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OPTIMAL SWITCH BETWEEN TWO FUNDS

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Abstract

The option to switch between two funds is analyzed with the real world characteristics. For a risk averse investor an increment of price volatility of one fund promotes the incentive to switch to another fund. However the real option approach takes into account the opportunity cost of lost option to switch later on and justifies the delay of investment. It is shown that the standard real option result of negative uncertainty-investment relationship is obtained when the fund prices are negatively correlated. A positive correlation between prices reduces the likelihood of funds drifting apart and the uncertainty of relative price of funds. Now the optimal price trigger value for investment is a decreasing function of the price uncertainty and positive uncertainty-investment relationship is obtained. The numerical results of model solution point out that the uncertainty-investment relation gradually tends to negative as the price correlation decreases.

Keywords: Fund prices, real options, trigger values, Brownian Motion-Poisson jump process

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1. INTRODUCTION

Nearly a half of households owned stocks directly or indirectly in U.S.A. in 1999 because of their high liquidity and return (Fabozzi et al. 2002). Particularly, funds have become popular because they satisfy investors with diversification, convenience, and professional portfolio management. However, the uncertainty of fund prices and irreversibility of transaction costs make their returns quite risky. Note that an uncertain price implies also that an investor can make a profit by switching among funds under suitable conditions. What is the optimal strategy to switch between the chosen two funds so that the investor can maximize the profits under uncertain prices? This is the problem we try to solve.

According to the arbitrage principles the investor should switch to the other fund (P_{other}) once the relative price (P_{own} / P_{other}) of the owned fund is high enough and the net present value (NPV) of switch is positive. However, the NPV rule is misleading to some extent as it pays too little attention to risk and uncertainty of market. For example, since the price uncertainty of funds varies across the market, the relative price of a particular fund may increase continuously. Thus investor can gain more if he waits for a longer time instead of switching immediately.

The investor who waits for an investment is holding an opportunity to invest. This opportunity is like an American option: the investor has the right but no the obligation to invest into an asset at some time in the future. Once the investment occurs, the lost option value is an opportunity cost that must be included as a part of the sunk cost. In order to distinguish it from a financial option, the option to the investment is called a "real option". The real option approach takes into account the opportunity cost and justifies the wait for investment even when the NPV is positive. The present model is based on the real option theory.

Previous research has applied the real option approach to various fields of investment decision-making (e.g. Schwartz & Trigeorgis 2004). Recently, Yap (2004) analyzed the Philippine forest plantation leases and evaluated lease policies. Strobel (2003) applied the method to non-economic phenomena. He examined the value of an option to marry by maximizing singles' utility and determined their optimal decision of if they should marriage or not. Lin et al. (2005) developed a model to evaluate the optimal trigger value for entry or exit in the Internet securities trading business in the face of uncertainties of two factors.

As for the standard results in the real options pricing theory, all writers advocate that higher volatility increases the value of the option, and also the incentive to wait instead of investing. Sarkar (2000) pointed out that this negative uncertainty-investment relationship is not always correct. However he attributed the result only to the increasing probability of investing. Moreover, he did not point out in which case the positive relationship exists and which case it disappears. Typically he also noted that the trigger value "is always an increasing function of σ (volatility of the price), as predicted".

The real option theory is also applied to a switch between two assets. For example, Arak & Taylor (1996) analyzed the solution of a switch between two similar funds. They considered the difference of prices as only stochastic variable in order to simplify computations. However the simplification ignored the influence of the degree of non-similarity between the funds on the optimal strategy. Typically the transaction cost is in proportion to the value of investment, and the cost of switching from the fund A to B is different from the switching from B to A. They assumed that the transaction cost per unit of a fund is constant, and the transaction cost is the same in both directions between A and B. As a result, the value of an option to switch from A to B is equivalent to that of an option to switch from B to A.

Next we build a model of optimal switch strategy without this equivalency condition. We study the role of price correlation in optimal switch strategy therein in switch-uncertainty relationship. Our assumptions are more consistent with the empirical observations compared to the earlier literature. Note also that an emergent event might occur so that the investor has to sell the fund immediately for the cash. This is introduced also in the model and it is called as liquidity event.

2. THE MODEL

2. 1. Notations and assumptions

X_A: the price of fund *A*, X_B: the price of fund *B*

P: relative price (X_A/X_B)

c_{sA} (c_{bA}): the cost ratio to sell (buy) fund *A*

c_{sB} (c_{bB}): the cost ratio to sell (buy) fund *B*

r: the risk-adjusted discount rate per a unit of time

ρ_{AB}: coefficient of correlation between prices of *A* and *B*

$F(X_A, X_B)$: the value of option to switch from *A* to *B*

$V(X_A, X_B)$: the value of option to switch from *B* to *A*.

Next assume the prices of funds A and B follow *Geometric Brownian Motion*

$$\frac{dX_A}{X_A} = \alpha_A dt + \sigma_A dz_A \qquad (1)$$

$$\frac{dX_B}{X_B} = \alpha_B dt + \sigma_B dz_B \qquad (2)$$

where dz is an increment of standard Wiener process and satisfies the condition

$$dz_i = \varepsilon_{it}\sqrt{dt} \qquad (3)$$

where ε_{it} is a normally distributed independent random variable with a mean of zero and a standard deviation of one, i.e. $N(0,1)$. The price of fund *A* (*B*) is expected to grow at a rate of α_A (α_B) per a time interval *dt.* σ is the standard deviation of price growth rate, and the level of uncertainty in the investment can be measured by this volatility term. Random elements, ε_{it}, influence the growth process. Both funds are assumed not to pay dividends.

2.2. Cutoff Strategy

Cutoff strategy means that only two strategies are available in any period of time: stop and continue (Dixit & Pindyck 1994). If stop is chosen at some period, then the process ends and the termination payoff $\Omega(x)$ is made, where *x* is a state variable. If continuation is chosen at some period, then the instant payoff is $\pi(x)$, and another similar binary choice will be available in next period. Let $G(x_0)$ denote the maximal discounted expected payoff given x_0. The G may be found by solving the Bellman equation (Dixit & Pindyck 1993):

$$G_t(x) = Max\left\{\Omega(x), \quad \pi(x) + \frac{1}{1+rdt} E_t\left[(G_t + dG_t)\middle|G_t\right]\right\} \qquad (4)$$

where $\frac{1}{1+rdt} E_t\left[(G_t + dG_t)\middle|G_t\right]$ is the discounted expected future profit flow, and therefore $\pi(x) + \frac{1}{1+rdt} E_t\left[(G_t + dG_t)\middle|G_t\right]$ is the discounted expected profit flow from the time period *t+dt*.

2. 3. Basic model

This cutoff strategy is quite general and it can be applied to a variety of investment problems. In this study, the strategy is revised to apply to switch between two funds.

First we assume that some critical relative price (say, P_A^*) exists so that the investor with fund *A* should switch to fund *B* once the relative price *P* exceeds this critical level. In turn, the investor with fund *B* should switch to fund *A* once the relative price *P* falls below some other trigger level P_B^*. Therefore, such cutoff strategy can be expressed as follows:

For the investor with fund *A* the optimal strategy is

$$\begin{cases} Wait \quad (retain \quad holding \quad fund \quad A) \quad if \quad P = X_A / X_B \le P_A * \\ Switch \quad to \quad fund \quad B \quad immediatly \quad if \quad P = X_A / X_B > P_A * \end{cases} \tag{5}$$

For the investor with fund *B* the optimal strategy is

$$\begin{cases} Wait \quad (retain \quad holding \quad fund \quad B) \quad for \quad P = X_A / X_B \ge P_B * \\ Switch \quad to \quad fund \quad A \quad immediatly \quad for \quad P = X_A / X_B < P_B * \end{cases} \tag{6}$$

where P_A^* and P_B^* are two trigger values to be solved.

An investor with fund *A* has an option $F(X_A,X_B)$ to switch to fund *B*. While he exercises the option to sell *A* and hold *B*, he gets simultaneously another option $V(X_A,X_B)$ to switch back to fund *A*. Thus the value of option to switch from *A* to *B* can be expressed in the form of Bellman equation as

$$F_t(X_A, X_B) = Max\left\{(1-c_{sA})X_A - (1+c_{bB})X_B + V(X_A, X_B),\ \frac{1}{1+rdt}E_t\left[(F_t + dF_t)\middle|F_t\right]\right\} \quad , (7)$$

where the value of option of immediate switch to fund *B* is

$$(1-c_{sA})X_A-(1+c_{bB})X_B+V(X_A,X_B),$$

and

$$E_t\left[(F_t+dF_t)\middle|F_t\right]$$

is the expected value of option to switch at the next time interval $t+dt$ on the condition of its current value F_t . $\frac{1}{1+rdt}$ is discount factor.

The expected present value of the option in the case of wait and switch later is measured by

$$\frac{1}{1+rdt}E_t\left[(F_t+dF_t)\middle|F_t\right].$$

Similarly an investor with fund B will not get only fund A but also the option to switch back to B, $F(X_A, X_B)$, after he switched from B to A. Therefore, the value of option to switch from B to A is

$$V_t(X_A,X_B)=Max\left\{(1-c_{sB})X_B-(1+c_{bA})X_A+F(X_A,X_B),\ \frac{1}{1+rdt}E_t\left[(V_t+dV_t)\middle|V_t\right]\right\}. \quad (8)$$

Using Ito's Lemma (see Appendix A), the expected values of differentials dF and dV can be expressed as

$$\frac{1}{dt}E[dF]=\tfrac{1}{2}\sigma_A^2X_A^2F_{X_AX_A}+\tfrac{1}{2}\sigma_B^2X_B^2F_{X_BX_B}+\rho_{AB}\sigma_A\sigma_BX_AX_BF_{X_AX_B}+\alpha_AX_AF_{X_A}+\alpha_BX_BF_{X_B} \quad (9)$$

$$\frac{1}{dt}E[dV]=\tfrac{1}{2}\sigma_A^2X_A^2V_{X_AX_A}+\tfrac{1}{2}\sigma_B^2X_B^2V_{X_BX_B}+\rho_{AB}\sigma_A\sigma_BX_AX_BV_{X_AX_B}+\alpha_AX_AV_{X_A}+\alpha_BX_BV_{X_A} \quad (10)$$

2.4. Model with the jump

Now we introduce a liquidity event into the model. Assume that the occurrence of this event follows a Poisson process. A Poisson process is a process subject to jump of fixed or random size and the jump occurs with mean arrival rate. Let λ denote the mean arrival rate of the event during a time interval *dt*, so the probability that the event will occur is λ*dt* and the probability that the event will not occur is *1-λdt.* When the event has happened, the value of an option $F(X_A,X_B)$ is lost and liquidation of fund incurs a cost, i.e. $-c_{sA}X_A$, the cost of a sale of fund *A*, with the probability of *λdt*. In the case of non-event there are no any addition costs or returns. Hence, the expected value of $dF_{PoissonJump}$ is given by

$$E\left[dF_{PoissonJump}\right] = \lambda E\left[-c_{sA}X_A - F(X_A, X_B)\right]dt \tag{11}$$

Thus, the stochastic process of price is a combination of Brownian Motion and Poisson Process. Assume that there is no correlation between these processes, so the Ito's Lemma is a straight combination of these, that is,

$$dF = dF_{BrownianMotion} + dF_{PoissonJump} \tag{12}$$

Therefore, for the Brownian Motion-Poisson jump process, the Ito's Lemma is

$$\begin{aligned}\frac{1}{dt}E\left[dF\right] &= \tfrac{1}{2}\sigma_A^2 X_A^2 F_{X_A X_A} + \tfrac{1}{2}\sigma_B^2 X_B^2 F_{X_B X_B} + \rho_{AB}\sigma_A\sigma_B X_A X_B F_{X_A X_B} + \alpha_A X_A F_{X_A} + \alpha_B X_B F_{X_B} \\ &\quad + \lambda E\left[-c_{sA}X_A - F(X_A, X_B)\right]\end{aligned} \tag{13}$$

Likewise the expected value of differential *dV* is

$$\frac{1}{dt}E\left[dV\right] = \tfrac{1}{2}\sigma_A^2 X_A^2 V_{X_A X_A} + \tfrac{1}{2}\sigma_B^2 X_B^2 V_{X_B X_B} + \rho_{AB}\sigma_A \sigma_B X_A X_B V_{X_A X_B} + \alpha_A X_A V_{X_A} + \alpha_B X_B V_{X_A}$$

$$+\lambda E\left[-c_{sB} X_A - V(X_A, X_B)\right]$$

(14)

We will focus on the region of wait in the two Bellman equations (7) and (8):

$$F_t(X_A, X_B) = \frac{1}{1+rdt} E_t\left[(F_t + dF_t)\middle| F_t\right] \tag{15}$$

$$V_t(X_A, X_B) = \frac{1}{1+rdt} E_t\left[(V_t + dV_t)\middle| V_t\right] \tag{16}$$

Substitution of (13) into (15) and (14) into (16) gives differential equations:

$$\tfrac{1}{2}\sigma_A^2 X_A^2 F_{X_A X_A} + \tfrac{1}{2}\sigma_B^2 X_B^2 F_{X_B X_B} + \rho_{AB}\sigma_A \sigma_B X_A X_B F_{X_A X_B} + \alpha_A X_A F_{X_A} + \alpha_B X_B F_{X_B}$$

$$-(r+\lambda)F - \lambda c_{sA} X_A = 0$$

(17)

and

$$\tfrac{1}{2}\sigma_A^2 X_A^2 V_{X_A X_A} + \tfrac{1}{2}\sigma_B^2 X_B^2 V_{X_B X_B} + \rho_{AB}\sigma_A \sigma_B X_A X_B V_{X_A X_B} + \alpha_A X_A V_{X_A} + \alpha_B X_B V_{X_A}$$

$$-(r+\lambda)V - \lambda c_{sB} X_B = 0$$

(18)

Using $P = X_A/X_B$ and after a series of mathematical transformations combining with the boundary conditions, i.e. value matching and smooth pasting conditions (see Appendix B), we can deduce equations (19) - (22) from equations (17) and (18). The values of four unknowns, A_1, B_2, P_A^*, P_B^*, can be determined by solving the four equations:

$$A_1 P_A^{\beta_1} + \lambda c_{sA} P_A /(\alpha_A - r - \lambda) = -(1 + c_{bB}) + (1 - c_{sA}) P_A + B_2 P_A^{\beta_2} + \lambda c_{sB} /(\alpha_B - r - \lambda) \quad (19)$$

$$A_1 \beta_1 P_A^{(\beta_1 - 1)} + \lambda c_{sA} /(\alpha_A - r - \lambda) = +(1 - c_{sA}) + B_2 \beta_2 P_A^{(\beta_2 - 1)} \quad (20)$$

$$B_2 P_B^{\beta_2} + \lambda c_{sB} /(\alpha_B - r - \lambda) = (1 - c_{sB}) - (1 + c_{bA}) P_B + A_1 P_B^{\beta_1} + \lambda c_{sA} P_B /(\alpha_A - r - \lambda) \quad (21)$$

$$B_2 \beta_2 P_B^{(\beta_2 - 1)} = -(1 + c_{bA}) + A_1 \beta_1 P_B^{(\beta_1 - 1)} + \lambda c_{sA} /(\alpha_A - r - \lambda) \quad (22)$$

where β_1 and β_2 are the two roots in the below quadratic equation and $\beta_1 > \beta_2$:

$$\tfrac{1}{2}(\sigma_A^2 - 2\rho_{AB}\sigma_A\sigma_B + \sigma_B^2)(\beta - 1)\beta + (\alpha_A - \alpha_B)\beta - (r + \lambda - \alpha_B) = 0 \quad . \quad (23)$$

The output of the model is P_A^* and P_B^*, i.e. the optimal switch points of fund prices. The equations are nonlinear in the thresholds, so that analytical solutions are unavailable and we have use numerical methods for solutions.

3. NUMERICAL ANALYSIS

There are 11 parameters in this model: α_A, α_B, σ_A, σ_B, c_{sA}, c_{sB}, c_{bA}, c_{bB}, r, ρ_{AB}, λ. In order to determine the numerical solutions, the time interval dt is set as one week and the base values of parameters are set as in Table 1. These values are selected from some related reports (see Appendix C). The solutions are obtained by mathematical software *Maple VI*. The trigger value of switch to fund B $(P_A{*})$ was 1.4485 and the trigger value of switch to fund A $(P_B{*})$ was 0.5887 in the base case.

Table 1: Base case

α_A	α_B	σ_A	σ_B	r	λ	ρ_{AB}	c_{sA}	c_{sB}	c_{bA}	c_{bB}	$P_A{*}$	$P_B{*}$
0. 01	0. 012	0.03	0. 04	0.005	0.05	0. 3	0. 1	0. 08	0. 001	0. 01	1.207 2	0.7765

A set of solutions was solved by changing values of parameters. The sensitivity of the model to each parameter was analyzed, i.e. the variation of output, $P_A{*}$ and $P_B{*}$, accounted for the changes of parameter values (Table 2). According to the strategy of switch in section 2, an increase in P_A* reduces the incentive to switch to fund B, and an increase in P_B* increases the incentive to switch to fund A. Thus, the influence of the parameters on the incentive to switch can be derived from the variation of P_A* and P_B* . This is illustrated in Table 3.

Table2: Influence of the parameters on $P_{A,}$, P_B

	α_A	α_B	σ_A	σ_B	r	λ	ρ_{AB}	c_{sA}	c_{sB}	c_{bA}	c_{bB}
P_A	+	−	±	±	±	—	−	+	+	+	+
P_B	+	−	±	±	±	+	+	−	−	−	−

+ monotonically increasing; − monotonically decreasing; ± unclear

Table3: Influence of the parameters on the incentive to switch

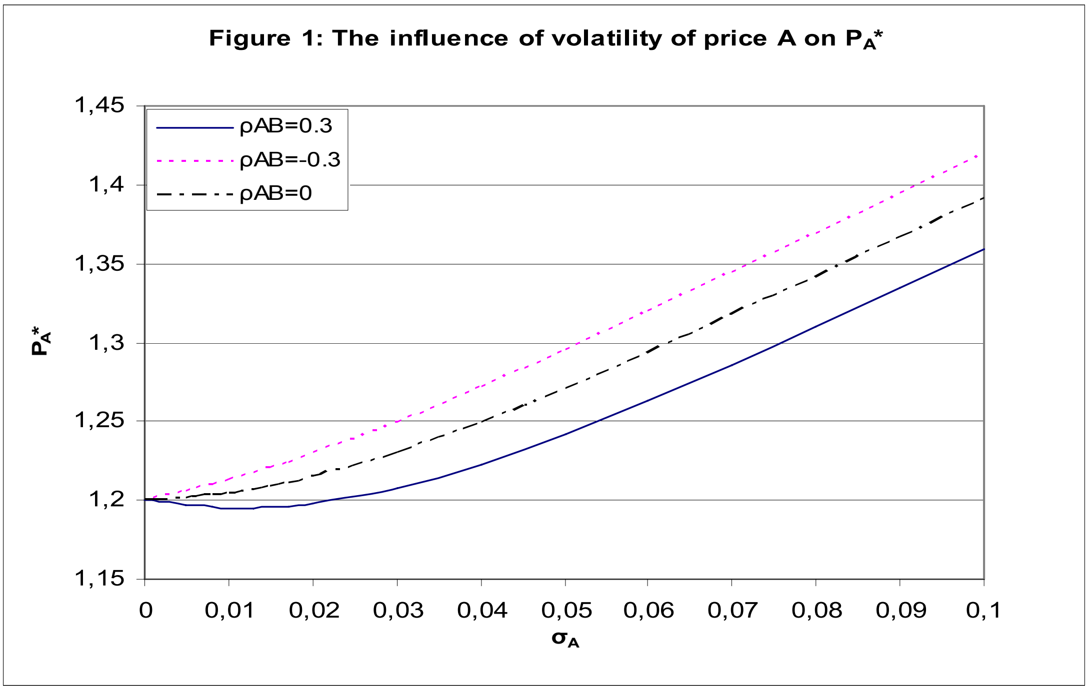
	α_A	α_B	σ_A	σ_B	r	λ	ρ_{AB}	c_{sA}	c_{sB}	c_{bA}	c_{bB}
$A \rightarrow B$	−	+	±	±	±	+	+	−	−	−	−
$B \rightarrow A$	+	−	±	±	±	+	+	−	−	−	−

+ monotonically increasing; − monotonically decreasing; ± unclear

An increase in the growth rate of a fund, a_A and a_B, increases the value of option to switch to the other fund and the incentive to holding the fund. There are two opposite effects resulting from the discount rate r: the trigger value effect and the incentive to switch effect. An increase in discount rate implies the increment of the opportunity cost of leaving option to switch unexercised, so the incentive to switch increases. On the other hand, the increased discount rate increases the future value of transaction costs, so the incentive to switch decreases.

The higher correlation of the prices on two funds, ρ_{AB}, reduces the likelihood of two funds drifting apart and uncertainty of the relative price. This implies that the value of option to switch decreases and the incentive to switch increases. The incentive to switch is reduced when any transaction cost increases since all costs are sunk costs.

Figure 1. illustrates the influence of the price volatility of the fund A (σ_A) on P_A^*



The curve is partly decreasing when ρ_{AB} is positive and the curve turns straighter when the value of correlation (ρ_{AB}) decreases. For example, when the correlation is positive (e.g. ρ_{AB} *=0.3*), P_A decreases until σ_A reaches a certain value (approximate *0.013 ~ 0.015* in the base case) and after it increases with increasing σ_A. There exists a monotonic increasing relation between P_A and σ_A when ρ_{AB} is *0,* and the monotonic relation become stronger when ρ_{AB} is *–0.3*. ρ_{AB} has the similar influence on P_B* (see Figure 2). The impact of volatility of fund B (σ_B) on trigger value P_B^* mirrors the effects of σ_A.

The fund price effects are related to the following three factors:

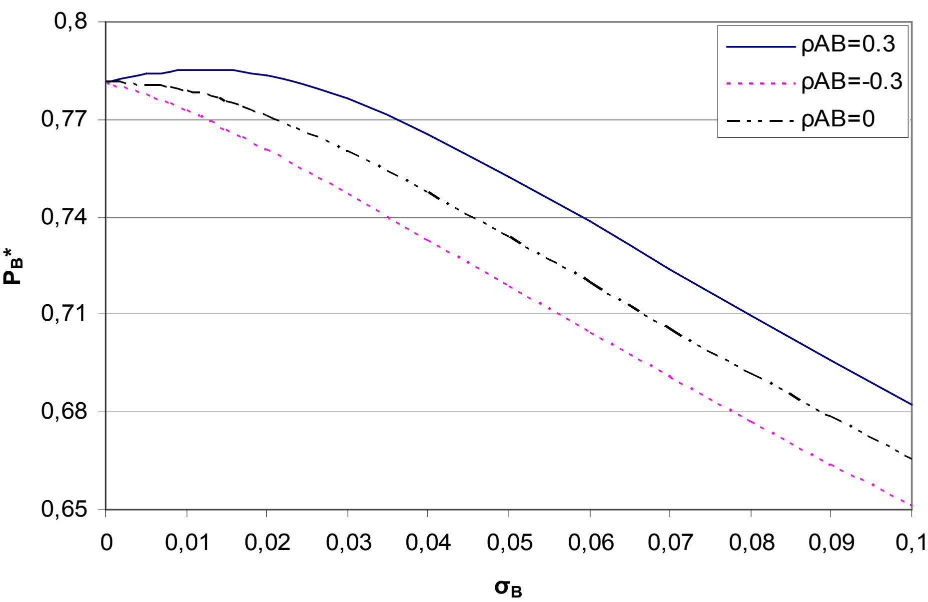
1. An increase in volatility increases the risk of fund holding, so a precautious investor requires more time to obtain more information for making decision, i.e. investor is willing to hold the option to switch and she/he waits.

2. Contrary to this an increasing volatility implies the increasing risk of holding the fund, so it promotes the incentive to switch to the other fund.

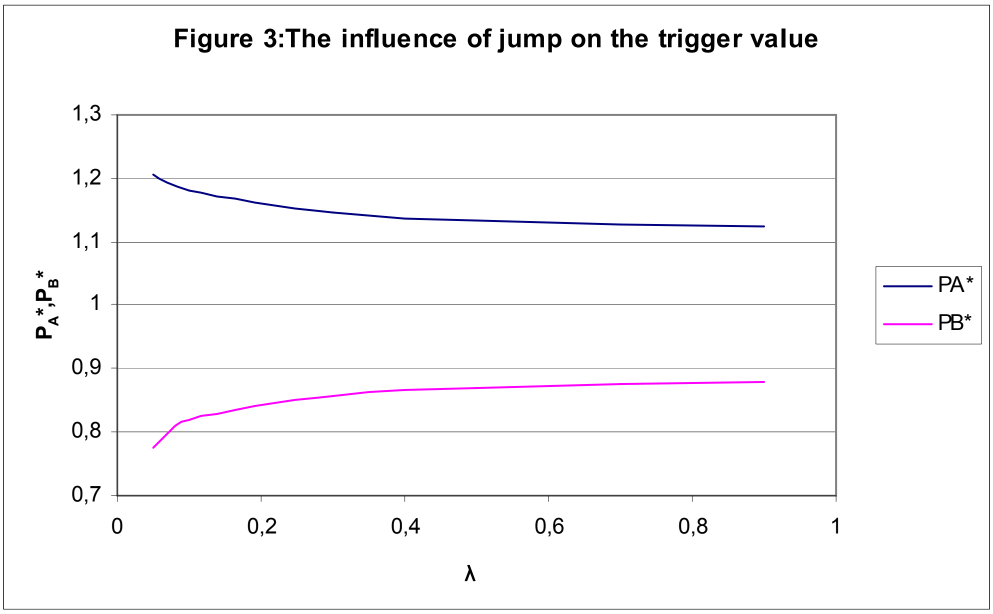
The net effect of these factors depends on the correlation between the fund prices.

3. The higher positive correlation between the prices of the two funds, the smaller the likelihood that the two funds drifts apart. This reduces the uncertainty in the relative price when the volatility of price increases. This leads to a decrease in P_A* and an increase in P_B*, and also promotes the incentive to switch. However, when the prices of two funds have negative correlation, it has an opposite influence that increases the uncertainty in the relative price and therefore it increases P_A*, decreases P_B* and reduces the incentive to switch.

Figure 2: The influence of volatility of price on P_B*



Furthermore, an increase in the probability of occurrence of liquidity event λ decreases the value of wait, and consequently promotes the switch. Even an event with small occurrence probability has a significant effect on the trigger value (Figure. 3).



Note that whit NPV rule, an investor with fund *A* should switch to fund *B* as soon as the net present value of switch, $(1-c_{sA})X_A-(1+c_{bB})X_B$, is positive. Similarly an investor with fund *B* switches to *A* once the present value of profit, $(1-c_{sB})X_B-(1+c_{bA})X_A$, is positive. Note that the trigger values are affected only by the transaction costs.

4. CONCLUSIONS

The standard results in real options pricing theory (Dixit & Pindyck 1993) advocates that higher volatility increases the value of option and also the incentive of wait instead of switch. Sarkar (2000) pointed out that the negative uncertainty-investment relationship is not always correct. He did not consider the correlation between the funds when analyzing the relation between trigger values (P_A*,P_B*) and price volatilities. We showed that in the case of positive correlation between the prices of two funds, when the volatility of the price of fund *A* or fund *B* is very small, an increase in the corresponding volatility decreases P_A* and increases P_B*. This implies a decrease in the value of waiting and a promotion of the incentive to switch, i.e. the positive uncertainty-investment relationship prevails. The present study not only emphasizes how the correlation between two variables influence the switch-uncertainty relationship but also clearly point out that the uncertainty-investment relation gradually tends to negative one as the correlation decreases. Furthermore, as opposed to Sarkar (2000), it is possible that the trigger value is a decreasing function of the uncertainty of the price.

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Appendix A

Ito's Lemma: Assume Xt_t is an Ito-process $dx_t = adt + bdz_t$. Let $F(X, t)$ be a twice continuously differentiable function. Then $F(x, t)$ is also an Ito process which satisfies

$$dF = \left[\frac{\partial F}{\partial t} + a\frac{\partial F}{\partial x} + \tfrac{1}{2} b^2 \frac{\partial^2 F}{\partial x^2} \right] dt + b\frac{\partial F}{\partial x} dz \quad . \qquad \text{(A1)}$$

It can be extended to the function of several Ito processes (Dixit & Pindyck 1993): suppose that $F = F(x_1, \ldots, x_m, t)$ is a function of time and of the m Ito processes $x_i, \ldots, x_m$, where

$$dx_i = a_i dt + b_i dz_i, \qquad i = 1, \ldots, m \qquad \text{(A2)}$$

When $E(dz_i dz_j) = \rho_{ij} dt$, dF has form of

$$dF = \left[\frac{\partial F}{\partial t} + \sum_i a_i \frac{\partial F}{\partial x_i} + \tfrac{1}{2}\sum_i b_i^2 \frac{\partial^2 F}{\partial x_i^2} + \tfrac{1}{2}\sum_{i \neq j} \rho_{ij} b_i b_j \frac{\partial^2 F}{\partial x_i \partial x_j} \right] dt + \sum_i b_i \frac{\partial F}{\partial x_i} dz_i \ . \qquad \text{(A3)}$$

In fund switch context F is a function of the two Ito processes X_A, X_B, where

$$dX_A = \alpha_A X_A dt + \sigma_A X_A dz_A \quad , \qquad \text{(A4)}$$

$$dX_B = \alpha_B X_B dt + \sigma_B X_B dz_B \quad , \qquad \text{(A5)}$$

$$E(dz) = 0 \quad . \qquad \text{(A6)}$$

By substituting Ito processes (A4) (A5) and condition (A6) into the expression (A3), we get

$$\frac{1}{dt} E\left[dF \right] = \tfrac{1}{2}\sigma_A^2 X_A^2 F_{X_A X_A} + \tfrac{1}{2}\sigma_B^2 X_B^2 F_{X_B X_B} + \rho_{AB}\sigma_A \sigma_B X_A X_B F_{X_A X_B} + \alpha_A X_A F_{X_A} + \alpha_B X_B F_{X_B}$$

For the other function $V(X_A,X_B)$, whereby, we can get similar result :

$$\frac{1}{dt} E[dV] = \tfrac{1}{2}\sigma_A^2 X_A^2 V_{X_A X_A} + \tfrac{1}{2}\sigma_B^2 X_B^2 V_{X_B X_B} + \rho_{AB}\sigma_A \sigma_B X_A X_B V_{X_A X_B} + \alpha_A X_A V_{X_A} + \alpha_B X_B V_{X_A}$$

Appendix B

Denote $P = \dfrac{X_A}{X_B}$ and the new function $f\,(\dfrac{X_A}{X_B})$ satisfies

$$F(X_A, X_B) = X_B f(\frac{X_A}{X_B}) = X_B f(P) \quad . \tag{B1}$$

The partial derivations of function $F(X_A, X_B)$ are

$$F_{X_A} = \frac{\partial(X_B f(P))}{\partial X_A} = f^{'}(P) \quad , \tag{B2}$$

$$F_{X_B} = \frac{\partial(X_B f(P))}{\partial X_B} = f(P) - Pf^{'}(P) \quad , \tag{B3}$$

$$F_{X_A X_A} = \frac{f^{"}(P)}{X_B} \quad , \tag{B4}$$

$$F_{X_A X_B} = \frac{P^2 f^{"}(P)}{X_B} = F_{X_B X_A} \quad . \tag{B5}$$

$$F_{X_B X_B} = \frac{P^2 f^{"}(P)}{X_B} \tag{B6}$$

Substituting these in the equation (17) and grouping terms, we get a new equation for the function $f(P)$:

$$\tfrac{1}{2}(\sigma_A^2 - 2\rho_{AB}\sigma_A\sigma_B + \sigma_B^2)P^2 f^{"}(P) + (\alpha_A - \alpha_B)Pf^{'}(P) - (r + \lambda - \alpha_B)f(P) - \lambda c_{sA}P = 0 \,. \tag{B7}$$

By using the guess -approach for the homogenous part of the above equation gives a solution of form $f(P) = AP^{\beta}$. By substituting it to the homogenous part we get the equation

$$\tfrac{1}{2}(\sigma_A^2 - 2\rho_{AB}\sigma_A\sigma_B + \sigma_B^2)\beta(\beta - 1) + (\alpha_A - \alpha_B)\beta - (r + \lambda - \alpha_B) = 0 \quad . \tag{B8}$$

That is, the homogeneous part of the equation (B7) has solution of the form AP^{β}, provided that β is the root of the quadratic equation (B8). Denote the left part of the quadratic equation by $Q(\beta)$.

Now assume that $\alpha < \lambda + r$, so that there exists finite time when it is optimal to adjust (see Dixit&Pindyck 1994,pp.171-173). So *Q(1)=*α_A*-r-*λ*<0* and *Q(0)=*α_B*-r-*λ*<0* .

Since $\rho_{AB} \leq 1$, the coefficient of β^2 in *Q (β)* is positive

$$\tfrac{1}{2}(\sigma_A^2 - 2\rho_{AB}\sigma_A\sigma_B + \sigma_B^2) \geq \tfrac{1}{2}(\sigma_A^2 - 2\sigma_A\sigma_B + \sigma_B^2) = \tfrac{1}{2}(\sigma_A - \sigma_B)^2 \geq 0 \qquad \text{(B9)}$$

and now $\underset{\beta \to \pm\infty}{Q(\beta)} \to \infty$.

Therefore, the curve of *Q (β)* is convex (Figure A.1) with $\beta_1 > 1$ and $\beta_2 < 0$

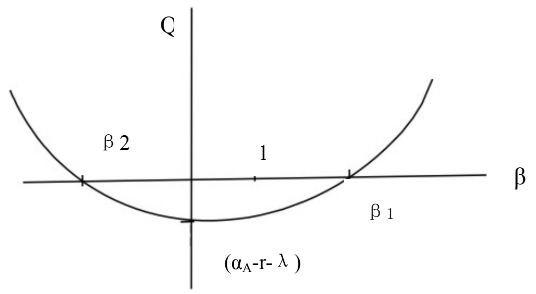

Figure A.1. The quadratic function

The general solution of the homogenous part of equation (B7) is a linear combination of the two independent solutions $A_1P^{\beta_1}$ and $A_2P^{\beta_2}$. $\dfrac{\lambda(c_{sA}-1)P}{\alpha_A-(r+\lambda)}$ is the specific solution to non-homogeneous differential equation (B7). Thus, the general solution of (B7) can be written as

$$f(P) = A_1P^{\beta_1} + A_2P^{\beta_2} + \frac{\lambda c_{sA}P}{\alpha_A-(r+\lambda)} \qquad \text{(B10)}$$

where both A_1 and A_2 are constants.

As the relative price *P* becomes very small, the likelihood of switch to fund B in the not-too-distant future becomes extremely small, so the option to switch to B should go to zero. However, P^{β_2} goes to ∞ as *P* goes to *0* with negative β_2. So A_2 should be zero. Thus the economic solution is

$$f(P) = A_1P^{\beta_1} + \frac{\lambda c_{sA}P}{\alpha_A-(r+\lambda)} \qquad \text{(B11)}$$

Similarly we obtain for $V(X_A, X_B)$

$$v(P) = B_2 P^{\beta_2} + \frac{\lambda c_{sB}}{\alpha_B - (r+\lambda)} \tag{B12}$$

by defining a new function $v(P)$ satisfying

$$V(X_A, X_B) = X_B v(P) \tag{B13}$$

The value matching and smooth pasting conditions for the functions $f(P)$ and $v(P)$ are (see Dixit & Pindyck 1994)

$$f(P_A) = -(1+c_{bB}) + (1-c_{sA})P_A + v(P_A) \quad , \tag{B14}$$

$$f'(P_A) = (1-c_{sA}) + v'(P_A) \quad , \tag{B15}$$

$$v(P_B) = (1-c_{sB}) - (1+c_{bA})P_B + f(P_B) \quad , \tag{B16}$$

$$v'(P_B) = -(1+c_{bA}) + f'(P_B) \quad . \tag{B18}$$

By substituting equations (B11) and (B13) into (B14)-(B18) we obtain

$$A_1 P_A^{\beta_1} + \lambda(c_{sA}-1)P_A/(\alpha_A - r - \lambda) = -(1+c_{bB}) + (1-c_{sA})P_A + B_2 P_A^{\beta_2} + \lambda c_{sB}/(\alpha_B - r - \lambda), \tag{19}$$

$$A_1 \beta_1 P_A^{(\beta_1-1)} + \lambda c_{sA}/(\alpha_A - r - \lambda) = (1-c_{sA}) + B_2 \beta_2 P_A^{(\beta_2-1)} \quad , \tag{20}$$

$$B_2 P_B^{\beta_2} + \lambda c_{sB}/(\alpha_B - r - \lambda) = (1-c_{sB}) - (1+c_{bA})P_B + A_1 P_B^{\beta_1} + \lambda(c_{sA}-1)P_B/(\alpha_A - r - \lambda), \tag{21}$$

$$B_2 \beta_2 P_B^{(\beta_2-1)} = -(1+c_{bA}) + A_1 \beta_1 P_B^{(\beta_1-1)} + \lambda c_{sA}/(\alpha_A - r - \lambda) \quad . \tag{22}$$

Appendix C

The parameter values were obtained from a variety of sources:

1. http://research.stlouisfed.org/fred2/data/PRIME.txt
2. http://www.investopedia.com/articles/04/021804.asp
3. http://www.ici.org/factbook/05_fb_sec3.html
4. Fabozzi.F.J., Modigliani.F, Jones.F.J & Ferri.M.G. (2002): Foundations of Financial Markets and Institutions. Prentice Hall Press. pp.130-131