Proportiones Perfectus Law and the Physics of the Golden Section

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Author’s contribution
The sole author designed, analyzed and interpreted and prepared the manuscript.

Abstract

The proportiones perfectus law is introduced. Let \( \sigma_x^y = \frac{x^y + \sqrt{x^{2y} + 4y}}{2} \). By definition, in the spectrum \( 1 \leq y \leq x, x \geq 1, \sigma_x^y \) is a proportione perfectus. With \( \sigma_x^y \) so defined, for an arbitrary positive integer \( h_1 \), it is shown that there exists an integer sequence \( H_n \) satisfying the quasi-geometric relation \( h_{n+1} = \text{round}(\sigma_x^y h_n), n \geq 1 \) such that the arithmetic relation \( h_{n+2} = xh_{n+1} + yh_n \) holds. The golden mean, designated \( \sigma_1^1 \) or \( \phi \), becomes the most basic and fundamental of proportiones perfectus. New concepts to the study of the golden section are presented: chirality, number genetics and law of polarity, special numerical harmony, and chemical geometry. A geometrical basis for the fine-structure constant in the golden section is established. Our stating of over forty theorems in this reading serves no other purpose than that of expanding the theory of the golden section while equipping the interested reader with instruments for further research and development of this science of number.

Keywords: Chemical geometry; chirality and homochirality; fine-structure constant; golden section; number genetics and law of polarity; proportiones perfectus; special numerical harmony.
1 Introduction

1.1 General

“The Golden Section as a concept and a term has had a long history in the arts as well as the sciences” – McWhinnie [1]. Boeyens and Thackeray [2] say, “We suggest that there is a strong case that the so-called ‘Golden Ratio’ (1.61803…) can be related not only to aspects of mathematics but also to physics, chemistry, biology and the topology of space-time.” They further state: “Less well known is the way in which the crystallographic structure of DNA, stress patterns in nanomaterials, the stability of atomic nuclides and the periodicity of atomic matter depend on the Golden Ratio.” Unfortunately, as Stakhov [3] points out, “During historical development the modern “materialistic” pedagogics had thrown out the golden section and the “harmony idea” on the dump of the doubtful scientific concepts together with astrology and other esoteric sciences. As a result, a majority of people well know “Pythagorean Theorem” but have rather dim idea about the golden section, one of the “treasures” of geometry.”

The role of the golden section in the physical universe goes beyond simply amazing. Olsen [4] says, “Even the heavens seem to be structured according to Golden Fibonacci “phyllotactic” relationships. When viewing Venus from Earth a pentagonal rosette forms every eight years (or thirteen Venerian years) i.e. 13:8:8.5. Also both the mean orbits and mean diameters of Earth and Mercury are in a $\phi^2:1$ relationship.” Yalta et al. [5] state, “Currently, there exists a growing interest towards the concepts of golden ratio and Fibonacci sequence along with their potential implications in the cardiovascular field. … a variety of studies associating these concepts with anatomy, physiology, electrocardiogram (ECD) and echocardiogram of the heart have been published in the literature, all exclusively yielding positive results.”

Akhtaruzzaman and Shafie [6] point out that “Midhat Gazale says that until Euclid the golden ratio’s mathematical properties were not studied.” Even today the golden section remains a “canned” concept. In order to extend the theory of the golden section, the present author, see [7,8], has introduced a new kind of number: parent number. This number is the seed value of a Fibonacci sequence.

1.2 Definition of a Fibonacci sequence

Definition 1.1

An integer sequence $H_n$ is a Fibonacci sequence if it satisfies the relations

\[ h_n = h_{n-1} + h_{n-2}, \ n \geq 3 \]  \hspace{1cm} (1.1)

\[ h_n = \text{round}(\phi h_{n-1}), \ n \geq 2 \] \hspace{1cm} (1.2)

simultaneously.

The “traditional” Fibonacci sequence

\[ 1,2,3,5,8,\ldots \] \hspace{1cm} (1.3)

is one member of the family of these “Fibonacci” sequences. It is for this reason, namely that the sequence (1.3) is a member of a family of infinite size, that the term Fibonacci sequence is used to describe these sequences, not to introduce any confusion. A sequence like

\[ 7,11,18,29,47,\ldots \] \hspace{1cm} (1.4)
for instance, being governed by relations (1.1) and (1.2), is duly called a Fibonacci sequence, and this nomenclature shall not confuse any serious reader. The sequence (1.4) is a regeneration of Lucas numbers. One may recall the well-known Lucas sequence

\[2,1,3,4,7,11,\ldots\] (1.5)

We refer to the Turing Sunflower Project [9] for the less known existence of “double Fibonacci numbers” in nature as observed in sunflower phyllotaxis. This is simply the Fibonacci sequence

\[4,6,10,16,26,\ldots\] (1.6)

Bolotin [10] says that “… Fibonacci numbers occur in Nature often enough to prove that they reflect some naturally occurring patterns …” One such pattern is the logarithmic spiral e.g. in spiral galaxies; fractal in character since it exhibits self-similarity.

Invoking 2 and 2 as seed values to the relation (1.1), El Naschie [11] assembles the sequence

\[2,2,4,6,10,\ldots\] (1.7)

which is an extension of the Fibonacci sequence (1.6) two steps backward. Falcon and Plaza [12] point out, “In the present days there is a huge interest of modern science in the application of the golden section and Fibonacci numbers.” Stewart [13], cited by Kata [14] says, “By using mathematics to organize and systematize our ideas about patterns, we have discovered a great secret: nature’s patterns are not just there to be admired, they are vital clues to the rules that govern natural processes.” Fowler [15] notes, “That there appears to be a connection between the Fibonacci numbers (and hence the golden section) and phyllotaxis (i.e., the arrangement of leaves on a stem, scales on a pine cone, florets on a sunflower, inflorescences on a cauliflower, etc.) is an old and tantalizing observation.” Stakhov [16] says that the golden section “was raised in the ancient Greece to the level of aesthetic canon and main constant of the universe.”

### 1.3 Parent number

The seed value of a Fibonacci sequence is called a parent number. There exist algorithms for detecting parent numbers.

**Definition 1.2**

An integer \(x\) is a parent number if:

\[
\begin{align*}
\text{round}\left(\frac{x}{\phi}\right) &= y, \\
\text{round}(y\phi) &\neq x
\end{align*}
\] (1.8)

The abovemented definition is one such algorithm for detecting parent numbers in the domain of natural numbers. It must be noted therefore that Fibonacci sequences only require one seed value, the parent number. Only after the second term is generated using relation (1.2) can it be decided to employ relation (1.1).

It is the object of this thesis to introduce some new concepts to the study of the golden section while equipping the reader with tools for further development of this geometry. Before we proceed there is need to introduce proportiones perfectus.
1.4 Proportiones perfectus law

De Spinadel [17] introduced two families of metallic means. Let

\[ \sigma_X^Y = \frac{x + \sqrt{x^2 + 4y}}{2} \]  

(1.9)

Family 1 (our designation) is defined by \( \sigma_X^x, x \geq 1 \). The golden mean, silver mean, etc. are siblings in this family. Family 2 is defined by \( \sigma_X^y, y \geq 1 \). The golden mean, copper mean, etc. are siblings in this family. In passing we introduce Family 3 in which the golden section also claims full membership. This family is defined by \( \sigma_X^z, x \geq 1 \). Now,

Definition 1.3

\( \sigma_X^z \) is a proportione perfectus if \( x \geq 1, 1 \leq y \leq x \).

Axiom 1.1

When \( 1 \leq y \leq 0.5x, x \geq 1 \), then round\( (\sigma_X^z) = x \).

Axiom 1.2

When \( 0.5x < y \leq x, x \geq 1 \), then round\( (\sigma_X^z) = x + 1 \).

Axiom 1.3

Let \( H_n \) be defined by \( h_{n+1} = \text{round}(\sigma_X^z h_n), h_1 = 1, n \geq 1 \). For an arbitrary positive integer \( h_1 \), let \( H_n \) be defined by \( h_{n+1} = \text{round}(\sigma_X^z h_n), n \geq 1 \). When \( 1 \leq y \leq 0.5x, x \geq 1 \), then \( h_{n+1} - h_1 h_n = y h_{n-1} \).

Axiom 1.4

Let \( H'_n \) be defined by \( h'_{n+1} = \text{round}(\sigma_X^z h'_n), h'_1 = 1, n \geq 1 \). For an arbitrary positive integer \( h_1 \), let \( H'_n \) be defined by \( h'_{n+1} = \text{round}(\sigma_X^z h'_n), n \geq 1 \). When \( 0.5x < y \leq x, x \geq 1 \), then \( h_2 h_n - h_{n+1} = rh_{n-1} + y h_{n-2} \), where \( r = x - y \).

Corollary 1.1 arises from the abovestated axioms.

Corollary 1.1: proportiones perfectus law

Let \( \sigma_X^z \) be a proportione perfectus. For an arbitrary positive integer \( h_1 \), there exists an integer sequence \( H_n \) such that when

\[ h_{n+1} = \text{round}(\sigma_X^z h_n), n \geq 1 \]  

(1.10)

Then

\[ h_{n+2} = x h_{n+1} + y h_n, n \geq 1 \]  

(1.11)
We find Falcon and Plaza [12] immediately worth discussing. We here quote verbatim the definition of the k-th Fibonacci sequence:

“For any integer number \( k \geq 1 \), the k-th Fibonacci sequence, say \( \{F_{k,n}\}_{n \in \mathbb{N}} \) is defined recurrently by \( F_{k,0} = 0, F_{k,1} = 1, \) and \( F_{k,n+1} = kF_{k,n} + F_{k,n-1} \) for \( n \geq 1 \).”

One may notice that if we allow \( \sigma^1_k, \sigma^2_k, \geq 1 \), the k-th Fibonacci sequences of Falcon and Plaza are governed by proportiones perfectus. Here, \( \sigma^1_k \) is Family 1 metallic means of de Spinadel [17]. A result of particular interest is obtained at \( \sigma^1_k = \varphi^3 \). Using relation (1.12) the sequence

\[
H_n = 1,4,17,72,305,1292,\ldots
\]

is assembled. Recall \( F_n = (1.3) \). Notice that with \( H_n = (1.13) \),

\[
h_i = \frac{f_{3i-1}}{2}, \; i \geq 1
\]

Again let 3 be a seed value to relation (1.12) still with \( \sigma^1_k = \sigma^2_k \). We obtain the sequence

\[
G_n = 3,13,55,233,987,4181,\ldots
\]

where

\[
g_i = f_{3i}
\]

Many interesting results are obtained from studying \( \sigma^1_k \) alongside \( \sigma^1_k \) and this should be taken as an important area of further research by the interested reader.

In line with the above, consider \( \sigma^1_k, \; x \geq 1 \).

Let

\[
H_n = 1, h_2,xh_2 + 1,\ldots
\]

First consider \( x \geq 2 \). From axiom 1.1,

\[
H_n = 1, x,x^2 + 1, x^3 + 2x, x^4 + 3x^2 + 1,\ldots
\]

In the sequence (1.18), for \( n = 1,3,5,7, \text{ etc.} \) the constant 1 and the coefficients of powers of \( x \) are:

\[
1; \; 1,1; 1,3,1; 1,5,6,1; 1,7,15,10,1;\ldots
\]

These constants are obtained from Pascal triangle as marked in Table 1.1.

The summation of these constants yields

\[
1,2,5,13,34,89,\ldots
\]
Table 1.1. Generalized Sequence (1.18) in the Pascal triangle

Now for \( n = 2,4,6,8,\text{etc.} \) in the sequence (1.18), coefficients of powers of \( x \) are given by

\[
1; 1,2; 1,4,3; 1,6,10,4; 1,8,21,20,5; \ldots
\]

(1.21)

The unmarked diagonals in Table 1.1 clearly show these constants. Their summation yields

\[
1,3,8,21,55, \ldots
\]

(1.22)

The sequences (1.20) and (1.22) are the well-known Fibonacci numbers.

Now consider the sequence (1.17) with \( x = 1 \). From axiom 1.2,

\[
H_n = 1, x + 1, x^2 + x + 1, x^3 + x^2 + 2x + 1, \ldots
\]

(1.23)

Again we are interested in the constant 1 and the coefficients of powers of \( x \). In the sequence (1.23) we have, for odd \( n \),

\[
1; 1,1,1; 1,1,3,2,1; 1,1,5,4,6,3,1; \ldots
\]

(1.24)

Table 1.2 shows how these constants are remarkably obtained from the Pascal triangle.

Table 1.2. Generalized Sequence (1.23) in the Pascal triangle (odd \( n \))

In the sequence (1.23) for even \( n \) we have

\[
1,1,1,2,1; 1,1,4,3,3,1; \ldots
\]

(1.25)
Table 1.3 shows these constants in the Pascal triangle. Notice that the summation of elements of the terms of the sequence (1.24) yields the sequence (1.22) and the summation of the elements of the terms of the sequence (1.25) yields the sequence (1.20).

Table 1.3. Generalized Sequence (1.23) in the Pascal triangle (even n)

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<td>1680</td>
<td>1120</td>
<td>720</td>
<td>480</td>
</tr>
</tbody>
</table>

The depiction in Table 1.1 is a well-known result in the literature, first reported by George Polya (1887 – 1985), see Stakhov and Aranson [18]. However Tables 1.2 and 1.3 show yet another novel way of obtaining Fibonacci numbers from the Pascal triangle. The “Fibonacciism” in the Pascal triangle is remarkable and noteworthy. The ubiquity of the so-called Fibonacci numbers in both physical and abstract systems leads to the inevitable conclusion that phi, the golden section, is one of the most important intelligence signatures in the universe. This constant is the most basic and fundamental of proportiones perfectus. Corollary 1.1 holding, a new kind of calculus has emerged and the specialist in mathematical analysis may find this concept interesting.

This work is dedicated to the study of $H_n$ satisfying both relations (1.10) and (1.11) with $\sigma_n^2 = \varphi$, and $H_n$ is a Fibonacci sequence.

2 Number Genetics

In this section the theory of number genetics is introduced as a system of logic for the golden section. Number genetics in the theory of the golden section shows how Fibonacci sequences, though assembled independently of each other, are interconnected – mathematical entanglement.

2.1 Computing machines for number genetics

2.1.1 Machine A

Given an arbitrary Fibonacci sequence $H_n$, compute

$$2h_1 - h_2 = g_{n-2}$$ (2.1)

such that

$$h_1 = g_n \pm 1; n \geq 4$$ (2.2)

Notice that $g_n = \text{round}(\varphi^2 g_{n-2})$. We say that $h_1$ is a daughter cell of $g_n$, and $g_n$ is a mother cell of $h_1$.

If $h_1 = g_n + 1$, then

$$H_n = g_n + 1, g_{n+1} + 2, g_{n+2} + 3, g_{n+3} + 5, g_{n+4} + 8, ...$$ (2.3)
If \( h_1 = g_n - 1 \), then

\[
H_n = g_n - 1, g_{n+1} - 2, g_{n+2} - 3, g_{n+3} - 5, g_{n+4} - 8, \ldots
\]  
(2.4)

2.1.2 Machine B

Given an arbitrary Fibonacci sequence \( H_n \), compute

\[
2h_2 - h_3 = l_{i-2}
\]  
(2.5)
such that

\[
h_2 = l_i \pm 1 ; 3 \leq i \leq 4
\]  
(2.6)

Notice that \( l_i = \text{round}(\varphi^2l_{i-2}) \).

2.2 Mechanical representation of a Fibonacci sequence

For purposes of this reading there is need to represent a Fibonacci sequence \( H_n \) in the form

\[
h_i = g_{n+i-1} \pm f_i , n \geq 4
\]  
(2.7)

where \( F_n = (1.3) \) and \( G_n \) is a Fibonacci sequence. This becomes a straightforward way of understanding Fibonacci sequences mechanically.

2.2.1 Type I sequence

This type of \( H_n \) is such that

\[
h_1 = g_{n+i-1} + f_i ; i \geq 1, n \geq 4
\]  
(2.8)

where \( G_n \) is a Fibonacci sequence and \( F_n = (1.3) \).

2.2.2 Type II sequence

This type of \( H_n \) is such that

\[
h_1 = g_{n+i-1} - f_i ; i \geq 1, n \geq 4
\]  
(2.9)

where \( G_n \) is a Fibonacci sequence and \( F_n = (1.3) \).

The notations (2.8) and (2.9) are used hereinafter without repeating the definitions. It follows therefore that throughout this manuscript the designation \( F_n \) is reserved for the sequence (1.3).

2.3 Proof of the law of polarity

The law of polarity states that for any arbitrary Fibonacci sequence \( H_n \), if

\[
h_1 = g_n + 1 ; n \geq 4
\]

then

\[
h_2 = l_i - 1 ; 3 \leq i \leq 4.
\]
If\[ h_1 = g_n - 1 ; n \geq 4 \]
then\[ h_2 = l_i + 1 ; 3 \leq i \leq 4; \]
where $G_n$ and $L_n$ are Fibonacci sequences.

We prove the law of polarity by way of theorems 2.1 and 2.2. Theorems 2.3 to 2.16 are also stated to this effect and help the reader appreciate the fundamental premises of number genetics.

**Theorem 2.1**

*Given a Fibonacci sequence $H_n$, if $h_1 = g_n + 1 ; n \geq 4$, then $h_2 = l_i - 1 , 3 \leq i \leq 4, G_n$ and $L_n$ are Fibonacci sequences.*

**Proof**

$H_n = g_n + 1, g_{n+1} + 2, g_{n+2} + 3, g_{n+3} + 5, g_{n+4} + 8, ...$

Since $h_2 = l_i - 1$, it follows $l_i - 1 = g_{n+1} + 2$, thus $l_i = g_{n+1} + 3$.

If $n = 4$, then $l_{i-2} = g_3 + 1$. But $n = 4$ implies $g_n = p_{j+n-1} - f_{n}$ where $P_n$ is a Fibonacci sequence and $F_n = (1.3)$ and from theorem 2.3, $g_3 + 1$ is not a parent number. This means $l_1 = g_2 + 1 = l_{i-3}$; therefore $i = 4$.

If $n = 5$, then $l_{i-2} = g_4 + 1$.

When $g_n = p_{j+n-1} - f_{n}$, then $l_{i-2} = l_1$ because $g_4 + 1$ is a parent number, theorem 2.4. This means $i = 3$.

When $g_n = p_{j+n-1} + f_{n}$, then $g_4 + 1$ is not a parent number, but $g_3 + 1$ is, theorems 2.5 and 2.6, thus $l_{i-3} = g_3 + 1 = l_1$.

$\therefore i = 4$.

For $n \geq 6$,

$l_{i-2} = g_{n-1} + 1$.

Let $n - 1 = u$. This means $u \geq 5$, and from theorem 2.7, $g_u + 1$ is a parent number. It follows $l_{i-2} = l_1$, thus $i = 3$.

**Theorem 2.2**

*Given a Fibonacci sequence $H_n$, if $h_1 = g_n - 1 ; n \geq 4$, then $h_2 = l_i + 1 , 3 \leq i \leq 4, G_n$ and $L_n$ are Fibonacci sequences.*
Proof

\[H_i = g_n - 1, g_{n+1} - 2, g_{n+2} - 3, g_{n+3} - 5, g_{n+4} - 8, \ldots\]
Since \(h_2 = l_1 + 1\), it follows

\[l_1 + 1 = g_{n+1} - 2, \text{ thus} \]
\[l_i = g_{n+1} - 3.\]

If \(n = 4\), then \(l_{i-2} = g_2 - 1\). But \(n = 4\) implies \(g_n = p_{j+n-1} + f_n\) where \(P_n\) is a Fibonacci sequence and \(F_n = 1, 2, 3, 5, 8, \ldots\) and from theorem 2.13, \(g_3 - 1\) is not a parent number. This means

\[l_1 = g_2 - 1 = l_{i-3}; \text{ therefore } i = 4.\]
If \(n = 5\), then

\[l_{i-2} = g_4 - 1.\]

When \(g_n = p_{j+n-1} + f_n\), then \(l_{i-2} = l_1\) because \(g_4 - 1\) is a parent number, theorem 2.5. This means \(i = 3\).

When \(g_n = p_{j+n-1} - f_n\), then \(g_4 - 1\) is not a parent number, but \(g_3 - 1\) is, theorems 2.4 and 2.15, thus

\[l_{i-3} = g_3 - 1 = l_1,\]

\[\therefore i = 4.\]

For \(n \geq 6,\)

\[l_{i-2} = g_{n-1} - 1.\]

Let \(n - 1 = u. \text{ This means } u \geq 5, \text{ and from theorem 2.7, } g_u - 1\) is a parent number. It follows \(l_{i-2} = l_1, \text{ thus } i = 3.\)

Theorem 2.3

When a Fibonacci sequence \(H_i\) is such that \(h_i = g_{n+i-1} - f_i; n \geq 4, \text{ then } h_3 - 1\) is a parent number and \(h_3 + 1\) is not a parent number.

Proof

\[h_3 = g_{n+2} - f_3 = g_{n+2} - 3\]
\[h_3 - 1 = g_{n+2} - 4.\]

Let \(n + 2 = u. \text{ Since } n \geq 4, \text{ it means } u \geq 6. \text{ From theorem 2.8, } g_u - 4\) is a parent number, therefore \(h_3 - 1\) is a parent number.

\[h_3 + 1 = g_u - 2. \text{ From theorem 2.7, since } u \geq 6, \text{ it follows } g_u - 1\) is a parent number, say \(l_1, \text{ } L_u\) is a Fibonacci sequence.

But \(l_1 \pm 1\) is not a parent number, theorem 2.11, therefore \(g_u - 2 = g_u - 1 - 1 = l_1 - 1\) is not a parent number.
Theorem 2.4

When a Fibonacci sequence $H_n$ is such that $h_i = g_{n+i-1} - f_i; n \geq 4$, then $h_4 + 1$ is a parent number and $h_4 - 1$ is not a parent number.

Proof

\[ h_4 = g_{n+3} - f_4 = g_{n+3} - 5 \]

Let $n + 3 = u$. Since $n \geq 4$, it means $u \geq 7$. From theorem 2.8, $g_u - 4$ is a parent number, therefore $h_4 + 1$ is a parent number.

\[ h_4 - 1 = g_{n+3} - 6. \] When $n \geq 5$, theorem 2.9 suggests that $g_{n+3} - 6$ is not a parent number. When $n = 4$, $h_4 - 1 = g_7 - 6$.

But $n = 4$ also implies that $g_n = p_{j+n-1} + f_n, j \geq 4$, $P_j$ is a Fibonacci sequence, theorem 2.5. From theorem 2.14, $g_7 - 6$ is not a parent number.

Theorem 2.5

When $H_n$ is such that $h_i = g_{n+i-1} + f_i; n \geq 4$, then $h_4 - 1$ is a parent number and $h_4 + 1$ is not a parent number.

Proof

\[ h_4 = g_{n+3} + f_4 = g_{n+3} + 5 \]

Since $n \geq 4$, from theorem 2.8, $g_{n+3} + 4$ is a parent number, therefore $h_4 - 1$ is a parent number.

\[ h_4 + 1 = g_{n+3} + 6. \] For $n \geq 5$, theorem 2.9 ensures that $g_{n+3} + 6$ is second term of a Fibonacci sequence and is therefore not a parent number. Theorem 2.3 suggests that $n = 4$ here means

\[ g_n = l_{j+n-1} - f_n; j \geq 4, L_n \text{ is a Fibonacci sequence and } F_n = 1, 2, 3, 5, 8, \ldots \text{ It follows from theorem 2.10 that } g_7 + 6 \text{ is not a parent number.} \]

Theorem 2.6

When $H_n$ is such that $h_i = g_{n+i-1} + f_i; n \geq 4$, then $h_3 + 1$ is a parent number.

Proof

\[ h_3 = g_{n+2} + f_3 = g_{n+2} + 3 \]

Since $n \geq 4$, from theorem 2.8, $g_{n+3} + 4$ is a parent number, therefore $h_3 + 1$ is a parent number.

Theorem 2.7

For any arbitrary Fibonacci sequence $H_n$ $h_i \pm 1; i \geq 5$, is a parent number.
Proof

Geometric stability, theorem 2.12, means that \((h_i, h_{i+1}) \pm (1, 2) = p_j, p_{j+1}\) where \(P_j\) is a Fibonacci sequence, holds for all \(i \geq 4\). It follows from this result that since \(p_i(h_i \pm 1) \neq (h_{i+1} \pm 1)\), \(i \geq 4\), then \(h_{i+1} \pm 1\) is a parent number. Since \(i \geq 4\), it means \(h_i \pm 1, i \geq 5\), is a parent number.

**Theorem 2.8**

For any arbitrary Fibonacci sequence \(H_n, h_i \pm 4, i \geq 4\), is a parent number.

**Proof**

\(h_i = g_{n+i-1} \pm f_i, i \geq 1, n \geq 4\), where \(G_n\) is a Fibonacci sequence and \(F_n = (1.3)\). Let us investigate the case \(h_i = g_{n+i-1} + f_i\). For \(i \geq 5\), \(h_i \pm 1\) is a parent number. This means \(h_i + 1 + 3 = h_i + 4\) and \(h_i - 1 - 3 = h_i - 4\) are parent numbers. Now for \(i = 4\), since \(h_i = g_{n+i-1} + f_i\), it follows \(h_i - 1\) is a parent number. It also follows \(h_i - 1 = h_i - 4\) is a parent number. Now \(h_i + 1\) is not a parent number, but second term of a Fibonacci sequence. Let \(h_i + 1 = l_i, L_n\) is a Fibonacci sequence. It follows from law of polarity that \(l_i = r_j - 1\), \(j \geq 4\). \(R_j\) is a Fibonacci sequence. It therefore follows \(l_i + 1 = h_i + 2\) is a parent number. We need now to prove that \(h_i + 2 = q_m - 1\), \(m \geq 5\). \(Q_m\) is a Fibonacci sequence.

\[h_i = g_{n+3} + f_i = g_{n+3} + 5\]

\[h_i + 2 = g_{n+3} + 7\]

Let \(g_{n+3} = g_u\).

It follows \(h_i + 2 = g_u + 7\).

Since \(n \geq 4\), it follows \(u \geq 7\). Because \(h_i + 2\) is a parent number, it implies \(g_u + 7\) is a parent number. Here, notice that \(G_u\) is such that \(g_u = s_{u+1} - f_i, S_u\) is a Fibonacci sequence and \(F_u = (1.3)\). Now

\[g_u + 7 = g_u + 1 + 3 + 3 = p_i\]

This means from corollary 3.1 that the next parent number is \(g_u + 1 + 3 + 3 + 2 = p_i\). Here, \(p_i\) and \(p_i'\) are symmetrical about

\[q_m = g_u + 1 + 3 + 3 + 1\]

It follows therefore that

\[p_i = q_m - 1\]

\[p_i' = q_m + 1,\]

which symmetry implies \(m \geq 5\).

The same procedure is repeated for the case \(h_i = g_{n+i-1} - f_i\).

**Theorem 2.9**

For any arbitrary Fibonacci sequence \(H_n, (h_i, h_{i+1}) \pm (4, 6) = p_j, p_{j+1}\); \(P_j\) is a Fibonacci sequence, holds for all \(i \geq 7\).

**Proof**

We only require to prove the critical case \(i = 7\).
Let $H_n$ be such that $h_i = g_{n+i-1} + f_i$. It follows $h_{i+1} = g_{n+i} + f_{i+1}$ and

$$(h_i, h_{i+1}) + (4, 6) = g_{n+i-1} + f_i + 4, g_{n+i} + f_{i+1} + 6. \text{ When } i = 7, \text{ we have } g_{n+6} + 21 + 4, g_{n+7} + 34 + 6 = g_{n+6} + 25, g_{n+7} + 40.$$

Since $\geq 4, \varphi(g_{n+6} + 25) = g_{n+7} + 40$, and it follows

$$(h_7, h_8) + (4, 6) = p_j, p_{j+1}; P_j \text{ is a Fibonacci sequence.}$$

Now consider

$$g_{n+6} + 21 - 4, g_{n+7} + 34 - 6 = g_{n+6} + 17, g_{n+7} + 28.$$ Since $\geq 4, \varphi(g_{n+6} + 17) = g_{n+7} + 28$, and it follows

$$(h_7, h_8) - (4, 6) = p_j, p_{j+1}; P_j \text{ is a Fibonacci sequence.}$$

Now let $h_i = g_{n+i-1} - f_i$.

$$(h_i, h_{i+1}) + (4, 6) = g_{n+i} - 21 + 4, g_{n+i+1} - 34 + 6 = g_{n+6} - 17, g_{n+7} - 28.$$ Since $\geq 4, \varphi(g_{n+6} - 17) = g_{n+7} - 28$, and it follows

$$(h_7, h_8) + (4, 6) = p_j, p_{j+1}; P_j \text{ is a Fibonacci sequence.}$$

Now consider

$$g_{n+6} - 21 - 4, g_{n+7} - 34 - 6 = g_{n+6} - 25, g_{n+7} - 40.$$ Since $\geq 4, \varphi(g_{n+6} - 25) = g_{n+7} - 28$, and it follows

$$(h_7, h_8) - (4, 6) = p_j, p_{j+1}; P_j \text{ is a Fibonacci sequence.}$$

Since result is true for the critical case, proof is complete.

**Theorem 2.10**

*If a Fibonacci sequence $H_n$ is such that $h_i = g_{n+i-1} - f_i$, then $h_7 + 6$ is not a parent number.*

**Proof**

$$h_7 = g_{n+6} - f_7 = g_{n+6} - 21$$

$$h_7 + 6 = g_{n+6} - 21 + 6 = g_{n+6} - 15 = (g_{n+6} - 14) - 1.$$ Let $n + 6 = u$. Since $n \geq 4$, it follows $u \geq 10$, and from postulate of special numerical harmony, $g_u - 14$ is a parent number. This means $(g_u - 14) - 1$ is not a parent number, theorem 2.11.

**Theorem 2.11**

*If an integer $x$ is a parent number, then $x \pm 1$ is not a parent number.*

**Proof**

Let $x = h_1$; $H_1$ is a Fibonacci sequence.
$h_1 = g_n + 1, \ n \geq 4, \ G_n$ is a Fibonacci sequence. Consider $h_1 = g_n + 1$.

$h_1 - 1 = g_n$. Since $n \geq 4, \ g_n$ is not a parent number; thus $h_1 - 1$ is not a parent number.

$h_1 + 1 = g_n + 2$, but

$\frac{g_n + 2}{\phi} = g_{n-1} + 1$,

therefore $g_n + 2$ is not a parent number; thus $h_1 + 1$ is not a parent number.

The same logical procedure is followed when $h_1 = g_n - 1$.

**Theorem 2.12**

For any given Fibonacci sequence $H_n$, geometric stability begins at $n = 4$, that is to say, $(h_i, h_{i+1}) \pm (1,2) = p_j, p_{j+1}; \ j \geq 1, \ P_j$ is a Fibonacci sequence, holds for all $i \geq 4$.

**Proof**

Let $h_1 + 1 = p_1$. Here, let's assume $h_i = g_n + f_i; \ i \geq 1; \ n \geq 4$ where $G_n$ is an arbitrary Fibonacci sequence.

Let $h_4, h_5, h_6, \ldots = l_1, l_2, l_3, \ldots$. This means $l_i = h_{i+3}; \ r \geq 1$. We can thus prove the lemma by induction on $r$.

We first consider $r = 1$.

$(l_1, l_2) = (h_4 + 1, h_5 + 2) = (g_n + 5 + 1, g_{n+4} + 8 + 2); \ n \geq 4$

$= (g_n + 6, g_{n+4} + 10)$

$= q_m q_{m+1}; \ Q_m$ is a Fibonacci sequence.

$(l_1, l_2) = (h_4 - 1, h_5 - 2) = (g_n + 5 - 1, g_{n+4} + 8 - 2); \ n \geq 4$

$= (g_n + 4, g_{n+4} + 6)$

$= p_j, p_{j+1}; \ P_j$ is a Fibonacci sequence.

Result is true for $r = 1$. We now consider $r = k; \ k \geq 2$; arbitrary.

$(l_k, l_{k+1}) = (h_{k+3} + 1, h_{k+4} + 2)$

$= (g_{n+k+2} + f_{k+3} + 1, g_{n+k+3} + f_{k+4} + 2); \ n \geq 4$

Since $k \geq 2$, it follows $g_{n+k+2} + f_{k+3} + 1$ is a parent number. This also means $g_{n+k+3} + f_{k+4} + 1$ is a parent number. It follows therefore that

$(g_{n+k+2} + f_{k+3} + 1, g_{n+k+3} + f_{k+4} + 2); = q_m q_{m+1}; \ Q_m$ is a Fibonacci sequence.

$(l_k, l_{k+1}) = (h_{k+3} - 1, h_{k+4} - 2)$

$= (g_{n+k+2} + f_{k+3} - 1, g_{n+k+3} + f_{k+4} - 2); \ n \geq 4$
Since \( k \geq 2 \), it follows \( g_{n+k+2} + f_{k+3} - 1 \) is a parent number. This also means \( g_{n+k+3} + f_{k+4} - 1 \) is a parent number. It follows therefore that

\[
(g_{n+k+2} + f_{k+3} - 1, g_{n+k+3} + f_{k+4} - 2) = p_j p_{j+1}, P_j \text{ is a Fibonacci sequence.}
\]

Result is therefore true for \( r = k \).

The same argument is provided for \( r = k+1 \). The same procedure is repeated for \( h_i = g_{n+i} - 1 + f_i ; i \geq 1; n \geq 4 \) with the same results obtained. Proof of lemma is therefore complete.

This is the essence of the concept of geometric stability: it provides that

\[
(h_1 - 1) = h_{i+1} - 2; \quad \varphi(h_1 + 1) = h_{i+1} + 2; \quad i \geq 4.
\]

(2.10)

Theorem 2.13

When a Fibonacci sequence \( H_i \) is such that \( h_1 = g_{n+i-1} + f_i \) then \( h_3 + 1 \) is a parent number and \( h_3 - 1 \) is not a parent number.

Proof

\[
h_3 = g_{n+2} + f_3 = g_{n+2} + 3
\]

\[
h_3 + 1 = g_{n+2} + 4.
\]

Let \( n + 2 = u \). Since \( n \geq 4 \), it means \( u \geq 6 \). From theorem 2.8, it follows \( g_u + 4 \) is a parent number.

Now consider \( h_3 - 1 = g_u + 3 - 1 = g_u + 2 \). Since \( u \geq 6 \), it follows \( g_u + 1 \) is a parent number, theorem 2.7. This means \( g_u + 2 \) is not a parent number, theorem 2.11.

Theorem 2.14

If a Fibonacci sequence \( H_i \) is such that \( h_1 = g_{n+i-1} + f_i \), then \( h_3 - 6 \) is not a parent number.

Proof

\[
h_3 = g_{n+6} + f_7 = g_{n+6} + 21
\]

\[
h_3 - 6 = g_{n+6} + 21 - 6 = g_{n+6} + 15 = (g_{n+6} + 14) + 1.
\]

Let \( n + 6 = u \). Since \( n \geq 4 \), it follows \( u \geq 10 \), and from postulate of special numerical harmony, \( g_u + 14 \) is a parent number. This means \( (g_u + 14) + 1 \) is not a parent number, theorem 2.11.

Theorem 2.15

When \( H_i \) is such that \( h_1 = g_{n+i-1} - f_i ; n \geq 4 \), then \( h_3 - 1 \) is a parent number.

Proof

\[
h_3 = g_{n+2} - f_3 = g_{n+2} - 3
\]

\[
h_3 - 1 = g_{n+2} - 4.
\]
Since \( n \geq 4 \), from theorem 2.8, \( g_{n+3} - 4 \) is a parent number, therefore \( h_3 - 1 \) is a parent number.

**Theorem 2.16**

Let \( F_n = 1, 2, 3, 5, 8, \ldots \) and \( G_n, H_i, \) and \( L_j \) be arbitrary Fibonacci sequences. If \( (g_n, g_{n+1}, g_{n+2}, g_{n+3}, \ldots) - (f_1, f_2, f_3, f_4, \ldots) = h_i, h_{i+1}, h_{i+2}, h_{i+3}, \ldots \); then \( (g_{n+1} - 1) \) is a parent number. If \( (g_n, g_{n+1}, g_{n+2}, g_{n+3}, \ldots) + (f_1, f_2, f_3, f_4, \ldots) = L_j, L_{j+1}, L_{j+2}, L_{j+3}, \ldots \); then \( (g_{n+1} + 1) \) is a parent number.

**Proof**

\[
\begin{aligned}
(g_n, g_{n+1}, g_{n+2}, g_{n+3}, \ldots) - (f_1, f_2, f_3, f_4, \ldots) &= h_i, h_{i+1}, h_{i+2}, h_{i+3}, \ldots \\
&= g_n - 1 = h_i \\
g_{n+1} - 2 &= h_{i+2} \\
\end{aligned}
\]  

(2.11)

We need to prove that this implies that \( g_{n+1} - 1 \) is a parent number. We prove this from brute-force logical argumentation. Relations (2.11) imply that

\[
\begin{aligned}
\varphi(g_n - 1) &= g_{n+1} - 2 \\
\varphi(g_n - 1) &\neq g_{n+1} - 1 \\
\end{aligned}
\]  

(2.12)

but

\[
\frac{g_{n+1} - 1}{\varphi} = g_n - 1
\]  

(2.13)

It follows from (2.12) and (2.13) that \( g_{n+1} - 1 \) is a parent number since \( \frac{g_{n+1} - 1}{\varphi} = g_n - 1 \) while \( \varphi(g_n - 1) \neq g_{n+1} - 1 \).

The same reasoning is employed in proving the second part of this theorem.

**3 Phi and Symmetry: Chirality and Homochirality**

**3.1 The parent number line**

An array of parent numbers in ascending order forms the parent number line.

\[
\begin{array}{cccccccccc}
1 & 4 & 7 & 9 & 12 & 14 & 17 & 20 & 22 \\
\end{array}
\]

Fig. 3.1. The parent number line

In constructing the parent number line, let parent a number \( x = h_i \). These rules apply, stated as corollary 3.1:

**Corollary 3.1**

i. If \( h_i = g_4 \pm 1 \), then the parent numbers \( h_i - 3 \) and \( h_i + 3 \) are its immediate neighbours,

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If \( h_1 = g_n + 1; \) \( n \geq 5, \) then the parent numbers \( h_1 - 2 \) and \( h_1 + 3 \) are its immediate neighbours.

If \( h_1 = g_n - 1; \) \( n \geq 5, \) then the parent numbers \( h_1 - 3 \) and \( h_1 + 2 \) are its immediate neighbours.

### 3.2 Chirality

Phi comes with myriad properties of symmetry. In this section we only study a few, those due to the first three Fibonacci sequences, viz:

\[
F_n = (1.3);
H_n = (1.6);
\]

and

\[
L_n = 7,11,18,29,47,\ldots
\]

#### 3.2.1 Symmetry due to \( F_n \)

Theorem 2.12 states that for any given Fibonacci sequence \( H_n \), \( (h_i, h_{i+1}) \pm (1,2) = p_j, p_{j+1}; j \geq 1, P_j \) is a Fibonacci sequence, holds for all \( i \geq 4 \). Theorem 2.7 combined with theorem 2.12 ensures that corollary 3.2 below holds.

**Corollary 3.2**

Given any Fibonacci sequence \( H_n \), \( (h_i, h_{i+1}) \pm (1,2) = p_j, p_{j+1}; P_n \) is a Fibonacci sequence, holds for \( i \geq 5 \).

For instance, take the Fibonacci sequence

\[
H_n = 20,32,52,84,136,\ldots
\]

Take \( h_5 \) for example and we have

\[
136,220,356,576,\ldots
\]

This means

\[
(3.3) - (1.3) = 135,218,353,571,\ldots
\]

\[
(3.3) + (1.3) = 137,222,359,581,\ldots
\]

are Fibonacci sequences symmetrical about the sequence (3.3).

#### 3.2.2 Symmetry due to \( H_n = 4,6,10,16,26,\ldots \)

Theorems 2.8 and 2.9 provides the basis of the formulation of corollary 3.3.

**Corollary 3.3**

Given any Fibonacci sequence \( H_n \), \( (h_i, h_{i+1}) \pm (4,6) = p_j, p_{j+1}; P_n \) is a Fibonacci sequence, holds for \( i \geq 7 \).

For example, use the sequence

\[
H_n = 14,23,37,60,97,157,254,411,665,1076,\ldots
\]

Now take \( h_7 \) for instance and we have

\[
254,411,665,1076,1741,\ldots
\]
It follows

\[(3.7) - (1.6) = 250,405,655,1060,1715,\ldots \quad (3.8)\]

\[(3.7) + (1.6) = 258,417,675,1092,1767,\ldots \quad (3.9)\]

are Fibonacci sequences symmetrical about the sequence (3.7).

Further symmetry within this symmetry is stated in theorem 3.1 below.

**Theorem 3.1**

For any arbitrary Fibonacci sequence \( H_n \), \( h_i - 4 = p_4 + 1 \) and \( h_i + 4 = q_4 - 1, i \geq 8 \), \( P_n \) and \( Q_n \) are Fibonacci sequences.

**Proof**

Consider \( H_n \) such that \( h_i = g_{n+i-1} + f_i; n \geq 4 \).

\[ h_i - 4 = g_{n+i-1} + f_i - 4; i \geq 8 \]

Let \( n + i - 1 = u. \) Since \( i \geq 8 \) and \( n \geq 4 \), it follows \( u \geq 11 \).

\[ h_i - 4 = g_u + f_i - 4 \]

Let

\[ g_u + f_i - 4 = p_j + 1 \quad (3.10) \]

We have

\[ p_j = g_u + f_i - 5 \quad (3.11) \]

Assuming \( P_n \) is a Fibonacci sequence, equation (3.11) implies \( j = 4 \) since it can be written as

\[ p_j = (g_u + f_i) - f_i \]

\[ \therefore h_i - 4 = p_4 + 1 \quad (3.12) \]

Let \( h_i + 4 = q_r - 1 \)

\[ q_r = h_i + 5 \]

\[ = g_u + f_i + 5 \]

which can be written in the form

\[ q_r = (g_u + f_i) + f_4 \]

implying \( r = 4 \),

\[ \therefore h_i + 4 = q_4 - 1 \quad (3.13) \]

Now equations (3.12) and (3.13) need to be validated. From equation (3.12), it follows

\[ p_4 = (g_{u-3} + f_{i-3}) - 1 \quad (3.14) \]
and from equation (3.13), it follows

\[ q_1 = (g_{u-3} + f_{i-3}) + 1 \]  

(3.15)

For equations (3.14) and (3.15) to hold corollary 3.2 demands that

\[ g_{u-3} + f_{i-3} = l_r; r \geq 5 \]  

(3.16)

But equation (3.16) holds if and only if \( i - 3 = r = 5 \) \( \vdash i \geq 8 \).

The above procedure is repeated when \( H_4 \) is such that \( h_4 = \pm 7 \), \( i \geq 8 \), and the same result is established.

3.2.3 Symmetry due to \( L_{08} = 7,11,18,29,47, \ldots \)

Theorem 3.2

For any arbitrary Fibonacci sequence \( H_4 \), \( h_4 \pm 7, i \geq 8 \), is a parent number.

Proof

Since \( i \geq 8 \), it follows from theorem 2.7 that \( h_4 + 1 \) is a parent number. It also follows from theorem 2.8 that \( h_4 + 4 = h_4 + 1 + 3 \) is a parent number. But \( h_4 + 4, i \geq 8 \), gives \( l_4 = p_4 - 1; P_n \) is a Fibonacci sequence, see theorem 3.1. This means from corollary 3.1 that \( h_4 + 1 + 3 + 3 = h_4 + 7 ; i \geq 8 \), is a parent number. Also, theorem 2.7 ensures that \( h_4 - 1 \) is a parent number since \( i \geq 8 \).

\( h_4 - 1 - 3 = h_4 - 4 \) is a parent number, theorem 2.8. But \( h_4 - 4, i \geq 8 \), gives \( l_4 = p_4 + 1; P_n \) is a Fibonacci sequence, see theorem 3.1. This means from corollary 3.1 that \( h_4 - 1 - 3 - 3 = h_4 - 7 ; i \geq 8 \), is a parent number.

Theorem 3.3

When \( H_n \) is such that \( h_i = g_{n+i-1} \pm f_i; n = 4 \), then \( (h_i, h_{i+1}) \pm (7, 11) = p_j, p_{j+1}; j \geq 1 \), \( P_j \) is a Fibonacci sequence, holds for \( i \geq 4 \).

Proof

Let \( h_i = g_{4+i-1} + f_i \). This means from theorem 2.4 that \( g_n = q_{m+n-1} - f_n; m \geq 4 \) where \( Q_m \) is a Fibonacci sequence.

\[ h_4 = g_{4+3} + f_4 = g_7 + f_4 \]

\[ = q_{m+6} - 21 + 5 = q_{m+6} - 16 \]

\[ h_5 = q_{m+7} - 34 + 8 = q_{m+7} - 26. \]

Notice that

\[ h_1 = q_{m+3} - 4. \]

Let \( m + 3 = u \). It follows \( h_1 = q_u - 4 \). Since \( m \geq 4 \), it follows \( u \geq 7 \), and from theorem 2.8, \( q_u - 4 \) is a parent number, and from theorem 2.9, \( H_n \) is a Fibonacci sequence. Here, the critical case is \( i = 4 \).
\[(h_4, h_5) + (7, 11) = q_{m+6} + 16 + 7, q_{m+7} = 26 + 11 = q_{m+6} - 9, q_{m+7} - 15 \quad (3.17)\]

Since \(m \geq 4\), it is a result of theorem 2.10 that \(\varphi(q_{m+6} - 9) = q_{m+7} - 15\), and it follows \((h_4, h_5) + (7, 11) = p_1, p_2, P_n\) is a Fibonacci sequence.

\[(h_4, h_5) - (7, 11) = q_{m+6} - 16 - 7, q_{m+7} - 26 - 11 = q_{m+6} - 23, q_{m+7} - 37 \quad (3.18)\]

Since \(\varphi(q_{m+6} - 23) = q_{m+7} - 37\), it follows \((h_4, h_5) - (7, 11) = l_r, l_{r+2} ; r \geq 2, L_n\) is a Fibonacci sequence.

The case \(h_i = g_{n+i-1} - f_i\) follows the same procedure and is not here repeated.

**Theorem 3.4**

When \(H_n\) is such that \(h_i = g_{n+i-1} \pm f_i ; n \geq 5\), then \((h_i, h_{i+1}) \pm (7, 11) = p_j, p_{j+1} ; j \geq 1, P_j\) is a Fibonacci sequence, holds for \(i \geq 3\).

**Proof**

Consider \(H_n\) such that \(h_i = g_{n+i-1} - f_i\) with \(g_n = q_{m+n-1} + f_n ; n \geq 5\).

Here, the critical case is \(n = 5\) and \(i = 3\).

\[h_3 = g_{5+3-1} - f_3\]
\[= g_7 - 3\]
\[g_7 = q_{m+6} + f_7\]
\[= q_{m+6} + 21\]
\[\therefore h_3 = q_{m+6} + 18, h_4 = q_{m+7} + 29\]

Notice in passing that \(h_1 = q_{m+4} + 7\). Let \(m + 4 = u\). Since \(m \geq 4\), it follows \(u \geq 8\). From theorem 3.2, \(q_u + 7\) is a parent number, here \(h_5\).

Now

\[(h_3, h_4) + (7, 11) = q_{m+6} + 18 + 7, q_{m+7} + 29 + 11 = q_{m+6} + 25, q_{m+7} + 40 \quad (3.19)\]

Since \(m \geq 4\), \(\varphi(q_{m+6} + 25) = q_{m+7} + 40\) and from postulate of special numerical harmony, \(q_{m+6} + 25\) is a parent number, therefore

\[(h_3, h_4) + (7, 11) = p_1, p_2 ; P_n\) is a Fibonacci sequence.

\[(h_3, h_4) - (7, 11) = q_{m+6} + 18 - 7, q_{m+7} + 29 - 11 = q_{m+6} + 11, q_{m+7} + 18 \quad (3.20)\]

But \(\varphi(q_{m+6} + 11) = q_{m+7} + 18\). But from corollary 3.1 it can be deduced that \(q_{m+6} + 11\) is not a parent number. This means

\[(h_3, h_4) - (7, 11) = l_2, l_3 ; L_n\) is a Fibonacci sequence.
The condition \( h_i = g_{n+i-1} - f_i \) with \( g_n = q_{m+n-1} - f_n \); \( n \geq 5 \) is similarly proved. The same procedure is followed for the case \( h_i = g_{n+i-1} + f_i \) and we here pursue the proof no farther.

Theorems 3.2, 3.3, and 3.4 give rise to corollary 3.4.

**Corollary 3.4**

*Given any Fibonacci sequence \( H_n, (h_i, h_{i+1}) \pm (7,11) = p_1, p_2; P_n \) is a Fibonacci sequence, holds for all \( i \geq 8 \).*

For illustration consider the sequence

\[
H_n = 9,15,24,39,63,102,165,267,432,\ldots
\]  

(3.21)

Take \( h_8 \) and we have

\[
267,432,699,1131,1830,\ldots
\]  

(3.22)

Now

\[
(3.22) - (3.1) = 260,421,681,1102,1783,\ldots
\]  

(3.23)

\[
(3.22) + (3.1) = 274,443,717,1160,1877,\ldots
\]  

(3.24)

such that (3.23) and (3.24) are Fibonacci sequences symmetrical about (3.22).

We have just demonstrated three instances of symmetry in the theory of the golden section. As earlier said, there are myriads of symmetries as there are myriads of Fibonacci sequences. These symmetries are obviously revealing a conservation law at work. We have therefore introduced the concept of chirality in Fibonacci sequences, the idea of handedness and gender prevalent in the universe. In point of fact, at the most basic and fundamental level, this handedness/gender is manifested in the mechanical representation of any given Fibonacci sequence as presented earlier on in this work. Elsewhere the concept of chirality has been reported. Li et al. [19] report a scenario whereby Fibonacci spiral patterns develop in both sinister and dexter forms on stressed Ag core/SiO\(_2\) shell microstructures (also as in pinecones). Livio [20] says, “*The importance of mirror-reflection symmetry to our perception and aesthetic appreciation, to the mathematical theory of symmetries, to the laws of physics, and to science in general, cannot be overemphasized...*”

### 3.3 Homochirality

Consider the symmetry due to \( F_n \) presented in section 3.2.1.

**Definition 3.1**

A Fibonacci sequence \( H_n \) has first order wiring (f.o.w.) if \( 2(h_i \pm 2, h_{i+1} \pm 3) = q_m, q_{m+1} \) where \( Q_n \) is a Fibonacci sequence, holds for \( i \geq 8 \).

**Definition 3.2**

A Fibonacci sequence \( H_n \) has second order wiring (s.o.w.) if \( 2(h_i \pm 2, h_{i+1} \pm 3) = q_m, q_{m+1} \) where \( Q_n \) is a Fibonacci sequence, holds for \( i \geq 9 \).
When \( H_n \) has f.o.w. it means \( h_1 \pm 1, i \geq 7 \) gives parent numbers \( p_i \) and \( l_i \) such that both \( P_n \) and \( L_n \) have a discontinuity at \( n = 5 \).

When \( H_n \) has s.o.w. it means \( h_1 \pm 1, i \geq 8 \) gives parent numbers \( p_i \) and \( l_i \) such that both \( P_n \) and \( L_n \) have a discontinuity at \( n = 5 \).

Wiring is essentially an exhibition of homochirality: the production/existence of enantiomorphs of the same kind by/in a system.

**Definition 3.3**

A Fibonacci sequence \( H_n \) has discontinuity at \( n = x + 1 \) if, for the continuous range \( 1 \leq n \leq x \), the formula

\[
\left( h_1, h_2, h_3, h_4, h_5, h_6, \ldots \right) = \left( w_1, w_2, w_3, w_4, w_5, \ldots \right)
\]

where \( W_n \) is a Fibonacci sequence holds.

**Theorem 3.5**

Let \( H_n \) be an arbitrary Fibonacci sequence. If \( 2(h_2, h_3) = p_1, p_2, P_n \) is a Fibonacci sequence, then \( H_n \) has f.o.w. If \( 2(h_2, h_3) \neq p_1, P_2, P_n \) is a Fibonacci sequence, then \( H_n \) has s.o.w.

**Proof**

From definitions 3.1 and 3.2, consider

\[
2(h_i \pm 2, h_{i+1} \pm 3) = 2h_i \pm 4, 2h_{i+1} \pm 6 \quad (3.25)
\]

Let

\[
2h_i, 2h_{i+1} = l_r, l_{r+1} \quad (3.26)
\]

where \( L_n \) is a Fibonacci sequence. The sequence (3.25) becomes

\[
l_r \pm 4, l_{r+1} \pm 6 \quad (3.27)
\]

From theorem 2.9, the sequence (3.27) is a Fibonacci sequence if \( r \geq 7 \).

**Scenario I:** \( H_n \) is such that (3.26) holds for \( i \geq 2 \).

The critical case becomes

\[
2(h_2, h_3, h_4, h_5, h_6, \ldots) = l_1, l_2, l_3, l_4, l_5, \ldots
\]

such that \( l_2 = 2h_8 \). This means equation (3.25) holds for \( i \geq 8 \), and by definition 3.1, \( H_n \) has f.o.w.

**Scenario II:** \( H_n \) is such that equation (3.26) holds for \( i \geq 3 \).

The critical case becomes

\[
2(h_3, h_4, h_5, h_6, h_7, \ldots) = l_1, l_2, l_3, l_4, l_5, \ldots
\]

such that \( l_2 = 2h_9 \). This means equation (3.25) holds for \( i \geq 9 \), and by definition 3.2, \( H_n \) has s.o.w.

Theorems 3.6 and 3.7 below directly prove the concept of wiring.

**Theorem 3.6**

If a Fibonacci sequence \( H_n \) has f.o.w. then \( h_1 \pm 1 = q_1, i \geq 7, Q_n \) is a Fibonacci sequence with discontinuity at \( n = 5 \).
Proof

$H_n$ has f.o.w. if $2(h_2, h_3) = p_{i+2}, p_{i+1}$; $P_n$ is a Fibonacci sequence.

Here, $2h_i = p_{i-1}$
$\therefore h_i = \frac{p_{i-1}}{2}$

Let $h_i - 1 = l_i$
$\therefore l_1 = \frac{p_{i-1}}{2} - 1; l_2 = \frac{p_i}{2} - 2; l_3 = \frac{p_{i+1}}{2} - 3$

$$2l_2 = p_i - 4$$
$$2l_3 = p_{i+1} - 6$$
$$2l_2, 2l_3 = (p_i, p_{i+1}) - (4, 6) \quad (3.28)$$

Now let $h_i + 1 = w_i$, $W_n$ is a Fibonacci sequence. It follows

$$w_1 = \frac{p_{i-1}}{2} + 1; w_2 = \frac{p_i}{2} + 2; w_3 = \frac{p_{i+1}}{2} + 3$$

$$2w_2 = p_i + 4$$
$$2w_3 = p_{i+1} + 6$$
$$2w_2, 2w_3 = (p_i, p_{i+1}) + (4, 6) \quad (3.29)$$

From corollary 3.3, both equations (3.28) and (3.29) hold for $i \geq 7$. These equations also show that $L_n$ and $W_n$ do not have a discontinuity at $n = 2$. By showing that they do not have a discontinuity at $n = 4$ we prove that they have a discontinuity at $n = 5$. This we accomplish by only showing that $4l_4$ and $4w_4$ are parent numbers.

$$4l_4 = 4 \left(\frac{p_{i+2}}{2} - 5\right) = 2p_{i+2} - 20 \quad (3.30)$$

Since $i \geq 7$, the critical value of $p_{i+2}$ is $p_9$. The critical case is when $P_n$ has s.o.w. This means

$$2p_9 = v_{n-2} = v_7, V_n \text{ is a Fibonacci sequence. Equation (3.30) becomes}$$

$$4l_4 = v_n - 20, n \geq 7 \quad (3.31)$$

Corollary 3.1 shows that $v_n - 20, n \geq 7$ is a parent number for any arbitrary Fibonacci sequence $V_n$. This means $4l_4$ is a parent number.

$$4w_4 = 4 \left(\frac{p_{i+2}}{2} + 5\right) = 2p_{i+2} + 20 \quad (3.32)$$

Let $2p_{i+2} = z_n; n \geq 7$

$$4w_4 = z_n + 20 \quad (3.33)$$

Corollary 3.1 shows that $z_n + 20, n \geq 7$ is a parent number for any arbitrary Fibonacci sequence $Z_n$. This means $4w_4$ is a parent number. This completes the proof.
Theorem 3.7

If a Fibonacci sequence $H_n$ has s.o.w. then $h_i \pm 1 = q_i, i \geq 8$, $Q_n$ is a Fibonacci sequence with discontinuity at $n = 5$.

Proof

Follow proof to theorem 3.6.

The phenomenon of wiring is an exhibition of homochirality. It is a concept which must improve our understanding of higher geometries. Homochirality is a phenomenon observable in the universe. Bonner [21], cited by Cronin and Reisse [22], tends to argue that homochirality (of amino acids) is a condition for the existence of life.

4 Special Numerical Harmony

Special Numerical Harmony (SNH) is here defined as the condition that given an integer $c$ and a parent number $x$, both $\vert c - x \vert$ and $c + x$ are parent numbers. Using an arbitrary Fibonacci sequence

$$H_n = h_1,h_2,h_3,h_4,h_5,...$$

(4.1)

it is found that, when parent numbers are considered in ascending order, SNH fails by the positive and/or negative action of the parent number 1 from $h_1$ to $h_5$, by the positive or negative action of the parent number 7 from $h_5$ to $h_7$, by the positive or negative action of the parent number 27 from $h_8$ to $h_{10}$, etc. Set Y gives these (lowest) parent numbers violating SNH in successive regions/energy levels of any given Fibonacci sequence (4.1). Set Y is given by:

$$Y = \{1;7;27;117;493;2091;...\}$$

(4.2)

We also make an interesting observation. Let $F_n = (1.3)$.

It is found that:

$$\begin{align*}
    y_1 &= f_1 = 1; \\
    y_2 &= f_5 - f_4 = 8 - 1 = 7; \\
    y_3 &= f_9 - f_8 + f_7 = 34 - 8 + 1 = 27; \\
    y_4 &= f_{13} - f_{12} + f_{11} - f_{10} = 144 - 34 + 8 - 1 = 117; \\
    y_5 &= f_{17} - f_{16} + f_{15} - f_{14} + f_{13} = 610 - 34 - 8 + 1 = 117;
\end{align*}$$

etc.

(4.3)

By way of equation (4.3) we show the sinusoidal character of this phenomenon. There are surprisingly many ways of creating set Y. Let $F_n = (1.3)$. To compute $y_n, n \geq 1$, let $\frac{f_{2n-1}}{2} = h_i$; $H_n$ is a Fibonacci sequence. It follows:

$$\begin{align*}
    y_n &= h_{i+4} + 1; \text{odd } n \\
    y_n &= h_{i+4} - 1; \text{even } n
\end{align*}$$

(4.4)

Notice that when $n = 1, f_{2n-4} = f_{-1}$. Extend $F_n$ backward by two steps to get $0,1,1,2,3,5,\ldots$ so that $f_{-1} = 0$. Now

$$\frac{f_{-1}}{2} = -1 = h_i.$$
This means $H_n$ is the null Fibonacci sequence

$$0,0,0,0,0,...$$  \hspace{1cm} (4.5)

$$\therefore h_{i+4} = 0.$$  

Since $n = 1 = \text{odd}$, from equation (4.4), $y_1 = 0 + 1 = 1$.

Other elements of Y are similarly computed. From this the following theorem is stated.

**Theorem 4.1**

*All elements of set Y are parent numbers.*

**Proof**

Take equation (4.4):

$$y_n = h_{i+4} + 1; \text{odd } n \}
\quad y_n = h_{i+4} - 1; \text{even } n \}.$$  

Since $i \geq 1$, theorem 2.7 implies that $y_n$ is a parent number.

Having shown how to create energy levels in Fibonacci sequences and how to construct set Y, we duly state the SNH Postulate.

**Postulate 4.1**

*Let $x$ be an arbitrary parent number such that $x < y$, $x$ is in SNH with all members of the $i^{th}$ energy level of $H_n$.*

We intend to prove the Postulate of SNH. However, because it is very extensive in nature, we shall only seek to prove the first few and basic components, viz:

i. $h_i \pm 1, i \geq 5$, is a parent number;
ii. $h_i \pm 4, i \geq 4$, is a parent number;
iii. $h_i \pm 7, i \geq 8$, is a parent number;

where $H_n$ is an arbitrary Fibonacci sequence. The above follow these premises:

i. the parent number 1 is in SNH with $h_i, i \geq 5$;
ii. the parent number 4 is in SNH with $h_i, i \geq 4$;
iii. the parent number 7 is in SNH with $h_i, i \geq 8$.

The reader shall affect to recall theorems 2.7, 2.8, and 3.2 respectively in proving the above three propositions.

The SNH postulate is an important tool in the expansion of the theory of the golden section as it presents a Fibonacci sequence as a numeric atomic model. In $H_n, h_1$ is the *nucleus* because SNH is violated by both the positive and negative action of the parent number 1, that is, both $h_1 - 1$ and $h_1 + 1$ are not parent numbers, theorem 4.2.

**Theorem 4.2**

*Given an arbitrary Fibonacci sequence $H_n, h_1 \pm 1$ is not a parent number.*
Proof

See theorem 2.11.

It follows therefore that albeit the first energy level is from \( h_1 \) to \( h_4 \), this region includes the nucleus, \( h_1 \). Being closest to the nucleus, \( h_2 \) to \( h_4 \) are in the ground state and SNH is violated by the positive or negative action of the same parent number that violates SNH on the nucleus.

Set \( Y \) is intricately connected to the sequence (1.3). This formula holds:

\[
y_{n+1} = f_{2n+2} - y_n; \ n \geq 1; \ y_1 = f_1 = 1
\]

(4.6)

It may also be useful to note that \( y_1 + y_2 = 8; \ y_2 + y_3 = 34; \ y_3 + y_4 = 144; \) etc. i.e. the even terms of the sequence (1.3) for \( n \geq 5 \). That set \( Y \) is derived from the sequence (1.3) and applies to every Fibonacci sequence proves that the sequence (1.3) is not only ubiquitous in the physical universe, but in the universe of the golden section itself. We say that the sequence (1.3) on one extremity is a representation of measure and on the other, a representation of concept. In other words, it is the principle of the golden section. We give the first ten energy levels in Table 4.1.

We see that all except the first energy level are comprised of 3 elements, i.e. from \( f_n \) to \( f_{n+2} \). The nucleus included, the values of \( n \) at the end of energy levels are

\[
T_n = 1,4,7,10,13,16,19,\ldots
\]

(4.7)

where

\[
t_n = t_{n-1} + 3; \ n \geq 2; \ t_1 = 1
\]

(4.8)

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Energy level & Starts at \( n \) = & Ends at \( n \) = & Fails SNH by action of \( y \) = \\
\hline
1 & 1 & 4 & 1 \\
2 & 5 & 7 & 7 \\
3 & 8 & 10 & 27 \\
4 & 11 & 13 & 117 \\
5 & 14 & 16 & 493 \\
6 & 17 & 19 & 2091 \\
7 & 20 & 22 & 8855 \\
8 & 23 & 25 & 37513 \\
9 & 26 & 28 & 158905 \\
10 & 29 & 31 & 673135 \\
\hline
\end{tabular}
\caption{Energy levels in Fibonacci sequences}
\end{table}

The sequence (4.7) is called the Teleois system. That Teleois numbers perfectly fit the Fibonacci sequences is suggestive. Landone [23] says, “These Teleois proportions are found in the structure of the solar system ... in musical scales, in designs inside of snowflakes, in the human skeleton, in geometric forms, in the grain of wood, in time units of earth and sun, catenary of mathematics, et cetera.” Hardy et al. [24], cited by Sherbon [25] says, “The Teleois proportions are used by the creative force because they best fit the electromagnetic energy fields of the atom.”

From set \( Y \), we create set \( X \) such that

\[
x_n = y_{n+1} - y_n; \ n \geq 1
\]

(4.9)

i.e.

\[
X = \{6,20,90,376,1598,6764,28658,\ldots\}
\]

(4.10)
Given $F_n = (1, 3)$, we find that:

\[ x_1 = f_4 + 1; \]
\[ x_2 = f_7 - 1; \]
\[ x_3 = f_{10} + 1; \]
\[ x_4 = f_{13} - 1; \]

etc. These positions $f_4, f_7, f_{10}, f_{13}$, etc. are, as said above, Teleois positions. Sherbon [25] says, "As part of a series based on modulo 3 arithmetic, the Teleois proportions are easily noticeable in the Queen’s Chamber of the Great Pyramid of Giza, designed with seven Teleois spheres that also correspond to the geometry included in the cosmological circle" and "William Conner also referenced the Teleois as a “cosmic formula behind form in the physical world”".

## 5 Chemical Geometry

Chemical geometry is primarily concerned with the study of (imaginary) numerical chemical compounds for the sole purpose of penetrating the very heart of the golden section.

### 5.1 The $Z$-class

**Theorem 5.1**

If a Fibonacci sequence $H_n$ is such that $h_i = g_{n+i-1} - f_i, n = 5$, and $g_n = q_{m+n-1} + f_n$, then $10(h_5, h_6) = p_1, p_2, P_n$ is a Fibonacci sequence.

**Proof**

\[ h_i = g_{n+i-1} - f_i, n = 5 \]
\[ h_5 = g_{5+5-1} - f_5 = g_9 - 8 \]
\[ h_6 = g_{10} - 13 \]

But $g_n = q_{m+n-1} + f_n$ means

\[ g_9 = q_{m+8} + 55 \]
\[ g_{10} = q_{m+9} + 89 \]

It follows

\[ h_5 = q_{m+8} + 55 - 8 = q_{m+8} + 47 \]  \hspace{1cm} (5.1)
\[ h_6 = q_{m+9} + 89 - 13 = q_{m+9} + 76 \]  \hspace{1cm} (5.2)
\[ 10(h_5, h_6) = 10q_{m+8} + 470, 10q_{m+9} + 760 \]  \hspace{1cm} (5.3)

Since $m \geq 4$, it follows $\varphi(10q_{m+8} + 470) = 10q_{m+9} + 760$

\[ \therefore 10(h_5, h_6) = p_j, p_{j+1}; P_n \text{ is a Fibonacci sequence.} \]

\[ \text{Since } \frac{10h_5}{\varphi} = 10h_4 \quad \text{but } \varphi(10h_4) \neq 10h_5 \]

This implies $j = 1$.  

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Theorem 5.2

If a Fibonacci sequence $H_n$ is such that $h_i = g_{n+i-1} + f_i, n = 5$, and $g_n = q_{m+n-1} - f_n, n = 5$, then $10(h_5, h_6) = p_1, p_2, P_n$ is a Fibonacci sequence.

Proof

$$h_i = g_{n+i-1} + f_i, n = 5$$
$$h_5 = g_5 = g_5 + 8$$
$$h_6 = g_{10} + 13$$

But $g_n = q_{m+n-1} - f_n$ and

$$g_9 = q_{m+8} - 89, \text{ it follows } g_9 = q_{m+8} - 89$$

$$h_5 = q_{m+8} - 55 + 8 = q_{m+8} - 47 \quad (5.4)$$
$$h_6 = q_{m+9} - 89 + 13 = q_{m+9} - 76 \quad (5.5)$$

$$10(h_5, h_6) = 10q_{m+8} - 470, 10q_{m+9} - 760 \quad (5.6)$$

Since $m \geq 4$, it follows $\varphi(10q_{m+8} - 470) = 10q_{m+9} - 760$

$: 10(h_5, h_6) = p_1, p_2, P_n$ is a Fibonacci sequence.

Since $\frac{10h_5}{\varphi} = 10h_4$ but $\varphi(10h_4) \neq 10h_5$

it implies $j = 1$.

Theorem 5.3

If a Fibonacci sequence $H_n$ is such that $h_i = g_{n+i-1} - f_i, n = 6$, and $g_n = q_{m+n-1} - f_n, m = 4$, then $10(h_5, h_6) = p_1, p_2, P_n$ is a Fibonacci sequence.

Proof

$$h_5 = g_{6+5-1} - f_5 = g_5 - 8$$
$$h_6 = g_{11} - 13.$$  

With $m = 4$,

$$g_10 = q_{13} - 89; g_{11} = q_{14} - 144$$

It follows

$$h_5 = q_{13} - 97; h_6 = q_{14} - 157$$

But $m = 4$ implies that $q_m = l_{r+m-1} + f_m$, therefore

$$q_{13} = l_{r+12} + 377; q_{14} = l_{r+13} + 610$$
$$h_5 = q_{13} - 97 = l_{r+12} + 280$$
$$h_6 = q_{14} - 157 = l_{r+13} + 453$$
Since $r \geq 4$, it means

$$\varphi 10h_5 = 10h_6$$

$$\frac{10h_5}{\varphi} = 10h_4 \text{ but } \varphi 10h_4 \neq 10h_5. \text{ This means}$$

$$10(h_5, h_6) = p_1, p_2; P_n \text{ is a Fibonacci sequence.}$$

**Theorem 5.4**

If a Fibonacci sequence $H_n$ is such that $h_i = g_{n+i-1} + f_{i}, n = 6$, and $g_n = q_{m+n-1} + f_n$, $m = 4$, then $10(h_5, h_6) = p_1, p_2, P_n$ is a Fibonacci sequence.

**Proof**

$$h_5 = g_{6+5-1} + f_5 = g_{10} + 8$$

$$h_6 = g_{11} + 13.$$  

With $m = 4$,

$$g_{10} = q_{13} + 89; g_{11} = q_{14} + 144$$

It follows

$$h_5 = q_{13} + 97; h_6 = q_{14} + 157.$$  

But $m = 4$ implies that $q_m = l_{r+m-1} - f_m$, therefore

$$q_{13} = l_{r+12} - 377; q_{14} = l_{r+13} - 610$$

$$h_5 = q_{13} + 97 = l_{r+12} - 280$$

$$h_6 = q_{14} + 157 = l_{r+13} - 453.$$  

Since $r \geq 4$, it means

$$\varphi 10h_5 = 10h_6$$

$$\frac{10h_5}{\varphi} = 10h_4 \text{ but } \varphi 10h_4 \neq 10h_5. \text{ This means}$$

$$10(h_5, h_6) = p_1, p_2; P_n \text{ is a Fibonacci sequence.}$$

**Theorem 5.5**

If a Fibonacci sequence $H_n$ is such that $h_i = g_{n+i-1} \pm f_{i}, n = 6$, and $g_n = q_{m+n-1} \pm f_n$, $m = 5$, then $10(h_5, h_6) = p_1, p_2, P_n$ is a Fibonacci sequence conditional upon the value of $r$ in $q_m = l_{r+m-1} \pm f_m, r \geq 4$.

**Proof**

Notice that $h_5 = g_{6+5-1} \pm f_5 = g_{10} \pm 8$

$$g_{10} = q_{5+10-1} \pm f_{10} = q_{14} \pm 89$$

Let $q_m = l_{r+m-1} \pm f_m, r \geq 4$
It follows

$$q_{14} = l_{r+14-1} \pm f_{14} = l_{r+13} \pm 610$$

$$\therefore h_5 = l_{r+13} \pm 707$$

At this level of implementation, the behaviour of $L_n$ is critical. The value of $r$ is important as it governs whether or not $10q_h = 10h_0$.

The abovementioned theorems give the mathematical background in the creation of the $Z$-class. A Fibonacci sequence $H_n$ is a member of the $Z$-class if $10(h_{5}, h_{6}) = p_{1}, p_{2}$; $P_{n}$ is a Fibonacci sequence. The first four members of the $Z$-class are:

$$\begin{align*}
7, 11, 18, 29, 47, 
27, 44, 71, 115, 186, 
41, 66, 107, 173, 280, 
48, 78, 126, 204, 330,
\end{align*}$$

(5.7)

Notice that the sequences

$$\begin{align*}
27, 44, 71, 115, 186, \\
41, 66, 107, 173, 280,
\end{align*}$$

(5.8)

are symmetrical about

$$34, 55, 89, 144, 233,$$

(5.9)

and the symmetry is due to $7, 11, 18, 29, 47, \ldots$ In point of fact, this is the kind of symmetry in the $Z$-class: whenever two successive sequences in this class have the same arrangement of odd and even terms, then they are symmetrical about $l_{i}, l_{i+1}, l_{i+2}, l_{i+3}, l_{i+4}, \ldots$, $i \geq 8$. $L_{n}$ is a Fibonacci sequence; corollary 3.4 acts as guarantor to this proposition. When $i \geq 9$, $l_{i} - 7 = p_{i}$ and $l_{i} + 7 = q_{i}$ such that both $P_{n}$ and $Q_{n}$ are members of the $Z$-class. We say that $P_{n}$ and $Q_{n}$ are $Z_{in}$. For $i = 8$, some sequences enrol both $l_{i} \pm 7$ in the $Z$-class while others only enrol one of these. Theorems 5.1 to 5.5 read in conjunction with theorems 5.6 to 5.11 below should assist in appreciating this concept.

**Theorem 5.6**

Given an arbitrary Fibonacci sequence $H_{n}$, $h_{i} + 7 = q_{5} - 1$, $Q_{n}$ is a Fibonacci sequence, $i \geq 9$.

**Proof**

Since $i \geq 9$, from theorem 3.1, $h_{i} + 4 = p_{4} - 1$, $P_{n}$ is a Fibonacci sequence. From corollary 3.1, this means $h_{i} + 1 + 3 + 3$ is a parent number. This also means that $h_{i} + 1 + 3 + 3 + 2$ is a parent number. This simply means

$$h_{i} + 7 = q_{m} - 1, i \geq 9, m \geq 5$$

(5.11)

where $Q_{n}$ is a Fibonacci sequence. Equation (5.11) is a consequence of theorem 2.7. It follows from equation (5.11) that

$$q_{m} = h_{i} + 8$$

(5.12)

From this representation, $m$ is conditionally equal to 5. The critical case comes when $h_{i} = g_{i+1} + f_{i}$ because $h_{i} = 1$ is not a parent number, theorem 2.5. This means the lowest parent number for the critical case is $h_{5} + 1$. With this parent number, we form the sequence
\[ L_n = h_5 + 1, h_6 + 2, h_7 + 3, h_8 + 5, h_9 + 8, \ldots \] (5.13)

from which we see that \( L_5 = h_9 + 8 \). But \( L_5 - 1 \) is a parent number, theorem 2.7. This means \( h_9 + 8 - 1 = h_9 + 7 \) is a parent number. It follows that our minimum value of \( i \) is 9, thus legitimately for arbitrary \( H_n, h_1 + 7 = q_5 - 1, Q_n \) is a Fibonacci sequence, \( i \geq 9 \).

**Theorem 5.7**

If a Fibonacci sequence \( H_n \) is such that \( h_i = g_{n+i-1} + f_i \), then \( h_8 + 7 = q_6 - 1, Q_n \) is a Fibonacci sequence.

**Proof**

Following proof to theorem 5.5, \( h_8 + 7 = q_m - 1, m \geq 5 \). This means \( q_m = h_8 + 8 \). But \( h_8 + 8 \) appears in the sequence

\[ h_4 + 1, h_5 + 2, h_6 + 3, h_7 + 5, h_8 + 8, \ldots \] (5.14)

Since \( h_i = g_{n+i-1} + f_i \), \( h_4 + 1 \) is not a parent number, theorem 2.5. This means \( h_4 + 1 \) is second term of the Fibonacci sequence \( Q_n \). Therefore, in the sequence (5.14),

\[ h_8 + 8 = q_6 \]
\[ \therefore h_8 + 7 = q_6 - 1 \]

Following theorems 5.6 and 5.7 we state theorems 5.8 to 5.11 with no further proof.

**Theorem 5.8**

If a Fibonacci sequence \( H_n \) is such that \( h_i = g_{n+i-1} - f_i \), then \( h_8 + 7 = q_5 - 1, Q_n \) is a Fibonacci sequence.

**Theorem 5.9**

If a Fibonacci sequence \( H_n \) is such that \( h_i = g_{n+i-1} + f_i \), then \( h_8 - 7 = q_5 + 1, Q_n \) is a Fibonacci sequence.

**Theorem 5.10**

If a Fibonacci sequence \( H_n \) is such that \( h_i = g_{n+i-1} - f_i \), then \( h_8 - 7 = q_5 + 1, Q_n \) is a Fibonacci sequence.

**Theorem 5.11**

Given an arbitrary Fibonacci sequence \( H_n, h_1 - 7 = q_5 + 1, Q_n \) is a Fibonacci sequence, \( i \geq 9 \).

### 5.2 Tetrad and pentad chemistry

The concept of discontinuity of Fibonacci sequences plays a central role in Chemical geometry. The definition of discontinuity has been given in definition 3.3. As a rule, if \( H_n \) is \( Z_n \) then it has f.o.w. This means \( H_n \) can have discontinuity at \( n = 4 \) (denoted \( d_4 \)) or at \( n = 5 \) (denoted \( d_5 \)). In the \( Z \)-class, if two successive sequences \( H_n \) and \( G_n \) have the same arrangement of odd and even terms, then

\[ G_n = H_n = 2(7,11,18,29,47,\ldots) \] (5.15)
It also follows that
\[
\frac{G_n + H_n}{2} = l_i, l_{i+1}, l_{i+2}, \ldots \quad i \geq 8
\]
(5.16)

Take the sequence \( L_n \).

Let \( l_i - 7 = h_1 ; \quad l_i + 7 = g_1 \).

For \( 8 \leq i \leq 13 \), \( H_n \) and \( G_n \) are matched such that one is \( d_4 \) and the other is \( d_5 \). We assume that this matching introduces a force of attraction between the two sequences. Now consider four adjacent sequences in the \( Z \)-class:

\[
\begin{align*}
H_n & \\
G_n & \\
P_n & \\
Q_n & 
\end{align*}
\]
(5.17)

Let \( P_n \) and \( Q_n \) be constituted as \( H_n \) and \( G_n \), that is, let one be \( d_4 \) and the other be \( d_5 \). If \( G_n \) and \( P_n \) have the same kind of discontinuity, we assume there is repulsion between the two sequences. If \( G_n \) and \( P_n \) are of different discontinuities, we assume there is attraction between the two sequences. This results in coupling, that is, the four sequences are bound together by forces of attraction and they form a tetrad.

Now consider five adjacent sequences:

\[
\begin{align*}
H_n & \\
G_n & \\
R_n & \\
P_n & \\
Q_n & 
\end{align*}
\]
(5.18)

Let \( H_n \) and \( G_n \) have different discontinuities. Let \( P_n \) and \( Q_n \) also have different discontinuities, and in addition, let \( G_n \) and \( P_n \) have the same discontinuity but different from that of \( R_n \). This results in the five sequences being bound together by forces of attraction forming a pentad.

A very important role in tetrad and pentad chemistry is played by \( L_n \) with \( d_2 \).

Let an arbitrary Fibonacci sequence \( L_n \) be \( d_2 \). Let \( l_9 - 7 = r_1 ; \quad l_9 + 7 = t_1 \). When both \( G_n \) and \( H_n \) are enrolled in the \( Z \)-class, then they are involved in tetrad formation, and they form the tetrad (5.17).

Let \( l_9 - 7 = r_1 ; \quad l_9 + 7 = t_1 \). Let one of \( R_n \) and \( T_n \) be enrolled in the \( Z \)-class. For illustration, let \( R_n \) be \( Z \). This means that \( R_n \) will be involved in pentad formation and it forms the pentad (5.18).

The interested reader will find this chemistry both interesting and challenging.

**Axiom 5.1**

In \( F_n = (1.8) \), let \( f_n \) be even, \( n \geq 8 \). If \( n \) is even, then \( \frac{f_n}{2} = h_1 \) such that \( h_8 - 7 = p_1 \) and \( h_8 + 7 = q_1 \) where \( P_n \) is in the \( Z \)-class and \( Q_n \) is not; \( P_n \) is involved in pentad formation. If \( n \) is odd, then \( \frac{f_n}{2} = h_1 \) such that \( h_9 - 7 = p_1 \) and \( h_9 + 7 = q_1 \) where \( Q_n \) is in the \( Z \)-class and \( P_n \) is not; \( Q_n \) is involved in pentad formation.

We here give an example of a pentad:
6 Golden Section and Geometrical Basis for the Fine-structure Constant

There is a number which plays a central role in the theory of the golden section. This number is 117. In the subsequent presentation we show its significance to the theory of the golden section and its relationship with the fine-structure constant.

6.1 The number 117 in SNH (energy levels in Fibonacci sequences)

Recall set Y from section 4. As proved in theorem 4.1, all elements of Y are parent numbers. Firstly we need to show that \( y_n, n \geq 2 \), is a parent number \( p_n \) such that \( P_n \) is a Fibonacci sequence in the Z-class. Consider theorem 6.1.

**Theorem 6.1**

Given an arbitrary Fibonacci sequence \( H_n \) such that \( h_1 \) or \( h_2 \) or both are odd, when \( h_i \) is even, \( i \geq 2 \), then \( \frac{h_i}{2} \) is a parent number.

**Proof**

Let \( d = \text{odd} \) and \( e = \text{even} \). \( H_n \) can take any of the forms:

\[
\begin{align*}
&d, d, e, d, d, e, \ldots \\
&d, e, d, d, e, d, \ldots \\
&e, d, d, e, d, \ldots 
\end{align*}
\]

Let \( h_i \) be even.

Let \( \frac{h_i}{2} = l_n; \) \( l_n \) is a Fibonacci sequence. It follows

\[ h_i = 2l_n. \]

From equation (6.1), we see that \( h_{i+1} \) is odd. It follows

\[ h_{i+1} = 2l_{n+1} \pm 1 \]

\[ \therefore 2l_{n+1} = h_{i+1} \mp 1, \text{ thus} \]

\[ 2l_{n+1} - 2l_n = h_{i+1} - h_i \mp 1 \]

\[ = h_{i-1} \mp 1 \text{ (even)} \]

This means

\[ l_{n+1} - l_n = \frac{h_{i-1} \mp 1}{2} = l_{n-1} \]

If \( i \geq 2 \), then
\[ \varphi l_{n-1} = \varphi \left( \frac{h_i + 1}{2} \right) = \frac{h_i}{2} \pm 1 \neq l_n \]

\[ \therefore n = 1, \]

thus

\[ \frac{h_i}{2} = l_1, \text{ a parent number}. \]

It now suffices to recall equation (4.4). For \( n \geq 2, i = 1 \), theorem 6.1. Therefore,

\[
\begin{align*}
    y_{n} &= h_5 + 1; \text{odd } n \\
    y_{n} &= h_5 - 1; \text{even } n
\end{align*}
\]

where

\[ h_1 = \frac{f_{2n+4}}{2}, \]

noting that \( F_n = (1.3) \).

But \( y_n = h_5 + 1 \) when \( H_n \) is such that \( h_i = g_{r+i-1} - f_1 \), and
\( y_n = h_5 - 1 \) when \( H_n \) is such that \( h_i = g_{r+i-1} + f_1 \); theorem 6.3.

From theorems 5.1 and 5.2, this means \( y_n, n \geq 2 \), is a parent number \( p_1 \) such that the Fibonacci sequence \( P_n \) is in the \( Z \)-class. The behaviour of set \( Y \) is analogous to that of a Fibonacci sequence. Let \( H_n \) be an arbitrary Fibonacci sequence and \( Y_n \) be set \( Y \). As said in section 4, \( h_i \) is the nucleus of \( H_n \). In \( Y_n, y_1 \) is the only parent number outside the \( Z \)-class, and is thus analogous to \( h_1 \), the nucleus. In \( H_n \), geometric-stability begins at \( h_4 \), theorem 2.12, and for \( n \geq 5, h_n \pm 1 \) is a parent number. Now \( y_n, n \geq 4 \), is a parent number \( p_1 \) such that \( P_n \) is involved in compound (tetrad and pentad) formation – this is analogous to geometric stability in \( H_n \) which begins at \( h_4 \). Notice that \( y_4 \) is involved in tetrad formation while \( y_n, n \geq 5 \) is involved in pentad formation, theorem 6.2, - analogous to \( h_n \pm 1, n \geq 5 \) is a parent number (theorem 2.7). This analysis shows that \( y_4 = 117 \) is the point of transition in \( Y_n \).

**Theorem 6.2**

Given set \( Y = \{1,7,27,117,493,2091,\ldots\} \), \( y_n, n \geq 5 \), is involved in pentad formation.

**Proof**

Take \( F_n = (1.3) \). Let \( f_i \) be even. Let \( i \) be even, \( i \geq 2 \).

We have \( \frac{f_i}{2} = h_i \) such that \( h_8 - 7 = y_n, n \geq 3, \text{odd} \).

It follows \( \frac{f_{i+2}}{2} = g_8 \) such that \( g_8 + 7 = y_{n+1} \).

Since \( n \) is odd, \( n + 1 \) is even.

It follows from axiom 5.1 that for \( n \geq 5 \), when \( n \) is odd, \( y_n = h_8 - 7 = p_1 \) such that \( P_n \) is in the \( Z \)-class and involved in pentad formation.

Again from axiom 5.1, for \( n \geq 5 \), when \( n \) is even, \( y_n = g_8 + 7 = q_1 \) such that \( Q_n \) is in the \( Z \)-class and forms a pentad.

To validate that \( n \geq 5 \) in \( Y_n \), notice that axiom 5.1 sets \( f_8 \) as the critical. But \( \frac{f_8}{2} = \frac{34}{2} = 17 = h_1 \) such that \( h_8 = 500 \) and \( h_8 - 7 = 493 = y_5 \); therefore \( y_5 \) is the critical, thus \( n \geq 5 \) in \( y_n \).
Theorem 6.3

Let an arbitrary Fibonacci sequence \( H_n = h_1, h_2, h_3, h_4, \ldots \) be such that \( h_1 = g_n + 1, n \geq 4, G_n \) is a Fibonacci sequence. Let \( h_i \) be even where either \( h_1 \) or \( h_2 \) or both are odd. If \( i \) is even, then \( \frac{h_i}{2} = l_r + 1, r \geq 4. \) If \( i \) is odd, then \( \frac{h_i}{2} = l_r - 1, r \geq 4, L_r \) is a Fibonacci sequence.

Proof

Let \( h_i \) be the first even term of \( H_n \). Here, \( G_n = g_n + 2, g_{n+2} + 3, g_{n+3} + 5, \ldots \) is a Fibonacci sequence. Notice that this requires \( G_n = e, e, e, e, e, \ldots \)

This also means \( i = 2, \) thus

\[ h_2 = g_{n+1} + 2 \]

\[ \frac{h_2}{2} = \frac{g_{n+1} + 2}{2} \]

Let \( \frac{g_{n+1}}{2} = l_r. \)

It follows \( \frac{h_2}{2} = l_r + 1. \)

Let \( n + 1 = u. \) This means \( G_n = g_n + 2, q_3, q_4, \ldots \) depending on whether the Fibonacci sequence \( Q_m \) has first or second order wiring. Our critical case is therefore \( g_5 = 2q_6. \) This means \( l_r = \frac{g_5}{2} = \frac{2q_6}{2} = q_6 \) is the critical. Here, allow \( q_m = l_r. \) This means \( \frac{h_3}{2} = l_r + 1, r \geq 6. \) Since \( r \geq 6, \) the condition \( r \geq 4 \) is satisfied.

When \( i \) is odd, two scenarios ensue:

Scenario I: \( i = 1: \)

\[ \frac{h_1}{2} = \frac{g_{n+1}}{2}. \) Here, \( g_n \) is odd. Let \( \frac{g_n}{2} = x - 0.5. \) This means

\[ \frac{h_1}{2} = x - 0.5 + 0.5 = x. \]

From theorem 6.4,

\[ x = l_r - 1, r \geq 4. \]

Scenario II: \( i = 3: \)

\[ \frac{h_3}{2} = \frac{g_{n+2} + 3}{2}. \) Here, \( g_{n+2} \) is odd.

\[ \frac{g_{n+2} + 3}{2} = x - 0.5 + 1.5 = x + 1 \text{ where } x = p_1 - 1 \text{ and } p_1 = q_m - 1, \text{ theorem 5.16.} \]

Theorem 6.4

Let a Fibonacci sequence \( H_n \) be such that \( h_i \) and \( h_{i+1} \) are odd. \( \frac{h_i}{2} = x - 0.5 \) where \( x = l_r - 1, r \geq 4, L_r \) is a Fibonacci sequence.
Proof

Let $\frac{h_i}{2} = x - 0.5$

Let $x = p_j$. It follows

$\frac{h_i}{2} = p_j - 0.5$

$h_i = 2p_j - 1$.

But $h_{i+1}$ is odd. This means

$h_{i+1} = 2p_{j+1} - 1$.

Let $p_j = l_r + 1, r \geq 4$.

$h_i = 2(l_r + 1) - 1 = 2l_r + 1$

$h_{i+1} = 2(l_{r+1} + 2) - 1 = 2l_{r+1} + 3$

Now $\varphi h_i \neq h_{i+1} \cdot p_j \neq l_r + 1, r \geq 4$.

Let $p_j = l_r - 1, r \geq 4$.

$h_i = 2(l_r - 1) - 1 = 2l_r - 3$

$h_{i+1} = 2(l_{r+1} - 2) - 1 = 2l_{r+1} - 5$

Since $\varphi h_i = h_{i+1}$ it means $p_j = l_r - 1, r \geq 4, l_n$ is a Fibonacci sequence.

6.2 The number 117 in chemical geometry

Axiom 6.1

Given an arbitrary Fibonacci sequence $H_n$ let $h_i - 7 = p_i, h_i + 7 = q_i$. For $8 \leq i \leq 13, P_x$ and $Q_x$ are such that one is d4 and the other is d5. For $14 \leq i \leq 28$, both $P_x$ and $Q_x$ are d4 (homochirality). $H_n$ behaves in this manner if $1 \leq h_1 \leq 117$. When $119 \leq h_1 \leq 1130, H_n$ begins homochirality at $n = 14$ or at $n = 15$.

In axiom 6.1 we find the number 117 at the first point of transition. Notice that $1 \leq h_1 \leq 117$ is a pure region, that is, all $H_n$ in this region begins homochirality (defined in axiom 6.1) at $n = 14$. The region $119 \leq h_1 \leq 1130$ is a “mixed” region – some $H_n$ begins homochirality at $n = 14$ while some begins at $n = 15$. Figure 6.1 illustrates.

![Fig. 6.1. Illustration of axiom 6.1](image-url)
6.3 The number 117 in expansion of $F_n = (1.3)$

The above two instances give the role of the number 117 in special numerical harmony and chemical geometry; several such instances could be given, here omitted due to space restrictions. It shall be made accessible to the critic that even outside chemical geometry, the number 117 still marks transition, and this third instance demonstrates this fact.

From the Fibonacci sequence $(1.3)$, we assemble other Fibonacci sequences by multiplying segments of this sequence by an integer constant. $2(2,3,5,8,...) = 4,6,10,16,...$ is an example. More than one constant can multiply the segment $f_nf_{n+1}, f_{n+2}, f_{n+3},...$ to yield Fibonacci sequences. In Table 6.1 we summarize this result for the first 20 terms of $F_n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f_n$</th>
<th>Number of multipliers ($w$)</th>
<th>Min. multiplier ($x$)</th>
<th>Max. multiplier ($y$)</th>
<th>Number of multipliers $&gt; f_n$ ($z$)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1</td>
<td>1</td>
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<td>1</td>
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<td>7640</td>
<td>11015</td>
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</tbody>
</table>

In Table 6.1, $f_n$ is the integer at the beginning of the segment from $(1.3)$ e.g. for the segment $8,13,21,34,55,..., f_n = 8$. It can be noticed that for $1 \leq n \leq 14, w_n = x_i$ or $y_i$, where $1 \leq i \leq 12$. For instance $w_{14} = y_{12} \rightarrow w_0 = x_5$, etc. For $n \geq 15, w_n$ is no longer given by $x_i$ or $y_i$, that is, $w$ is no longer maximum or minimum value of the multipliers. It is interesting to note that $z_{15} = 117$, again at point of transition.

6.4 The number 117 and the fine-structure constant

That the number 117 is always notoriously marking (energy) transitions in the theory of the golden section is an important and perhaps disturbing signal. We duly call attention to the overwhelming fact that this number, in essence, may provide the geometrical basis of the fine-structure constant in the golden section. Is it by “squared” numerical coincidence that

$$117^2 = 13689 \quad (6.3)$$

and
\[
\frac{117^2}{100} = 136.89 \quad (6.4)
\]
such that the inverse square law yields
\[
100 \left( \frac{1}{117} \right)^2 = 0.007305135 \quad (6.5)
\]
a value greater than the CODATA 2014 estimate of \( \alpha \) at
\[
\alpha = 0.0072973525664(17) \quad (6.6)
\]
see Mohr et al. [26], by 0.1%? Due to this high degree of precision, we find no incentive in seeking further agreement with the experimental value, that is, equation (6.5) is scientifically acceptable for estimating the fine-structure constant. Varlaki et al. [27] state that, “We can measure precisely neither the speed of light, nor the Planck constant and the elementary charge of the electron, furthermore, the accuracy of spectroscopic measurements of hydrogen spectrum are also very limited, not to mention the “higher members” of the intermediate calculations”. This notwithstanding, and for interest’s sake, further agreement with the experimental value is achieved by
\[
\alpha = 100 \left( \frac{1}{117.0623647} \right)^2 = 0.007297353 \quad (6.7)
\]
In equations (6.5) and (6.7) we have used the constant 100 to show that the fine-structure constant is based “purely” on the number 117 without engineering. Having demonstrated the central role played by this number in the theory of the golden section, we witness here a remarkable intersection of geometry and atomic physics. Immediately the critic might need to know if there is any relationship between the golden section and hydrogen spectra. We quote Heyrovská [28]: "The surprising discovery that Bohr radius is divided at the Golden Point into two sections pertaining to the electron and proton due to electrostatic reasons led to the general finding that it is a geometrical constant in atomic and ionic radii, bond lengths and bond angles.” Furthermore, Haight [29] says, “Lest it be forgotten, the fine-structure constant of the prime element hydrogen is 137+, and the pathways of its electrons follow the Fibonacci sequence of numbers exactly.”

Physicists like Wolfgang Pauli (1900-1958) and Richard Feynman (1918-1988) took the fine-structure constant problem seriously. Pauli, quoted by Varlaki et al. [27] says, “… The smallness of these new effects is a consequence of the smallness of the so-called fine structure constant, which is often linked with Sommerfeld’s name, since its fundamental significance first came clearly to light through his theory of 1916 of the fine structure of hydrogen spectra. The theoretical interpretation of its numerical value is one of the most important unsolved problems of atomic physics.” Sherbon [25] says, “David Lindorff [30] comments, “Pauli’s sense that number in itself had a deep psychological significance is striking; it would later become of singular importance to him. ... He wrote, ‘Here new Pythagorean elements are at play, which can perhaps be still further researched.’”

It should be noted that from an examination of literature on the variation of the fine-structure constant, see e.g. [31–34], such “variation” cannot move the “constant” away from 117, i.e. it can only affect the value 117.0623647 in equation (6.7) beginning at the third decimal place. Pinho and Martins [31] say, “A recent analysis by Webb et al. of a large archival dataset has provided some evidence for spatial variations of the fine-structure constant, at the level of a few parts per million (ppm), [32,33].”

Thus done and said, we leave the question: Was Wolfgang Pauli, in close contact with the Platonic origins of the fine-structure constant, mistaken in “searching everywhere” for the number 137?
7 Conclusion

This paper may be used by the interested researcher to appreciate and hence further develop not only the theory of the golden section, but of proportiones perfectus as a whole. Green [35] says, “The neo-Platonism of the age [medieval] supported, and was supported by, the interest in the myriad and fascinating properties and uses of the golden section. With the rise of empiricism, however, in the 17th century, interest in such matters came to be actively discouraged. It was not until the 19th century that significant interest in the golden section (and formal logic, for that matter) was revived.”

That we have established a geometrical basis for the fine-structure constant in the golden section must not come as a surprise. Sherbon [36] says, “In the Timaeus, Plato “considered the golden section to be the most binding of all mathematical relationships and the key to the physics of the cosmos,” quoted by Robert Schoch and Robert McNally in Pyramid Quest [37].” Chown [38], cited by Sherbon [25], states that “Perhaps the most surprising place the golden ratio crops up is in the physics of black holes, a discovery made by Paul Davies.” Petukhov and He [39] say, “… many physiological systems and processes are connected with the golden section … proportions of the golden section characterize cardio-vascular processes, respiratory processes, electric activity of the brain, locomotion activity, aesthetic phenomena, etc.” Dattoli [40] “remains convinced that the value of the coupling constant, the number we measure or better, we observe, is just what it is and cannot be different.” Under the circumstances of this presentation, and given the irrefutable ubiquity of the golden section in physical and abstract systems alike, we agree with Max Born that the number 137 “is the dominating factor of all natural phenomena”. Perhaps that which we call number is the software of the universe.

Competing Interests

Author hereby declares that no competing interests exist in the publication of this thesis.

References


[34] Le TD. A stringent limit on variation of the fine-structure constant using absorption line multiplets in the early universe. Astrophysics. 2016;59(285).


[40] Dattoli G. The fine-structure constant and numerical alchemy. Servizio Edizioni Scientifiche – ENEA Centro Ricerche Frascati, Rome, Italy; 2009.

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