On Graph Contractions and Induced Minors*

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Abstract. The Induced Minor Containment problem takes as input two graphs G and H, and asks whether G has H as an induced minor. We show that this problem is fixed parameter tractable in $|V_H|$ if G belongs to any minor-closed graph class and H is a planar graph. For a fixed graph H, the H-Contractibility problem is to decide whether a graph can be contracted to H. The computational complexity classification of this problem is still open. So far, H has a dominating vertex in all cases known to be polynomially solvable, whereas H does not have such a vertex in all cases known to be NP-complete. Here, we present a class of graphs H with a dominating vertex for which H-Contractibility is NP-complete. We also present a new class of graphs H for which H-Contractibility is polynomially solvable. Finally, we study the (H, v)-Contractibility problem, where v is a vertex of H. The input of this problem is a graph G and an integer k, and the question is whether G is H-contractible such that the "bag" of G corresponding to v contains at least k vertices. We show that this problem is NP-complete whenever H is connected and v is not a dominating vertex of H.

1 Introduction

There are several natural and elementary algorithmic problems that check if the structure of some fixed graph H shows up as a *pattern* within the structure of some input graph G. This paper studies the computational complexity of two

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such problems, namely the problems of deciding if a graph G can be transformed into a graph H by performing a sequence of edge contractions and vertex deletions, or by performing a sequence of edge contractions only. Theoretical motivation for this research can be found in [3, 8, 14, 15] and comes from hamiltonian graph theory [12] and graph minor theory [16], as we will explain below. Practical applications include surface simplification in computer graphics [1, 4] and cluster analysis of large data sets [5, 11, 13]. In the first practical application, graphic objects are represented using (triangulated) graphs and these graphs need to be simplified. One of the techniques to do this is by using edge contractions. In the second application, graphs are coarsened by means of edge contractions.

Basic Terminology. All graphs in this paper are undirected, finite, and have neither loops nor multiple edges. Let G and H be two graphs. The *edge contraction* of edge $e = uv$ in G removes u and v from G, and replaces them by a new vertex adjacent to precisely those vertices to which u or v were adjacent. If H can be obtained from G by a sequence of edge contractions, vertex deletions and edge deletions, then G contains H as a *minor*. If H can be obtained from G by a sequence of edge contractions and vertex deletions, then G contains H as an *induced minor*. If H can be obtained from G by a sequence of edge contractions, then G is said to be *contractible to* H and G is called H*-contractible*. This is equivalent to saying that G has a so-called H*-witness structure* $\mathcal{W}$, which is a partition of V_G into $|V_H|$ sets $W(h)$, called H*-witness sets*, such that each $W(h)$ induces a connected subgraph of G and for every two $h_i, h_j \in V_H$, witness sets $W(h_i)$ and $W(h_j)$ are adjacent in G if and only if h_i and h_j are adjacent in H. Here, two subsets A, B of V_G are called *adjacent* if there is an edge $ab \in E_G$ with $a \in A$ and $b \in B$. Clearly, by contracting the vertices in the witness sets $W(h)$ to a single vertex for every $h \in V_H$, we obtain the graph H. See Figure 1 for an example that shows that in general the witness sets $W(h)$ are not uniquely defined.

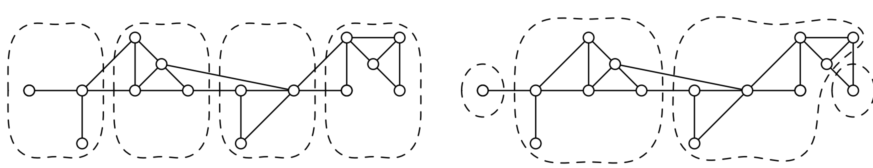

Fig. 1. Two P_4-witness structures of a graph.

The problems H-Minor Containment, H-Induced Minor Containment and H-Contractibility ask if an input graph G has H as a minor, has H as an induced minor or is H-contractible, respectively. When H is part of the input, we denote the three problems by Minor Containment, Induced Minor Containment and Contractibility.

Known Results. A celebrated result by Robertson and Seymour [16] states that H-Minor Containment can be solved in polynomial time for every fixed graph H. The complexity classification of the other two problems is still open. Fellows, Kratochvíl, Middendorf, and Pfeiffer [8] give both polynomially solvable and NP-complete cases for the the H-Induced Minor Containment problem. They also prove the following.

Theorem 1 ([8]). *For every fixed H, the H-Induced Minor Containment problem is polynomially solvable for planar input graphs.*

Brouwer and Veldman [3] initiated the research on the H-Contractibility problem. Their main result is stated below. A *dominating* vertex is a vertex adjacent to all other vertices.

Theorem 2 ([3]). *Let H be a connected triangle-free graph. The H-Contractibility problem is in P if H has a dominating vertex, and is NP-complete otherwise.*

Note that a connected triangle-free graph with a dominating vertex is a star and that $H = P_4$ (path on four vertices) and $H = C_4$ (cycle on four vertices) are the smallest graphs H for which H-Contractibility is NP-complete. The research of [3] was continued in [14, 15].

Theorem 3 ([14, 15]). *Let H be a connected graph with $|V_H| \leq 5$. The H-Contractibility problem is in P if H has a dominating vertex, and is NP-complete otherwise.*

The NP-completeness results in Theorem 2 and 3 can be extended using the notion of degree-two covers. Let $d_G(x)$ denote the degree of a vertex x in a graph G. A graph H' with an induced subgraph H is called a *degree-two cover* of H if the following two conditions both hold. First, for all $x \in V_H$, if $d_H(x) = 1$ then $d_{H'}(x) \geq 2$, and if $d_H(x) = 2$ and its two neighbors in H are adjacent then $d_{H'}(x) \geq 3$. Second, for all $x' \in V_{H'} \setminus V_H$, either x' has one neighbor and this neighbor is in H, or x' has two neighbors and these two neighbors form an edge in H.

Theorem 4 ([14]). *Let H' be a degree-two cover of a connected graph H. If H-Contractibility is NP-complete, then so is H'-Contractibility.*

In [3, 14] a number of other results are shown. To discuss these we need some extra terminology (which we will use later in the paper as well). For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \emptyset$, we denote their *join* by $G_1 \bowtie G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{uv \mid u \in V_1, v \in V_2\})$, and their *disjoint union* by $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. For the disjoint union $G \cup G \cup \cdots \cup G$ of k copies of the graph G, we write kG; for $k = 0$ this yields the empty graph $(\emptyset, \emptyset)$. For integers $a_1, a_2, \ldots, a_k \geq 0$, we let $H_i^*(a_1, a_2, \ldots, a_k)$ be the graph $K_i \bowtie (a_1P_1 \cup a_2P_2 \cup \cdots \cup a_kP_k)$, where K_i is the complete graph on i vertices and P_i is the path on i vertices. Note that $H_1^*(a_1)$ denotes a star on a_1+1 vertices. Brouwer and Veldman [3] show that H-Contractibility is polynomially solvable for $H = H_1^*(a_1)$ or $H = H_1^*(a_1, a_2)$ for any $a_1, a_2 \geq 0$. Observe that $H_i^*(0) = K_i$ and that K_i-Contractibility is equivalent to K_i-Minor Containment, and hence polynomially solvable, by the previously mentioned result of Robertson and Seymour [16]. These results have been generalized in [14] leading to the following theorem.

Theorem 5 ([14]). *The H-Contractibility problem is in* P *for:*

1. $H = H_1^*(a_1, a_2, \ldots, a_k)$ *for any* $k \geq 1$ *and* $a_1, a_2, \ldots, a_k \geq 0$
2. $H = H_2^*(a_1, a_2)$ *for any* $a_1, a_2 \geq 0$
3. $H = H_3^*(a_1)$ *for any* $a_1 \geq 0$
4. $H = H_i^*(0)$, *for any* $i \geq 1$.

Our Results and Paper Organization. In Section 2 we first recall some basic notions in parameterized complexity. Then we consider the Induced Minor Containment problem, where we assume that G belongs to some fixed *minor-closed* graph class $\mathcal{G}$ (i.e., contains every minor of every member) and that H is planar. We prove that under these assumptions this problem becomes fixed parameter tractable in $|V_H|$. Note that the graph H in Theorem 1 may be assumed to be planar, as otherwise any (planar) input graph is a no-instance. This observation, together with the fact that the class of planar graphs is minor-closed, implies that our aforementioned result generalizes Theorem 1.

The presence of a dominating vertex seems to play an interesting role in the complexity classification of the H-Contractibility problem. So far, in all polynomially solvable cases of this problem the pattern graph H has a dominating vertex, and in all NP-complete cases H does not have such a vertex. Following this trend, we extend Theorem 5 in Section 3.1 by showing that $H_4^*(a_1)$-Contractibility is polynomially solvable for all $a_1 \geq 0$. In Section 3.2 however we present the first class of graphs H with a dominating vertex for which H-Contractibility is NP-complete.

In Section 4 we study the following problem.

(H, v)-Contractibility
Instance: A graph G and a positive integer k.
Question: Does G have an H-witness structure $\mathcal{W}$ with $|W(v)| \geq k$?

We show that (H, v)-Contractibility is NP-complete whenever H is connected and v is not a dominating vertex of H. For example, let $P_3 = p_1p_2p_3$. Then the (P_3, p_3)-Contractibility problem is NP-complete (whereas P_3-Contractibility is polynomially solvable). Section 5 contains the conclusions and mentions a number of open problems.

2 Induced Minors in Minor-Closed Classes

We start this section with a short introduction on the complexity classes XP and FPT. Both classes are defined in the framework of parameterized complexity as developed by Downey and Fellows [7]. The complexity class XP consists of parameterized decision problems Π such that for each instance (I, k) it can be decided in $\mathcal{O}(f(k)|I|^{g(k)})$ time whether $(I, k) \in \Pi$, where f and g are computable functions depending only on k. So XP consists of parameterized decision problems which can be solved in polynomial time if the parameter is considered to be a constant. A problem is *fixed parameter tractable* in k if an instance (I, k) can be solved in time $\mathcal{O}(f(k)n^c)$, where f denotes a computable function and c a constant independent of k. Therefore, such an algorithm may provide a solution to the problem efficiently if the parameter is reasonably small. The complexity class $\mathsf{FPT} \subseteq \mathsf{XP}$ is the class of all fixed-parameter tractable decision problems.

We show that Induced Minor Containment is fixed parameter tractable in $|V_H|$ on input pairs (G, H) with G from any fixed minor-closed graph class $\mathcal{G}$ and H planar. Before doing this we first recall the following notions. A *tree decomposition* of a graph $G = (V, E)$ is a pair $(\mathcal{X}, T)$, where $\mathcal{X} = \{X_1, \ldots, X_r\}$ is a collection of *bags*, which are subsets of V, and T is a tree on vertex set $\mathcal{X}$ with the following three properties. First, $\bigcup_{i=1}^{r} X_i = V$. Second, for each $uv \in E$, there exists a bag X_i such that $\{u, v\} \subseteq X_i$. Third, if $v \in X_i$ and $v \in X_j$ then all bags in T on the (unique) path between X_i and X_j contain v. The *width* of a tree decomposition $(\mathcal{X}, T)$ is $\max\{|X_i| - 1 \mid i = 1, \ldots, r\}$, and the *treewidth* $\mathrm{tw}(G)$ of G is the minimum width over all possible tree decompositions of G.

Our proof idea is as follows. We check if the input graph G has sufficiently large treewidth. If not, then we apply the monadic second-order logic result of Courcelle [6]. Otherwise, we show that G always contains H as an induced minor. Before going into details, we first introduce some additional terminology.

The $k \times k$ *grid* M_k has as vertex set all pairs (i, j) for $i, j = 0, 1, \ldots, k-1$, and two vertices (i, j) and (i', j') are joined by an edge if and only if $|i-i'|+|j-j'| = 1$. For $k \geq 2$, let Γ_k denote the graph obtained from M_k by triangulating its faces as follows: add an edge between vertices (i, j) and (i', j') if $i - i' = 1$ and $j' - j = 1$, and add an edge between corner vertex $(k-1, k-1)$ and every external vertex, i.e., every vertex (i, j) with $i \in \{0, k-1\}$ or $j \in \{0, k-1\}$. We let Π_k denote the graph obtained from Γ_k by adding a new vertex s that is adjacent to every vertex of Γ_k. See Figure 2 for the graphs M_6, Γ_6, and Π_6.

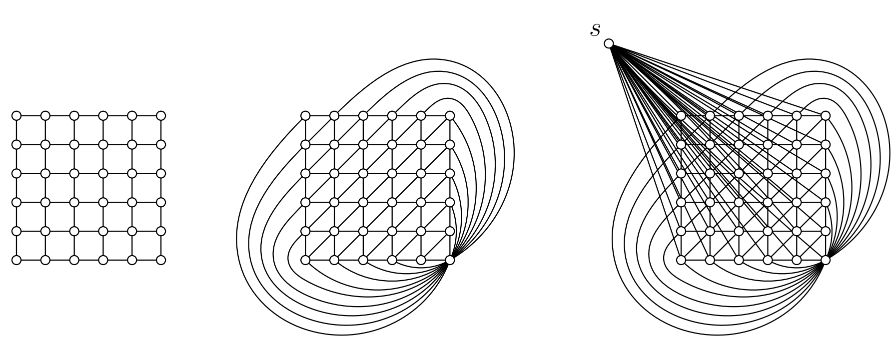

Fig. 2. The graphs M_6, Γ_6, and Π_6, respectively.

Let $\mathcal{F}$ denote a set of graphs. Then a graph G is called *$\mathcal{F}$-minor-free* if G does not contain a graph in $\mathcal{F}$ as a minor. If $\mathcal{F} = \{F\}$ we say that G is F-minor-free. We need the following results from [9] and [8], respectively.

Theorem 6 ([9]). *For every graph F, there is a constant c_F such that every connected F-minor-free graph of treewidth at least $c_F \cdot k^2$ is Γ_k-contractible or Π_k-contractible.*

Theorem 7 ([8]). *For every planar graph H, there is a constant b_H such that every planar graph of treewidth at least b_H contains H as an induced minor.*

We also recall the well-known result of Robertson and Seymour [17] proving Wagner's conjecture.

Theorem 8 ([17]). *A graph class $\mathcal{G}$ is minor-closed if and only if there exists a finite set $\mathcal{F}$ of graphs such that $\mathcal{G}$ is equal to the class of $\mathcal{F}$-minor-free graphs.*

We are now ready to prove our generalization of Theorem 1 (recall that the graph H in this theorem may be assumed to be planar and that the class of planar graphs is minor-closed).

Theorem 9. *Let $\mathcal{G}$ be any minor-closed graph class. Then the* Induced Minor Containment *problem is fixed parameter tractable in $|V_H|$ on input pairs (G, H) with $G \in \mathcal{G}$ and H planar.*

Proof. Let H be a fixed planar graph with constant b_H as defined in Theorem 7. Let G be a graph on n vertices in a minor-closed graph class $\mathcal{G}$. From Theorem 8 we deduce that there exists a finite set $\mathcal{F}$ of graphs such that G is $\mathcal{F}$-minor-free. By Theorem 6, for each $F \in \mathcal{F}$, there exists a constant c_F such that every connected F-minor-free graph of treewidth at least $c_F \cdot b_H^2$ is Γ_{b_H}-contractible or Π_{b_H}-contractible. Let $c := \max\{c_F \mid F \in \mathcal{F}\}$. We first check if $\text{tw}(G) < c \cdot b_H^2$. We can do so as recognizing such graphs is fixed parameter tractable in $c \cdot b_H^2$ due to a result of Bodlaender [2].

Case 1. $\text{tw}(G) < c \cdot b_H^2$. The property of having H as an induced minor is expressible in monadic second-order logic (cf. [8]). Hence, by a well-known result of Courcelle [6], we can determine in $\mathcal{O}(n)$ time if G contains H as an induced minor.

Case 2. $\text{tw}(G) \geq c \cdot b_H^2$. We will show that in this case G is a yes-instance. By Theorem 6, we find that G is Γ_{b_H}-contractible or Π_{b_H}-contractible.

First suppose G is Γ_{b_H}-contractible. Then G has Γ_{b_H} as an induced minor. It is easy to prove that M_{b_H} has treewidth b_H. It is clear from the definition of treewidth that any supergraph of M_{b_H}, and Γ_{b_H} in particular, has treewidth at least b_H. Note that Γ_{b_H} is a planar graph. Then, by Theorem 7, Γ_{b_H} has H as an induced minor. Consequently, by transitivity, G has H as an induced minor.

Now suppose G is Π_{b_H}-contractible. Let $\mathcal{W}$ be a Π_{b_H}-witness structure of G. We remove all vertices in $W(s)$ from G. We then find that G has Γ_{b_H} as an induced minor and return to the previous situation. □

3 The H-Contractibility Problem

As we mentioned in Section 1, the presence of a dominating vertex seems to play an interesting role in the complexity classification of the H-Contractibility problem. So far, in all polynomially solvable cases of this problem the pattern graph H has a dominating vertex, and in all NP-complete cases H does not have such a vertex. The first result of this section follows this pattern: we prove in Section 3.1 that $H_4^*(a_1)$-Contractibility is polynomially solvable for all $a_1 \geq 0$. In Section 3.2 however we present the first class of graphs H with a dominating vertex for which H-Contractibility is NP-complete.

3.1 Polynomial Cases With Four Dominating Vertices

In this section, we extend Theorem 5 by showing that H-Contractibility is polynomially solvable for $H = H_4^*(a_1)$ for any integer $a_1 \geq 0$.

Let H and G be graphs such that G is H-contractible. Let $\mathcal{W}$ be an H-witness structure of G. We call the subset of vertices in a witness set $W(h_i)$ that are adjacent to vertices in some other witness set $W(h_j)$ a *connector* $C_{\mathcal{W}}(h_i, h_j)$. We use the notion of connectors to simplify the witness structure of an $H_4^*(a_1)$-contractible graph. Let $G[U]$ denote the subgraph of G induced by $U \subseteq V_G$. Let $y_1, \ldots, y_4$ denote the four dominating vertices of $H_4^*(a_1)$ and let $x_1, \ldots, x_{a_1}$ denote the remaining vertices of $H_4^*(a_1)$.

Lemma 1. *Let $a_1 \geq 0$. Every $H_4^*(a_1)$-contractible graph has an $H_4^*(a_1)$-witness structure $\mathcal{W}'$ such that $1 \leq |C_{\mathcal{W}'}(x_i, y_j)| \leq 2$ for all $1 \leq i \leq a_1$ and for all $1 \leq j \leq 4$.*

Proof. Let $\mathcal{W}$ be an $H_4^*(a_1)$-witness structure of an $H_4^*(a_1)$-contractible graph G. Below we transform $\mathcal{W}$ into a witness structure $\mathcal{W}'$ that satisfies the statement of the lemma.

From each $W(x_i)$ we move as many vertices as possible to $W(y_1) \cup \cdots \cup W(y_4)$ in a greedy way and without destroying the witness structure. This way we obtain an $H_4^*(a_1)$-witness structure $\mathcal{W}'$ of G. We claim that $1 \leq |C_{\mathcal{W}'}(x_i, y_j)| \leq 2$ for all $1 \leq i \leq a_1$ and for all $1 \leq j \leq 4$.

Suppose, for contradiction, that $|C_{\mathcal{W}'}(x_i, y_j)| \geq 3$ for some x_i and y_j. Let u_1, u_2, u_3 be three vertices in $C_{\mathcal{W}'}(x_i, y_j)$. Then $G[W'(x_i) \setminus \{u_1\}]$ has at least one component that contains a vertex of $C_{\mathcal{W}'}(x_i, y_1) \cup \cdots \cup C_{\mathcal{W}'}(x_i, y_4)$. Let $L_1, \ldots, L_p$ denote the vertex sets of these components. Observe that each L_q must be adjacent to at least two witness sets from $\{W'(y_1), \ldots, W'(y_4)\}$ that are not adjacent to $W'(x_i) \setminus L_q$, since otherwise we would have moved L_q to $W'(y_1) \cup \cdots \cup W'(y_4)$. Since u_1 is adjacent to at least one witness set, we deduce that $p = 1$. The fact that $p = 1$ implies that u_1 must even be adjacent to at least two unique witness sets from $\{W'(y_1), \ldots, W'(y_4)\}$, i.e., that are not adjacent to $W'(x_i) \setminus \{u_1\}$; otherwise we would have moved u_1 and all components of $G[W'(x_i) \setminus \{u_1\}]$ not equal to L_1 to $W'(y_1) \cup \cdots \cup W'(y_4)$. By the same arguments, exactly the same fact holds for u_2 and u_3. This is not possible, as three vertices cannot be adjacent to two unique sets out of four. □

We need one additional result, which can be found in [14] but follows directly from the polynomial time result on minors in [16].

Lemma 2 ([14]). *Let G be a graph and let $Z_1, \ldots, Z_p \subseteq V_G$ be p specified non-empty pairwise disjoint sets such that $\sum_{i=1}^{p} |Z_i| \leq k$ for some* fixed *integer k. The problem of deciding whether G is K_p-contractible with K_p-witness sets $U_1, \ldots U_p$ such that $Z_i \subseteq U_i$ for $i = 1, \ldots, p$ is polynomially solvable.*

Using Lemma 1 and Lemma 2 we can prove the following result.

Theorem 10. *The $H_4^*(a_1)$-Contractibility problem is solvable in polynomial time for any fixed non-negative integer a_1.*

Proof. Let $G = (V, E)$ be a connected graph. We guess a set $\mathcal{S} = \{C_{\mathcal{W}}(x_i, y_j) \mid 1 \leq i \leq a_1, 1 \leq j \leq 4\}$ of connectors of size at most two. For each vertex u in each connector $C_{\mathcal{W}}(x_i, y_j) \in \mathcal{S}$ we pick a neighbor of u that is not in $\mathcal{S}$ and place it in a set Z_j. This leads to four sets $Z_1, \ldots, Z_4$. We remove $\mathcal{S}$ from G and call the resulting graph G'. We check the following. First, we determine in polynomial time whether $Z_1 \cup \cdots \cup Z_4$ is contained in one component T of G'. If so, we check whether T is K_4-contractible with K_4-witness sets $U_1, \ldots, U_4$ such that $Z_i \subseteq U_i$ for $i = 1, \ldots, 4$. This can be done in polynomial time due to Lemma 2. We then check whether the remaining components of G' together with the connectors $C_{\mathcal{W}}(x_i, y_j) \in \mathcal{S}$ form witness sets $W(x_i)$ for $i = 1, \ldots, a_1$. Also, this can be done in polynomial time; there is only one unique way to do this because witness sets $W(x_i)$ are not adjacent to each other. If somewhere in the whole process we get stuck, we check another set $\mathcal{S}$ of connectors and start all over. Due to Lemma 1, it indeed suffices to consider only sets of connectors that have size at most two. Hence, the total number of different 5-tuples $(\mathcal{S}, Z_1, \ldots, Z_4)$ is bounded by a polynomial in a_1, and consequently, the polynomial time result follows. □

3.2 NP-Complete Cases With a Dominating Vertex

We show the existence of a class of graphs H with a dominating vertex such that H-Contractibility is NP-complete. To do this we need the following.

Proposition 1. *Let H be a graph. If H-Induced Minor Containment is NP-complete, then so are $(K_1 \bowtie H)$-Contractibility and $(K_1 \bowtie H)$-Induced Minor Containment.*

Proof. Let H and G be two graphs. Write $K_1 = (\{x\}, \emptyset)$. We claim that the following three statements are equivalent.

(i) G has H as an induced minor;
(ii) $K_1 \bowtie G$ is $(K_1 \bowtie H)$-contractible;
(iii) $K_1 \bowtie G$ has $K_1 \bowtie H$ as an induced minor.

"$(i) \Rightarrow (ii)$" Suppose G has H as an induced minor. Then, by definition, G contains an induced subgraph G' that is H-contractible. We extend an H-witness structure $\mathcal{W}$ of G' to a $(K_1 \bowtie H)$-witness structure of $K_1 \bowtie G$ by putting x and all vertices in $V_G \setminus V_{G'}$ in $W(x)$. This shows that $K_1 \bowtie G$ is $(K_1 \bowtie H)$-contractible.

"$(ii) \Rightarrow (iii)$" Suppose $K_1 \bowtie G$ is $(K_1 \bowtie H)$-contractible. By definition, $K_1 \bowtie G$ contains $K_1 \bowtie H$ as an induced minor.

"$(iii) \Rightarrow (i)$" Suppose $K_1 \bowtie G$ has $K_1 \bowtie H$ as an induced minor. Then $K_1 \bowtie G$ contains an induced subgraph G^* that is $K_1 \bowtie H$-contractible. Let $\mathcal{W}$ be a $(K_1 \bowtie H)$-witness structure of G^*. If $x \in V_{G^*}$, then we may assume without loss of generality that $x \in W(x)$. We delete $W(x)$ and obtain an H-witness structure of the remaining subgraph of G^*. This subgraph is an induced subgraph of G. Hence, G contains H as an induced minor. □

Fellows et al. [8] showed that there exists a graph $\bar{H}$ on 68 vertices such that $\bar{H}$-Induced Minor Containment is NP-complete; this graph is depicted in Figure 3. Combining their result with Proposition 1 (applied repeatedly) leads

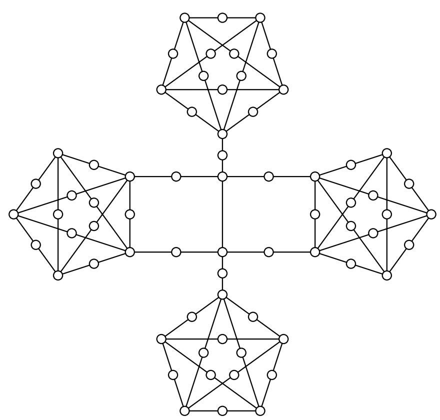

Fig. 3. The graph $\bar{H}$.

to the following corollary.

Corollary 1. *For any $i \geq 1$, $(K_i \bowtie \bar{H})$-*Contractibility *is* NP-*complete.*

4 The (H, v)-Contractibility Problem

We start with an observation. A *star* is a complete bipartite graph in which one of the partition classes has size one. The unique vertex in this class is called the *center* of the star. We denote the star on $k + 1$ vertices with center c and leaves $b_1, \ldots, b_p$ by $K_{p,1}$.

Observation 1 *The* $(K_{p,1}, c)$-Contractibility *problem is polynomially solvable for all* $p \geq 1$.

Proof. Let graph $G = (V, E)$ and integer k form an instance of the $(K_{p,1}, c)$-Contractibility problem. We may without loss of generality assume that $|V| \geq k + p$. If G is $K_{p,1}$-contractible, then there exists a $K_{p,1}$-witness structure $\mathcal{W}$ of G such that $|W(b_i)| = 1$ for all $1 \leq i \leq k$. This can be seen as follows. As long as $|W(b_i)| \geq 2$ we can move vertices from $W(b_i)$ to $W(c)$ without destroying the witness structure. Our algorithm would just guess the witness sets $W(b_i)$ and check whether $V \setminus (W(b_1) \cup \cdots W(b_p))$ induces a connected subgraph. As the total number of guesses is bounded by a polynomial in p, this algorithm runs in polynomial time. □

We expect that there are relatively few pairs (H, v) for which (H, v)-Contractibility is in P (under the assumption P $\neq$ NP). This is due to the following observation and the main result in this section that shows that (H, v)-Contractibility is NP-complete whenever v is not a dominating vertex of H.

Observation 2 *Let* H *be a graph. If* H-Contractibility *is* NP-*complete, then* (H, v)-Contractibility *is* NP-*complete for every vertex* $v \in V_H$.

Theorem 11. *Let* H *be a connected graph and let* $v \in V_H$. *The* (H, v)-Contractibility *problem is* NP-*complete if* v *does not dominate* H.

Proof. Let H be a connected graph, and let v be a vertex of H that does not dominate H. Let $N_H(v)$ denote the neighborhood of v in H. We partition $V_H \setminus \{v\}$ into three sets $V_3 := V_H \setminus (N_H(v) \cup \{v\})$, $V_2 := \{w \in N_H(v) \mid w \text{ is not adjacent to } V_3\}$ and $V_1 := \{w \in N_H(v) \mid w \text{ is adjacent to } V_3\}$. Note that neither V_1 nor V_3 is empty because H is connected and v does not dominate H; V_2 might be empty.

Clearly, (H, v)-Contractibility is in NP, because we can verify in polynomial time whether a given partition of the vertex set of a graph G forms an H-witness structure of G with $|W(v)| \geq k$. In order to show that (H, v)-Contractibility is NP-complete, we use a reduction from 3-SAT, which is well-known to be NP-complete (cf. [10]). Let $X = \{x_1, \ldots, x_n\}$ be a set of variables and $C = \{c_1, \ldots, c_m\}$ be a set of clauses making up an instance of 3-SAT. Let $\overline{X} := \{\overline{x} \mid x \in X\}$. We introduce two additional literals s and t, as well as $2n$ additional clauses $s_i := (x_i \vee \overline{x}_i \vee s)$ and $t_i := (x_i \vee \overline{x}_i \vee t)$ for $i = 1, \ldots, n$. Let $S := \{s_1, \ldots, s_n\}$ and $T := \{t_1, \ldots, t_n\}$. Note that all $2n$ clauses in $S \cup T$ are satisfied for any satisfying truth assignment for C. For every vertex $w \in V_1$ we create a copy X^w of the set X, and we write $X^w := \{x_1^w, \ldots, x_n^w\}$. The literals s^w, t^w and the sets $\overline{X}^w$, C^w, S^w and T^w are defined similarly for every $w \in V_1$.

We construct a graph G such that C is satisfiable if and only if G has an H-witness structure $\mathcal{W}$ with $|W(v)| \geq k$. In order to do this, we first construct a subgraph G^w of G for every $w \in V_1$ in the following way:

- every literal in $X^w \cup \overline{X}^w \cup \{s^w, t^w\}$ and every clause in $C^w \cup S^w \cup T^w$ is represented by a vertex in G^w

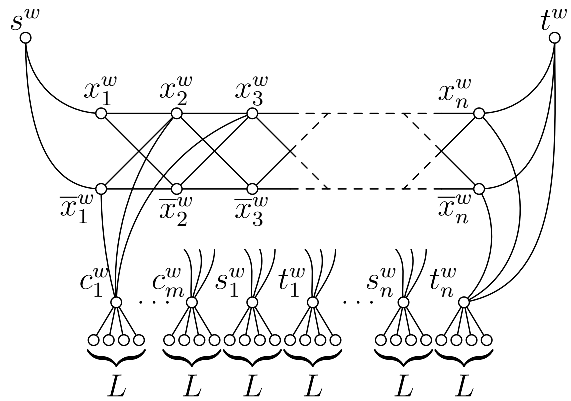

Fig. 4. A subgraph G^w, where $c_1^w = (\overline{x}_1^w \vee x_2^w \vee x_3^w)$.

- we add an edge between $x \in X^w \cup \overline{X}^w \cup \{s^w, t^w\}$ and $c \in C^w \cup S^w \cup T^w$ if and only if x appears in c;
- for every $i = 1, \ldots, n-1$, we add edges $x_i^w x_{i+1}^w$, $x_i^w \overline{x}_{i+1}^w$, $\overline{x}_i^w x_{i+1}^w$, and $\overline{x}_i^w \overline{x}_{i+1}^w$
- we add edges $s^w x_1^w$, $s^w \overline{x}_1^w$, $t^w x_n^w$, and $t^w \overline{x}_n^w$
- for every $c \in C^w \cup S^w \cup T^w$, we add L vertices whose only neighbor is c; we determine the value of L later and refer to the L vertices as the *pendant vertices*.

See Figure 4 for a depiction of subgraph G^w. For clarity, most of the edges between the clause vertices and the literal vertices have not been drawn. We connect these subgraphs to each other as follows. For every $w, x \in V_1$, we add an edge between s^w and s^x in G if and only if w is adjacent to x in H. Let v^* be some fixed vertex in V_1. We add an edge between $s_1^{v^*}$ and s_1^w for every $w \in V_1 \setminus \{v^*\}$. No other edges are added between vertices of two different subgraphs G^w and G^x.

We add a copy of $H[V_2 \cup V_3]$ to G as follows. Vertex $x \in V_2$ is adjacent to s^w in G if and only if x is adjacent to w in H. Vertex $x \in V_3$ is adjacent to both s^w and t^w in G if and only if x is adjacent to w in H. Finally, we connect every vertex $x \in V_2$ to $s_1^{v^*}$. See Figure 5 for an example.

We define $L := (2 + 2n)|V_1| + |V_2| + |V_3| + 1$ and $k := (L + 1)(m + 2n)|V_1|$. We prove that G has an H-witness structure $\mathcal{W}$ with $|W(v)| \geq k$ if and only if C is satisfiable.

Suppose $t : X \rightarrow \{T, F\}$ is a satisfying truth assignment for C. Let X_T (respectively X_F) be the variables that are set to true (respectively false) by t. For every $w \in V_1$, we define $X_T^w := \{x_i^w \mid x_i \in X_T\}$ and $\overline{X}_T^w := \{\overline{x} \mid x \in X_T^w\}$; the sets X_F^w and $\overline{X}_F^w$ are defined similarly. We define the H-witness sets of G as follows. Let $W(w) := \{w\}$ for every $w \in V_2 \cup V_3$, and let $W(w) := \{s^w, t^w\} \cup X_F^w \cup \overline{X}_T^w$ for every $w \in V_1$. Finally, let $W(v) := V_G \setminus (\bigcup_{w \in V_1 \cup V_2 \cup V_3} W(w))$. Note that for every $w \in V_1$ and for every $i = 1, \ldots, n$, exactly one of $x_i^w, \overline{x}_i^w$ belongs to $X_F^w \cup \overline{X}_T^w$. Hence, $G[W(w)]$ is connected for every $w \in V_1$.

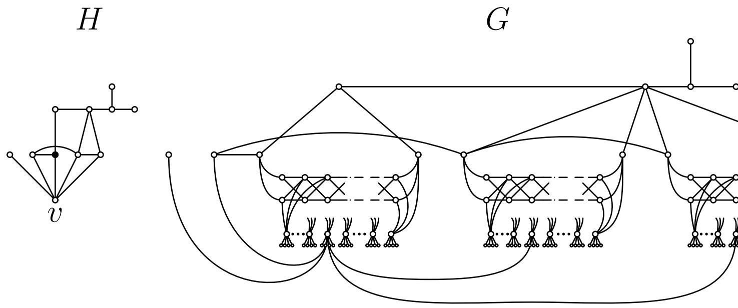

Fig. 5. A graph H, where v^* is the black vertex, and the corresponding graph G.

Since t is a satisfying truth assignment for C, every c_i^w is adjacent to at least one vertex of $X_T^w \cup \overline{X}_F^w$ for every $w \in V_1$; by definition, this also holds for every s_i^w and t_i^w. This together with the edges between $s_1^{v^*}$ and s_1^w for every $w \in V_1 \setminus \{v^*\}$ assures that $G[W(v)]$ is connected. So the witness set $G[W(w)]$ is connected for every $w \in V_H$. By construction, two witness sets $W(w)$ and $W(x)$ are adjacent if and only if w and x are adjacent in H. Hence $\mathcal{W} := \{W(w) \mid w \in V_H\}$ is an H-witness structure of G. Witness set $W(v)$ contains $n|V_1|$ literal vertices, $(m+2n)|V_1|$ clause vertices and L pendant vertices per clause vertex, i.e., $|W(v)| = (L+1)(m+2n)|V_1| + n|V_1| \geq k$.

Suppose G has an H-witness structure $\mathcal{W}$ with $|W(v)| \geq k$. We first show that all of the $(m+2n)|V_1|$ clause vertices must belong to $W(v)$. Note that for every $w \in V_1$, the subgraph G^w contains $2 + 2n + (L+1)(m+2n)$ vertices: the vertices s^w and t^w, the $2n$ literal vertices in $X^w \cup \overline{X}^w$, the $m+2n$ clause vertices and the $L(m+2n)$ pendant vertices. Hence we have

$$|V_G| = (2 + 2n + (L+1)(m+2n))|V_1| + |V_2| + |V_3|.$$

Suppose there exists a clause vertex c that does not belong to $W(v)$. Then the L pendant vertices adjacent to c cannot belong to $W(v)$ either, as $W(v)$ is connected and the pendant vertices are only adjacent to c. This means that $W(v)$ can contain at most $|V_G| - (L+1) = (L+1)(m+2n)|V_1| - 1$ vertices, contradicting the assumption that $W(v)$ contains at least $k = (L+1)(m+2n)|V_1|$ vertices. So all of the $(m+2n)|V_1|$ clause vertices, as well as all the pendant vertices, must belong to $W(v)$.

We define $W_i := \bigcup_{w \in V_i} W(w)$ for $i = 1, 2, 3$ and prove four claims.

Claim 1: $V_3 = W_3$.

The only vertices of G that are not adjacent to any of the clause vertices or pendant vertices in $W(v)$ are the vertices of V_3. As W_3 contains at least $|V_3|$ vertices, this proves Claim 1.

Claim 2: For any $w \in V_1$, both s^w and t^w belong to W_1.

Let w be a vertex in V_1, and let $w' \in V_3$ be a neighbor of w in H. Recall that both s^w and t^w are adjacent to w' in G. Suppose that s^w or t^w belongs to

$W(v) \cup W_2$. By Claim 1, $w' \in W_3$. Then $W(v) \cup W_2$ and W_3 are adjacent. By construction, this is not possible. Suppose that s^w or t^w belongs to W_3. Then W_3 and $W(v)$ are adjacent, as s^w and t^w are adjacent to at least one clause vertex, which belongs to $W(v)$. This is not possible.

Claim 3: For any $w \in V_1$, at least one of each pair $x_i^w, \overline{x}_i^w$ of literal vertices belongs to $W(v)$.

Let $w \in V_1$. Suppose there exists a pair of literal vertices $x_i^w, \overline{x}_i^w$ both of which do not belong to $W(v)$. Apart from its L pendant vertices, the vertex t_i^w is only adjacent to x_i^w, $\overline{x}_i^w$ and t^w. The latter vertex belongs to W_1 due to Claim 2. Hence t_i^w and its L pendant vertices induce a component of $G[W(v)]$. Since $G[W(v)]$ contains other vertices as well, this contradicts the fact that $G[W(v)]$ is connected.

Claim 4: There exists a $w \in V_1$ for which at least one of each pair $x_i^w, \overline{x}_i^w$ of literal vertices belongs to W_1.

Let $S' := \{s^w \mid w \in V_1\}$ and $T' := \{t^w \mid w \in V_1\}$. By Claim 2, $S' \cup T' \subseteq W_1$. Suppose, for contradiction, that for every $w \in V_1$ there exists a pair $x_i^w, \overline{x}_i^w$ of literal vertices, both of which do not belong to W_1. Then for any $x \in V_1$, the witness set containing t^x does not contain any other vertex of $S' \cup T'$, as there is no path in $G[W_1]$ from t^x to any other vertex of $S' \cup T'$. But that means W_1 contains at least $|V_1| + 1$ witness sets, namely $|V_1|$ witness sets containing one vertex from T', and at least one more witness set containing vertices of S'. This contradiction to the fact that W_1, by definition, contains exactly $|V_1|$ witness sets finishes the proof of Claim 4.

Let $w \in V_1$ be a vertex for which of each pair $x_i^w, \overline{x}_i^w$ of literal vertices exactly one vertex belongs to W_1 and the other vertex belongs to $W(v)$; such a vertex w exists as a result of Claim 3 and Claim 4. Let t be the truth assignment that sets all the literals of $X^w \cup \overline{X}^w$ that belong to $W(v)$ to true and all other literals to false. Note that the vertices in C^w form an independent set in $W(v)$. Since $G[W(v)]$ is connected, each vertex $c_i^w \in C^w$ is adjacent to at least one of the literal vertices set to true by t. Hence t is a satisfying truth assignment for C. □

5 Open Problems

The most challenging task is to finish the computational complexity classification of both the H-Induced Minor Containment problem and the H-Contractibility problem. With regards to the second problem, all previous evidence suggested some working conjecture stating that this problem is polynomially solvable if H contains a dominating vertex and NP-complete otherwise. However, in this paper we presented a class of graphs H with a dominating vertex for which H-Contractibility is NP-complete. This sheds new light on the H-Contractibility problem and raises a whole range of new questions.

1. What is the smallest graph H that contains a dominating vertex for which H-Contractibility is NP-complete?

The smallest graph known so far is the graph $K_1 \bowtie \bar{H}$, where $\bar{H}$ is the graph on 68 vertices depicted in Figure 3. By Observation 2, we deduce that $(K_1 \bowtie \bar{H}, v)$-Contractibility is NP-complete for all $v \in V_{K_1 \bowtie \bar{H}}$. The following question might be easier to answer.

*2. What is the smallest graph H that contains a dominating vertex v for which (H, v)-*Contractibility *is* NP*-complete?*

We showed that (H, v)-Contractibility is NP-complete if H is connected and v does not dominate H. We still expect a similar result for H-Contractibility.

*3. Is the H-*Contractibility *problem* NP*-complete if H does not have a dominating vertex?*

Lemma 1 plays a crucial role in the proof of Theorem 10 that shows that $H_4^*(a_1)$-Contractibility is polynomially solvable. This lemma cannot be generalized such that it holds for the $H_i^*(a_1)$-Contractibility problem for $i \geq 5$ and $a_1 \geq 2$. Hence, new techniques are required to attack the $H_i^*(a_1)$-Contractibility problem for $i \geq 5$ and $a_1 \geq 2$.

*4. Is $H_5^*(a_1)$-*Contractibility *in* P *for all $a_1 \geq 0$?*

We expect that the (H, v)-Contractibility problem is in P for only a few target pairs (H, v). One such class of pairs might be (K_p, v), where v is an arbitrary vertex of K_p. Using similar techniques as before (i.e., simplifying the witness structure), one can easily show that (K_p, v)-Contractibility is polynomially solvable for $p \leq 3$.

*5. Is (K_p, v)-*Contractibility *in* P *for all $p \geq 4$?*

We finish this section with some remarks on fixing the parameter k in an instance (G, k) of the (H, v)-Contractibility problem.

Proposition 2. *The (P_3, p_3)-*Contractibility *problem is in* XP.

Proof. We first observe that any graph G that is a yes-instance of this problem has a P_3-witness structure $\mathcal{W}$ with $|W(p_1)| = 1$. This is so, as we can move all but one vertex from $W(p_1)$ to $W(p_2)$ without destroying the witness structure (see also Figure 1). Moreover, such a graph G contains a set $W^* \subseteq W(p_3)$ such that $|W^*| = k$ and $G[W^*]$ is connected. Hence we act as follows.

Let G be a graph. We guess a vertex v and a set V^* of size k. We put all neighbors of v in a set W_2. We check if $G[V^*]$ is connected. If so, we check for each $y \in V_G \setminus (V^* \cup N(v) \cup \{v\})$ whether it is separated from $N(v)$ by V^* or not. If so, we put y in V^*. If not, we put y in W_2. In the end we check if $G[W_2]$ and $G[V^*]$ are connected. If so, G is a yes-instance of (P_3, p_3)-Contractibility, as $W(p_1) = \{v\}$, $W(p_2) = W_2$ and $W(p_3) = V^*$ form a P_3-witness structure of G with $|W(p_3)| \geq k$. If not, we guess another pair (v, V^*) and repeat the steps above. Since these steps can be performed in polynomial time and the total number of guesses is bounded by a polynomial in k, the result follows. □

*6. Is (P_3, p_3)-*Contractibility *in* FPT*?*

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