v' \in V \backslash D$ is called a *D-private neighbor* (or simply private neighbor if no confusion is possible) of a vertex $v \in D$ if $N_D(v') = \{v\}$.

Let u, v be a pair of adjacent vertices in a dominating set D of a graph G such that $\{u, v\}$ dominates D. We call a dominating set $D' \subseteq D$ of G a *minimizer of D for uv* if $\{u, v\} \subseteq D'$ and each vertex of $D' \backslash \{u, v\}$ has a D'-private neighbor in G. We can obtain such a minimizer D' from D in polynomial time by repeatedly removing vertices without private neighbor from $D \backslash \{u, v\}$. This can be seen as follows. It is clear that D' dominates all vertices in $V(G) \backslash D$, since we only remove a vertex from $D \backslash \{u, v\}$ if all its neighbors outside D are dominated by remaining vertices in D. Moreover, since $\{u, v\}$ dominates D, all vertices removed from $D \backslash \{u, v\}$ are dominated by $\{u, v\}$. Note that the fact that u and v are adjacent means that the graph $G[D']$ is connected. We point out that D may have several minimizers for the same edge uv depending on the order in which its vertices are considered.

Example. Consider the graph G and its connected dominating set D in the left-hand side of Figure 2. All private neighbors are colored black. The set D' in the right-hand side is a minimizer of D for uv obtained by removing w_4 from D. Note that u does not have a D'-private neighbor but v does. Instead of removing w_4 we could also have chosen to remove w_2 first, since w_2 does not have a D-private neighbor. Let $D^1 := D \backslash \{w_2\}$. Since w_3 does not have a D^1-private neighbor, we can remove w_3 from D^1. The resulting set $D^2 := D^1 \backslash \{w_3\}$ is a minimizer of D for uv in which every vertex of D^2 (including u) has a D^2-private neighbor.

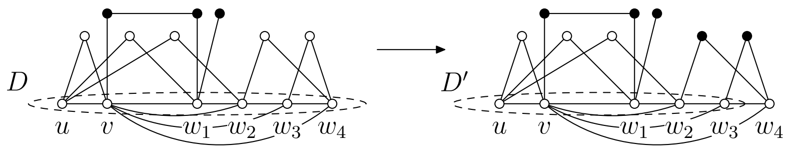

Fig. 2. A dominating set D and a minimizer D' of D for uv.

3 Finding connected dominating subgraphs in P_6-free graphs

Let G be a connected P_6-free graph. We say that D is a *type 1 dominating set* of G if D dominates G and $G[D]$ is an induced C_6. We say that D is a *type 2 dominating set* of G *defined by* $A(D)$ and $B(D)$ if D dominates G and $G[D]$ contains a spanning complete bipartite subgraph with partition classes $A(D)$ and $B(D)$.

Theorem 7. *If G is a connected P_6-free graph, then we can find a type 1 or type 2 dominating set of G in polynomial time.*

Proof. Let $G = (V, E)$ be a connected P_6-free graph with connected order $\pi = x_1, \ldots, x_{|V|}$. Recall that we write $G_i := G[\{x_1, \ldots, x_i\}]$, and note that G_i is connected and P_6-free for every i. For every $2 \leq i \leq n$ we want to find a type 1 or type 2 dominating set D_i of G_i. Let $D_2 := \{x_1, x_2\}$. Suppose $i \geq 3$. Assume D_{i-1} is a type 1 or type 2 dominating set of G_{i-1}. We show how we can use D_{i-1} to find D_i in polynomial time. Since the total number of iterations is $|V|$, we then find a desired dominating subgraph of $G_{|V|} = G$ in polynomial time. We write $x := x_i$. If $x \in N(D_{i-1})$, then we set $D_i := D_{i-1}$. Suppose otherwise. Since π is connected, G_i contains a vertex y (not in D_{i-1}) adjacent to x.

Case 1. D_{i-1} is a type 1 dominating set of G_{i-1}.

We write $G[D_{i-1}] = c_1c_2c_3c_4c_5c_6c_1$. We claim that $D := N_{D_{i-1}}(y) \cup \{x, y\}$ dominates G_i, which means that $D_i := D$ is a type 2 dominating set of G_i defined by $A(D_i) := \{y\}$ and $B(D_i) := \{x\} \cup N_{D_{i-1}}(y)$. Suppose D does not dominate G_i, and let $z \in V(G_i)$ be a vertex not dominated by D. Since D_{i-1} dominates G_{i-1}, we may without loss of generality assume that $yc_1 \in E(G_i)$.

Suppose $yc_4 \in E(G_i)$. Note that z is dominated by G_{i-1}. Without loss of generality, assume z is adjacent to c_2. Consequently, y is not adjacent to c_2. Since z is not adjacent to any neighbor of y and the path $zc_2c_1yc_4c_5$ cannot be induced in G_i, either z or y must be adjacent to c_5. If $zc_5 \in E(G_i)$, then $xyc_4c_5zc_2$ is an induced P_6 in G_i. Hence $zc_5 \notin E(G_i)$ and $yc_5 \in E(G_i)$. In case $zc_6 \in E(G_i)$ we obtain an induced path $xyc_5c_6zc_2$ on six vertices, and in case $zc_6 \notin E(G_i)$ we obtain an induced path $zc_2c_1c_6c_5c_4$. We conclude $yc_4 \notin E(G_i)$.

Suppose y is not adjacent to any vertex in $\{c_3, c_5\}$. Since G_i is P_6-free and $xyc_1c_2c_3c_4$ is a P_6 in G_i, y must be adjacent to c_2. But then $xyc_2c_3c_4c_5$ is an induced P_6 in G_i, a contradiction. Hence y is adjacent to at least one vertex in $\{c_3, c_5\}$, say $yc_5 \in E(G_i)$. By symmetry (using c_5, c_2 instead of c_1, c_4) we find $yc_2 \notin E(G_i)$.

Suppose z is adjacent to c_2. The path $zc_2c_1yc_5c_4$ on six vertices and the P_6-freeness of G_i imply $zc_4 \in E(G_i)$. But then $c_2zc_4c_5yx$ is an induced P_6. Hence $zc_2 \notin E(G_i)$. Also $zc_4 \notin E(G_i)$ as otherwise $zc_4c_5yc_1c_2$ would be an induced P_6, and $zc_3 \notin E(G_i)$ as otherwise $zc_3c_2c_1yx$ would be an induced P_6. Then z must be adjacent to c_6 yielding an induced path $zc_6c_1c_2c_3c_4$ on six vertices. Hence we may choose $D_i := D$.

Case 2. D_{i-1} is a type 2 dominating set of G_{i-1}.

Since D_{i-1} dominates G_{i-1}, we may assume that y is adjacent to some vertex $a \in A(D_{i-1})$. Let $b \in B(D_{i-1})$. Let D be a minimizer of $D_{i-1} \cup \{y\}$ for ab (note that $\{a, b\}$ dominates $D_{i-1} \cup \{y\}$). By definition, D dominates G_i. Also, $G[D]$ contains a spanning (not necessarily complete) bipartite graph with partition classes $A \subseteq A(D_{i-1}), B \subseteq B(D_{i-1}) \cup \{y\}$. Note that we have $y \in D$, because x is not adjacent to D_{i-1} and therefore is a D-private neighbor of y. Since y might not have any neighbors in B but does have a neighbor (vertex a) in A, we chose $y \in B$.

Claim 1. If $G[D]$ contains an induced P_4 starting in y and ending in some $r \in A$, then we can find a type 1 or a type 2 dominating set D_i of G_i in polynomial time.

We prove Claim 1 as follows. Suppose $ypqr$ is an induced path in $G[D]$ with $r \in A$. Since D is a minimizer of $D_{i-1} \cup \{y\}$ for ab and $r \in D \backslash \{a, b\}$, r has a D-private neighbor s by definition. Since $xypqrs$ is a path on six vertices and $x \notin N(D_{i-1})$ holds, x must be adjacent to s. We first show that $D^1 := N_D(y) \cup \{x, y, q, r, s\}$ dominates G_i. See Figure 3 for an illustration of the graph $G[D^1]$. Suppose D^1

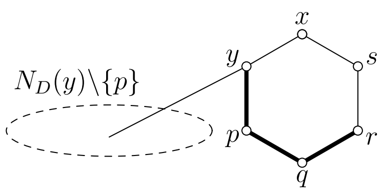

Fig. 3. The graph $G[D^1]$.

does not dominate G. Then there exists a vertex $z \in N(D) \backslash N(D^1)$. Note that $G[(D \backslash \{y\}) \cup \{z\}]$ is connected because the edge ab makes $D \backslash \{y\}$ connected and $\{a, b\}$ dominates D. Let P be a shortest path in $G[(D \backslash \{y\}) \cup \{z\}]$ from z to a vertex $p_1 \in N_D(y)$ (possibly $p_1 = p$). Since $z \notin N(D^1)$ and $p_1 \in D^1$, we have $|V(P)| \geq 3$. This means that $Pyxs$ is an induced path on at least six vertices, unless $r \in V(P)$ (since r is adjacent to s). However, if $r \in V(P)$, then the subpath $z\overrightarrow{P}r$ of P from z to r has at least three vertices (because $z \notin N(D^1)$). This means that $z\overrightarrow{P}rsxy$ contains an induced P_6, a contradiction. Hence D^1 dominates G_i.

To find a type 1 or type 2 dominating set D_i of G_i, we transform D^1 into D_i as follows. Suppose q has a D^1-private neighbor q'. Then $q'qpyxs$ is an induced P_6 in G_i, a contradiction. Hence q has no D^1-private neighbor and the set $D^2 := D^1 \backslash \{q\}$ still dominates G_i. Similarly, r has no D^2-private neighbor r', since otherwise $r'rsxyp$ would be an induced P_6 in G_i. So the set $D^3 := D^2 \backslash \{r\}$ also dominates G_i. Now suppose s does not have a D^3-private neighbor. Then the set $D^3 \backslash \{s\}$ dominates G_i. In that case, we find a type 2 dominating set D_i of G_i defined by $A(D_i) := \{y\}$ and $B(D_i) := N_D(y) \cup \{x\}$. Assume that s has a D^3-private neighbor s' in G_i. Let $D^4 := D^3 \cup \{s'\}$.

Suppose $N_D(y) \backslash \{p\}$ contains a vertex p_2 that has a D^4-private neighbor p_2'. Then $p_2'p_2yxss'$ is an induced P_6, contradicting the P_6-freeness of G_i. Hence we can remove all vertices of $N_D(y) \backslash \{p\}$ from D^4, and the resulting set $D^5 := \{p, y, x, s, s'\}$ still dominates G_i. We claim that $D^6 := D^5 \cup \{q\}$ is a type 1 dominating set of G_i. Clearly, D^6 dominates G_i, since $D^5 \subseteq D^6$. Since $qpyxss'$

is a P_6 and $qpyxs$ is induced, q must be adjacent to s'. Hence D^6 is a type 1 dominating set of G_i, and we choose $D_i := D^6$. This proves Claim 1.

Let $A_1 := N_A(y)$ and $A_2 := A \backslash A_1$. Let $B_1 := N_B(y)$ and $B_2 := B \backslash (B_1 \cup \{y\})$. Since $a \in A_1$, we have $A_1 \neq \emptyset$. If $A_2 = \emptyset$, then we define a type 2 dominating set D_i of G_i by $A(D_i) := A$ and $B(D_i) := B$. Suppose $A_2 \neq \emptyset$. Note $|B| \geq 2$, because $\{b, y\} \subseteq B$. If $B_2 = \emptyset$, then we define D_i by $A(D_i) := A \cup \{y\}$ and $B(D_i) := B_1 = B \backslash \{y\}$. Suppose $B_2 \neq \emptyset$. If $G[A_1 \cup A_2]$ contains a spanning complete bipartite graph with partition classes A_1 and A_2, we define D_i by $A(D_i) := A_1$ and $B(D_i) := A_2 \cup B$. Hence we may assume that there exist two non-adjacent vertices $a_1 \in A_1$ and $a_2 \in A_2$. Let $b^* \in B_2$. Then $ya_1b^*a_2$ is an induced P_4 starting in y and ending in a vertex of A. By Claim 1, we can find a type 1 or type 2 dominating set D_i of G_i in polynomial time. This finishes the proof of Theorem 7. □

We will now prove our main theorem.

Theorem 6. *A graph G is P_6-free if and only if each connected induced subgraph of G contains a dominating induced C_6 or a dominating (not necessarily induced) complete bipartite graph. Moreover, we can find such a dominating subgraph in polynomial time.*

Proof. Let G be a graph. Suppose G is not P_6-free. Then G contains an induced P_6 which contains neither a dominating induced C_6 nor a dominating complete bipartite graph. Suppose G is P_6-free. Let H be a connected induced subgraph of G. Then H is P_6-free as well. We apply Theorem 7 to H. □

The characterization in Theorem 6 is minimal due to the existence of the following family $\mathcal{F}$ of P_6-free graphs. For each $i \geq 2$, let $F_i \in \mathcal{F}$ be the graph obtained from a complete bipartite subgraph with partition classes $X_i = \{x_1, \ldots, x_i\}$ and $Y_i = \{y_1, \ldots, y_i\}$ by adding the edge x_1x_2 as well as for each $h = 1, \ldots, i$ a new vertex x'_h only adjacent to x_h and a new vertex y'_h only adjacent to y_h (see Figure 4 for the graph F_3).

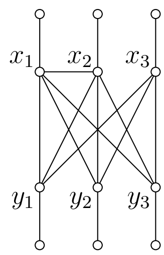

Fig. 4. The graph F_3.

Note that each F_i is P_6-free and that the smallest connected dominating subgraph of F_i is $F_i[X_i \cup Y_i]$, which contains a spanning complete bipartite subgraph.

Also note that none of the graphs F_i contain a dominating *induced* complete bipartite subgraph due to the edge x_1x_2.

We conclude this section by characterizing the class of graphs for which each connected induced subgraph contains a dominating induced C_6 or a dominating *induced* complete bipartite subgraph. Again, we will show how to find these dominating induced subgraphs in polynomial time. Let C_3^L denote the graph obtained from the cycle $c_1c_2c_3c_1$ by adding three new vertices b_1, b_2, b_3 and three new edges c_1b_1, c_2b_2, c_3b_3 (see Figure 5).

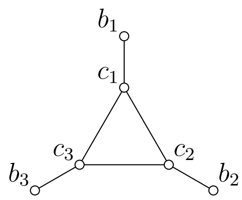

Fig. 5. The graph C_3^L.

Theorem 8. *If G is a connected graph in Forb($\{C_3^L, P_6\}$), then we can find a dominating induced C_6 or a dominating induced complete bipartite subgraph of G in polynomial time.*

Proof. Let $G = (V, E)$ be a connected graph in *Forb*($\{C_3^L, P_6\}$) with connected order $\pi = x_1, \ldots, x_{|V|}$. Recall that we write $G_i := G[\{x_1, \ldots, x_i\}]$, and note that $G_i \in$ *Forb*($\{C_3^L, P_6\}$) for every i. For every $2 \leq i \leq n$ we want to find a dominating set D_i of G_i that either induces a C_6 or a complete bipartite subgraph in G_i. Let $D_2 := \{x_1, x_2\}$. Suppose $i \geq 3$. Assume D_{i-1} induces a dominating C_6 or a dominating complete bipartite subgraph in G_{i-1}. We show how we can use D_{i-1} to find D_i in polynomial time. Since the total number of iterations is $|V|$, we find a desired dominating subgraph of $G_{|V|} = G$ in polynomial time. We write $x := x_i$. If $x \in N(D_{i-1})$, then we set $D_i := D_{i-1}$. Suppose otherwise. Since π is connected, G_i contains a vertex y (not in D_{i-1}) adjacent to x. We first prove a useful claim.

Claim 1. If $N_{D_{i-1}}(y) \cup \{x, y\}$ dominates G_i, then we can find a dominating induced C_6 or a dominating induced complete bipartite subgraph of G in polynomial time.

We prove Claim 1 as follows. Suppose $D^* := N_{D_{i-1}}(y) \cup \{x, y\}$ dominates G_i. We check whether $G[D^*]$ is complete bipartite. If so, then we choose $D_i := D^*$ and we are done. Otherwise y has a neighbor u in D_{i-1} with $N_{D^*}(u) \backslash \{y\} \neq \emptyset$. If u has no D^*-private neighbor, then we remove u from D^* and perform the same check in the smaller set $D^* \backslash \{u\}$. Let u' be a D^*-private neighbor of u in G_i. Let $v \in N_{D^*}(u) \backslash \{y\}$. Then u' is adjacent to any D^*-private neighbor

v' of v, as otherwise $G[\{u, v, y, u', v', x\}]$ is isomorphic to C_3^L. So we find that $D^1 := (D^* \backslash N_{D^*}(u)) \cup \{y, u'\}$ dominates G_i. If u' does not have a D^1-private neighbor, then we remove u' from D^1, check if y is adjacent to two neighbors in the smaller set $D^1 \backslash \{u'\}$ and repeat the above procedure. Let u'' be a D^1-private neighbor of u'. Suppose $N_{D^1}(y) = \{x, u\}$. Then $D^1 = \{x, y, u, u'\}$. If x does not have a D^1-private neighbor, then we choose $D_i := \{y, u, u'\}$. If x has a D^1-private neighbor x', then the P_6-freeness of G_i implies that x' is adjacent to u'', and we choose $D_i := \{x', x, y, u, u', u''\}$.

Suppose $N_{D^1}(y) \backslash \{x, u\} \neq \emptyset$, say y is adjacent to some vertex $t \in D^1 \backslash \{x, u\}$. If t does not have a D^1-private neighbor, then we remove t from D^1 and check if y is adjacent to some vertex in the smaller set $D^1 \backslash \{x, u, t\}$. Let t' be a D^1-private neighbor of t. Then the path $u''u'uytt'$ is an induced P_6 of G_i, unless u'' is adjacent to t'. However, in that case $xyuu'u''t'$ is an induced P_6. This finishes the proof of Claim 1.

Case 1. D_{i-1} **induces a dominating** C_6 **in** G_{i-1}**.**

Since D_{i-1} is a type 1 dominating set of G_{i-1}, we know from the corresponding Case 1 in the proof of Theorem 7 that $D := N_{D_{i-1}}(y) \cup \{x, y\}$ dominates G_i. By Claim 1, we can find a dominating induced C_6 or a dominating induced complete bipartite subgraph of G in polynomial time.

Case 2. D_{i-1} **induces a dominating complete bipartite subgraph in** G_{i-1}**.**

Let $A(D_{i-1})$ and $B(D_{i-1})$ denote the partition classes of D_{i-1}. Note that both $A(D_{i-1})$ and $B(D_{i-1})$ are independent sets. Since D_{i-1} dominates G_{i-1}, we may without loss of generality assume that y is adjacent to some vertex $a \in A(D_{i-1})$. Let $b \in B(D_{i-1})$. Let D be a minimizer of $D_{i-1} \cup \{y\}$ for ab. By definition, D dominates G_i. Also, $G[D]$ contains a spanning (not necessarily complete) bipartite graph with partition classes $A \subseteq A(D_{i-1})$ and $B \subseteq B(D_{i-1}) \cup \{y\}$. Note that $y \in D$, because x is not adjacent to D_{i-1} and therefore is a D-private neighbor of y, and consequently, $y \in B$ because y is adjacent to $a \in A$ (and y might not have any neighbors in B). Let $A_1 := N_A(y)$ and $A_2 := A \backslash A_1$. Let $B_1 := N_B(y)$ and $B_2 := B \backslash (B_1 \cup \{y\})$. Since $a \in A_1$, we have $A_1 \neq \emptyset$.

Suppose $G[D]$ contains an induced P_4 starting in y and ending in a vertex in A. Just as in the proof of Theorem 7 we can obtain (in polynomial time) a dominating C_6 of G_i or else we find that $N_D(y) \cup \{x, y\}$, and consequently, $N_{D_{i-1}}(y) \cup \{x, y\}$ dominates G_i. In the first case, we choose D_i to be the obtained dominating induced C_6. In the second case, we can find a dominating induced C_6 or a dominating induced complete bipartite subgraph of G in polynomial time by Claim 1. So we may assume that $G[D]$ does not contain such an induced P_4. This means that at least one of the sets A_2, B_2 is empty, as otherwise we find an induced path yab_2a_2 for any $a_2 \in A_2$ and $b_2 \in B_2$. We may without loss of generality assume that $A_2 = \emptyset$. (Otherwise, in case $B_2 = \emptyset$, we obtain $B = B_1$, which means that y is adjacent to b as well, so we can reverse the role of A and B.) If $B_2 = \emptyset$, then we find that $A_1 \cup B_1 \cup \{y\} \subset N_{D_{i-1}}(y) \cup \{x, y\}$ dominates G_i, and we are done as a result of Claim 1. So $B_2 \neq \emptyset$. Let $b_2 \in B_2$.

We claim that $D^2 := A_1 \cup B_2 \cup \{x, y\}$ dominates G_i. Suppose otherwise. Then there exists a vertex b'_1 adjacent to some vertex $b_1 \in B_1$ but not adjacent to D^2. Then $G[\{y, a, b_1, x, b_2, b'_1\}]$ is isomorphic to C_3^L, a contradiction. Hence D^2 dominates G_i. If x does not have a D^2-private neighbor, then we can choose $D_i := D^2 \backslash \{x\}$, since $G[D_i]$ is a complete bipartite graph with partition classes A_1 and $B_2 \cup \{y\}$. Suppose x has a D^2-private neighbor x'. If b_2 does not have a D^2-private neighbor, then we remove b_2 from D^2, and check whether B_2 contains another vertex (if not we are done, i.e., we can choose $D_i := A_1 \cup \{x, y\}$, since $G[D_i]$ is a complete bipartite graph with partition classes $A_1 \cup \{x\}$ and $\{y\}$). Suppose b_2 has a D^2-private neighbor b'_2. Then the path $x'xyab_2b'_2$ is a path on six vertices, so we must have $x'b'_2 \in E$.

We claim that $D^3 := \{x', x, y, a, b_2, b'_2\}$ dominates G_i. Suppose otherwise. Then there exists a vertex c' adjacent to some vertex c in $A_1 \cup B_2$ but not adjacent to a vertex in D^3. Suppose $c \in A_1$. Then $c'cb_2b'_2x'x$ is an induced P_6. Suppose $c \in B_2$. Then $c'cayxx'$ is an induced P_6. So D^3 dominates G_i. Since D^3 also induces a C_6 in G_i, we may choose $D_i := D^3$. This finishes the proof of Theorem 8. □

Theorem 9 is an immediate result of Theorem 8 together with the observation that neither the graph P_6 nor C_3^L has a dominating induced C_6 or a dominating induced complete bipartite subgraph.

Theorem 9. *A graph G is in Forb($\{C_3^L, P_6\}$) if and only if each connected induced subgraph of G contains a dominating induced C_6 or a dominating induced complete bipartite graph. Moreover, we can find such a dominating subgraph in polynomial time.*

Bacsó, Michalak and Tuza [2] prove (non-constructively) that a graph G is in *Forb*($\{C_3^L, C_6, P_6\}$) if and only if each connected induced subgraph of G contains a dominating induced complete bipartite graph. Note that Theorem 9 immediately implies this result.

4 The Hypergraph 2-Colorability problem

A *hypergraph* is a pair $(Q, \mathcal{S})$ consisting of a set $Q = \{q_1, \ldots, q_m\}$ and a set $\mathcal{S} = \{S_1, \ldots, S_n\}$ of subsets of Q. With a hypergraph $(Q, \mathcal{S})$ we associate its *incidence graph* I, which is a bipartite graph with partition classes Q and $\mathcal{S}$, where for any $q \in Q, S \in \mathcal{S}$ we have $qS \in E(I)$ if and only if $q \in S$. For any $S \in \mathcal{S}$, we write $H - S := (Q, \mathcal{S} \backslash S)$. A 2-*coloring* of a hypergraph $(Q, \mathcal{S})$ is a partition (Q_1, Q_2) of Q such that $Q_1 \cap S_j \neq \emptyset$ and $Q_2 \cap S_j \neq \emptyset$ for $1 \leq j \leq n$.

The Hypergraph 2-Colorability problem asks whether a given hypergraph has a 2-coloring. This is a well-known NP-complete problem (cf. [10]). Let $\mathcal{H}_6$ denote the class of hypergraphs with P_6-free incidence graphs.

Theorem 10. *The* Hypergraph 2-Colorability *problem restricted to $\mathcal{H}_6$ is polynomially solvable. Moreover, for any 2-colorable hypergraph $H \in \mathcal{H}_6$ we can find a 2-coloring of H in polynomial time.*

Proof. Let $H = (Q, \mathcal{S}) \in \mathcal{H}_6$, and let I be the (P_6-free) incidence graph of H. We assume that I is connected, as otherwise we just proceed component-wise.

Claim 1. We may without loss of generality assume that $\mathcal{S}$ does not contain two sets S_i, S_j with $S_i \subseteq S_j$.

We prove Claim 1 as follows. Suppose $S_i, S_j \in \mathcal{S}$ with $S_i \subseteq S_j$. Note that we can check in polynomial time whether such sets S_i, S_j exist. We show that H is 2-colorable if and only if $H - S_j$ is 2-colorable. Clearly, if H is 2-colorable then $H - S_j$ is 2-colorable. Suppose $H - S_j$ is 2-colorable. Let (Q_1, Q_2) be a 2-coloring of $H - S_j$. By definition, $S_i \cap Q_1 \neq \emptyset$ and $S_i \cap Q_2 \neq \emptyset$. Since $S_i \subseteq S_j$, we also have $S_j \cap Q_1 \neq \emptyset$ and $S_j \cap Q_2 \neq \emptyset$, so (Q_1, Q_2) is a 2-coloring of H. This proves Claim 1.

By Theorem 6, we can find a type 1 or type 2 dominating set D of I in polynomial time. Since I is bipartite, $G[D]$ is bipartite. Let A and B be the partition classes of $G[D]$. Since I is connected, we may without loss of generality assume $A \subseteq Q$ and $B \subseteq \mathcal{S}$. Let $A' := Q \backslash A$ and $B' := \mathcal{S} \backslash B$. We distinguish two cases.

Case 1. D is a type 1 dominating set of I.

We write $I[D] = q_1 S_1 q_2 S_2 q_3 S_3 q_1$, so $A = \{q_1, q_2, q_3\}$ and $B = \{S_1, S_2, S_3\}$. Suppose $A' = \emptyset$, so $Q = \{q_1, q_2, q_3\}$. Obviously, H has no 2-coloring. Suppose $A' \neq \emptyset$ and let $q' \in A'$. Since D dominates I, q' has a neighbor, say S_1, in B. If S_2 and S_3 both have no neighbors in A', then $q' S_1 q_2 S_2 q_3 S_3$ is an induced P_6 in I, a contradiction. Hence at least one of them, say S_2, has a neighbor in A'.

We claim that the partition (Q_1, Q_2) of Q with $Q_1 := A' \cup \{q_1\}$ and $Q_2 := \{q_2, q_3\}$ is a 2-coloring of H. We have to check that every $S \in \mathcal{S}$ has a neighbor in both Q_1 and Q_2. Recall that S_1 has neighbors q_1 and q_2 and S_3 has neighbors q_1 and q_3, so S_1 and S_3 are OK. Since S_2 is adjacent to q_2 and has a neighbor in A', S_2 is also OK. It remains to check the vertices in B'. Let $S \in B'$. Since D dominates I and I is bipartite, S has at least one neighbor in A. Suppose S has exactly one neighbor, say q_1, in A. Then $S q_1 S_1 q_2 S_2 q_3$ is an induced P_6 in I, a contradiction. Hence S has at least two neighbors in A. The only problem occurs if S is adjacent to q_2 and q_3 but not to q_1. However, since S_2 is adjacent to q_2 and q_3, S must have a neighbor in A' due to Claim 1. Hence (Q_1, Q_2) is a 2-coloring of H.

Case 2. D is a type 2 dominating set of I.

Suppose $A' = \emptyset$. Then $|B| = 1$ as a result of Claim 1. Let $B = \{S\}$ and $q \in A$. Since S is adjacent to all vertices in A, we find that $B' = \emptyset$ as a result of Claim 1. Hence H has no 2-coloring if $|A| = 1$, and H has a 2-coloring $(\{q\}, A \backslash \{q\})$ if $|A| \geq 2$. Suppose $A' \neq \emptyset$. We claim that (A, A') is a 2-coloring of H. This can be seen as follows. By definition, each vertex in $\mathcal{S}$ is adjacent to a vertex in A. Suppose $|B| = 1$ and let $B = \{S\}$. Since S dominates Q and $A' \neq \emptyset$, S has at least one neighbor in A'. Suppose $|B| \geq 2$. Since every vertex in B is adjacent to all vertices in A, every vertex in $\mathcal{S}$ must have a neighbor in A' as a result of Claim 1. □

5 Conclusions

The key contributions of this paper are the following. We presented a new characterization of the class of P_6-free graphs, which strengthens results of Liu and Zhou [14] and Liu, Peng and Zhao [15]. We used an algorithmic technique to prove this characterization. Our main algorithm efficiently finds for any given connected P_6-free graph a dominating subgraph that is either an induced C_6 or a (not necessarily induced) complete bipartite graph. Besides these main results, we also showed that our characterization is “minimal” in the sense that there exists an infinite family of P_6-free graphs for which a smallest connected dominating subgraph is a (not induced) complete bipartite graph. We also characterized the class $Forb(\{C_3^L, P_6\})$ in terms of connected dominating subgraphs, thereby generalizing a result of Bacsó, Michalak and Tuza [2].

Our main algorithm can be useful to determine the computational complexity of decision problems restricted to the class of P_6-free graphs. To illustrate this, we applied this algorithm to prove that the Hypergraph 2-Colorability problem is polynomially solvable for the class of hypergraphs with P_6-free incidence graphs. Are there any other decision problems for which the algorithm is useful? In recent years, several authors studied the classical k-Colorability problem for the class of P_ℓ-free graphs for various combinations of k and ℓ [13, 16, 18]. The 3-Colorability problem is proven to be polynomially solvable for the class of P_6-free graphs [16]. Hoàng et al. [13] show that for all fixed $k \geq 3$ the k-Colorability problem becomes polynomially solvable for the class of P_5-free graphs. They pose the question whether there exists a polynomial time algorithm to determine if a P_6-free graph can be 4-colored. We do not know yet if our main algorithm can be used for simplifying the proof of the result in [16] or for solving the open problem described above. We leave these questions for future research.

The next class to consider is the class of P_7-free graphs. Recall that a graph G is P_7-free if and only if each connected induced subgraph of G contains a dominating subgraph of diameter at most three [1]. Using an approach similar to the one described in this paper, it is possible to find such a dominating subgraph in polynomial time. However, a more important question is whether this characterization of P_7-free graphs can be narrowed down. Also determining the computational complexity of the Hypergraph 2-Colorability problem for the class of hypergraphs with P_7-free incidence graphs is still an open problem.

Finally, a natural problem for a given graph class deals with its recognition. We are not aware of any recognition algorithms for (even bipartite or triangle-free) P_7-free graphs that have a better running time than the trivial algorithm that checks for every 7-tuple of vertices whether they induce a path. This might be another interesting direction for future research, considering the following results on recognition of (subclasses of) P_6-free graphs. Fouquet [9] presents a cubic recognition algorithm for the class of P_6-free graphs (in an internal report). Giakoumakis and Vanherpe [11] show that bipartite P_6-free graphs can be recognized in linear time. They do this by extending the techniques developed in [6] for linear time recognition of P_4-free graphs (also see [12]) and by using a characterization of P_6-free graphs in terms of canonical decomposition trees (which is not related to our characterization) from [9]. Brandstädt, Klembt and Mahfud [4] show that triangle-free P_6-free graphs have bounded clique-width. The recognition algorithm they obtain from this result runs in quadratic time. Since the class of P_6-free graphs has unbounded clique-width (cf. [3]), their technique cannot be applied to find a quadratic recognition algorithm for the class of P_6-free graphs.

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