FORESTRY AND NATURAL SCIENCES

Anssi Lehikoinen

Modeling Uncertainties in Process Tomography and Hydrogeophysics

Publications of the University of Eastern Finland
Dissertations in Forestry and Natural Sciences

UNIVERSITY OF EASTERN FINLAND

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Publications of the University of Eastern Finland
Dissertations in Forestry and Natural Sciences
No 82

Academic Dissertation
To be presented by permission of the Faculty on Natural Sciences and Forestry
for public examination in the Auditorium ML3 in Medistudia Building at the
University of Eastern Finland, Kuopio, on November, 2, 2012,
at 12 o'clock noon.

Department of Applied Physics

Kopijyv Oy
Kuopio, 2012
Editor: Prof. Pertti Pasanen, Prof. Pekka Kilpelinen, Prof. Kai Peiponen,
Prof. Matti Vornanen

Distribution:
Eastern Finland University Library / Sales of publications
P.O. Box 107, FI-80101 Joensuu, Finland
tel. +358-50-3058396
http://www.uef.fi/kirjasto

ISBN: 978-952-61-0882-7
ISSNL: 1798-5668
ISSN: 1798-5668
ISBN: 978-952-61-0883-4 (pdf)
ISSN: 1798-5676 (pdf)

Author's address:
University of Eastern Finland
Department of Applied Physics
P.O.Box 1627
70211 KUOPIO
FINLAND
email: anssi.lehikoinen@uef.fi

Supervisors:
Professor Jari Kaipio, Ph.D.
University of Eastern Finland
Department of Applied Physics

Professor Marko Vauhkonen, Ph.D.
University of Eastern Finland
Department of Applied Physicsi

Professor Jouko Lampinen, Ph.D.
Aalto University School of Science and Technology
Lab. of Computational Engineering

Reviewers:
Associate Professor Raul Lima, Ph.D.
University of So Paulo
Department of Mechanical Engineering

Associate Professor Tuomo Kauranne, Ph.D.
Lappeenranta University of Technology
Department of Mathematics and Physics

Opponent:
Academy Research Fellow Nuutti Hyvnen, Ph.D.
Aalto University School of Science and Technology
Department of Mathematics and System Analysis

ABSTRACT

In the majority of real world problems, the interesting quantities cannot be measured directly. Instead, some measurable quantities are usually related to the interesting quantities via mathematical models, and thus information on the interesting quantities can be obtained. With stable problems, one can perform a more or less straightforward model fitting procedure to gain this information. Technically, this fitting is usually carried out by minimizing the difference between the measurements and the model predictions.

With unstable problems, which are also called inverse problems, such straightforward model fitting cannot be employed. Such problems can, however, be tackled by using so-called deterministic regularization appoaches, or by formulating the problem in the statistical Bayesian framework. The latter approach is feasible also with problems in which the models themselves are only partially known or contain errors.

This thesis considers tomographic problems, which are one of the largest classes of inverse problems. In particular, we consider a soft-field tomographic modality which probes the unknown object via electric fields. As specific technical model uncertainties, we consider the domain truncation problem in which the computations are carried out in a small region of interest, and the problem in which some uninteresting unknown quantities need to be handled in an efficient manner. Furthermore, the focus in this thesis is on models and approaches that facilitate efficient computations. Thus, simultaneous model reduction is also considered. As particular applications, we consider specific problems in hydrogeophysics and process tomography.

The methods and results in this thesis show that the recently proposed approximation error approach is a feasible one for the considered model uncertainties, as well as simultaneous model reduction. Most of the studies address only the feasibility of the computational approaches but in one case we also show that the framework is feasible with dynamical laboratory measurement setup.

Overall, the results of the thesis suggest that several practical industrial and similar inverse problems can be successfully handled when the ubiquitous modeling errors and uncertainties are treated properly. Furthermore, making the applications industrially feasible by employing simultaneous model reduction is possible by using the same formalism. This property reduces the overhead of developing the approach for specific industrial problems.

Universal Decimal Classification: 537.311.6, 550.3, 556.012, 621.3.011.2, 621.317.33
PACS Classification: 02.30.Zz, 06.20.Dk, 81.70.Tx, 84.37.+q, 87.63.Pn
INSPEC Thesaurus: inverse problems; tomography; electric resistance measurement; electric impedance measurement; process monitoring; hydrology; geophysics; uncertainty handling; errors; modelling
Yleinen suomalainen asiasanasto: tomografia; prosessit; monitorointi; tarkkailu; hydrologia; geofysiikka; epvarmuus; virheet; mallintaminen

ABBREVIATIONS

2D	Two-dimensional
3D	Three-dimensional
CEM	Complete electrode model
ERT/EIT	Electrical resistance (impedance) tomography
EKF	Extended Kalman filter
MCMC	Markov Chain Monte Carlo
MAP	Maximum a posteriori

NOTATIONS

V	Data vector (voltage observations)
σ	Electrical conductivity
π	Probapility density function
$\pi(\sigma, V)$	Joint density of parameters and data
$\pi(\sigma\|V)$	Posterior density
$\pi(V\|\sigma)$	Likelihood density
$\pi(\sigma)$	Prior density
v	Gaussian observation noise
S_t	Time-varying variable
h_t	Evolution model
g_t	Observation model
ω_t	State noise
ϵ_t	Observation noise
$\bar{U}$	Accurate complete electrode model also referred as forward model for EIT/ERT
z	Contact impedance

$\bar{\bar{\xi}}$	Boundary conditions
e	Additive errors
P	Projection operator
U	Computationally reduced forward model
e_*	Mean value of e
Γ	Covariance matrix
$\vec{r}$	Position vector
u	Potential distribution
Ω	Domain
$\partial\Omega$	Boundary of Ω
N_{el}	Number of electrodes
e_ℓ	lth electrode
z_ℓ	Contact impedance on lth electrode
U_ℓ	Potential on the ℓ^{th} electrode
$\mathbb{R}^N$	N-dimensional real space
I_ℓ	Injected current through electrode e_ℓ
$\mathcal{Z}$	Augmented variable
Ξ_t	Augmented variable
ϕ	Porosity
K	Unsaturated hydraulic conductivity
P_c	Capillary pressure
ρ_w	Water density
g	Gravitational constant
$\hat{z}$	Unit vector
k	Absolute permeability
k_{rel}	Relative permeability
μ_w	Dynamic viscosity of water
m	Soil-specific parameter
α	Soil-specific parameter
S_e	Effective water saturation
S_{wr}	Residual water saturation
σ_{w}	Electrical conductivity of the liquid phase
b	Cementation index
n	Saturation index
c	Concentration distribution
$\vec{v}$	Velocity field
κ	Diffusion coefficient

Preface

This thesis was carried out under the supervision of Professor Jari Kaipio in the Department of Applied Physics at the University of Eastern Finland during the years 2004–2012. I want to express my gratitude to Jari Kaipio and Stefan Finsterle for their encouragement, support and friendship. The author also thanks Professor Jouko Lampinen for support in the initial phase of the study. Furthermore, the author wants to express his gratitude to Professor Marko Vauhkonen for all the help that I have received during these years, and to my co-authors Arto Voutilainen, Janne Huttunen, Mike Kowalsky and Lasse Heikkinen for many valuable discussions on inverse problems and numerics. Special thanks go to the people behind the main members of Numcore Ltd and Rocsole Ltd for their encouragement and support during finalizing this thesis.

Special thanks go also to my parents Raili and Viljo for their encouragement and support during all these years. Finally, I want to express my deepest gratitude and love to my wife Anni and my children Hilla and Hermanni for their love and support during the writing process of this thesis.

Kuopio, November 2, 2012 *Anssi Lehikoinen*

LIST OF PUBLICATIONS

This thesis consists of an overview and the following four original articles which are referred to in the text by their Roman numerals **I-IV**:

I A. Lehikoinen, S. Finsterle, A. Voutilainen, L.M. Heikkinen, M. Vauhkonen and J.P. Kaipio. Approximation Errors and Truncation of Computational Domains with Application to Geophysical Tomography. *Inverse Problems and Imaging* **1**:371-389, 2007.

II A. Lehikoinen, S. Finterle, A. Voutilainen, M.B. Kowalsky and J.P. Kaipio. Dynamical Inversion of Geophysical ERT Data: State Estimation in the Vadose Zone. *Inverse Problems in Science and Engineering* **17**:715-736, 2009.

III A. Lehikoinen, J.M.J. Huttunen, S. Finsterle, M.B. Kowalsky and J.P. Kaipio. Dynamic Inversion for Hydrological Process Monitoring with Electrical Resistance Tomography Under Model Uncertainties. *Water Resources Research* **46**:W04513, 2010.

IV A. Voutilainen, A. Lehikoinen, M. Vauhkonen and J.P. Kaipio. Three-dimensional Nonstationary Electrical Impedance Tomography with a Single Electrode Layer. *Measurement Science and Technology* **21**:035107, 2010.

Throughout the overview, these papers will be referred to by Roman numerals.

AUTHOR'S CONTRIBUTION

The publications selected in this thesis are original research papers on statistical inversion methods applied in electrical resistance tomography. All publications are result of the joint work with co-authors and the contribution of co-authors has been significant in all papers. The author is the lead author in Publications I, II and III. In these publications, the author has been also responsible for developing the methods presented in the publications and has carried out all the implementations of numerical simulations and results. As an exception, in publication IV, the writing task and numerical simulations were divided among the authors. While the specific problems behind publications I - III were suggested by the coauthors, the author suggested the main idea behind publication IV.

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1 Introduction

Tomography, in the wide sense, refers to the construction of the internal structure of an object based on measurement that are carried out only outside of the object. The internal structure can be probed, for example, using X-rays or electromagnetic fields. The most common and well known tomographic modality is X-ray tomography, which is widely used in biomedical imaging [1]. Different probing fields interact with the target material in different ways and convey comlementary information on the target structure. For example, X-ray tomography essentially probes the distribution of the mass density. On the other hand, the interaction of electromagnetic fields with the target depends significantly on the used frequency, or wavelength of the field. In this thesis, we use low frequency electric fields, using a modality called electrical resistance tomography (ERT), or electrical impedance tomography (EIT).

In hydrogeophysics, one uses geophysical methods, such as ERT or seismic methods, to answer questions that are related to subsurface water distribution, or the flow of water or contaminants, see for further information [2]. These questions may be related to the distribution or flow itself, or to the characteristics of the subsurface that permit the flow. If, for example, we are interested in how a particular ground patch would conduct contaminated water spill and whether the contamination would eventually end in an aquifer, we would typically need to know the spatial distribution of hydraulic parameters first. Geophysical methods would in this case be needed for the estimation of these parameters. In this thesis, we consider a particular case in which the water is not driven by hydraulics, but by capillary and other similar physical processes. Such cases are called vadoze (zones) [3].

In process tomography, the task is to probe the innards of process vessels, mixers, pipelines, reactors and other targets used in process industry. Typical tasks are to monitor the state of mix-

ing, detection of air or gas, progress of chemical reactions and estimation of mass flow. Very often, the (possibly multi-phase) fluid should be homogeneous on some scale, and the detection of deviation from homogeneity is an intermediate task. With respect to modalities, electromagnetic modalities such as ERT, EIT and electrical capacitance tomography (ECT), are by far the most common ones [4,5].

Modeling is needed when the entities (variables) that we are interested in, are not directly observable. On the other hand, we are able to observe some other variables, which are connected to the interesting variables through models. The straightforward classical use of the models would then be to carry out the observations, and find such parameters (interesting entities) that best fit to the observations via the model.

One of the key issues in modeling is that models are always mere approximations to, not exact representation of physical reality, and are always subject to uncertainties and errors. The largest class of models that relate the interesting and observable variables, are partial differential equations (PDE's) and the related initial-boundary value problems. In some cases, PDE models can be claimed to be quite accurate, in other, they are known to be highly approximate idealizations. In addition to the PDE models being approximate themselves, further approximations and errors are introduced via different types of sources. For example, numerical approximations have almost always to be used, which induces discretization errors, some of the conventionally required boundary data (conditions) are not known, and the geometry of the target may only be approximatively known. Furthermore, in some problems, the locations of where the measurements were made might not be exactly known.

To further complicate the overall problem, many practical applications presume that all computations are carried out with very limited computational arsenal and possibly in a time frame of a millisecond, while conventional numerical considerations would often require using several seconds for the particular computations.

Thus, even in cases in which a very good model would be available, this model could not be used in practice. This is particularly common in process tomography applications. Whereas, in hydrogeophysics, the processes are typically temporally very slow, in process tomography the processes are often very fast. Other than this difference, the modeling and computational problems that are related to hydrogeophysical and process tomography are either the same or at least similar.

Computational models have been successfully used in science and engineering for decades and even centuries. Why, then, are uncertainties claimed to be particular problems in the above? The answer here is that tomographic problems are a typical example of inverse problems, which by loosely speaking are defined to be problems that tolerate errors and uncertainties poorly [6,7]. Classical successful examples of modeling are stable problems.

In many fields of science and engineering, proprietary, and often very approximate, methods have been developed and used since the 1930s, the classical theory of inverse problems can be said to have been originated in the 1960s and to have been well established in the 80s and 90s [8–13].

The classical theory considers the models as known and accurate, and the measurement (observation) errors to be small or even infinitesimal. As note above, in most practical inverse problems, the models are not accurate, and furthermore, the measurement errors can often not be considered to be small. The classical theory for the solution of inverse problems, the regularization theory, is often not well suited to dealing with inverse problems with approximative and uncertain models [14–16].

The most natural approach to model errors and uncertainties, is obviously statistics. This means that originally deterministic models, such as the convection-diffusion model, are turned to their stochastic counterparts. This, then, requires the further modeling of the underlying statistics of the models. With respect to choosing between the frequentist and Bayesian frameworks for statistics, the instability of inverse problems points directly to the Bayesian

framework. In this framework, all primary and secondary uncertainties and errors are modelled explicitly. This is an essential feature when the computational models are constructed so that they tolerate unavoidable modeling related uncertainties. In this thesis, the so-called approximation error approach is used as a building block in the overall construction of the computational models.

Aims of the thesis

In this thesis, we consider four typical sources of errors and uncertainties and typical related cases. The first uncertainty, the partially unknown boundary data (or conditions), can be argued to be the most common one. In almost all computational inverse problems that are governed by partial differential equations and the related boundary value problems, the whole domain can not be modelled. Instead, only a subdomain around the region of interest is modelled, and the computational domain is truncated to include this subdomain only. On these truncation boundaries, the boundary conditions are not known. Employing standard off-the-shelf boundary conditions will in most cases render the solutions useless and meaningless. Paper I deals with this particular problem. The approximation error approach is employed to construct the statistical model for the observation errors that are related to the unknown boundary conditions. The application that is considered is geophysical ERT.

Papers II and III deal with nonstationary hydrogeophysical problems, in which the unknowns are modelled as stochastic processes. State estimation approaches, such as Kalman filters, are typically used for the solution of such problems. The uncertainties here are related to the modeling of the statistics of these processes. The feasibility of these models is shown to be an essential requirement for the overall solvability of the problem. In addition, in Paper III, we consider the additional uncertainty of not knowing the central hydrological parameter (permittivity) when estimating the flow of water in a simulated injection case. Furthermore, we consider an approach to update the statistics of the related stochastic processes as time evolves and further observations are acquired.

In Paper IV, we again consider the nonstationary inversion problem, here in the context of process tomography. In this case, the principal uncertainty is related to symmetry, and is not inherent to the overall problem. Instead, the symmetry problem is induced via the practical problem of constructing a two-dimensional measurement sensor in a three-dimensional situation. The approach here is to construct a special nonstationary model which breaks the symmetry problem. The modeling of the unknown boundary conditions and other uncertainties as a stochastic process plays a central role in this task.

2 Inverse Problems

2.1 BAYESIAN FRAMEWORK FOR INVERSE PROBLEMS

The classical theory of ill-posed problems has been developed since the 60's [8–11] and produced a number of different approaches, such as truncated singular value decomposition, Tikhonov regularization, stopped iterative methods, to accompany the obvious projection approaches, see for example [6,7,12,13,17]. These methods are referred to as *regularization methods*.

Regularization methods are not based on explicit models for the unknowns, except in the case of some projection methods [18, 19]. However, these methods can be argued to employ implicit models, which is clearest in the case of truncated singular value decomposition [14]. Also, the deterministic methods usually seek only to find a single solution for the problem, possibly with some error estimates based, for example, on sensitivity analysis. These error estimates do not, however, generally bear any clear (statistical) interpretation. Also, regularization methods are implicitly based on a number of assumptions which may not be valid [14]. For example, using 2-norm based functionals (to be minimized) usually refers to an additive noise model with independent identically distributed Gaussian errors.

In statistical (Bayesian) inversion theory all variables are modelled explicitly and are considered as random variables, see [14–16, 20,21]. Given the measurements, the primary objective in Bayesian inversion is to determine the posterior probability density of the quantity of interest, given the measurements. The posterior density can be understood as the solution of the inverse problem, providing the basis for computing various point estimates as well as spread and interval estimates, and posterior marginal distributions.

Bayesian inversion is a hierarchical process which first calls for the modeling of the measurement process and the unknown, with special reference to the actual uncertainties of the models. These

models, the likelihood model and the prior model, respectively, together with the measurements fix the uncertainty of the unknowns given the measurements. Formally, this uncertainty is given in the form of the *posterior distribution* which is then subject to exploration, for example, using MCMC sampling [14, 16, 22, 23]. It is of central importance that the modeling of the measurement process and the modeling of the unknowns are carried out separately. This is not usually the case with regularization methods in which a change in measurement setting may change the implicit model for unknowns.

In the following, we use notations that are most common for the electrical impedance tomograhy case, that is, the measurements are voltages on electrodes and are denoted by V, and the primary unknown is (a parametrization of) the resistivity or conductivity distribution and is denoted by σ. Probability density functions are denoted by π. It is central to bear in mind that all distributions are to be understood as models, although they would occasionally be referred simply to as distributions.

Let us assume that the continuous random variables $\sigma \in \mathbb{R}^N$ and $V \in \mathbb{R}^{N_V}$ have a joint probability density $\pi(\sigma, V)$. According to the definition of conditional densities, the joint density can be expressed as

$$\pi(\sigma, V) = \pi(\sigma|V)\pi(V) = \pi(V|\sigma)\pi(\sigma), \tag{2.1}$$

where the marginal densities are

$$\pi(V) = \int_{\mathbb{R}^N} \pi(\sigma, V)\, \mathrm{d}\sigma \quad \text{and} \quad \pi(\sigma) = \int_{\mathbb{R}^{N_V}} \pi(\sigma, V)\, \mathrm{d}V.$$

From (2.1) we obtain the conditional probability density

$$\pi(\sigma|V) = \frac{\pi(V|\sigma)\pi(\sigma)}{\pi(V)} \propto \pi(V|\sigma)\pi(\sigma),$$

which is the well-known Bayes' rule. Once observations are obtained, Bayes' formula allows us to calculate the posterior density.

Above, we assumed that the joint probability density $\pi(\sigma, V) = \pi(V|\sigma)\pi(\sigma)$ is known. The density $\pi(V|\sigma)$, which is called the

likelihood model, which typically consists of a deterministic forward model and a statistical model of the observation noise e. In addition, the marginal density (model) $\pi(\sigma)$ must be constructed. In statistical inversion, $\pi(\sigma)$ is called the prior density (model). The model $\pi(\sigma)$ is constructed on the basis of knowledge on the quantity of interest before any measurements are carried out. The construction of the prior model is an important step, and care must be taken so that the model is not too restrictive, and that it is feasible. For discussion on feasibility of models in Bayesian inversion, see for example, [24].

With inverse problems, the construction of the likelihood model is much more critical. This is due to the fact that the likelihood densities are typically very narrow, and mismodeling, and especially underestimating, the uncertainties and errors will almost invariably lead to an infeasible posterior model. This means that the actual unknown can have essentially vanishing probability with respect to the posterior model. This property makes inverse problems a special class of Bayesian inference problems. See [21, 24] for further discussion on this topic.

In the case of additive Gaussian observation noise which is mutually independent with the primary unknown, we can write $v = V - R(\sigma) \sim \mathcal{N}(v_*, \Gamma_v)$, where $\mathcal{N}$ denotes a Gaussian density function, the likelihood density is of the form

$$\pi(V|\sigma) \propto \exp\left\{-\frac{1}{2}(V - R(\sigma) - v_*)^{\mathrm{T}}\Gamma_v^{-1}(V - R(\sigma) - v_*)\right\}.$$

Further, assuming that $\pi(\sigma) = \mathcal{N}(\sigma_*, \Gamma_\sigma)$ and assuming that v and σ are mutually independent, the posterior density is

$$\begin{aligned}\pi(\sigma|V) \propto \exp\{ &-\frac{1}{2}(V - R(\sigma) - v_*)^{\mathrm{T}}\Gamma_v^{-1}(V - R(\sigma) - v_*) \\ &-\frac{1}{2}(\sigma - \sigma_*)^{\mathrm{T}}\Gamma_\sigma^{-1}(\sigma - \sigma_*)\Big\}.\end{aligned}$$

In high-dimensional problems, posterior densities are difficult to illustrate; the solution is thus reported by computing point estimates.

For example, the maximum a posteriori (MAP) estimate is given by

$$\hat{\sigma}_{\mathrm{MAP}} = \arg \max_{\sigma} \pi(\sigma|V).$$

and the conditional mean (CM) by

$$\hat{\sigma}_{\mathrm{CM}} = \mathrm{E}\{\sigma|V\} = \int_{\mathbb{R}^N} \sigma\pi(\sigma|V)\,\mathrm{d}\sigma.$$

In addition to point estimates such as the MAP and CM estimates, marginal densities of single variables or pairs of variables are often of interest. Furthermore, some spread estimates are also typically computed. In particular, (an approximation for) the conditional covariance is almost always computed.

Giving an answer to a general statistical question regarding the posterior uncertainty, in general, requires the implementation of a Markov chain Monte Carlo (MCMC) algorithm that attempts to yield a representative ensemble of samples from the posterior distribution. The MCMC based inference is almost invariably an infeasible alternative when end applications are considered. In this thesis, we do not consider MCMC sampling and general inference. Rather, the focus is on Bayesian modeling and carrying out approximate inference that is based on a *feasible posterior model*.

2.2 NONSTATIONARY INVERSE PROBLEMS AND STATE ESTIMATION

Inverse problems in which the unknowns are time-varying, are referred to as dynamic, time-varying, or *nonstationary inverse problems* [14, 19, 25]. Several inverse problems are nonstationary in the sense that the unknown is naturally a time-varying variable. These problems are also naturally considered in the Bayesian framework. Nonstationary inverse problems are usually written as evolution-observation models (or evolution-measurement models) in which the evolution of the unknown is typically modelled as a (vector-valued) stochastic process.

Problems in which the primary unknowns are explicitly time-varying and can be modelled as stochastic processes, are called *state*

estimation problems. The related algorithms are sequential and in the most general form are of the Monte Carlo type [26]. However, the most commonly used algorithms are based on the Kalman recursions [14, 27–29], see also the review on state estimation in process tomography in [30].

The sequential Bayesian filtering approach to solve the state estimation problem can be described as follows. Let S_t be a finite dimensional time-varying variable. If conductivity is time varying and the only time-dependent unknown, we would write $S_t = \sigma_t$. Often, however, we would write $S_t = (\sigma_t, \xi_t)$ where ξ_t is another unknown (as in **[IV]**), or S_t could be a variable that depends on σ_t (as in **[II,III]**).

The task is to estimate S_t based on some observations V_t. More specifically, let S_t and V_t be stochastic processes defined for discrete time indices $t = 1, 2, \ldots, T$. Let the evolution of the state be modeled as a first order Markov process

$$S_{t+1} = h_t(S_t, \omega_t) \tag{2.2}$$

and the state is observed as described by the observation model:

$$V_t = g_t(S_t, \epsilon_t). \tag{2.3}$$

Here, h_t and g_t are assumed to be known, possibly nonlinear, functions. The temporally uncorrelated random vectors ω_t and ϵ_t represent the state noise and observation noise, respectively. The conditional probability densities $p(V_t|S_t)$ and $p(S_t|S_{t-1})$ are related to the observation and evolution models, and are called the likelihood density and the evolution (or prediction) density, respectively. The objective of filtering is to determine the posterior distribution of the state S_t conditioned on a set of the observations $D_t = \{V_1, V_2, \ldots, V_t\}$ obtained by that time. Filtering can be understood as a process in which the knowledge of the system is updated each time a new observation is made.

The updating process is a recursive scheme in which the evolution updating step and the observation updating step alternate. The density for the first state, $p(S_0|D_0) = p(S_0)$, reflects the uncertainty

in the initial conditions. The posterior densities are then obtained recursively with the following updating formulas:

1. Evolution (time) update

$$p(S_t|D_{t-1}) = \int p(S_t|S_{t-1})p(S_{t-1}|D_{t-1})dS_{t-1} \tag{2.4}$$

2. Observation (measurement) update

$$p(S_t|D_t) = \frac{p(V_t|S_t)p(S_t|D_{t-1})}{p(V_t|D_{t-1})} \tag{2.5}$$

where

$$p(V_t|D_{t-1}) = \int p(V_t|S_t)p(S_t|D_{t-1})dS_t. \tag{2.6}$$

The probability density obtained from the evolution updating can be interpreted as the sequentially generated, time-evolving prior model for the state S_t.

For Gaussian linear systems, the mean square state estimates as well as the conditional covariances and marginal densities can be computed using the classical Kalman filter algorithm. Approximate solutions to nonlinear problems are typically computed using different versions of the extended Kalman filter (EKF), which are based on sequential linearization of the observation model, and sometimes also the evolution model. Furthermore, the linearization can be carried out around different states. In highly nonlinear problems – such as those involving flow of water in unsaturated porous media – care must be taken in choosing the version of the extended Kalman filter. In this thesis, we employ the extended Kalman filter (EKF) that is sequentially linearized in the predictor estimate. For texts in state estimation, we refer to [29, 31] in general, to [14, 32] in connnection to nonstationary inverse problems, and to [30] in connection to general state space modeling.

2.3 TOMOGRAPHIC MODALITIES

In practice, tomography refers to estimating the internal structure of an object when measurements can be carried out only outside

the object, or at least outside the region of interest. The information about the interior can be obtained, for example, by probing the object with "hard field" radiation: high energy photons such as X-rays, or particles, such as neutrons. The fluxes of outgoing photons or neutrons are then measured at some locations around the object. Excluding biomedical applications, the "soft-field" modalities such as electrical impedance (resistance) tomography, optical (diffusion) tomography, ulrasound and thermal tomography are more common. The soft-field modalities are usually also called diffuse modalities since, generally, all measurements depend on the object properties in all of the object domain. For reviews of different tomographic modalities, see, for example, [1,4,5,33].

In this thesis, we deal exclusively with the electrical resistance tomography (ERT), which is the same as electrical impedance tomography (EIT) except that the phase shift of measurements is ignored. It is a common practice, however, to refer to EIT also when the phase shifts are ignored. The currently best known physical model for EIT/ERT is called he *complete electrode model* and is treated in Section 4.3.

3 Uncertainties and approximation errors

3.1 UNCERTAINTIES IN MATHEMATICAL MODELING

In the general field of inverse problems, the modeling of uncertainties has been gaining systematic attention only recently, and is mainly considered within the Bayesian framework [14]. Model errors have only seldom been considered in the deterministic framework, for an example, see [34]. In the deterministic framework it is possible to derive error bounds under model errors but it is very difficult to take any of these errors into account in the reconstruction itself.

Distributed parameter estimation problems induced by partial differential equations and the related initial-boundary value problems (such as EIT) constitute perhaps the largest class of inverse problems. Such models provide always a more or less simplified approximation for the physical reality. For example, the following modeling problems are common:

- The domain (geometry) is unknown or poorly known [35–38] and **[I]**
- The location or other properties of measurement sensors is only approximately known [39] and **[IV]**
- Measurement noise statistics is poorly known [15, 40]
- Simplified or approximate physical measurement models are used [41] and **[III]**
- Models for the unknown variables (prior models) are uncertain [42, 43] and **[III,IV]**
- Boundary conditions are partially unknown [44–46] and **[I,III,IV]**

Technically, there are two main approaches to model the uncertainties: small-dimensional uncertainties can often be modelled

using the hyperprior approach [22], but large dimensional uncertianties (especially when computational efficiency is important) can be handled using the approximation error approach, which is also used in this thesis, see Section 3.4.

3.2 UNCERTAINTIES IN HYDROGEOPHYSICS

In hydrogeophysics, one is typically interested in a single distributed parameter, such as saturation, while a large number of unknown secondary distributed parameters are typically handled by inserting a nominal spatially constant value for the variables. Since the uninteresting parameters may occupy several orders of magnitude, such as permeability does, the induced errors in the estimates may be significant. Basically, there are three sets of model parameters that may contain significant uncertainties. The first two ones are the hydro(geo)logical parameters that describe the flows, and the petrophysical parameters that are related to the mappings between the flows and the measurable variables, as discussed, for example in [47,48] .

The geometry itself, however, can be taken to form a third set of parameters in field measurements. The sensor locations are typically uncertain with both surface and borehole measurements [49, 50], and the ground surface topography is seldom modelled properly. When ground penetrating radar (GPR) and ray tracing forward models are used, It has been noted that the modeling uncertainties may induce significant bias to the estimates [51]. Similar observations have been made with respect to errors in petrophysical parameters [52,53]

There have been some attempts to take the modeling errors into account, such as weighting the measurements [54] and modeling errors as colored noise [55]. Systematic comprehensive approaches have not been introduced to model errors in hydrogeophysical imaging.

3.3 UNCERTAINTIES IN PROCESS TOMOGRAPHY

In process tomography, the geometry of the entire object (such as a pipeline) is typically very well known. In addition, the location of the electrodes (or other sensors) are nowadays known accurately since machining is computer controlled. In pipeline flow situations, however, the computational domain is typically truncated to include only the electrode array and it's immediate surroundings. Thus, the boundary conditions of the PDE models on the boundaries of the computational domain are (again) unknown [14, 44].

In process vessels such as flotation cells and some types of bioreactors, the height of the surface of the liquid or suspension in the vessel can be unknown. Also, the upper part of the vessel may be occupied by a very poorly conducting froth so that the practical upper level of the domain is the surface of the liquid phase. This level, furthermore, can be time-varying on both the fast and slow time scales [4].

The properties of the sensors can also be both unknown and time-varying, for example, due to corrosion or contamination. In EIT, in particular, the contact impedances depend on the composition of the target (on a short time scale) and also on contamination (on a long time scale), and these impedances cannot be measured directly and independently of the primary unknown. Thus, in practical process tomography using EIT measurements, the contact impedances also have to be modeled as (time-varying) uncertainties **[IV]**, or marginalized.

The typical EIT measurement models are valid for low frequencies and non-magnetic targets. In some mining-related processes, the ores may contain ferromagnetic particles but the typically in such small relative quantities that this is probably not a major problem. The main sources of errors are probably model reduction and contact impedances (if they are not estimated simutaneously).

The evolution models in nonstationary (time-varying) cases, however, are practically always highly approximative. In theory, multiphase and turbulent computational fluid dynamics (CFD) models

could be used. In practice in most practical applications, however, the computational models have to be kept extremely efficient using very limited computational resources, and evolution models using high end CFD models may not be practicable. Most evolution models that have been employed, are either stationary or nostationary Navier-Stokes models, or even simpler ones such as plug flow models [44, 56–58]. In practice, even stationary single phase models have been shown to be feasible even for nonstationary multiphase flows [59–61], [**IV**].

In pipeline flows, the boundary conditions of the evolution models on the input boundary is the most crucial uncertainty [56, 60]. Moreover, these conditions are time varying on all time scales. Philosophically, if we knew the conditions on the input boundary, we would not need to carry out any measurements. Thus, the (Dirichlet) boundary conditions need to be modelled as stochastic processes, and typically marginalized, see [44], [**IV**].

Whereas the number of secondary unknowns in a hydrogeological state evolution model can be large, there is a dominant distributed parameter with typical pipeline flow problems: the velocity field. This has been treated as a fixed (incorrect) nominal field (typically parabolic flow profile) [44], estimated as a profile only [61] marginalized entirely and embedded in the (approximation error) state noise [62] or marginalized partially, while the main principal components have been estimated simultaneously with the conductivity [58]. The appropriate approach depends naturally heavily on the flow dynamics.

3.4 APPROXIMATION ERROR APPROACH

The approximation error approach was introduced in [14, 63] originally to handle pure model reduction errors. For example, in electrical impedance (resistance) tomography (EIT, ERT) and deconvolution problems, it was shown that significant model reduction is possible without essentially sacrificing the feasibility of estimates. With EIT, for example, this means that very low dimensional finite

element approximations can be used. Later, the approach has also been applied to handle other kinds of approximation and modeling errors as well as other inverse problems: model reduction, unknown anisotropy structures and approximate marginalization of unknown but uninteresting scattering coefficient in diffuse optical tomography were treated in [35, 36, 43, 64]. Missing boundary data in geophysical ERT/EIT was considered in [**I**]. Furthermore, in [39,65] the problem of recovery from simultaneous domain truncation and model reduction was found to be possible, and in [37,38] the recovery from the errors related to inaccurately known body shape was shown to be feasible. The most comprehensive treatments of the approximation error theory in the stationary case can be found in [24, 43].

In the following, we consider the case in which we have auxiliary (uninteresting) unknowns in addition to the interesting unknowns, and we also wish to use a (possibly significantly) approximative model in the construction of the likelihood model. We follow [43] and [24] in which the details can be found.

In what follows, an overbar refers to an accurate model. Let

$$V = \bar{U}(\bar{\sigma}, z, \xi) + e \in \mathbb{R}^m$$

denote an accurate model for the relation between the potential measurements and the unknowns (in the context of EIT, the variables can be interpreted as) conductivity σ, contact impedances z, boundary conditions ξ, and let the additive errors e be mutually independent with $(\bar{\sigma}, z, \xi)$. The (accurate) complete electrode model $\bar{U}$ and the related variables of the forward model for EIT/ERT are described in Section 4.3.

Below, we approximate the accurate representation of the primary unknown $\bar{\sigma}$ by $\sigma = P\bar{\sigma}$ where P is a projection operator. Let $\pi(\sigma, z, \xi, e)$ be the model for the joint distribution of the unknowns.

Instead of using the accurate forward model $(\bar{\sigma}, z, \xi) \mapsto \bar{U}(\bar{\sigma}, z, \xi)$ with $(\bar{x}, z, \xi)$ as the unknowns, we fix the random variables $(z, \xi) \leftarrow (z_0, \xi_0)$ and use a computationally reduced model

$$\sigma \mapsto U(\sigma, z_0, \xi_0)$$

Thus, we write the measurement model in the form

$$\begin{aligned} V &= \bar{U}(\bar{\sigma}, z, \xi) + e && (3.1)\\ &= U(\sigma, z_0, \xi_0) + \left(\bar{U}(\bar{\sigma}, z, \xi) - U(\sigma, z_0, \xi_0)\right) + e && (3.2)\\ &= U(\sigma, z_0, \xi_0) + \varepsilon + e && (3.3) \end{aligned}$$

where we define the *approximation error* $\varepsilon = \varphi(\bar{\sigma}, z, \xi) = \bar{U}(\bar{\sigma}, z, \xi) - U(\sigma, z_0, \xi_0)$. The expresion (3.3) is exact but not (yet) generally useful.

Using the Bayes' formula we get

$$\begin{aligned} \pi(V, \sigma, z, \xi, e, \varepsilon) &= \pi(V \,|\, \sigma, z, \xi, e, \varepsilon)\pi(\sigma, z, \xi, e, \varepsilon)\\ &= \delta(V - U(\sigma, z_0, \xi_0) - e - \varepsilon)\\ &\qquad \pi(e, \varepsilon \,|\, \sigma, z, \xi)\pi(z, \xi \,|\, \sigma)\pi(\sigma)\\ &= \pi(V, z, \xi, e, \varepsilon \,|\, \sigma)\pi(\sigma) \end{aligned}$$

giving the likelihood

$$\begin{aligned} \pi(V \,|\, \sigma) &= \iiiint \pi(V, z, \xi, e, \varepsilon \,|\, \sigma)de\, d\varepsilon\, dz\, d\xi\\ &= \iint \delta(V - U(\sigma, z_0, \xi_0) - e - \varepsilon)\pi(e, \varepsilon|\sigma)de\, d\varepsilon\\ &= \int \pi_e(V - U(\sigma, z_0, \xi_0) - \varepsilon)\pi_{\varepsilon|\sigma}(\varepsilon \,|\, \sigma)\, d\varepsilon \qquad (3.4) \end{aligned}$$

Next, both π_e and $\pi_{\varepsilon|\sigma}$ are approximated with normal distributions. Let the normal approximation for the joint density $\pi(\varepsilon, \sigma)$ be

$$\pi(\varepsilon, \sigma) \propto \exp\left\{ -\frac{1}{2} \begin{pmatrix} \varepsilon - \varepsilon_* \\ \sigma - \sigma_* \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \Gamma_\varepsilon & \Gamma_{\varepsilon\sigma} \\ \Gamma_{\sigma\varepsilon} & \Gamma_\sigma \end{pmatrix}^{-1} \begin{pmatrix} \varepsilon - \varepsilon_* \\ \sigma - \sigma_* \end{pmatrix} \right\} \qquad (3.5)$$

Thus we write $e \sim \mathcal{N}(e_*, \Gamma_e)$ and $\varepsilon \,|\, \sigma \sim \mathcal{N}(\varepsilon_{*,\sigma}, \Gamma_{\varepsilon|\sigma})$ where

$$\begin{aligned} \varepsilon_{*,\sigma} &= \varepsilon_* + \Gamma_{\varepsilon\sigma}\Gamma_\sigma^{-1}(\sigma - \sigma_*) && (3.6)\\ \Gamma_{\varepsilon|\sigma} &= \Gamma_\varepsilon - \Gamma_{\varepsilon\sigma}\Gamma_\sigma^{-1}\Gamma_{\sigma\varepsilon} && (3.7) \end{aligned}$$

Define $\nu = e + \varepsilon$ so that $\nu \,|\, \sigma = e + \varepsilon \,|\, \sigma$ and $\nu \,|\, \sigma \sim \mathcal{N}(\nu_{*|\sigma}, \Gamma_{\nu|\sigma})$ where

$$\begin{aligned} \nu_{*|\sigma} &= e_* + \varepsilon_* + \Gamma_{\varepsilon\sigma}\Gamma_\sigma^{-1}(\sigma - \sigma_*) && (3.8) \\ \Gamma_{\nu|\sigma} &= \Gamma_e + \Gamma_\varepsilon - \Gamma_{\varepsilon\sigma}\Gamma_\sigma^{-1}\Gamma_{\sigma\varepsilon} && (3.9) \end{aligned}$$

Thus, we obtain the approximate likelihood model

$$V \,|\, \sigma \sim \mathcal{N}(V - U(\sigma, z_0, \xi_0) - \nu_{*|\sigma}, \Gamma_{\nu|\sigma})$$

Assuming that we have a normal prior model $\mathcal{N}(\sigma_*, \Gamma_\sigma)$, we obtain the approximation for the posterior distribution

$$\pi(\sigma \,|\, V) \propto \pi(V \,|\, \sigma)\pi(\sigma) \propto \exp\left(-\frac{1}{2}\Psi(\sigma)\right)$$

where $\Psi(\sigma)$ is

$$\begin{aligned} \Psi(\sigma) &= (V - U(\sigma, z_0, \xi_0) - \nu_{*|\sigma})^{\mathrm{T}}\Gamma_{\nu|\sigma}^{-1}(V - U(\sigma, z_0, \xi_0) - \nu_{*|\sigma}) \\ &\quad + (\sigma - \sigma_*)^{\mathrm{T}}\Gamma_\sigma^{-1}(\sigma - \sigma_*) \\ &= \|L_{\nu|\sigma}(V - U(\sigma, z_0, \xi_0) - \nu_{*|\sigma})\|^2 + \|L_\sigma(\sigma - \sigma_*)\|^2 \quad (3.10) \end{aligned}$$

and where $\Gamma_{\nu\,|\,\sigma}^{-1} = L_{\nu|\sigma}^{\mathrm{T}} L_{\nu|\sigma}$, $\Gamma_\sigma^{-1} = L_\sigma^{\mathrm{T}} L_\sigma$ and $\nu_{*|\sigma} = \nu_{*|\sigma}(\sigma)$.

The MAP estimate of σ with the approximation error model is obtained by

$$\hat{\sigma} = \arg\min_\sigma \left\{ \|L_{\nu|\sigma}(V - U(\sigma, z_0, \xi_0) - \nu_{*|\sigma})\|^2 + \|L_\sigma(\sigma - \sigma_*)\|^2 \right\} \quad (3.11)$$

An estimate for the posterior covariance is computed by linearizing $U(\sigma, z_0, \xi_0)$ at $\sigma = \hat{\sigma}$

$$\hat{\Gamma}_{\sigma|d} \approx \left(\tilde{J}^{\mathrm{T}}\Gamma_{\nu|\sigma}^{-1}\tilde{J} + \Gamma_\sigma^{-1}\right)^{-1} \quad (3.12)$$

where $\tilde{J} = J + \Gamma_{\varepsilon\sigma}\Gamma_\sigma^{-1}$ and J is the Jacobian of $U(\sigma, z_0, \xi_0)$ evaluated at $\sigma = \hat{\sigma}$.

In practice, one proceeds as follows. Denote $\eta = (\sigma, z, \xi)$. Once the model $\pi(\sigma, z, \xi)$ is constructed, one computes an ensemble of draws $\{\eta^{(\ell)}, \ell = 1, \ldots, M\}$ and then computes the respective approximation errors $\varepsilon^{(\ell)}$. All second order statistics is then estimated via sample averages.

3.5 APPROXIMATION ERRORS IN STATE ESTIMATION

The approximation error approach was extended to nonstationary inverse problems in [66] in which linear nonstationary (heat transfer) problems were considered, and in [67] and [68] in which nonlinear problems and state space identification problems were considered, respectively.

The earliest similar but partial treatment in the framework on nonstationary inverse problems was considered in [44] in which the the boundary data that is related to stochastic convection diffusion models was partially unknown. A modification in which the approximation error statistics can be updated with accumulating information was proposed in [**III**] in the context of hydrogeophysical monitoring and in the context of flow monitoring in [69].

The treatment of modeling and approximation errors for nonstationary inverse problems was initiated in [44]. The analysis based on the semigroup formulation was treated in [70]. The systematic approach for numerical and computational implementation was developed in [66–68]. Recently, uncertainties in the hydrogeophysical Kalman filtering have been considered in [71–73]. They implemented an ad hoc stochastic forcing term to model the uncertainties. In the present paper we propose a more systematic approach to the modeling of the stochastics of the evolution model.

In a nutshell, if the accurate state space model is of the form

$$\begin{aligned} \bar{\sigma}_{t+1} &= \bar{F}_t(\bar{\sigma}_t, z_t, \xi_t) + w_t \\ V_t &= = \bar{U}_t(\bar{\sigma}_t, z_t, \xi_t) + e_t \end{aligned}$$

this is turned to an approximate model

$$\begin{aligned} \sigma_{t+1} &= F_t(\sigma_t, z_0, \xi_0) + w_t + h_t \\ V_t &= U_t(\sigma_t, z_0, \xi_0) + e_t + q_t \end{aligned}$$

where the stochastic processes h_t and q_t are introduced by the approximation process, and (z_0, ξ_0) are some fixed values, possibly related to contact impedances, boundary values, and/or uninteresting distributed parameters (especially in hydrogeophysics).

State space models that are relevant in hydrogeophysics (in the vadose, or unsaturated, conditions) and process tomography , are discussed in Sections 5.2, [**II,III**], and 6.1, [**IV**], respectively. An extension to the approximation error approach that uses importance sampling to update the statistics of the approximation error processes, is discussed in Section 5.4 and in [**III**].

4 Electrical impedance tomography and the truncation problem

4.1 ELECTRICAL IMPEDANCE TOMOGRAPHY

Electrical resistance (impedance) tomography is an imaging method in which the internal structure of the target of interest is probed by conducting electric (alternating small frequency) currents into the target and by measuring the resulting potentials on the boundary. In practice, both the current injections and the potential measurements are carried out through (and on) discrete electrodes that are placed in contact with the target [74]. The current injections induce a current and potential distribution in the target that depends on the spatial conductivity distribution. In practice, several different current patterns (different combinations of currents on the electrodes) and the coresponding measurements are used to obtain a single reconstruction.

The mapping from currents to (measured) potentials is linear, but the mapping from the conductivity distribution to the measurements is nonlinear. The objective of EIT/ERT is to reconstruct, or estimate, the electrical conductivity distribution within the target based on the measured potentials on the boundary.

4.2 GENERAL CHOICES IN EIT IMAGING

Depending on the situation (the properties of the target, general modeling issues, available computational resources), there is a large number of choices that have to be made in choosing the reconstruction approach and mode.

Absolute vs. difference imaging

In difference imaging, two sets of data are collected from the target in two different states. For example, a set of measurements are obtained from a tank filled with saline, and another set when an object is inserted into the tank. The difference on the conductivity between these two states is then estimated based on the difference between the two sets of measurements. This is the classical (engineering) approach to EIT imaging and they are referred to as the backprojection or the sensitivity approach [74]. No information on the absolute values of conductivity can be obtained.

Absolute imaging is based on a single set of measurements, and attempts to provide the actual conductivity values. The absolute reconstructions are based on different PDE models, such as the complete electrode model, and are either single step or iterative algorithms. Both deterministic regularization approaches or the Bayesian framework usualy need to be employed.

The difference modality has been the prevalent modality over absolute imaging since it is relatively tolerant to various modeling errors and uncertainties. Also, the reconstructions are very fast since they amount to a multiplication by a small dimensional matrix only, with typically around 20,000 – 500,000 multiplications and additions. The cons are, of course, that absolute values are not accessed and that the reconstructions are more or less qualitative in any case. The grand challenge with absolute imaging is that the overall computational model has to be comprehensive: no unhandled uncertainties or model errors are tolerated, and the models have to be feasible. In this thesis and in [**I-IV**], we consider mainly absolute imaging.

Iterative vs. single step approaches

Absolute imaging can be carried out using a single step or an iterative approach. The single step algorithms are related to sensitivity approaches: if the sensitivity matrix is computed in the correct geometry and a feasible (usually spatially homogeneous) conductivity value, this matrix corresponds to the Jacobian matrix of single step algorithms such as NOSER [75].

The single step reconsructions tend, however, to be blurry and spatially qualitative, and they always correspond to basic normal assumptions for measurement errors and target model [14,76].

Since the mapping from the conductivity distribution to the measurements is nonlinear, both the determination of regularized least squares estimates (such as Tikhonov estimates) and the computation of the MAP estimate are almost exclusively carried out iteratively, see [77] for an exception in which the absolut image can be computed without iterations in a 2D domain.

The iterative approaches are invariably quite slow, especially since the Jacobian matrix has to be recomputed iteratively. The state of the art approach to this subtopic is to use the so-called adjoint methods.

Stationary vs nonstationary imaging

One of the first questions in choosing the reconstruction approach is whether an adequate number of current patterns can be injected before the target changes. In low noise environments, a single current pattern and the measurements can be applied in about a millisecond. In 3D and a system that can measure all voltages simultaneously, the complete linearly independent set of current patterns can be carried out in about 50 - 100 milliseconds. If the target changes during this period, the measurements don't correspond to the same conductivity distribution and the reconstructions are typically meaningless.

In process tomography, when fast flows in pipelines are considered, the target can typically be assumed to be constant (with respect to time) for a few milliseconds only. Usually, a 3D reconstruction is not feasible when data from a few injections only are used, at least when standard binary current pattern types (current is fed only through electrode pairs) are used. See [78] for the determination of so-called optimal current patterns that enable feasible reconstructions using 1-3 current patterns only.

If the target (primary and secondary unknowns) can be modelled as a stochastic process, this leads directly to the framework of state estimation. State estimation in the context of inverse problems

was developed, for example, in [14,25,44,79], especially in the context of process tomography. Earlier, deterministic evolution models were used in hydrogeophysics and some other fields [71,80], but the models were taken as deterministic, that is they were (implicitly) assumed to be an exact represention of the underlying physical processes. The deterministic evolution models are technically treated as constraints. Such deterministic models are difficult to sustain in practical situations. The nonstationary EIT imaging is usually based on the formulation of the problems as a sequential Bayesian estimation problem, and then carrying out a number of (more or less controlled) approximations so that extended Kalman filter can be employed. This is the approach, for example, in [**II,III,IV**].

Modeling and handling of uncertainties

As claimed above, in absolute imaging with real data, the overall model has to be feasible and the uncertainties cannot generally be ignored. After a feasible model has been constructed, there are basically two alternatives: either marginalize over the secondary unknowns, or estimate these simultaneously with the primary unknown. The latter usually leads to an infeasible *computational* model, and can, of course, be a mathematically unidentifiable model too. For example, in process tomography using the Navier-Stokes flow model in an convection-diffusion model, the unknwons would typically be the (time-varying) conductivity distribution, (time-varying) flow field, (time-varying) boundary conditions on all truncation boundaries, diffusion coefficient and the (possibly time-varying) contact impedances. All these would be needed to be estimated typically based on around 10-50 real numbers at a time. Clearly (especially the general time varying flow field), all the unknowns cannot be estimated based on such data.

Thus, at least some of the uncertainties – and at least some components of these – need to be (approximatively) marginalized. Significant progress can, however, be made if feasible low dimensional approximations can be constructed for the unknowns. Typically, proper orthogonal decompositions (principal component analysis) provide such low dimensional approximations, but this, again, pre-

sumes feasible prior models for the unknowns, see, for example, [62], in which only a few components of a time-varying Navier-Stokes flow need to be estimated simultaneously with the conductivity, and the remainder together with the unknown time-varying boundary data processes are embedded in the (approximation error) state noise process. In [**II**], we estimate the time-varying saturation and take all other parameters as known.

In nonstationary situations, there is a major difference between stationary and nonstationary uncertainties. As data is accumulated, the conditional uncertainty of stationary uncertainties given the data reduces (or at least does not increase). A solution to this problem is considered in [**III**], in which we use the nonstationary approximation error approach and marginalize over the unknown (stationary) permeability distribution, and refine the approximation error statistics continuously when more data is accumulated. The approach is based on importance sampling of the approximation errors.

In [**IV**], we consider real pipeline flow data and estimate the time-varying contact impedances and boundary conditions simultaneously with the conductivity distribution. The additional complexity here is that the electrodes are positioned on a single ring, which in a stationary case would induce a theoretically unidentifiable problem. The evolution model and the state estimation formulation, however, break the unidentifiability.

4.3 COMPLETE ELECTRODE MODEL

The model (that is part of the model of the likelihood) that gives the errorless prediction of the measurements, is called the forward model. In some cases, such as Poisson distributed observations, the forward model provides the expectation of the measurements, see, for example [14].

With EIT, the most accurate known forward model is called the *complete electrode model* (CEM) [81–83]. The CEM is governed by the following elliptic partial differential equation and boundary condi-

tions:

$$\nabla \cdot (\sigma(\vec{r})\nabla u(\vec{r})) = 0, \quad \vec{r} \in \Omega \subset \mathbb{R}^{2,3} \tag{4.1}$$

$$u(\vec{r}) + z_\ell \sigma(\vec{r}) \frac{\partial u}{\partial \vec{n}} = U_\ell, \quad \vec{r} \in e_\ell,\ \ell = 1, 2, \ldots, N_{\text{el}} \tag{4.2}$$

$$\int_{e_\ell} \sigma(\vec{r}) \frac{\partial u(\vec{r})}{\partial \vec{n}} \mathrm{d}S = I_\ell, \quad \vec{r} \in e_\ell,\ \ell = 1, 2, \ldots, N_{\text{el}} \tag{4.3}$$

$$\sigma \frac{\partial u}{\partial \vec{n}} = 0, \quad \vec{r} \in \partial\Omega \setminus \cup_{\ell=1}^{N_{\text{el}}} e_\ell, \tag{4.4}$$

where $\vec{r}$ is the position vector, $u(\vec{r})$ is the potential distribution in Ω, $\sigma(\vec{r})$ is the conductivity distribution, N_{el} is the number of electrodes, e_ℓ is the patch on the (internal or external) boundary $\partial\Omega$ corresponding to the location of the ℓ^{th} electrode, z_ℓ is the contact impedance between the ℓ^{th} electrode and the object to be imaged, U_ℓ is the potential on the ℓ^{th} electrode, and I_ℓ the current injected through electrode e_ℓ. The currents must satisfy the following charge conservation law:

$$\sum_{\ell=1}^{N_{\text{el}}} I_\ell = 0.$$

The collection of these currents is called a *current pattern* $I = (I_1, \ldots, I_{N_{el}})^{\mathrm{T}}$, and usually several different current patterns are employed, and all the respective data is used to estimate a single spatial conductivity distribution.

The reference level for the potential is defined, for example, by setting

$$\sum_{\ell=1}^{N_{\text{el}}} U_\ell = 0.$$

Given the injected currents, the ERT forward model can be formally written using the CEM as

$$V = \bar{U}(\bar{\sigma}(\vec{r}); I) + e = \bar{R}(\bar{\sigma}(\vec{r}))I + e, \tag{4.5}$$

where the vector $V \in \mathbb{R}^{N_V}$ contains the measured voltages corresponding to current pattern I, and e is the measurement error resulting from non-ideal operation of the data collection system.

The inverse problem consists of estimating the conductivity distribution $\bar{\sigma}(\vec{r})$ given the data V. Since no analytical solutions exist for the geometric conditions encountered in practical applications, the solution of the inverse problem is sought numerically. For this numerical solution, the problem is conventionally approximated with a finite-dimensional model

$$V = U(\sigma; I) + e. \tag{4.6}$$

Here, $\sigma \in \mathbb{R}^N$ is a typically high-resolution discrete approximation for $\sigma(\vec{r})$ using a very large model domain, where N is the number of conductivity values. The approximate model $U(\sigma; I)$ can be constructed using, for example, the finite element method (FEM) [84], boundary element method (BEM) [85], or other suitable numerical schemes. In [**I-IV**], we use the finite element method that is based on the variational form derived in [82].

4.4 DYNAMICAL EIT

As given in Section 3.5, the state estimation problem is of the form (including the approximation error processes) when the secondary unknowns have been approximately marginalized

$$\begin{aligned} \sigma_{t+1} &= F_t(\sigma_t, z_0, \xi_0) + w_t + h_t \\ V_t &= U_t(\sigma_t, z_0, \xi_0) + e_t + q_t \end{aligned}$$

In the case of hydrogeopysical estimation of time-varying saturation, the evolution model is usually written in terms of the saturation S_t and the petrophysical model $S \mapsto \sigma$ is embedded in U, as is done in [**II**]. In [**II**], the variables (z_0, ξ_0) are taken as known, where these refer to the contact impedances and all hydrogeophysical and petrophysical parameters. In [**III**], the (time-invariant) permeability $\xi = k$ is taken as a secondary unknown, and the random variable k is embedded in the process h_t.

In [**IV**], we consider the single electrode ring case in which the marginalization on secondary unknowns is not a feasible approach.

Instead, we estimate both the contact impedances and the boundary conditions simultaneously with the conductivity. Here, we denote the (nonstationary Dirichlet) boundary condition with ξ_t. Thus, we write the model as

$$\begin{aligned} \mathcal{X}_{t+1} &= F_t(\mathcal{X}_t) + \mathcal{W}_t + \mathcal{H}_t \\ V_t &= \tilde{U}_t(\mathcal{X}_t) + e_t + q_t \end{aligned}$$

where $\mathcal{X} = (\sigma_t, \mathcal{Z}_t, \Xi_t)^{\mathrm{T}}$ and where $\mathcal{Z}_t = (z_t, z_{t-1})^{\mathrm{T}}$ and $\Xi_t = (\xi_t, \xi_{t-1})^{\mathrm{T}}$. Thus with respect to the contact impedances and bounday conditions, the evolution model is of the second order, but can be written as a first order model with respect to the augmented variables $\mathcal{Z}_t$ and Ξ_t. Furthermore, $\mathcal{W}_t$ and $\mathcal{H}_t$ are the augmented processes, see **[IV]** for the details on the construction of the models and statistics.

In all above models, the observation model may be explicitly time-varying, since a different set of current patterns may be injected at time t. This is the case in **[IV]**. In **[II,III]**, however, we deal with hydrogeophysical problems in which the time rate of change of the hydrogeophycial process such as saturation is typically in hours, so that all current patterns can be injected consecutively and can be assumed to correspond to the same target.

Finally, we note that it is possible also to decompose a single entity into two components, for example, $\xi_t = \xi_t^1 + \xi_t^2$ and the estimate ξ_t^1 while (the contribution of) ξ_t^2 is embedded in the approximation error processes, as was done in [62].

4.5 THE DOMAIN TRUNCATION PROBLEM

If the computational domain is truncated to include only the region of interest, the truncation boundaries are artificial and no standard deterministic model could be posed. A standard choice in such a case woud be to pose homogeneous Neumann conditions on the truncation boundaries and hope that the current densities are more or less vanishing on the boundary $\partial\Omega$. The size of the computational domain that would approximately guarantee that the current

densities vanish, however, can be quite large, roughly 2-3 times the distance of the furthermost electrodes [86]. Thus, in a borehole geometry, in which the region of interest is the domain between the boreholes, the safe computational domain (using Neumann conditions) is around 30-50 times the size of the actual region of interest.

The reason for truncation is, of course, the computational complexity but also stability of the inversion approach. The latter is particularly central in the case of many regularization approaches but also if improper prior models are used.

If homogeneous Neumann conditions are used and the computational domain is too small, reflection-type echoes appear in the reconstructions. these mask all actual sructures near the boundary of the computatonal domain. Furthermore, the edges tend to be overcompensated towards the center of the domain, which basically destroys any attempts at absolute imaging.

In [**I**], we deal with this problem, and construct an inhomogeneous Markov random field [87] in a large domain $\bar{\Omega}$, on the boundary of which it is safe to assume Neumann conditions. The approximation errors are computed as the discrepancy of the predicted measurement between the large domain and a small domain Ω which is used as the basis of the forward model in the likelihood model.

We note that it is also possible to model the truncation boundaries with the so-called Dirichlet-to-Neumann (DtN) operator, as was done in [88] in which the conductivity distribution was modeled as spatially known (or as a constant) outside the region of interest. It is to be noted that the DtN operator depend only on the conductivity outside the region of interest. The stochastic version of the DtN approach is currently being considered [89]. In the context of image processing, the problem of unknown boundary conditions and marginalization over these uncertainties was considered in [90].

5 Tomographic imaging of unsaturated flows

5.1 RELEVANT TOMOGRAPHIC MODALITIES

Monitoring subsurface systems and their temporal changes is a central task in understanding of hydrological processes and in the use and planning of, for example, use of water resources. Technically, hydrological processes are roughly divided into saturated and unsaturated (or vadoze) processes. The former are driven by pressure while the latter are mainly driven by capillary processes.

While the mathematical model that is commonly used to model the flows in the saturated zone, is a simple linear diffusion model, the set of models that govern the vadoze zone are both nonlinear and discontinuous. As for measurement modalities, the saturated zone is often observed directly through water level in measurement wells (referred to as ”head”). In the vadoze zone, such direct measurements are not possible in practice.

Electrical resistance tomography (ERT) has been used to monitor time- and space-varying targets such as water content distributions and solute concentrations [91–95], hydrological barriers [96], transport of tracers [53, 97, 98], leakage from underground tanks [99], remediation processes [100], and changes in resistivity caused by rainfall [101]. Furthermore, ERT and ground penetrating radar (GPR) have been combined to perform joint inversion to yield significantly enhanced water content estimates [102].

The value of constraining ERT models with hydrological process modeling is increasingly recognized [103]. Hydrogeophysical data fusion approaches within a stochastic framework that combines primary and secondary data of different types and iteratively updates state estimates, unsaturated hydraulic properties, and the related covariance structures, have been used in sequential (nonsta-

tionary) inversion of ERT data [95, 104, 105]. We note, again, that most studies take the evolution models as deterministic and thus accurate.

5.2 RICHARD'S AND ARCHIE'S MODELS

The evolution of water saturation in unsaturated porous media is usually modeled with the Richards equation [106]:

$$\phi \frac{\partial S}{\partial t} + \nabla \cdot \left[\frac{K(S)}{\rho_w g} \nabla P_c(S) - K(S)\hat{z} \right] = 0. \tag{5.1}$$

Here, ϕ is the porosity, S is the water saturation, K and P_c are the unsaturated hydraulic conductivity and the capillary pressure, respectively, both nonlinear functions of water saturation, ρ_w is the water density, g is the gravitational constant, and $\hat{z}$ is the unit vector, positive upward. The hydraulic conductivity is given by

$$K = k \frac{k_{rel}(S)\rho_w g}{\mu_w} \tag{5.2}$$

where k is the absolute permeability, k_{rel} is the relative permeability (a nonlinear function of water saturation S), and μ_w is the dynamic viscosity of water. For the relative permeability k_{rel} and capillary pressure P_c we use the parametric representation given by the van Genuchten model [107]:

$$P_c = -\alpha^{-1}(S_e^{-1/m} - 1)^{1-m}, \tag{5.3}$$

$$k_{rel} = \sqrt{S_e}(1 - (1 - S_e^{m^{-1}})^m)^2, \tag{5.4}$$

$$S_e = \frac{S - S_{wr}}{1 - S_{wr}} \tag{5.5}$$

where m and α are soil-specific parameters, S_e is the effective water saturation, and S_{wr} is the residual water saturation.

In the above, the saturation depends on both space and time. The relative permeability depends on space and time only via the saturation, but the parameters $(k, \rho_w, \alpha, m,)$ are spatially dependent. In particular, the absolute permeability is unknown and usually covers several orders of magnitude.

The Archie's model is usualy employed as the petrophysical model that relates other parameters to conductivity:

$$\sigma(S) = \sigma_w \phi^b S^n$$

where σ_w is the conductivity of the liquid phase (water and soluble ions), b is the cementation index and n is the saturation index. Both b and n depend on the type of soil and are thus spatially dependent.

5.3 STATE SPACE MODEL FOR UNSATURATED FLOW

Geophysical ERT data that are sensitive to changes in fluid properties or fluid saturations have been proven effective for monitoring spatial and temporal changes in subsurface conditions. For example, the sensitivity of the bulk electrical conductivity to water saturation enables the use of electrical resistance tomography (ERT) to monitor water in variably saturated soils in the vadose zone [108].

Kalman filters have seen widespread application in hydrogeophysics in prediction of water resources and estimation of aquifer characteristics since the 70's [109, 110], see also the review in [111]. These studies have almost exclusively treated saturated cases in which the water is driven by pressure. The measurements have mostly been direct well (head) measurements, so that the Kalman filter has effectively been used to interpolate between the well data both temporally and spatially. Mildly ill-posed non-diffuse measurement modalities, such as ray-tracing ground penetrating radar (GPR) have also been studied in relation with saturated cases [111].

The evolution models for the vadose, or unsaturated, zone are considerably more nonlinear and computationally more involved than models for the saturated zone. Hydrogeophysical applications involving evolution models and ERT have recently been discussed in [98] and [112]. Cross-borehole ERT was used earlier to image the resistivity distribution of the vadose zone before and during infiltration experiments [91].

In the earlier nonstationary inversion approaches [95, 104, 105], the evolution models were assumed to be deterministic, and no

statistical models were employed to comply with the approximative nature the models. In [**II**], the evolution model was taken as a stochastic process and the approximation error processes were modeled according to the assumed/modeled uncertainty of the secondary unknowns. Full approximate marginalization was employed. As the most important result in this feasibility study, the approximate posterior covariances were feasible, that is, the true values were within 2 standard deviations of the state estimates. Naturally, the sequential state estimates were superior to the snapshot estimates which were computed without an evolution model.

5.4 IMPORTANCE SAMPLING UPDATING OF THE APPROXIMATION ERROR MODEL

With stationary uncertainties, such as the permeability when the saturation is considered as the primary unknown, the initial, or prior, uncertainty can sometimes be considerable. With the primary (estimated) unknowns, the initial uncertainty is naturally decreased as the measurements are accumulated. But with the standard approximation error approach, the posterior uncertainty of the secondary unknowns is not updated. With some nonstationary problems such as the hydrogeological experiments, the covariance of the approximation errors can increase very fast with time.

In [**III**], this problem was addresses by employing an importance sampling type approach to update the approximation error means and covariances as measurement data is accumulated. Roughly speaking, the standard approximation error approach uses the unconditional normal models $h_t \sim \mathcal{N}(\mathbb{E}(h_t), \mathrm{cov}\,(h_t))$ and $q_t \sim \mathcal{N}(\mathbb{E}(q_t), \mathrm{cov}\,(q_t))$, while the importance sampling modification employs the corresponding conditional normal models $h_t \sim \mathcal{N}(\mathbb{E}(h_t|D_t), \mathrm{cov}\,(h_t|D_t))$ and $q_t \sim \mathcal{N}(\mathbb{E}(q_t|D_t), \mathrm{cov}\,(q_t|D_t))$, where $D_t = (V_1, \ldots, V_t)$. At first sight, this seems a plausible but not a computationally feasible idea. It turns out, however, that if the actual ensemble of precomputed approximation errors are stored, the importance weights and thus the conditional means and covari-

ances are very efficient to compute. See [**III**] for the details.

It is to be noted that while the permeability distribution is not directly estimated during the state estimation process, the importance weights can also be used to approximate the actual permeability.

6 Process tomography

Process tomography, in general, refers to tomographic problems that are encountered in chemical process industry, mining enrichment, oil industry, pulp and paper production, food processing and other such industries. We refer to [4,5] for reviews on methods and applications.

Technically, most process tomograhic problems are nonstationary, but often the rate of change of the target is slow enough to enable the use of stationary reconsruction algorithms. In spite of this, if feasible evolution models are consructed, the estimates will almost invariably be better than with snaphot estimates.

Although the geometry process pipelines and vessels is mostly accurately known, other uncertainties may still be significant enough to destroy absolute estimates unles they are modelled properly. Thus, most practical applications still today are using difference imaging type approaches.

The state of the art approaches that are relevant to process tomography and nostationary inverse problems have already been reviewed in Section 2.2, see also [30]. In this chapter, we construct the state space model that enables one to use a single electrode layer sensor and to break the related unidentifiability problem. The focus is on constructing feasible models for the nonstationary contact impedances and input boundary conditions. The details are given in **[IV]**.

6.1 STOCHASTIC CONVECTION-DIFFUSION MODELS

In what follows, we assume that we deal with a stationary (fully developed) flow, and we take the flow field $\vec{v}(\vec{r}, t) = \vec{v}(\vec{r})$ as known. For handling of nonstationary unknown velocity fields, see, for example, [45, 62].

The evolution of concentration distributions of some substance

in a flowing fluid can be described with the convection-diffusion (CD) model

$$\frac{\partial c(\vec{r},t)}{\partial t} = -\vec{v}\cdot\nabla c(\vec{r},t) + \nabla\cdot\kappa\nabla c(\vec{r},t),\ \vec{r}\in\Omega, \tag{6.1}$$

where $c(\vec{r},t)$ is the concentration distribution, $\vec{v}$ is the velocity field and κ is the diffusion coefficient. Possible volume sources are modelled by adding a separate source term to eq. (6.1). However, volume sources do not typically exist in pipe segments intended for conventional mass transport. To complete the CD model, we pose the following boundary conditions:

$$c(\vec{r},t) = c_{\text{in}}(\vec{r},t),\ \vec{r}\in\partial\Omega_{\text{inflow}} \tag{6.2}$$

$$\frac{\partial c(\vec{r},t)}{\partial n} = 0,\ \vec{r}\in\partial\Omega\setminus\partial\Omega_{\text{inflow}} \tag{6.3}$$

where $\partial\Omega_{\text{inflow}}$ is the inflow boundary. In continuous process monitoring, the initial conditions are irrelevant.

Assuming that the relation between the conductivity and concentration can be described with an affine model, the evolution model for the conductivity and the boundary conditions are

$$\frac{\partial \sigma(\vec{r},t)}{\partial t} = -\vec{v}\cdot\nabla\sigma(\vec{r},t) + \nabla\cdot\kappa\nabla\sigma(\vec{r},t),\ \vec{r}\in\Omega, \tag{6.4}$$

$$\sigma(\vec{r},t) = \bar{\xi}(\vec{r},t),\ \vec{r}\in\partial\Omega_{\text{inflow}} \tag{6.5}$$

$$\frac{\partial \sigma(\vec{r},t)}{\partial n} = 0,\ \vec{r}\in\partial\Omega\setminus\partial\Omega_{\text{inflow}}. \tag{6.6}$$

The finite element implementation of the convection-diffusion model (6.1-6.3) or (6.4-6.6) has been described in detail in [60], and it leads to the update equation

$$\tilde{\sigma}_{t+1} = A\tilde{\sigma}_t + Z\xi_t + R\xi_{t+1} + h_t, \tag{6.7}$$

where A, Z and R are the matrices resulting from the FEM scheme, $\xi_t\in\mathbb{R}^{N_{\text{in}}}$ contains the conductivities at nodes located at the inflow boundary $\partial\Omega_{\text{inflow}}$ and $\tilde{\sigma}\in\mathbb{R}^{N-N_{\text{in}}}$ contains the conductivities at all other nodes. The state noise term h_t is the approximation error process and here accounts mainly for the model reduction effects.

6.2 STATE SPACE MODEL WITH UNKNOWN TIME-VARYING BOUNDARY DATA AND CONTACT IMPEDANCES

In this section, we decompose the contact impedance and input boundary codition processes into two subprocesses and construct second order Markov processes for all these processes. Standard augmentation is then used to turn the models into a first order Markov process.

The unknown conductivity distribution at the inflow boundary is divided into two parts as

$$\zeta_t = s_t \mathbf{1}_{N_{\text{in}}} + u_t, \tag{6.8}$$

where the first term on the right-hand side is a spatially homogeneous term ($s_t \in \mathbb{R}$ and $\mathbf{1}_{N_{\text{in}}} \in \mathbb{R}^{N_{\text{in}}}$ is a column vector of ones) and u_t represents spatial inhomogeneities in the conductivity distribution at $\partial\Omega_{\text{inflow}}$. The evolution of the homogeneous part is modelled with a second-order Markov model in order to guarantee smoother temporal behaviour than with first-order models. The model is of the form

$$s_{t+1} = \alpha_1 s_t + \alpha_2 s_{t-1} + \eta_{t+1}, \tag{6.9}$$

where α_1 and α_2 are fixed model parameters and the noise term $\eta_{t+1} \sim N(0, \Gamma_\eta)$ (for all t) is the innovation process that determines the variance of s_t. Temporal changes of the inhomogeneous part are described with another second-order Markov model as

$$u_{t+1} = \beta_1 u_t + \beta_2 u_{t-1} + \varepsilon_{t+1}, \tag{6.10}$$

where β_1 and β_2 are fixed parameters and $\varepsilon_{t+1} \sim N(0, \Gamma_\varepsilon)$ (for all t) is another innovation process.

In addition to the evolution models related to conductivity distribution, we specify an evolution model also for the contact impedances. Again, we employ a second-order Markov model of the form

$$z_{t+1} = \gamma_1 z_t + \gamma_2 z_{t-1} + \nu_{t+1}, \tag{6.11}$$

with model parameters γ_1 and γ_2 and $\nu_{t+1} \sim N(0, \Gamma_\nu)$ for all t.

By combining all the above models we obtain

$$\begin{bmatrix} \tilde{\sigma}_{t+1} \\ u_{t+1} \\ u_t \\ s_{t+1} \\ s_t \\ z_{t+1} \\ z_t \end{bmatrix} = \begin{bmatrix} A & Z+\beta_1 R & \beta_2 R & (Z+\alpha_1 R)\mathbf{1}_{N_{\text{in}}} & \alpha_2 R \mathbf{1}_{N_{\text{in}}} & 0 & 0 \\ 0 & \beta_1 I_{N_{\text{in}}} & \beta_2 I_{N_{\text{in}}} & 0 & 0 & 0 & 0 \\ 0 & I_{N_{\text{in}}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_1 I_{N_{\text{el}}} & \gamma_2 I_{N_{\text{el}}} \\ 0 & 0 & 0 & 0 & 0 & I_{N_{\text{el}}} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\sigma}_t \\ u_t \\ u_{t-1} \\ s_t \\ s_{t-1} \\ z_t \\ z_{t-1} \end{bmatrix}$$
$$+ \underbrace{\begin{bmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{M_\eta} e_{t+1} + \underbrace{\begin{bmatrix} R \\ I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{M_\varepsilon} \varepsilon_{t+1} + \underbrace{\begin{bmatrix} R\mathbf{1}_{N_{\text{in}}} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{M_\eta} \eta_{t+1} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I \\ 0 \end{bmatrix}}_{M_\nu} \nu_{t+1}, \tag{6.12}$$

which system we denote by

$$\theta_{t+1} = F\theta_t + \omega_{t+1}. \tag{6.13}$$

The marginaliation over the input boundary conditions is feasible, but not over the contact impedances. Therefore, we estimate the contact impedances simultaneously with the conductivity $\tilde{\sigma}_t$.

Assuming that the noise processes can be approximated as mutually independent, the state noise ω_t is Gaussian with zero mean and time-invariant covariance, which can be written in the form $\Gamma_\omega = M_e\Gamma_e M_e^{\mathrm{T}} + M_\varepsilon\Gamma_\varepsilon M_\varepsilon + M_\eta\Gamma_\eta M_\eta^{\mathrm{T}} + M_\nu\Gamma_\nu M_\nu^{\mathrm{T}}$, see **[IV]** for details..

6.3 THE SINGLE ELECTRODE LAYER PROBLEM

The symmetry problem that is related to electrical impedance tomography (EIT) imaging with a single electrode layer is the following. Consider a symmetric volume $\Omega \subset \mathbb{R}^3$ with symmetry plane $x_3 = 0$. Let all electrodes be placed on the symmetry plane. Let $\sigma(\vec{r}) > 0$ be an arbitrary conductivity distribution within the volume and let V be the measurements. Then, irrespective of the employed measurement model, we have $V(\sigma(x_1, x_2, x_3)) = V(\sigma(x_1, x_2, -x_3))$,

which shows that the conductivity is not identifiable in the classical sense.

If we have prior information or models that are nonsymmetric, the problem might be solvable, but these are seldom realistic, especially in a nonstationary setting. One of the few interesting cases is the one in which the conductivity is assumed to be translationally invariant in x_3-direction, leading to the so-called $2^1/_2$-dimensional problem [113,114]. In this approach, the invariability in x_3-direction enables the transformation of the governing 3D model into a 2D model.

In [**IV**], the unidentifiability of the symmetry problem is broken by the evolution model. We note that in earlier studies, such as [56], the state space model does not need to be as involved as the present one. The reason is that with standard electrode placement using several electrode rings/layers, the information is richer and in principle the problem is identifiable. The main (industrial) motivations for using a single electrode ring sensor are that such a sensor can easily be placed in a pipeline when multilayer sensors would take too much space and such a sensor is a cost- effective manufacturable.

7 Conclusions

Inverse problems can be loosely characterized as problems that tolerate poorly measurement and modeling errors. While measurement errors are often easy, or at least straightforward to model and to take into account in the overall computational model, the modeling errors have received very little attention before the mid 2000's. The main reason for this is that handling the modeling errors is only feasible in the Bayesian framework for inverse problems, while the majority of the inverse problems community favors the deterministic framework for inverse problems. Furthermore, the modeling of uncertainties that are related to partial differential equations and the initial-boundary value problems, require elements in the PDE theory, their numerical methods and Bayesian statistics, and exact analytical treatment is usually evasive. Thus, many of the well-known, and even practically ubiquitous, problems such as the boundary truncation problem have received very little attention.

Furthermore, when all the respective uncertainty models have been constructed, the overall computational model would typically be computationally too complex to allow for practical implementations that are feasible for the respective applications. Thus, standard Bayesian inference approaches based on, for example, Markov chain Monte Carlo are usually not considered, except in academic investigations. For example, if process tomography is used in industrial control and one is dealing with fast flows, the 3-dimensional reconstructions have to be carried out typically in a time frame of 1-10 milliseconds. And furthermore, the reconstructions (state estimates) may drive (an optimal) controller and thus feasible estimates for the second order statistics is also needed. Highly approximate computational models have thus to be used, but the overall model needs to be a feasible one.

The introduction of the approximation error approach in the mid 2000's provided an overall framework that made it feasible to

model all uncertainties and to simultaneously use drastically reduced order models that are feasible for resource limited problems. Furthermore, although the setting up of the overall model may be a tedious task, it is often a straightforward one. In particular, with problems in which the uncertainties don't facilitate using very low dimensional parametrizations, the approximation error approach may be the only practically feasible approach.

In this thesis, we have dealt with some of the most common uncertainties that are encountered with inverse problems, such as the boundary truncation problem and handling unknown uninteresting distributed parameters. Furthermore, we have considered both stationary and nonstationary problems. As a result of this thesis, the author and his coworkers have been able to realize the first industrial mass production of EIT systems that are capable of absolute imaging in an actual industrial setting, at least according to the author's knowledge. This work was carried out in Numcore Ltd, which won several national and international awards during its four year history. When the products were developed to an industrialy relevant status, other types of uncertainties were also considered, but the framework was mostly the same that is used in this thesis.

While the approach facilitates straightforward treatment of any number of simultaneous uncertainties and model reduction, the approach, naturally, cannot work miracles. The biggest drawback is that, since analytical results cannot usually be derived, one simply has to try how far one can go with the approach. This will often turn out to be quite a tedious task. Nevertheless, there is art in addition to the science, and experience in setting up the approach for different problems will in most cases make it a feasible task to set it up for the next problem.

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Anssi Lehikoinen

Modeling Uncertainties in Process Tomography and Hydrogeophysics

This thesis considers tomographic problems, which are one of the largest classes of inverse problems. In particular we consider Electrical Impedance Tomography (EIT) which probes the unknown object via electric fields. As specific technical model uncertainties, we consider the domain truncation problem and the use of approximated models to facilitate efficient computations. As particular applications, we consider problems in hydrogeophysics and process tomography.

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Dissertations in Forestry and Natural Sciences

ISBN 978-952-61-0882-7