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Abstract

Interest rate is treated as the growth rate of a monetary quantity with dimension $1/\Delta t$. With this definition, we can analyze time in discrete and continuous time interest calculation in a proper manner, and the two interest rate concepts are then consistent too. We study differences in discrete and continuous time interest and discount calculation, and in using different time units in discrete time analysis. Measurable concepts corresponding to the instantaneous velocities of debt and deposit capitals are defined, and their usefulness in economic analysis is demonstrated. (JEL E40, G10)

Keywords: Discrete and continuous time interest rate, interest calculation, discounting, dimensional analysis, measurement unit.

1 Introduction

An overwhelming majority of scientific writings dealing with interest rate treat it as a dimensionless number[1]. It is still assumed in these studies that interest rate is measured per annum, which implies that it is related to time; see for instance Chiang (1974 p. 289–94) or Hull (2000 p. 51). Treating the interest rate as a pure number makes the comparing of interest rates defined

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[1]Exceptions exist also; for instance de Jong (1967) and Lehtonen et al. (1982) treat the discrete time interest rate as it is treated here.

for two days, three weeks, etc. ambiguous, because as pure numbers they ought to be comparable as such. But we know that 10 % in a week differs from 10 % in a year. Another problem this definition creates is in which units changes in interest rates are measured. If the annual interest rate changes from 10 % to 12 %, the change is 2 percent points or 20 %.

In economic analysis, interest calculation has been defined in discrete and in continuous time. What seems not to have been done accurately is the treating of time in the analysis. Interest calculation in continuous time is old enough for Samuelson (1937) to call it "*a well-known formula*". Samuelson treats the continuous time interest rate as "*a flow of dollars per unit time divided by the value of the investment*", which definition has not been changed since. Solow (1963) writes: "*The rate of return, as I have defined it, is the rate of interest paid by the bank on one year deposits... ...the rate of return on investment, a dimensionless number (per unit of time)...*" Here we generalize the above definitions so that the time interval the interest rate is measured can be of any length, that is, it does not have to be one year or unity. This generalization — suggested by de Jong (1967) — allows us to deal with interest rate as a dimensional concept.

We study differences in interest and discount calculation in discrete and continuous time, and in discrete time with varying time units. We search for analytic transformation rules between various interest rate concepts which give identical results by different calculation methods. Such rules are presented for compound interest calculation, but for discounting monetary flows and for the time paths of dept and deposit capitals such rules cannot be found. Although continuous time is widely used in economic analysis, so far measurable concepts corresponding to the instantaneous velocities of debt and deposit capitals have not been defined. Samuelson (ibid.) though uses these concepts, but does not define them as dimensional quantities. We show that they are important measurable quantities for the continuous time analysis of debt and deposit capitals.

The study is organized as follows. In section 2 we study ways to measure changes in quantities with time. Sections 3 and 4 contain discrete and continuous time interest calculation. In section 5 are given parities between the discrete and continuous time interest rate, and between discrete time interest rates for various time units. Section 6 studies the present values of monetary flows in discrete and continuous time. Section 7 contains economic analysis of the dynamics of dept and deposit capitals and section 8 gives a summary.

2 Measuring Changes in Quantities in Time

All monetary quantities belong in the dimension of *value*, and are thus additive (de Jong (1967) p. 8). Monetary quantities may not be directly additive, however, because adding 5 (\$) + 10 (£) does not make sense[2]. This adding can be executed if we have a transformation rule for these two measurement units of value; for instance 1 (£) = 2.5 (\$). Exchange rate $E = 1/2.5$ (£/\$) or $1/E = 2.5$ (\$/£) expresses this rule. Knowing the exchange rate we can add any two quantities x (\$) and y (£) as $x + y/E$ in units of \$ or $xE + y$ in units of £. With a fixed (floating) exchange rate regime, E is a dimensionless constant (varying quantity). These transformations are analogous to those made in adding quantities belonging in the dimension *length* with units *mile* and *metre* by using the rule 1 (mi) = 1609.38 (m).

We assume that time is divided into intervals of equal length Δt which can be measured in any time units: hours (h), days (d), weeks (w), years (y) etc. The transformation rules between the measurement units of the dimension *time* allow us to express Δt the time units we like. The only problem is that the number of days in one month varies. This can be solved by defining 1 (m) = 30 (d), which gives 1 (y) = 12 (m) $\times$ 30 (d/m) = 360 (d), or defining 1 (y) = 52 (w), which gives 1 (y) = 52 (w) $\times$ 7 (d/w) = 364 (d).

In the following we denote time moments by t_0, t_1, $t_2, \ldots$ and their distances as $\Delta t = t_1 - t_0 = t_2 - t_1 \cdots$ It thus holds that $t_0 + \Delta t = t_0 + t_1 - t_0 = t_1$, $t_0 + 2\Delta t = t_0 + \Delta t + \Delta t = t_1 + \Delta t = t_2$ etc., and the time intervals are named according to their ending moments. In discrete time analysis, the length of Δt does not matter; essential is that the quantities applied are measured only at moments t_0, $t_1, \ldots$ (or at periods t_1, $t_2, \ldots$) and not within these moments. This requirement holds for all economic time series. Continuous time is constructed from discrete time by letting $\Delta t \to 0$. This way obtained time units of zero length correspond to time moments, and continuous time is formulated by connecting the adjacent time moments.

Interest rate measures the change of a monetary quantity with time. Now, we have four possible ways to measure the change of quantity x during time unit $\Delta t = (t_0 + \Delta t) - t_0$, where t_0 is a fixed moment. These are presented in Table 1. From Table 1 we see that absolute change is measured in the units of x, relative change is dimensionless, average velocity is measured in the units of $x/\Delta t$ and growth rate in the units of $1/\Delta t$. Now, if $\Delta t =$ *1 time unit*, the numerical values of the first and third, and the second and fourth quantity are equal, although their measurement units differ. The common assumption $\Delta t =$ *1 time unit* is the reason that interest rate has many times

[2]Measurement units are in braces after the numerical values of the quantities.

been erroneously identified as the relative change, and not the growth rate of a monetary quantity, which is the correct definition (see sections 3, 4). In economic data, growth rates are many times presented as growth rates in per cent, which are simply growth rates multiplied by 100. Knowing the measurement units of economic quantities is a necessity for well-defined mathematical formulations with these quantities, as we will see later.

absolute change	$x(t_0 + \Delta t) - x(t_0)$,
relative change	$\frac{x(t_0+\Delta t)-x(t_0)}{x(t_0)}$,
average velocity	$\frac{x(t_0+\Delta t)-x(t_0)}{\Delta t}$,
growth rate	$\frac{[x(t_0+\Delta t)-x(t_0)]/\Delta t}{x(t_0)}$.

Table 1. Discrete time quantities measuring change

The continuous time correspondents of the above discrete time quantities are obtained as their limits with $\Delta t \to 0$. They are in Table 2.

instantaneous absolute change	$\lim_{\Delta t\to 0} [x(t_0 + \Delta t) - x(t_0)] = dx\|_{t=t_0}$,
instantaneous relative change	$\lim_{\Delta t\to 0} \left[\frac{x(t_0+\Delta t)-x(t_0)}{x(t_0)}\right] = \frac{dx}{x}\|_{t=t_0}$,
instantaneous velocity	$\lim_{\Delta t\to 0} \left[\frac{x(t_0+\Delta t)-x(t_0)}{\Delta t}\right] = \frac{dx}{dt}\|_{t=t_0}$,
instantaneous growth rate	$\lim_{\Delta t\to 0} \left[\frac{[x(t_0+\Delta t)-x(t_0)]/\Delta t}{x(t_0)}\right] = \frac{dx/dt}{x}\|_{t=t_0}$.

Table 2. Continuous time quantities measuring change

In mathematics, dx is the differential and dx/dt the time derivative of x. Pure mathematics differs from "real sciences", however, there, that mathematics does not operate with dimensional quantities. Theoretical physicists, on the other hand, have developed an algebraic theory for mathematical operations with dimensional quantities called *dimensional analysis*, which theoretical physics successfully applies. The core of the theory is given, for instance, in de Jong (1967). According to dimensional analysis, the measurement units of the above continuous time quantities are identical with their discrete correspondents, because taking the limit does not affect the unit.

Let us now assume that quantity x measures the amount of money on a bank account, on which the bank pays interest. Because the velocity of the deposited capital depends on the amount of the capital, it is not a good measure for the strength of change of the capital. This problem can be avoided by using measures such as relative change or growth rate of the capital. For example, with 4 % annual interest rate the annual interest earnings for capitals \$ 100 and \$ 1000 are 4 (\$/y) and 40 (\$/y), respectively, but the relative changes and growth rates of the two capitals are pairwise equal: 0.04 and 0.04 ($1/y$). The growth rates of any quantities measured from time intervals of length y all have unit $1/y$, and are thus comparable. For example, the growth rate of GDP, the growth rate of the average price level (inflation) etc. are comparable quantities with interest rate.

3 Interest Calculation in Discrete Time

Suppose that at time moment t_0 the amount $x(t_0)$ (\$) is deposited on a bank account with fixed discrete time interest rate i. Interest rate i is the following growth rate of the deposited capital with dimension $1/\Delta t$,

$$\frac{[x(t_0+\Delta t)-x(t_0)]/\Delta t}{x(t_0)} = i \quad \Leftrightarrow \quad x(t_0+\Delta t) = (1+i\Delta t)x(t_0).$$

The average velocity of i can then be expressed as

$$\frac{\Delta i}{\Delta t} = \frac{1}{\Delta t}\left(\frac{x(t_2)-x(t_1)}{x(t_1)\Delta t} - \frac{x(t_1)-x(t_0)}{x(t_0)\Delta t}\right) = \frac{1}{(\Delta t)^2}\left(\frac{x(t_2)}{x(t_1)} - \frac{x(t_1)}{x(t_0)}\right),$$

and the dimension of $\Delta i/\Delta t$ is $1/(\Delta t)^2$.

With compound interest calculation, the capital on the bank account after n time periods is

$$x(t_0+n\Delta t) = (1+i\Delta t)^n x(t_0) \quad \Leftrightarrow \quad x(t_0) = (1+i\Delta t)^{-n} x(t_0+n\Delta t), \tag{5}$$

where $(1+i\Delta t)^n$ and $(1+i\Delta t)^{-n}$ are the interest and discount factors between time moments t_0 and $t_0+n\Delta t = t_n$. The interest and discount factors are dimensionless quantities because the units of i and Δt cancel each other. If the interest rates of the time intervals differ, that is $i_1 \neq i_2 \neq i_3 \neq \cdots$ where the subindex refers to the time interval, the interest and discount factors are

$$[(1+i_1\Delta t)\cdots(1+i_n\Delta t)] \;\text{ and }\; [(1+i_1\Delta t)\cdots(1+i_n\Delta t)]^{-1}.$$

Monetary quantities at all time moments belong in the dimension of *value*, and are so additive. However, positive interest rate makes the value of a future dollar smaller than that of the current dollar. In discrete time, this comparability problem between monetary quantities at different time moments is solved by the interest and discount factors given above, similarly as exchange rate solves the comparability problem between two currencies.

4 Interest Calculation in Continuous Time

The continuous time interest rate r is defined analogously as that of discrete time as the instantaneous growth rate of a monetary quantity at moment t, $r(t) = x'(t)/x(t)$. This first order differential equation can be solved as

$$x'(t) = r(t)x(t) \quad \Leftrightarrow \quad x(t) = x(t_0)e^{\int_{t_0}^{t} r(s)ds}, \tag{6}$$

where by s is denoted running time during the interval (t_0, t) and e is the base of the natural logarithm. Eq. (6) expresses the money on a bank account at moment t when $x(t_0)$ was deposited at moment t_0 with interest rate r, which may vary during (t_0, t). In continuous time, interest earnings are added to the capital after every instant of time, which creates the exponential growth. The instantaneous velocity of $r(t)$ is

$$r'(t) = \frac{x''(t)}{x(t)} - \left(\frac{x'(t)}{x(t)}\right)^2 = \frac{x''(t)}{x(t)} - r^2(t).$$

Now $r'(t)$ has dimension $1/(\Delta t)^2$, and the formula shows that the acceleration of the capital $x''(t)$ with dimension $\$/(\Delta t)^2$ (the instantaneous velocity of $x'(t)$) positively, and the interest rate itself negatively, affects $r'(t)$. These results hold for all quantities expressed in growth rates which explains their stationary in time: $\partial r'(t)/\partial r(t) = -2r(t) < 0$. Constant interest rate $r'(t) = 0$ corresponds to $r(t) = +\sqrt{x''(t)/x(t)}$ where $x''(t) = r^2 x(t) > 0$.

Using the definition $r(s) = x'(s)/x(s)$, we get

$$\int_{t_0}^{t} r(s)ds = \int_{t_0}^{t} \frac{x'(s)}{x(s)}ds = \Big|_{t_0}^{t} ln(x(s)) = ln\left(\frac{x(t)}{x(t_0)}\right).$$

We can thus express (6) as

$$x(t) = x(t_0)e^{ln\left(\frac{x(t)}{x(t_0)}\right)},$$

which is trivially true because the inverse operations — *exp* and *ln* — cancel each other. If, however, r is constant, we get

$$\int_{t_0}^{t} rdt = r \times (t - t_0),$$

and the numerical value of $\int_{t_0}^{t} rdt$ equals with that of r $(1/\Delta t)$ when $t - t_0 = \Delta t$. The present value of $x(t)$ at moment t_0 is obtained from (6) as

$$x(t_0) = x(t)e^{-\int_{t_0}^{t} r(s)ds}. \tag{7}$$

If $r(s)$ is constant during the interval (t_0, t), (6) and (7) become

$$x(t) = x(t_0)e^{r\times(t-t_0)} \quad \Leftrightarrow \quad x(t_0) = x(t)e^{-r\times(t-t_0)}.$$

When r and $t - t_0$ are expressed in a common unit of time, the continuous time interest and discount factors, $exp(\int_{t_0}^{t_n} r(s)ds)$ and $exp(-\int_{t_0}^{t_n} r(s)ds)$, are dimensionless quantities as they should be.

One difference between the two interest calculations exists, however. In continuous time the length of the time interval $t-t_0$ is measured in time units, but the exponent n in discrete time is a pure number which represents the order of the corresponding time interval. If time is split into years in discrete time analysis, then $n = 3$ implies that the interest factor corresponds to the third year from the present moment etc.

5 Parities between Various Interest Rates

5.1 Discrete and Continuous Time Interest Rate

According to the previous sections, with compound interest calculation the discrete and continuous time calculated values of capitals at time moment t_n, deposited on a bank account at moment t_0, are

$$x(t_n) = x(t_0)(1 + i\Delta t)^n \quad \text{and} \quad x(t_n) = x(t_0)e^{\int_{t_0}^{t_n} r(t)dt}.$$

Solving $x(t_n)/x(t_0)$ from both equations and setting them equal, yields

$$(1 + i\Delta t)^n = e^{\int_{t_0}^{t_n} r(t)dt}. \tag{8}$$

Both methods thus give identical results if the two interest (and discount) factors are equal. Suppose then that the two interest rates with dimension $1/\Delta t$ are constant. The amount of money on the bank account after time interval Δt, calculated by both methods, is then

$$x(t_1) = x(t_0)(1 + i\Delta t) \quad \text{and} \quad x(t_1) = x(t_0)e^{r\times(t_1-t_0)}.$$

Setting these equal and dividing by $x(t_0)$, yields

$$ln(1 + i\Delta t) = r \times (t_1 - t_0) \ \text{ or } \ i\Delta t = e^{r\times(t_1-t_0)} - 1. \tag{9}$$

Using the transformation (9), the continuous and discrete time compound interest calculation give identical results (notice that $t_1 - t_0 = \Delta t$ and $r\Delta t$ is a dimensionless quantity with the numerical value of r). The following definition for the continuous time interest rate, conformal with the corresponding discrete time interest rate, can thus be made for time unit $t_1 - t_0$,

$$r_c = \left(\frac{1}{t_1 - t_0}\right) ln\left(\frac{x(t_1)}{x(t_0)}\right) = \frac{ln(1 + i\Delta t)}{t_1 - t_0}. \tag{10}$$

Notice that in (10), $x(t_1)$ is calculated by using the corresponding discrete time interest rate and the discrete time method of calculation.

Example 1. We calculate the capital on a bank account by discrete and continuous time methods when 100 (\$) is deposited at moment $t_0 = 0$, $t_n - t_0 = 20$ (y) and $i = r = 0.1$ $(1/y)$. The results are graphed in Figure 1. The time paths imply that the continuous time (the curve) somewhat overestimates the discrete time capital (the dots). However, using $r_c = ln(1+i\Delta t)/\Delta t = 0.0953$ $(1/y)$ in continuous time the two time paths coincide, see Figure 2. Hull (2000 p. 52) shows that even without the adjustment $r = r_c$, the continuous time interest calculation gives almost identical results as the daily discrete time analysis with 'normal' levels of interest rates.

5.2 Discrete Time Interest Rates of Varying Length

Another parity can be made between discrete time interest rates defined for various time units. Let us denote the discrete time interest rate for time unit h by i_h, and that for time unit p by i_p, where $s = h/p$ is a positive natural number. Interest returns are assumed to be added to the deposited capital at the ending moment of every time interval. We assume $s > 1$, which implies $p < h$, and so i_p represents a more dense splitting of time. The deposited capital $x(t_0)$ (\$) increases in both these regimes during time unit h as

$$x(t_h) = x(t_0)(1 + i_h \times (1\ h)) \quad \text{and} \quad x(t_{sp}) = x(t_0)(1 + i_p \times (1\ p))^s.$$

Setting these two capitals equal and dividing by $x(t_0)$, yields

$$i_{c,h} \times (1\ h) = (1 + i_p \times (1\ p))^s - 1 \quad \Leftrightarrow \quad i_{c,p} \times (1\ p) = (1 + i_h \times (1\ h))^{1/s} - 1. \tag{11}$$

These transformations make any two discrete time interest rates comparable with each other, because the relation $h = sp$ can be defined for every two time units h, p. Eq. (11) thus defines two discrete time interest rates conformal with each other in compound interest calculation, as the subscripts imply.

Example 2. Suppose $i_y = 0.1$ $(1/y)$, $i_m = 0.01$ $(1/m)$ and 1 $(y) = 12$ (m). Using rules (11) the corresponding conformal monthly and annual interest rates become $i_{c,m} = 0.00797$ $(1/m)$ and $i_{c,y} = 0.1268$ $(1/y)$, respectively. These somewhat differ from the approximate ones we get by transforming with the time units, $\widehat{i}_m = 1/12$ $(y/m) \times 0.1$ $(1/y) = 0.0083$ $(1/m)$ and $\widehat{i}_y = 12$ $(m/y) \times 0.01$ $(1/m) = 0.12$ $(1/y)$.

Example 3. Figure 3 shows the differences in the capitals during 20 years when 100 (\$) is deposited at moment t_0 and interest rates $i_y = 0.1$ $(1/y)$ and $\widehat{i}_m = 0.0083$ $(1/m)$ are used (the curve refers to monthly and the dots to annual analysis). These differences are shown to disappear in Figure 4 by using the corresponding conformal monthly interest rate, $i_{c,m} = 0.00797$ $(1/m)$.

6 Present Values of Money Flows

Suppose a money flow $N(t_0 + j\Delta t)$ $(\$/\Delta t)$, $j = 1, 2, \ldots$ and fixed interest rate i $(1/\Delta t)$. In discrete time, the present value of an n period flow is

$$H_n = \frac{N(t_0+\Delta t)\Delta t}{1+i\Delta t} + \cdots + \frac{N(t_0+n\Delta t)\Delta t}{(1+i\Delta t)^n} = \sum_{j=1}^{n} \frac{N(t_0+j\Delta t)\Delta t}{(1+i\Delta t)^j}, \quad (12)$$

where the measurement unit of $N(t_0 + j\Delta t)\Delta t$ is \$ $\forall j$.

Next we analyze the present value of the above flow in continuous time. Discrete time is transformed to continuous by dividing Δt into k equal subintervals and letting $k \to \infty$. At the time interval $t_0 + j\Delta t/k$, the interest rate $i(t_0 + j\Delta t/k)$ has dimension $1/(\Delta t/k)$ and the corresponding money flow is $N(t_0 + j\Delta t/k)$ with dimension $\$/(\Delta t/k)$, $j = 1, 2, \ldots, nk$. With fixed k, the present value of the above flow during nk periods is

$$H_{nk} = \sum_{j=1}^{nk} N\left(t_0 + j\frac{\Delta t}{k}\right)\frac{\Delta t}{k}\left(1 + i\left(t_0 + j\frac{\Delta t}{k}\right)\frac{\Delta t}{k}\right)^{-j}.$$

The discount factor in the above formula can be modified as

$$\left(1 + i\left(t_0 + j\frac{\Delta t}{k}\right)\frac{\Delta t}{k}\right)^{-j} = \left(\left(1 + \frac{1}{\frac{k/\Delta t}{i\left(t_0+j\frac{\Delta t}{k}\right)}}\right)^{\frac{k/\Delta t}{i\left(t_0+j\frac{\Delta t}{k}\right)}}\right)^{\frac{-ji\left(t_0+j\frac{\Delta t}{k}\right)}{k/\Delta t}},$$

see Chiang (1974 p. 290). Now we know that

$$\lim_{z\to\infty}\left(1+\frac{1}{z}\right)^z = e,$$

and because $\lim_{k\to\infty} k/(i(t_0 + j\Delta t/k)\Delta t) = \infty$, we can simplify the above formula by the definition of the number e. The limiting process transforms the exponent of e to $-r(t) \times (t - t_0)$. This occurs because with $k \to \infty$, $j\Delta t/k \to jdt$ and $i(t_0 + j\Delta t/k)j\Delta t/k \to r(t_0 + jdt)jdt$. The definition $t = t_0 + jdt$, $j = 1, 2, \ldots$ for continuous time completes the proof.

The limiting process $k \to \infty$ transforms the sum in (12) to an integral with integration limits t_0 and $t_0+nk\Delta t/k = t_0+n\Delta t = t_n$, and $N(t_0+j\Delta t/k)$ approaches the instantaneous flow $N(t)$ at moment t. We can thus write

$$\lim_{k\to\infty} H_{nk} = \int_{t_0}^{t_n} N(t)e^{-r(t)\times(t-t_0)}dt, \quad t_0 \le t \le t_n.$$

In order to compare the discrete and continuous time present values, we assume that the flow is fixed, that is, $N(t_0+j\Delta t) = N \ \forall j$. We can then take the factor $N\Delta t$ in front of the sum in (12) and study the obtained geometric series with positive terms a^j,

$$a^j = \left(\frac{1}{1+i\Delta t}\right)^j, \quad j = 1, 2, \ldots, n,$$

where $0 < a < 1$ because $i > 0$. The sum of the series with n terms is

$$S_n = \sum_{j=1}^{n} a^j = \frac{a(1-a^n)}{1-a} = \frac{1-(1+i\Delta t)^{-n}}{i\Delta t}.$$

The present value of the flow (12) with fixed N is then

$$H_n = \frac{N\Delta t[1-(1+i\Delta t)^{-n}]}{i\Delta t} = \frac{N}{i}[1-(1+i\Delta t)^{-n}].$$

If r is constant, in continuous time the present value of the fixed flow is

$$H(t_n) = \int_{t_0}^{t_n} Ne^{-r\times(t-t_0)}dt = \Big|_{t_0}^{t_n} -\frac{N}{r}e^{-r\times(t-t_0)} = \frac{N}{r}[1-e^{-r\times(t_n-t_0)}],$$

where r, t_0, t_n are bounded positive quantities. Because $t_n - t_0 = n\Delta t$, letting $n \to \infty$ we get $H_n \to N/i$ and $H(t_n) \to N/r$. Both methods thus give identical present values for the same infinite fixed flow if $i = r$. Notice that the measurement units of the present values are $(\$/\Delta t)/(1/\Delta t) = \$$.

The condition for equal present values of the corresponding finite flows is

$$\frac{1}{i}[1-(1+i\Delta t)^{-n}] = \frac{1}{r}[1-e^{-rn\Delta t}]. \tag{14}$$

An analytic solution for this equation with respect to i or r does not exist, however, but the equation can be solved numerically keeping n and the other interest rate fixed. It turns out that these solutions essentially depend on n. In Figure 5 are presented the solutions for $\widehat{r}$ with $i = 0.1$ $(1/y)$ when n goes from 5 to 40 with 5 years interval. An increase in n makes $\widehat{r}$ closer to $i = 0.1$ $(1/y)$, but the convergent is not fast. We can thus conclude that a unique conformal continuous time interest rate, which gives identical present values as a certain discrete time interest rate, does not exist. With known values of n and i, however, the corresponding $\widehat{r}$ can be found numerically.

Assuming $n \to \infty$, we get an asymptotic solution for (14): $r = i$. Further, the transformation rule for conformal interest rates in compound interest calculation, $r\Delta t = ln(1 + i\Delta t)$, does not improve the accuracy in calculating present values, and the simple rule $i = r$ gives more accurate results, see Figures 6,7. The reason for this is that the studied money flow is constant and not increasing as deposited money does in compound interest calculation. This asymptotic accuracy does not, however, help in calculating present values of money flows for relatively short time intervals.

Next we calculate the present value of a fixed money flow with varying time units in discrete time. Let the two money flows be N $(\$/h) = N$ $(\$/sp) = \frac{N}{s}$ $(\$/p) = M$ $(\$/p)$, where $h = sp$ and the corresponding interest rates are i_h and i_p, respectively. The present values of the flows are

$$\begin{aligned}
H_n &= \sum_{j=1}^{n} \frac{N \times (1\ h)}{[1 + i_h \times (1\ h)]^j} = \frac{N}{i_h}[1 - (1 + i_h \times (1\ h))^{-n}], \\
H_{ns} &= \sum_{j=1}^{ns} \frac{M \times (1\ p)}{[1 + i_p \times (1\ p)]^j} = \frac{M}{i_p}[1 - (1 + i_p \times (1\ p))^{-ns}].
\end{aligned}$$

Setting $H_n = H_{ns}$ and letting $n \to \infty$, we get the asymptotic solution $i_h = si_p$ for this equation because $N = sM$. An increase in n thus increases the accuracy of the simple transformation of the interest rates by time units. As in the previous case, applying the conformal discrete time interest rates defined in (11) does not improve the accuracy, see Figures 8,9.

Example 4. Suppose a four month monetary flow 40 $(\$/4m)$ in continuous time with $r = 0.1$ $(1/y)$. With the simple transformation by time units, the corresponding four and one month interest rates are $4/120$ $(1/4m)$ and $1/120$ $(1/m)$, respectively. Time is first measured in the units of $4m$ and then in the units of m. The present value of the flow is in the first case

$$\int_{t=t_0}^{t=t_0+1} 40e^{-r\times(t-t_0)}dt = -\frac{40}{r}\left(e^{-r} - e^0\right) = 1200\left(1 - e^{-\frac{4}{120}}\right) = 39.34\ (\$).$$

Notice that the marginal change in time, dt, is measured above in the units of $4m$. The complete form of the integrated factor is then 40 ($\$/4m$) $\times$ dt ($4m$) $e^{-r\times(t-t_0)} = 40\ dt\ e^{-r\times(t-t_0)}$ ($\$$), because $e^{-r\times(t-t_0)}$ is dimensionless.

Next time is measured in months. Flow 40 ($\$/4m$) corresponds to 10 ($\$/m$) and the present value is

$$\int_{t=t_0}^{t=t_0+4} 10e^{-r\times(t-t_0)}dt = -\frac{10}{r}\left(e^{-4r} - e^0\right) = 1200\left(1 - e^{-\frac{4}{120}}\right) = 39.34\ (\$),$$

where dt is measured in the units of m. Independent of the time unit applied, we get the same present value for the same flow in continuous time discounting. Essential is, that the limits of integration, the interest rate and the integrated flow are measured in the same time units.

Example 5. The present value of the money flow in Example 4 is analyzed in discrete time. Thus either 40 ($\$$) is received after four months (the flow is then 40 ($\$/4m$) and the money is received at the ending moment of the period), or 10 ($\$$) is received after every month in a time of four months. The interest rate is 0.1 ($1/y$) and we apply the simple transformation rule with time units. The present value of the flow with $i_{4m} = 4/120$ is

$$\frac{40\ (\$/\Delta t) \times (\Delta t)}{1 + i_{4m}\Delta t} = \frac{40\ (\$/4m) \times (4m)}{1 + \frac{4}{120}\left(\frac{1}{4m}\right) \times (4m)} = 38.71\ (\$).$$

Next we calculate the present value of the monthly flow:

$$\sum_{j=1}^{4} \frac{10\ (\$/\Delta t)\Delta t}{(1 + i_m\Delta t)^j} = \frac{10\ (\$/m) \times (1\ m)}{1 + \frac{1}{120}\left(\frac{1}{m}\right) \times (1\ m)} + \frac{10\ (\$/m) \times (1\ m)}{\left(1 + \frac{1}{120}\left(\frac{1}{m}\right) \times (1\ m)\right)^2}$$
$$+ \frac{10\ (\$/m) \times (1\ m)}{\left(1 + \frac{1}{120}\left(\frac{1}{m}\right) \times (1\ m)\right)^3} + \frac{10\ (\$/m) \times (1\ m)}{\left(1 + \frac{1}{120}\left(\frac{1}{m}\right) \times (1\ m)\right)^4} = 39.18\ (\$).$$

Thus the more frequently the interest earnings are added to the capital in discrete time, the greater present value we get for a fixed positive flow. The more dense the splitting of time in the discrete time analysis, the more accurate approximate of it the continuous time calculated value is.

7 Analysis of Debt and Deposit Capitals

In order to simplify the solutions of the difference equations to be defined, we assume $t_0 = 0$ and that in discrete time either $\Delta t = 1$ (y) or $\Delta t = 1$ (m). The time intervals $\ldots, t_0+(n-1)\Delta t, t_0+n\Delta t, t_0+(n+1)\Delta t, \ldots$ then become $\ldots, n-1, n, n+1, \ldots$, which simplifies the notation. We denote by $V(t_n)$

and V_n the debt and by $S(t_n)$ and S_n the deposit capital of a household in continuous and discrete time at moment t_n; these all have unit \$. The average and instantaneous velocities of the corresponding capitals are

$$\frac{\Delta V_n}{\Delta t}, \quad V'(t_n), \quad \frac{\Delta S_n}{\Delta t}, \quad S'(t_n);$$

these all have unit \$/y or \$/m. The interest costs (returns) of the debt (deposit) capitals during time unit y or m with unit \$/y or \$/m, are

$$iV_n, \quad rV(t_n), \quad iS_n, \quad rS(t_n).$$

Let us next assume that the household repays his loan (accumulates his savings) by a fixed amount of money B_y (\$/y) or B_m (\$/m) at every time unit Δt in all cases. For simplicity, the continuous and discrete time interest rates are assumed constant. Then we have four equations to describe the velocities of the corresponding capitals:

$$\begin{aligned} V'(t_n) &= rV(t_n) - B_b \quad \text{or} \quad \frac{\Delta V_n}{\Delta t} = iV_n - B_b, \\ S'(t_n) &= rS(t_n) + B_b \quad \text{or} \quad \frac{\Delta S_n}{\Delta t} = iS_n + B_b, \quad b = y, m, \end{aligned}$$

where both sides of all equations either have unit \$/y or \$/m. Because the equations are analogous, for shortness we study here only the debt equations.

The discrete time dept equation is modified by writing $\Delta V_n = V_{n+1} - V_n$. Multiplying the equation by Δt, yields

$$V_{n+1} = (1 + i\Delta t)V_n - B_b\Delta t. \tag{17}$$

Notice that the definition of $\Delta V_n = V_n - V_{n-1}$ would give

$$(1 - i\Delta t)\overline{V}_n = \overline{V}_{n-1} - B_b\Delta t \quad \Leftrightarrow \quad \overline{V}_n = \overline{V}_{n-1} + i\Delta t\overline{V}_n - B_b\Delta t, \tag{18}$$

where the bar above V separates the two equations. Now, (17) is the correct form for the difference equation, because there on the left hand side is the debt capital at moment $n+1$, and the right hand side consists of the capital at moment n added by the interest costs $i\Delta tV_n$ and subtracted by the repayment $B_b\Delta t$ at period n. Both sides thus equal the capital at moment $n+1$. In Eq. (18), the interest costs are calculated using the future, and not the current value of the dept capital, which overestimates the speed of repayment.

Because $\Delta t, i$ and B_b are constants, (17) and (18) are linear first order difference equations with constant coefficients. The solutions of the differential and difference equations are

$$V(t_n) = \left(V(0) - \frac{B_b}{r}\right) e^{rt_n} + \frac{B_b}{r}, \tag{19}$$

$$V_n = \left(V_0 - \frac{B_b}{i}\right) (1 + i\Delta t)^n + \frac{B_b}{i}, \tag{20}$$

$$\overline{V}_{\overline{n}} = \left(\overline{V}_0 - \frac{B_b}{i}\right) \left(\frac{1}{1 - i\Delta t}\right)^{\overline{n}} + \frac{B_b}{i}, \quad b = y, m, \tag{21}$$

where $V(0)$, V_0 and $\overline{V}_0$ are the initial conditions and the order of the time interval in the last equation is denoted by $\overline{n}$. The reader can check that these solutions are dimensionally well-defined. If the dept capital is required to be paid off at a certain future time moment N, the initial conditions must be solved by setting $V(N) = V_N = \overline{V}_N = 0$.

Setting $V(t_n) = V_n$, we get a condition for identical results for the time path of the dept capital in continuous and discrete time analysis,

$$\left(V(0) - \frac{B_b}{r}\right) e^{rt_n} + \frac{B_b}{r} = \left(V_0 - \frac{B_b}{i}\right) (1 + i\Delta t)^n + \frac{B_b}{i}, \quad b = y, m. \tag{22}$$

An analytic solution for r in terms of i does not exist for Eq. (22). However, setting the numerical values of t_n and n equal gives numerical solutions for (22), which show an essential dependence on n. The condition for equal time paths for the dept capital in discrete time with time units $1\ (y) = 12\ (m)$ is,

$$\left(V_0 - \frac{B_y}{i_y}\right) (1 + i_y\ (1y))^n + \frac{B_y}{i_y} = \left(V_0 - \frac{B_m}{i_m}\right) (1 + i_m\ (1m))^{12n} + \frac{B_m}{i_m}. \tag{23}$$

Again, no analytic solution for i_y in terms of i_m (or vice versa) exists for Eq. (23). However, numerical solutions can be found which essentially depend on n. We can thus conclude that for the dynamics of dept and deposit capitals, exact transformation rules between continuous and discrete time interest rates, and between discrete time interest rates corresponding to different time units, cannot be derived. The corresponding conformal interest rates must be solved from Eq. (22) and (23) with known values of n.

Finally, we analyze the accuracy of (19) concerning the real world discrete time calculation (20). Figures 10, 11 display the time paths of the debt capitals (dots refer to Eq. (20)) during 20 years (240 months) with annual and monthly data: $V(0) = V_0 = \overline{V}_0 = 5000\ (\$)$, $r = i = 0.1\ (1/y) = 1/120\ (1/m)$ and $B_y = 600\ (\$/y) = 600/12\ (\$/m) = B_m$. On annual basis, the difference of roughly one year exists between the two methods concerning the time moment the debt capital is zero. On monthly basis, the difference is less than a month. With discrete time using $i_m = 1/120$ $(1/m)$ and $i_y = 0.1$ $(1/y)$, the time paths of the dept capitals are displayed in Figure 12 (dots refer to annual analysis). The simple rules $i = r$ and $i_y = 12\ i_m$ turn out to produce more accurate results for the dept capitals than using rules (9) and (11).

Example 6. We study numerically the accuracy of (19) concerning (20) with $V(0) = V_0 = 5000$ (\$), $r = i = 1/120$ $(1/m)$, $B_m = 400$ $(\$/m)$, $t_n = 7$ (m) and $n = 7$. Then $V(7\ (m)) = 2417.1$ (\$) and $V_7 = 2428.1$ (\$). Setting $V(t_n) = V_n = 0$ and solving the time moment (the order of the time interval) this occurs, we get $t_n = 13.20$ (m) and $n = 13.26$. Eq. (19) thus produces results accurate enough for the monthly discrete time analysis.

8 Summary

We studied the discrete and continuous time interest calculation with the aim to find an exact way to treat time in the analysis. We also searched for transformation rules between various interest rate concepts which would give identical results by different calculation methods. Measurable concepts corresponding to the instantaneous velocities of dept and deposit capitals were defined and applied in the modelling of the time paths of these capitals. Clear transformation rules for compound interest calculation were found, which though is not a new result. However, it was shown that for discounting monetary flows and calculating the time paths of dept and deposit capitals, such rules cannot be found. Transforming the interest rates by using the time units gives more accurate results in these cases than applying the conformal interest rates defined for compound interest calculation.

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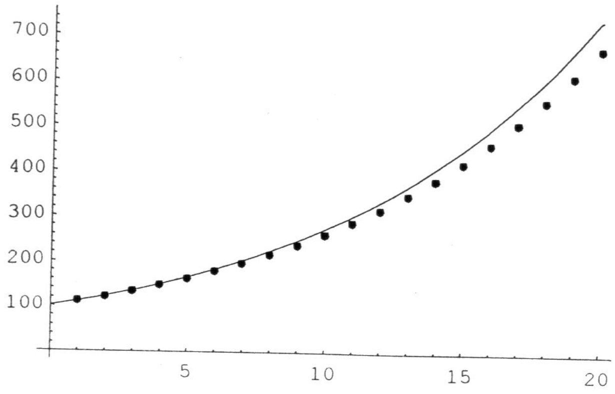

Figure 1. The capital $x(0) = 100(\$)$ in discrete and continuous time: $r = i = 0.1(1/y)$, $\Delta t = 1(y)$.

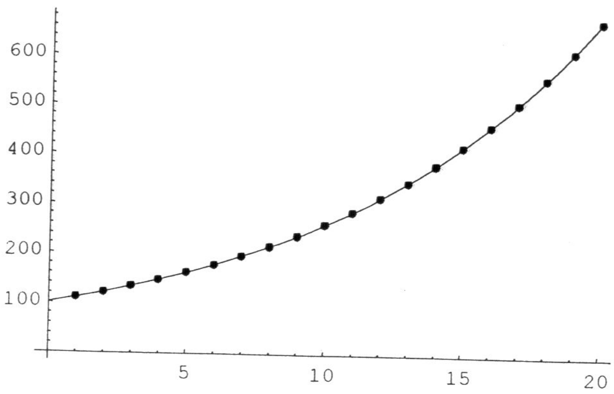

Figure 2. The capital $x(0) = 100(\$)$ in discrete and continuous time: $i \times (1y) = 0.1$, $r\Delta t = ln(1 + i\Delta t)$.

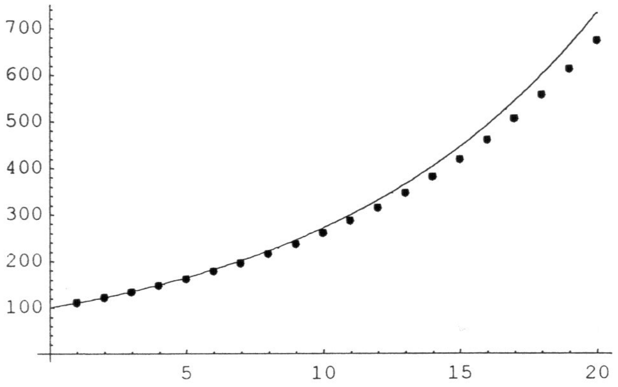

Figure 3. The capital $x(0) = 100(\$)$ with discrete time interest rates: $i_y \times (1y) = 0.1$, $i_m \times (1m) = 1/120$.

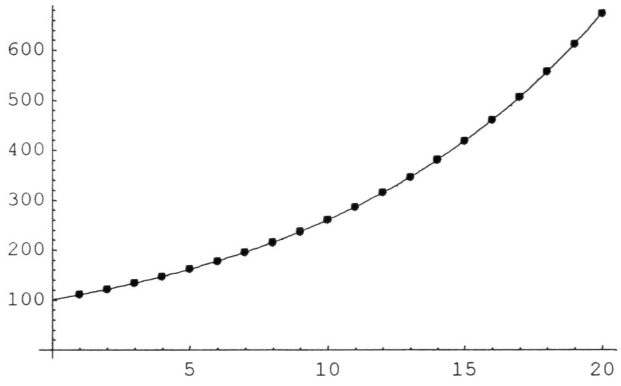

Figure 4. The capital $x(0) = 100(\$)$ in discrete time: $i_y \times (1y) = 0.1$, $i_m \times (1m) = (1 + i_y(1y))^{1/12} - 1$.

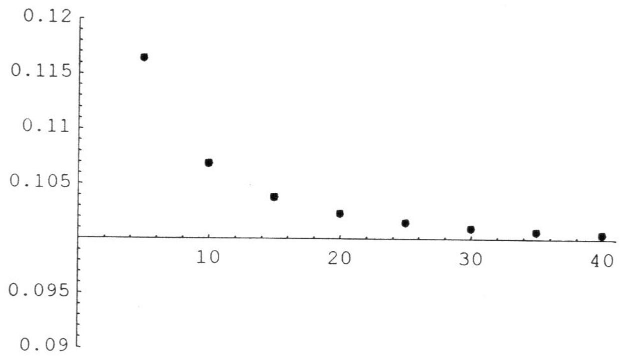

Figure 5. The continuous time conformal interest rates for $i_y \times (1y) = 0.1$ with $n = 5, 40$.

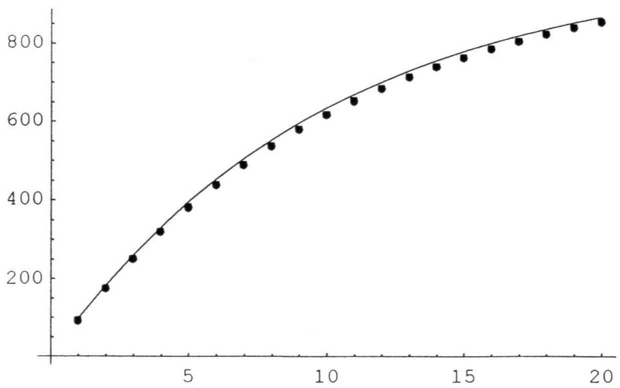

Figure 6. The present values for flow 100 ($\$/y$) in discrete and continuous time: $r = i = 0.1(1/y)$, $\Delta t = 1(y)$.

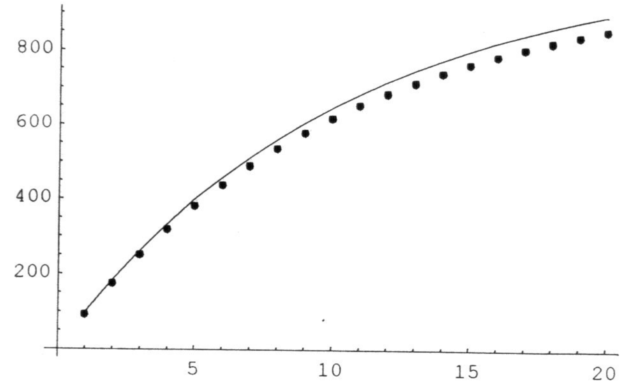

Figure 7. The present values for flow 100 ($\$/y$) in discrete and continuous time: $i = 0.1(1/y)$, $r\Delta t = ln(1 + i\Delta t)$.

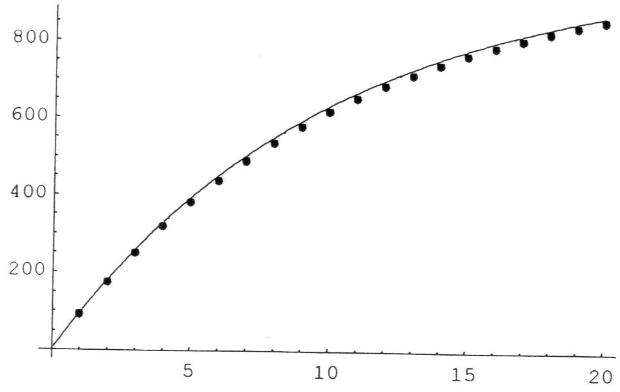

Figure 8. The present values for flow 100 ($\$/y$) in discrete time: $i_y \times (1y) = 0.1$, $i_m \times (1m) = 1/120$.

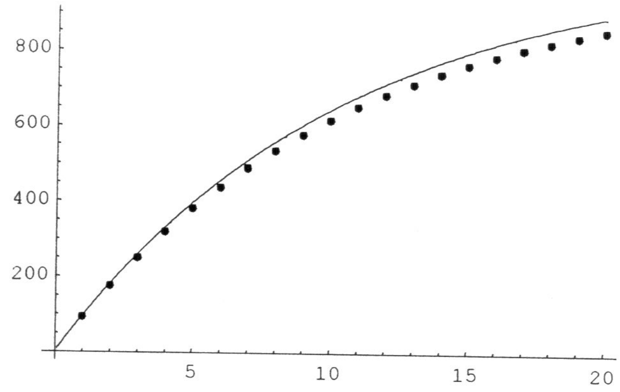

Figure 9. The present values for flow 100 ($\$/y$) in discrete time: $i_y = 0.1(1/y)$, $i_m = (1 + i_y(1y))^{1/12} - 1$.

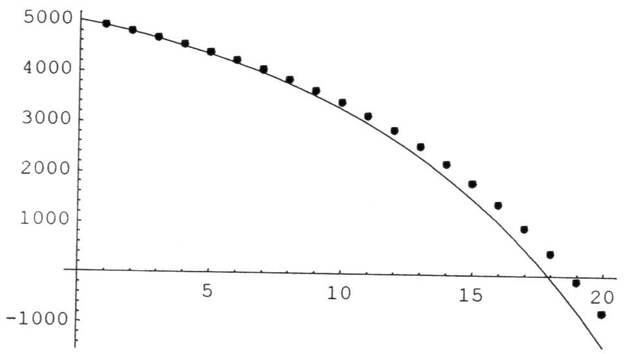

Figure 10. The debt capital in discrete and continuous time: $r = i = 0.1(1/y)$, $V(t_0) = V_0 = 5000$ ($\$$), $B = 600$ ($\$/y$).

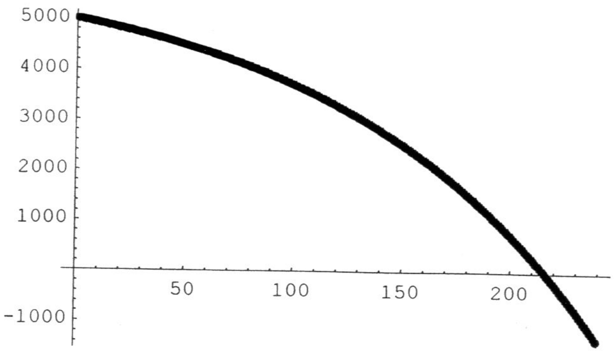

Figure 11. The debt capital in discrete and continuous time: $r = i = 1/120(1/m)$, $V(t_0) = V_0 = 5000(\$)$, $B = 50(\$/m)$.

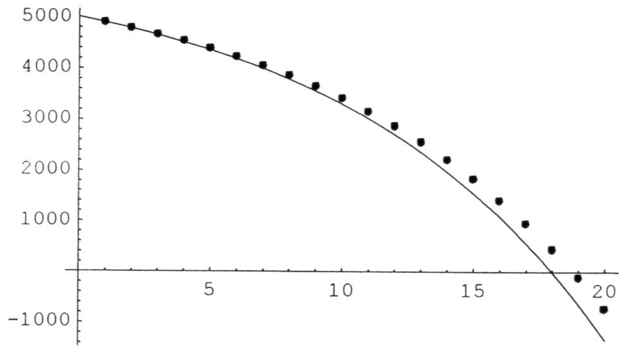

Figure 12. The debt capital in discrete time: $i_y \times (1y) = 0.1$, $i_m \times (1m) = 1/120$.