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ON THE ADJUSTMENT, STABILITY, AND GROWTH IN PERFECT COMPETITION

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Abstract. We present a model of the behavior of an industry in perfect competition. A new feature in the modeling is the consumers' role in the evolution of the industry. We show a link in between market behavior and economic growth at industry level. Growth may occur due to increases in technology or consumers' wealth, due to a positive change in consumers' preferences concerning this good or increasing returns to scale. The model is solved in a linear case; sufficient conditions for asymptotic stability in an autonomous nonlinear case are also given. In the autonomous linear case, the adjustment is exponentially damped where overshooting may occur. Samuelson's (1941) model of infinitely quickly adjusting consumers and producers is shown to be a limit case of our model in the nonlinear autonomous case. (JEL C62, D41)
Keywords: Perfect competition, adjustment, growth, stability.

1. Introduction

According to [14], neoclassical thinking is based on two distinct elements: egoistic economic agents by Smith (utility maximizing consumers by Jevons, Menger and Walras) and the mathematical metaphor of classical mechanics. The latter can be understood by the progenitors of neoclassical economics who were engineer level physicists. Concept equilibrium was borrowed from physics and introduced in economics by Canard at 1801 [15]. Although equilibrium is 'a balance of forces' situation, in economics the balancing 'forces' have not been defined. In order to efficiently exploit the concept of equilibrium, however, we should be able to know whether the equilibrium is stable or unstable, and which are the forces 'pushing' an economy toward the stable equilibrium.

The existence of forces acting upon economic quantities can be argued indirectly; every changing quantity (price, wage, exchange rate etc.) tells the existence of reasons (forces) causing these changes. This is analogous with arguing the existence of the gravitational force field by dropping a pen; without the force field the pen would not move. Fisher writes, [7, pp. 9–12]: "... I now briefly consider the features that a proper theory of disequilibrium adjustment should have ... if we

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are to show under what conditions the rational behavior of individual agents drives an economy to equilibrium. ... Such a theory must involve dynamics with adjustment to disequilibrium over time modeled. ...the most satisfactory situation would be one in which the equations of motion of the system permitted an explicit solution with the values of all the variables given as specific, known functions of time. ... Unfortunately, such a closed-form solution is far too much to hope for. ...the theory of the household and the firm must be reformulated and extended where necessary to allow agents to perceive that the economy is not in equilibrium and to act on that perception. ... A convergence theory that is to provide a satisfactory underpinning for equilibrium analysis must be a theory in which the adjustments to disequilibrium made by agents are made optimally."

According to [2, p. 12], the presently accepted definition for market stability is that of [17]. Samuelson writes: "In the history of mechanics, the theory of statics was developed before the dynamical problem was even formulated. But the problem of stability of equilibrium cannot be discussed except with reference to dynamical considerations ... we must first develop a theory of dynamics." Samuelson insists that the stability of a market equilibrium must be based on the dynamic adjustment of prices when the system is out of equilibrium. A generally accepted cause (force) behind price changes is the deviation between demand and supply. [13, p. 620] writes: "A characteristic feature that distinguishes economics from other scientific fields is that, for us, the equations of equilibrium constitute the center of our discipline. Other sciences, such as physics or even ecology, put comparatively more emphasis on the determination of dynamic laws of change. The reason, informally speaking, is that economists are good (or so we hope) at recognizing a state of equilibrium but are poor at predicting precisely how an economy in disequilibrium will evolve. Certainly there are intuitive dynamic principles: if demand is larger than supply then price will increase, if price is larger than marginal costs then production will expand..."

We base our modeling on the above principles. We define the forces acting upon the production, consumption and unit price of a homogeneous good in a perfectly competed industry, and apply these forces to model economic dynamics in real time. The possible asymptotic equilibrium of the industry is the neoclassical one: demand equals supply, price equals the average of firms' marginal costs and consumers' willingness to pay for one unit, and the equilibrium velocities of production (consumption) of firms (consumers) maximize their profit (utility). To define the 'economic forces', we assume that economic agents are not in their optimal situations, and they like to better their situation as Fisher required above. We believe that the *'economic agents' desire to better their situation'* is the cause for the dynamics in economies.

Because price affects the velocities of production and consumption, which both affect price, we cannot model the adjustment of any of these quantities separately but have to analyze them simultaneously. Firms and consumers take the price fixed still knowing that price adjusts at the market according to the aggregate excess demand. The information of the market firms and consumers have in their decision-making is the market price. The adjustment takes place in real time, and there exists inertial factors resisting changes in the adjusting quantities. [20] find evidence of price inertia in concentrated industries; the inertia of economic quantities is thus measurable.

This paper deviates from the existing models of adjustment in perfect competition in four ways. Firstly, the measurement units of the quantities involved are explicitly defined[1]. By this way and by seeking an analogy with Newtonian mechanics, we are guided to model production and consumption velocities (the produced and consumed amounts in a given time unit) rather than their volumes. Secondly, the existing models (for instance [12]) use a system of two differential equations where price adjusts according to the deviation between demand and supply, and production adjusts according to the deviation between price and marginal costs. We add to this a third equation describing consumer behavior, where consumers adjust their velocities of consumption according to the deviation between their willingness to pay for one unit of a good and its unit price. Thirdly, we not only study the stability of the adjustment, but also possible reasons for economic growth. Fourthly, the adjustment takes place in real and not imaginary time like tâtonnement, which allows us to study the speed of the adjustment.

The paper is organized as follows. In Sections 2 and 3 the dynamic behavior of an individual firm and consumer are defined. The industry level supply and demand of the studied good are defined in Section 4. The dynamic model is introduced in Section 5 and solved in a linear case. In Section 6 the system is assumed a non-autonomous nonlinear smooth one, and sufficient conditions for the asymptotic behavior and local stability are given. In Section 8 the smoothness conditions are replaced by monotonicity ones and sufficient conditions for asymptotic convergence are given. Conclusions summarizes the main results.

2. An Individual Firm

Firm i operates at a perfectly competed industry with n firms producing and m consumers consuming a homogeneous consumption good. We assume n, m fixed, for simplicity. Let $q_i(t)$ denote the velocity of production (production during time unit y) of firm i and $p(t)$ the unit price of the good at moment t. The measurement units for q_i and p

[1]A system of measurement units for economics is given in [10].

are $unit/y$ and $\$/unit$, respectively; y can be one *day*, *week* etc. For brevity, the dimensional constants and quantities are treated as real valued; all equations are still dimensionally well-defined.

The profit of the one-good firm during time unit y at moment t is

$$\Pi_i(t) = p(t)q_i(t) - C_i\big(t, q_i(t)\big), \tag{1}$$

where by $C_i(t, q_i)$ with unit $\$/y$ is denoted the production costs during time unit y at moment t at the velocity of production q_i. Due to technological progress — the firm's R & D activities or its workers' learning — the cost function and the marginal cost function are non-increasing in time. On the other hand, the cost function is increasing in the production velocity. The marginal cost function positively (negatively) depends on the velocity of production when decreasing (increasing) returns to scale prevail in the production. Hence

$$\frac{\partial C_i}{\partial t} \leq 0,\ \frac{\partial C_i}{\partial q_i} > 0,\ \frac{\partial^2 C_i}{\partial t \partial q_i} \leq 0. \tag{2}$$

Profit function (1) presupposes that the sold and unsold goods are of equal value, and price is exogenous for the firm. Firms like to sell their whole production, and they know that the price is determined according to demand and supply at the market. Firms know that they can sell their whole production at a unit price low enough, but unit price under average unit costs creates losses. Assuming that firms know their cost functions, the uncertainty in planning the velocity of production of a firm is focused on the maximum unit price the firm's whole production gets sold.

At the market, the price adjusts toward the maximal level by which the production of the industry gets sold. This occurs because excess demand allows some firms to raise their prices, and excess supply forces firms having unsold goods to decrease their prices. Due to the homogeneity of the good, consumers buy from the lowest price firm. This forces other firms follow a price decrease. The awareness of this negative relation between the industry level sales and unit price restricts the speeding up of production of a single firm even when market price exceeds the firm's marginal costs. This inertia in the firms' adjustment of production is included in our modeling. Other, more inherent factors for the inertial phenomena, are the inevitable delays in adjusting the use of manpower, finding finance for new machinery or production room, time needed for constructions etc.

Strictly taken, we ought to analyze the expected profits of firms because if production takes time, the realized profits depend on the future price. If, however, firms know their cost functions and expect the price to stay fixed, then the analysis is the same in both cases. Introducing price and cost uncertainties are possible future extensions of the present model.

The time derivative of the profit function is

$$\Pi_i'(t) = p'(t)q_i(t) - \frac{\partial C_i}{\partial t} + \Big[p(t) - \frac{\partial C_i}{\partial q_i}\Big] q_i'(t). \tag{3}$$

Because price changes $p'(t)$ are out of the control of the firm, and we do not model reasons for $\partial C_i/\partial t < 0$ — which would essentially complicate our model — but only study their role in the growth of the industry, a profit-seeking firm changes its only policy variable q_i as follows:

$$\begin{aligned}
q_i'(t) > 0 \quad &\text{when} \quad p(t) - \frac{\partial C_i}{\partial q_i}\big(t, q_i(t)\big) > 0,\\
q_i'(t) < 0 \quad &\text{when} \quad p(t) - \frac{\partial C_i}{\partial q_i}\big(t, q_i(t)\big) < 0,\\
q_i'(t) = 0 \quad &\text{when} \quad p(t) - \frac{\partial C_i}{\partial q_i}\big(t, q_i(t)\big) = 0.
\end{aligned}$$

The first two rules above make the third additive term in Eq. (3) positive thus increasing profit with time. The last rule means that the firm does not change $q_i(t)$ when this does not affect its profit[2].

Now, $q_i'(t)$ with unit $unit/y^2$ corresponds to the acceleration on production. Imitating Newtonian mechanics, we call $\partial\Pi_i/\partial q_i$ — the reason for this acceleration — the *'force' acting upon the velocity of production of firm i*. A relation, which fulfills the above adjustment rules, is

$$q_i'(t) = F_i\left(\frac{\partial \Pi_i}{\partial q_i}\right), \quad \frac{\partial \Pi_i}{\partial q_i} = p(t) - \frac{\partial C_i}{\partial q_i}\big(t, q_i(t)\big), \quad t \in R, \tag{4}$$

where $F_i \colon R \to R$ is strictly increasing with $F_i(0) = 0$. Firm i thus adjusts q_i according to the deviation between the (expected) price and marginal costs, which adjustment process [5] named 'myopic'. The first order Taylor approximation of function F_i in the neighborhood of the optimum point $\partial\Pi_i/\partial q_i = 0$ is

$$F_i\left(\frac{\partial \Pi_i}{\partial q_i}\right) = F_i(0) + F_i'(0)\left(\frac{\partial \Pi_i}{\partial q_i} - 0\right) + \epsilon_i \approx F_i'(0)\frac{\partial \Pi_i}{\partial q_i},$$

if the residual term ϵ_i is assumed negligible. With this approximation, we can write Eq. (4) as

$$m_{q_i} q_i'(t) = \frac{\partial \Pi_i}{\partial q_i} \quad \text{where} \quad m_{q_i} = \frac{1}{F_i'(0)}. \tag{5}$$

The last form of the equation exactly corresponds to the Newtonian formulation for dynamics where non-negative constant m_{q_i} with unit $\$ \times y^2/unit^2$ is the ratio between force and acceleration. Following Newton, we interpret m_{q_i} as the *inertial factor (the 'mass') of the velocity of production of firm i*. The magnitude of m_{q_i} measures the

[2]The proposed adjustment rules are explained verbally in elementary textbooks of economics: if marginal revenues > (<) marginal costs, a firm raises (lowers) its production. See, for example, [3, p. 138].

inertia in this adjustment. With these assumptions, Eq. (5) exactly corresponds to the Newtonian formulation for dynamics: $ma = F$, where $a = q_i'(t)$ and $F = \partial \Pi_i / \partial q_i$. The resulting equation is

$$m_{qi} q_i'(t) = p(t) - \frac{\partial C_i}{\partial q_i}\big(t, q_i(t)\big), \ t \in R. \tag{6}$$

Example. Let the cost function of a firm for time unit y be $C(t,q) = a + bq + (c/2)q^2$, where a, b, c are nonnegative constants with proper units, and the firm has no price setting power. Eq. (6) then becomes

$$m_q q'(t) = p - b - cq(t), \ t \in R, \tag{7}$$

and providing that the unit price p is constant, it has the solution

$$q(t) = \frac{p-b}{c} + \left(q(0) - \frac{p-b}{c}\right) e^{-ct/m_q}, \ t \in R.$$

With $t \to \infty$, $q(t) \to (p-b)/c$ which maximizes the firm's profit. Neoclassical theory thus corresponds to the asymptotic steady-state — the zero force situation — in the Newtonian formulation.

Example. Let us consider a physical analogy. A stone of mass m is dropped from an airplane at the height y_0. The height of the stone at time moment t is denoted by $y(t)$. The stone is moved downwards by the gravitation force $-mg$, $g = 9,81$ (m/s^2), and upwards by the air friction $-Cy'(t)$, where C is a positive constant. The Newtonian equation of movement reads as

$$my''(t) = -mg - Cy'(t), \ t \in R. \tag{8}$$

A comparison to (7) gives that $p-b$ is a gravitation like force and $-cq(t)$ is like a velocity dependent friction force. In both cases, the velocity functions $q(t)$ and $y'(t)$ adjust exponentially toward a stationary state, where the resultant of forces vanishes.

3. An Individual Consumer

We model consumer behavior analogously with that in the previous section. Consumer j is choosing between consumed magnitudes x_{j1}, x_{2j} of two goods labeled 1 and 2 for a time period of length y. Good 1 is a typical consumption good the consumer consumes every time unit. Because we analyze the demand of good 1, good 2 is assumed to be a basket of other goods the consumer spends money during the time unit. [19] proved the existence of an aggregate demand function for good 1 in this kind of a setting with a finite number of utility maximizing consumers. Unit prices p_1, p_2 are exogenous for the consumer, and we assume that the consumer has budgeted himself a fixed amount of money I_j for consumption for the time unit. Suitable measurement units for the quantities are x_{1j} : $unit/y$, x_{2j} : kg/y, p_1 : $\$/unit$, p_2 : $\$/kg$, I_j : $\$/y$. The budget equation is then

$$I_j(t) = p_1 x_{1j}(t) + p_2 x_{2j}(t).$$

Consumer j has utility function $\tilde{u}_j(t) = u_j\big(t, x_{1j}(t), x_{2j}(t)\big)$ measuring his satisfaction during time unit y in units $util/y$; time t in the function allows the consumer's preferences change with time. The first order partials of the utility function with respect to the consumption velocities are assumed continuous and positive, and the second order differential d^2u_j to be negatively definite everywhere[3]. Consumer j likes to increase his utility with time. We solve the budget equation as $x_{2j}(t) = \big(I_j(t) - p_1 x_{1j}(t)\big)/p_2$ and write the utility function as

$$\tilde{u}_j\big(t, x_{1j}(t), I_j(t), p_1, p_2\big) = u_j\left(t, x_{1j}(t), \frac{I_j(t) - p_1 x_{1j}(t)}{p_2}\right). \tag{9}$$

With fixed I_j, the constrained two variable utility maximization problem reduces to a one variable maximization problem. A necessary condition for its resolvability is

$$\frac{d\tilde{u}_j}{dx_{1j}} = \frac{\partial u_j}{\partial x_{1j}} - \frac{p_1}{p_2}\frac{\partial u_j}{\partial x_{2j}} = 0, \tag{10}$$

under which a sufficient condition for the existence of a local maximal utility is $d^2\tilde{u}_j/dx_1^2 < 0$. Since

$$\frac{d^2\tilde{u}_j}{dx_{1j}^2} = \frac{\partial^2 u_j}{\partial x_{1j}^2} - \frac{2p_1}{p_2}\frac{\partial^2 u_j}{\partial x_{1j}\partial x_{2j}} + \left(\frac{p_1}{p_2}\right)^2 \frac{\partial^2 u_j}{\partial x_{2j}^2} = d^2u_j\Big(1, \frac{p_1}{p_2}\Big) < 0,$$

the latter condition is satisfied. With fixed prices we get the following time derivative from (9):

$$\frac{d\tilde{u}_j}{dt} = \frac{\partial u_j}{\partial t} + \left(\frac{\partial u_j}{\partial x_{1j}} - \frac{p_1}{p_2}\frac{\partial u_j}{\partial x_{2j}}\right) x_{1j}'(t) + \left(\frac{1}{p_2}\frac{\partial u_j}{\partial x_{2j}}\right) I_j'(t).$$

Increasing utility with time corresponds to $d\tilde{u}_j/dt > 0$. Because consumer j has budgeted himself a fixed amount of money for the period, $I_j'(t) = 0$, and because we do not model changes in the consumer's preferences, $\partial u_j/\partial t$, the consumer has only one policy variable, x_{1j}. Consumer j changes x_{1j} to increase his utility with time as follows:

$$\begin{aligned} x_{1j}'(t) > 0 \quad &\text{when} \quad \frac{\partial u_j}{\partial x_{1j}} - \frac{p_1}{p_2}\frac{\partial u_j}{\partial x_{2j}} > 0, \\ x_{1j}'(t) < 0 \quad &\text{when} \quad \frac{\partial u_j}{\partial x_{1j}} - \frac{p_1}{p_2}\frac{\partial u_j}{\partial x_{2j}} < 0, \\ x_{1j}'(t) = 0 \quad &\text{when} \quad \frac{\partial u_j}{\partial x_{1j}} - \frac{p_1}{p_2}\frac{\partial u_j}{\partial x_{2j}} = 0. \end{aligned} \tag{11}$$

[3]Under our differentiability assumptions, $(x_1, x_2) \mapsto u_j(x_1, x_2)$ is a concave function if and only if d^2u_j is negatively semidefinite everywhere.

Now, $x'_{1j}(t)$ with unit $unit/y^2$ measures the 'change in the consumer's velocity of consumption of good 1' or his *'acceleration of consumption of good 1'*. Following Newton, we identify the reason for this acceleration, $\partial u_j/\partial x_{1j} - p_1/p_2\big(\partial u_j/\partial x_{2j}\big)$ with unit *util/unit*, as the *'force' acting upon the velocity of consumption of good 1 of consumer j*. Measuring this force is problematic, however, because measuring utility in units *util* is difficult[4]. Due to this we multiply the inequalities (11) by the positive factor $p_2/\big(\partial u_j/\partial x_{2j}\big)$ and get

$$x'_{1j}(t) > 0 \quad \text{when} \quad p_2\left(\frac{\partial u_j}{\partial x_{1j}}\Big/\frac{\partial u_j}{\partial x_{2j}}\right) - p_1 > 0 \quad \text{etc.} \tag{12}$$

Quantity

$$h_j\big(t, x_{1j}(t), I_j(t), p_1, p_2\big) = p_2\left(\frac{\partial u_j}{\partial x_{1j}}\Big/\frac{\partial u_j}{\partial x_{2j}}\right) \tag{13}$$

with unit \$/*unit* measures the consumer's *'willingness to pay for one unit of good 1'*. In (12) the force $h_j - p_1$ is measurable because p_1 is measurable and we can quantify h_j by a questionnaire. Eq. (12) implies that consumer j increases his consumption of good 1 if he is willing to pay more than the price p_1 and vice versa. Increases in $\partial u_j/\partial x_{1j}$ and p_2, and decreases in $\partial u_j/\partial x_{2j}$ increase h_j. The income and substitution effects of other goods are thus present in the formulation.

The above described dynamic behavior can be modeled as

$$x'_{1j}(t) = G_j\big(h_j - p_1\big),\ t \in R, \tag{14}$$

where $G_j\colon R \to R$ is strictly increasing with $G_j(0) = 0$. The first order Taylor approximation of function G_j in the neighborhood of the optimum point $h_j - p_1 = 0$ is

$$G_j\left(h_j - p_1\right) = G_j(0) + G'_j(0)\left(h_j - p_1 - 0\right) + \epsilon_j \approx G'_j(0)(h_j - p_1),$$

if the residual term ϵ_j is assumed negligible. With this approximation, we can write Eq. (14) as

$$m_{1j}x'_{1j}(t) = h_j - p_1, \quad m_{1j} = \frac{1}{G'_j(0)}, \quad t \in R, \tag{15}$$

where nonnegative constant m_{1j} with unit $(\$ \times y^2)/unit^2$ is the ratio between force and acceleration. The magnitude of m_{1j} measures the inertia in this adjustment. Following Newton, we call m_{1j} *the inertial factor ('mass') of the velocity of consumption of good 1 of consumer j*. The zero force situation in Eq. (15), $p_2\partial u_j/\partial x_{1j} = p_1\partial u_j/\partial x_{2j}$, corresponds to neoclassical theory.

Example. Suppose that the utility function of a consumer is of the form: $u(t) = z_1\ln\big(z_2x_1(t)x_2(t)\big)$, where x_1, x_2 are as above and z_1, z_2

[4]This problem disappears, if one uses \$ as the measure unit of utility. However, unit \$ may not be a proper one for measuring satisfaction, see [10].

are positive constants with units $util/y$ and $y^2/(kg \times unit)$, respectively. Utility is thus measured in units $util/y$ and the argument of the logarithmic function is dimensionless as it should, see [10, p. 141]. The budget equation is: $I = p_1x_1(t) + p_2x_2(t)$. Quantity $h_1 - p_1$ then takes the form: $I/(2x_1) - p_1$. Now, $h_1(x_1) - p_1 = 0$ when $p_1x_1 = I/2$. Setting $h_1 - p_1$ as the force in Eq. (15), a nonlinear differential equation results. Its implicit solutions

$$2p_1x_1(t) = I + K \exp\left(-\frac{4p_1t}{m_1}\right) \exp\left(-\frac{2p_1x_1(t)}{I}\right), \ K \in R,$$

can be found by separating the variables. The constant K is determined by the initial condition $x_1(0) = x_0$, $x_0 \in]0, I/p_1[$. Since $0 < \exp(-2p_1x_1(t)/I) < 1$, it follows that

$$\left|x_1(t) - \frac{I}{2p_1}\right| \le \frac{|K|}{2p_1} \exp\left(-\frac{4p_1t}{m_1}\right) \text{ for all } t > 0.$$

Hence, $x_1(t) \to I/(2p_1)$ exponentially, as $t \to \infty$, which zero-force situation corresponds to neoclassical theory. The time path for $x_2(t)$ can be obtained from $x_1(t)$ using the budget equation.

4. Average Firm and Consumer Behavior

For simplicity, the market is modeled on the basis of the firms' and consumers' average behavior. Analogous simplifications are made in physics. For example, the macroscopic laws of gases are written on the basis of the average behavior of molecules due to their huge number.

Because we study the market of good 1, we set $p_1(t) = p(t)$ and assume p_2 fixed. When every firm and consumer have adjusted optimally, we have

$$p(t) = \frac{\partial C_i}{\partial q_i}\big(t, q_i(t)\big), \quad i = 1, \dots n, \quad \text{and} \quad p(t) = h_j, \quad j = 1, \dots, m.$$

Adding the above n and m equations separately and dividing the results by n and m, respectively, we get

$$p(t) = \frac{1}{n}\sum_{i=1}^{n} \frac{\partial C_i}{\partial q_i}\big(t, q_i(t)\big) = \frac{1}{m}\sum_{j=1}^{m} h_j\big(t, x_{1j}(t), I_j(t), p, p_2\big), \quad (17)$$

where the middle term is the average of marginal costs of firms at the aggregate velocity of production $q_s(t) = \sum_{i=1}^{n} q_i(t)$, and $1/m \sum_{j=1}^{m} h_j$ is the consumers' average willingness to pay for one unit of good 1 at the aggregate velocity of consumption $q_d(t) = \sum_{j=1}^{m} x_{1j}(t)$; subscripts s, d refer to supply and demand[5]. Eq. (17) defines the inverse relations of market supply and demand. In the equilibrium, unit price equals the average of firms' marginal costs and consumers' willingness to pay for

[5] However, the last terms are functions of $(q_1, \dots, q_n)$ and $(x_{11}, \dots, x_{1m})$, respectively.

one unit, and no agent likes to change his behavior. This corresponds to neoclassical equilibrium.

The first order Taylor expansions of the firms' marginal cost functions in the neighborhood of the equilibrium velocities q_{i0} at t_0 are

$$\begin{aligned}\frac{\partial C_i}{\partial q_i}\big(t, q_i(t)\big) &= \frac{\partial C_i}{\partial q_i}\big(t_0, q_{i0}\big) + \frac{\partial^2 C_i}{\partial t \partial q_i}\big(t_0, q_{i0}\big)\big(t - t_0\big) \\ &+ \frac{\partial^2 C_i}{\partial q_i^2}\big(t_0, q_{i0}\big)\big(q_i(t) - q_{i0}\big) + \tilde{\epsilon}_i,\ i = 1, \dots, n;\end{aligned}$$

$\tilde{\epsilon}_i$ is the residual term. Assuming $\tilde{\epsilon}_i \approx 0$ and summing over i, we get

$$\begin{aligned}\sum_{i=1}^{n} \frac{\partial C_i}{\partial q_i}\big(t, q_i(t)\big) &\approx \sum_{i=1}^{n} \Big[\frac{\partial C_i}{\partial q_i}(t_0, q_{i0}) - \frac{\partial^2 C_i}{\partial t \partial q_i}\big(t_0, q_{i0}\big) t_0 \\ &- \frac{\partial^2 C_i}{\partial q_i^2}\big(t_0, q_{i0}\big) q_{i0} + \frac{\partial^2 C_i}{\partial t \partial q_i}\big(t_0, q_{i0}\big) t + \frac{\partial^2 C_i}{\partial q_i^2}\big(t_0, q_{i0}\big) q_i(t)\Big] \\ &\approx a_0 + a_1 t + \frac{a_2}{n} q_s(t),\end{aligned} \tag{18}$$

where[6]

$$a_0 = \sum_{i=1}^{n} \Big[\frac{\partial C_i}{\partial q_i}\,(t_0, q_{i0}) - \frac{\partial^2 C_i}{\partial t \partial q_i}\big(t_0, q_{i0}\big) t_0 - \frac{\partial^2 C_i}{\partial q_i^2}\,(t_0, q_{i0})\, q_{i0}\Big],$$

$$a_1 = \sum_{i=1}^{n} \frac{\partial^2 C_i}{\partial t \partial q_i}\big(t_0, q_{i0}\big) \quad \text{and} \quad a_2 = \sum_{i=1}^{n} \frac{\partial^2 C_i}{\partial q_i^2}\,(t_0, q_{i0})$$

are constants with units $\$/unit$, $\$/(unit \times y)$ and $(\$ \times y)/unit^2$, respectively, and $q_s(t) = \sum_{i=1}^{n} q_i(t)$.

Because marginal costs are positive at every t, q_s, then $a_0 > 0$ (take $t, q_s \to 0$ in (18)). The assumed technological progress means that $\partial^2 C_i/\partial t \partial q \leq 0$ for all $i = 1, \dots, n$, and so $a_1 \leq 0$. At the aggregate level increasing (decreasing) returns to scale in the industry correspond to $a_2 < 0$ ($a_2 > 0$). An approximate average of the firms' marginal costs thus linearly depends on the total velocity of production of the industry and time,

$$g\big(t, q_s(t)\big) \equiv \frac{1}{n} \sum_{i=1}^{n} \frac{\partial C_i}{\partial q_i}\big(t, q_i(t)\big) \approx \frac{a_0}{n} + \frac{a_1}{n} t + \frac{a_2}{n^2} q_s(t). \tag{19}$$

The average consumer behavior is defined similarly. The first order Taylor expansions of the consumers' willingness to pay functions in the neighborhood of the equilibrium velocities of consumption x_{1j0} and

[6]Because $\sum_{i=1}^{n} c_i q_i = \bar{c} \sum_{i=1}^{n} q_i + \sum_{i=1}^{n} (c_i - \bar{c}) q_i$ where $\bar{c} = (1/n) \sum_{i=1}^{n} c_i$, the approximation is the more accurate the less $c_i = \partial^2 C_i/\partial q_i^2$ or q_i vary, $i = 1, \dots, n$.

budgeted funds I_{j0} at time moment t_0, are

$$h_j\big(t, x_{1j}(t), I_j(t), p, p_2\big) = h_{j0} + \frac{\partial h_j}{\partial t}(z_0)(t - t_0) + \frac{\partial h_j}{\partial x_{1j}}(z_0)\big(x_{1j}(t) - x_{1j0}\big) + \frac{\partial h_j}{\partial I_j}(z_0)(I_j(t) - I_{j0}) + \tilde{\epsilon}_j,$$

where $(z_0) = (t_0, x_{1j0}, I_{j0}, p_0, p_{20})$ and $\tilde{\epsilon}_j$ is the residual term. Assuming $\tilde{\epsilon}_j \approx 0$, $j = 1, \dots, m$, and summing over the consumers, we get

$$\begin{aligned}\sum_{j=1}^{m} h_j &\approx \sum_{j=1}^{m}\Big[h_{j0} - \frac{\partial h_j}{\partial t}(z_0)t_0 - \frac{\partial h_j}{\partial x_{1j}}(z_0)x_{1j0} - \frac{\partial h_j}{\partial I_j}(z_0)I_{j0}\Big] \qquad (20)\\ &\quad + t\sum_{j=1}^{m}\frac{\partial h_j}{\partial t}(z_0) + \sum_{j=1}^{m}\frac{\partial h_j}{\partial x_{1j}}(z_0)x_{1j}(t) + \sum_{j=1}^{m}\frac{\partial h_j}{\partial I_j}(z_0)I_j(t)\\ &\approx b_0 + b_1 t + \frac{b_2}{m}q_d(t) + \frac{b_3}{m}I(t),\end{aligned}$$

where[7]

$$b_0 = \sum_{j=1}^{m}\Big[h_{j0} - \frac{\partial h_j}{\partial t}(z_0)t_0 - \frac{\partial h_j}{\partial x_{1j}}(z_0)x_{1j0} - \frac{\partial h_j}{\partial I_j}(z_0)I_{j0}\Big],$$

$$b_1 = \sum_{j=1}^{m}\frac{\partial h_j}{\partial t}(z_0), \quad b_2 = \sum_{j=1}^{m}\frac{\partial h_j}{\partial x_{1j}}(z_0), \quad b_3 = \sum_{j=1}^{m}\frac{\partial h_j}{\partial I_j}(z_0)$$

are constants with units $\$/unit$, $\$/(unit \times y)$, $(\$ \times y)/unit^2$ and $y/unit$, respectively, and $q_d(t) = \sum_{j=1}^{m} x_{1j}(t)$, $I(t) = \sum_{j=1}^{m} I_j(t)$.

Because the willingness to pay of every consumer is non-negative at every t, q_d, I, then $b_0 \geq 0$ (take $t, q_d, I \to 0$ in (20)). Increasing (decreasing) popularity of this good with time corresponds to $b_1 > 0$ ($b_1 < 0$). For normal goods $b_2 < 0$ and $b_3 > 0$, for Giffen goods $b_2 > 0$ and for inferior goods $b_3 < 0$. An approximate average of the consumers' willingness to pay for one unit of good 1 thus linearly depends on time, the total velocity of consumption of good 1 and the total flow of money the consumers have budgeted for consumption for the period,

$$h\big(t, q_d(t), I(t)\big) \equiv \frac{1}{m}\sum_{j=1}^{m} h_j \approx \frac{b_0}{m} + \frac{b_1}{m}t + \frac{b_2}{m^2}q_d(t) + \frac{b_3}{m^2}I(t). \qquad (21)$$

The existence of an aggregate demand function in an m -consumer case has been earlier proved by [19] and [11].

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[7] See the previous footnote.

5. Industry Level Analysis

The adjustment of production and consumption is modeled on the basis of the firms' and consumers' average behavior, and the law of demand and supply introduced by [17] is assumed to determine the velocity of the unit price $p'(t)$ with unit $\$/(unit \times y)$:

$$\begin{aligned} q_s'(t) &= \xi_s\Big(p(t) - g\big(t, q_s(t)\big)\Big),\ t \in R, && (22)\\ q_d'(t) &= \xi_d\Big(h\big(t, q_d(t), I(t)\big) - p(t)\Big),\ t \in R, && (23)\\ p'(t) &= \xi_p\big(q_d(t) - q_s(t)\big),\ t \in R, && (24)\end{aligned}$$

where $\xi_s, \xi_d, \xi_p \colon R \to R$ are strictly increasing with $\xi_s(0) = \xi_d(0) = \xi_p(0) = 0$. The economic content of Eq. (24) was explained in Section 2 when we discussed how excess demand and supply motivate firms to change their prices.

System (22-24) defines the time paths for q_s, q_d, p. The standard way to study the local stability of system (22-24) is to take its first order Taylor expansion in the neighborhood of the steady-state $q_s'(t) = q_d'(t) = p'(t) = 0$ and study that system. The first order Taylor approximations give the following linear system which fulfills the requirements for functions ξ_s, ξ_d, ξ_p:

$$\begin{aligned} m_s q_s'(t) &= p(t) - g\big(t, q_s(t)\big),\ t \in R, && (25)\\ m_d q_d'(t) &= h\big(t, q_d(t), I(t)\big) - p(t),\ t \in R, && (26)\\ m_p p'(t) &= q_d(t) - q_s(t),\ t \in R; && (27)\end{aligned}$$

the non-negative constants m_s, m_d, m_p can be identified[8] as the *'inertial factors ('masses') of aggregate supply, demand and the unit price'*, respectively[9]; m_p has unit $unit^2/\$$.

To get an analytic solution for the system (25)-(27), we assume $g(t, q_s(t))$ and $h(t, q_d(t), I(t))$ as in (19) and (21), and a linear time trend in $I(t) = b_4 t$ where parameter $b_4 \geq 0$ has unit $\$/y^2$. The solutions with specific parameter values are displayed in Figures 1-4 where unit price is the thickest and supply the thinnest curve. The solutions imply that the system converges with time to fixed values or steady-state paths. Growth in q_d, q_s may occur in situations: $b_1 > 0$, $b_3, b_4 > 0$, $a_1 < 0$ and $a_2 < 0$. These can be called preference, wealth, technology and increasing returns to scale based growth, respectively. The reason for the growth of the industry may thus be demand or supply. One clear difference between these cases exists, however. If the origin

[8]Notice that Eq. (27) does not exactly correspond to the Newtonian formulation.

[9]We can also interpret our modeling in probability terms. If every firm (consumer) has an equal probability $1/n$ $(1/m)$ to be the producer (consumer) of one unit of good 1 in the industry, then $p(t) - 1/n \sum_{i=1}^{n} \partial C_i/\partial q_i$ and $1/m \sum_{j=1}^{m} h_j - p(t)$, respectively, measure the expected value of the willingness of the firms (consumers) to expand the velocity of production (consumption) of good 1.

of the growth is demand, then p, q_s, q_d all have a positive time trend (cases $b_3, b_4 > 0$ and $b_1 > 0$ both give time paths as in Figure 3). On the other hand, if the origin of the growth is supply, then p will decrease and q_s, q_d increase with time (in Figure 2, $a_1 < 0$ and in Figure 4, $a_2 < 0$). These differences can be observed empirically. Technology based growth thus cannot last forever because the decreasing price level will cause bankruptcies of firms with time.

Next, we study the solutions of the general system (25) - (27). At time moment $t = 0$ the price, the consumption and the production velocities are known real numbers, say p_0, q_{d0} and q_{s0}, respectively. The system is described by

$$\begin{aligned} \frac{d}{dt}\Big(p(t), \eta q_d(t), \eta q_s(t)\Big) &= A(t)\Big(p(t), \eta q_d(t), \eta q_s(t)\Big) \\ &\quad + \mathbf{f}\Big(t, p(t), q_d(t), q_s(t)\Big), \quad t \in [0, \infty[, \end{aligned} \tag{28}$$

$$\big(p(0), q_d(0), q_s(0)\big) = (p_0, q_{d0}, q_{s0}), \tag{29}$$

where the linear mappings $A(t)\colon R^3 \to R^3$, $t \in R$, are given by their matrices

$$\mathrm{mat}A(t) = \begin{pmatrix} 0 & 1/(\eta m_p) & -1/(\eta m_p) \\ -\eta/m_d & \frac{\partial h}{\partial q_d}(t,0)/m_d & 0 \\ \eta/m_s & 0 & -\frac{\partial^2 C}{\partial q_s^2}(t,0)/m_s \end{pmatrix} \tag{30}$$

and $\mathbf{f}\colon \mathbf{R} \times \mathbf{R^3} \to \mathbf{R^3}$ is given by

$$\begin{cases} f_1(t, y_1, y_2, y_3) = 0 \\ f_2(t, y_1, y_2, y_3) = \frac{\eta}{m_d}\big(h(t, y_2) - \frac{\partial h}{\partial q_d}(t,0)y_2\big), \\ f_3(t, y_1, y_2, y_3) = \frac{\eta}{m_s}\big(\frac{\partial^2 C}{\partial q_s^2}(t,0)y_3 - \frac{\partial C}{\partial q_s}(t, y_3)\big). \end{cases} \tag{31}$$

For the dimensional homogeneity of (28), appears a positive constant η the dimension of which equals with that of p/q_d, cf. [10].

6. A Linear Autonomous Case

We assume that the average production costs and the willingness to pay are given by

$$C(t, q_s) = C_0 + aq_s + \frac{1}{2}bq_s^2, \quad h(t, q_d) = c - dq_d, \tag{32}$$

where a, C_0, b, c, d are dimensional constants all of which are positive except a, which may be any real number. Then $A(t) = A$ and $\mathbf{f} = (0, \eta c/m_d, -\eta a/m_s)$, given by (30)-(31), are constants. In this case the marginal costs are increasing and the willingness to pay is decreasing. Moreover, a, b, c, and d can be chosen such that (32) is an approximation of more general smooth $C(t, q_s)$ and $h(t, q_d)$. We identify the linear mappings and the corresponding matrices. Hence the solution of the initial value problem (28)-(29) is given by

$$\begin{pmatrix} p(t) \\ \eta q_d(t) \\ \eta q_s(t) \end{pmatrix} = \mathrm{e}^{At} \left(\begin{pmatrix} p_0 \\ \eta q_{d0} \\ \eta q_{s0} \end{pmatrix} + A^{-1}\mathbf{f} \right) - A^{-1}\mathbf{f}, \quad t \in [0, \infty[, \tag{33}$$

since $\det A = -(b+d)/(m_d m_p m_s) \neq 0$, i.e. A is invertible.

Next, we present some properties of the matrix A. Let us denote

$$\alpha = \frac{d m_s + b m_d}{m_d m_s}, \quad \beta = \frac{m_d + m_s + b d m_p}{m_d m_p m_s}, \quad \gamma = \frac{b+d}{m_d m_s m_p}. \tag{34}$$

Lemma 6.1. *The inverse of the matrix A is*

$$A^{-1} = \frac{1}{b+d} \begin{pmatrix} -bdm_p & -bm_d/\eta & dm_s/\eta \\ \eta b m_p & -m_d & -m_s \\ -\eta d m_p & -m_d & -m_s \end{pmatrix}. \tag{35}$$

Lemma 6.2. *The eigenvalues λ_1, λ_2 and λ_3 of the matrix A satisfy:*

$$\lambda_1 \in I_1 \colon = \left] -\frac{\alpha^3+\gamma}{\alpha^2+\beta}, -\frac{\gamma}{\beta} \right[\, ;$$

$$\operatorname{Re}\lambda_2, \ \operatorname{Re}\lambda_3 \in I_2 \colon = \left] -\frac{\alpha\beta-\gamma}{2\beta}, -\frac{\alpha\beta-\gamma}{2(\alpha^2+\beta)} \right[\, , \quad \textit{if } \operatorname{Im}\lambda_2 \neq 0;$$

$$\lambda_2, \lambda_3 \in I_3 \colon = \left] -\alpha, -\frac{\gamma}{\beta} \right[\ \textit{otherwise.}$$

The intervals I_1, I_2 and I_3 above are nonempty. Moreover, $\operatorname{Im}\lambda_2 \neq 0$, *if $\alpha^2 \leq 3\beta$.*

Proof. The characteristic polynomial of A is $f_A(\lambda) = \det(A - \lambda) = \lambda^3 + \alpha\lambda^2 + \beta\lambda + \gamma$ and it takes values of opposite signs at the endpoints of I_1. Thus $\lambda_1 \in I_1$. The two other eigenvalues are then

$$-\frac{\alpha+\lambda_1}{2} \pm \frac{1}{2}\sqrt{(\alpha+\lambda_1)^2 - 4\beta - 4\alpha\lambda_1 - 4{\lambda_1}^2}.$$

Thus Re λ_2, Re $\lambda_3 \in I_2$, if Im $\lambda_2 \neq 0$. Also the sufficient condition for imaginarity of λ_2 and λ_3 is obtained. If all the eigenvalues are real, they belong to I_3, since the characteristic polynomial does not take the value zero on $R \setminus I_3$. **q.e.d.**

Lemma 6.3. *There is a positive constant C_A, depending only on the matrix A, such that for each $t \in [0, \infty[$ and $y \in R^3$,*

$$\|e^{At}y\| \leq \begin{cases} C_A e^{-\frac{\gamma}{\beta}t}\|y\|, & \textit{if } \operatorname{Im}\lambda_2 = 0, \\ C_A e^{-\delta t}\|y\| & \textit{otherwise,} \end{cases}$$

where

$$\delta = \min\left\{ \frac{\gamma}{\beta}, \ \frac{\alpha\beta-\gamma}{2(\alpha^2+\beta)} \right\} > 0.$$

Proof. The proof is straightforward. It is based on the equivalence of A to its canonical Jordan's form and on the formula $e^{At} = \sum_{n=0}^{\infty} \frac{1}{n!}(At)^n$. **q.e.d.**

Remark 6.1. *There exist sequences of matrices of the form of (30), for which* $C_A \to \infty$.

By these lemmas we obtain the following results on the asymptotic behavior and the stability of the solution of (28)-(29).

Theorem 6.1. *(Asymptotic stability) For each* $t \in [0, \infty[$,

$$\Big|p(t) - \frac{ad+bc}{b+d}\Big| + \eta\Big|q_d(t) - \frac{c-a}{b+d}\Big| + \eta\Big|q_s(t) - \frac{c-a}{b+d}\Big|$$
$$\leq \sqrt{3}\ C_A e^{-\delta t}\left(\Big|p_0 - \frac{ad+bc}{b+c}\Big| + \eta\Big|q_{d0} - \frac{c-a}{b+d}\Big| + \eta\Big|q_{s0} - \frac{c-a}{b+d}\Big|\right).$$

Theorem 6.2.

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} h\big(t, q_d(t)\big) = \lim_{t\to\infty} \frac{\partial C}{\partial q_s}\big(t, q_s(t)\big),\ \lim_{t\to\infty} q_d(t) = \lim_{t\to\infty} q_s(t).$$

Theorem 6.3. *(Stability with respect to initial conditions). Assume that* (p, q_d, q_s) *and* $(\tilde{p}, \tilde{q}_d, \tilde{q}_s)$ *are two solutions of (28) with the initial values* (p_0, q_{d0}, q_{s0}) *and* $(\tilde{p}_0,\ \tilde{q}_{d0},\ \tilde{q}_{s0}) \in R^3$, *respectively. Then, for each* $t \in [0, \infty[$,

$$\Big|p(t) - \tilde{p}(t)\Big| + \eta\Big|q_d(t) - \tilde{q}_d(t)\Big| + \eta\Big|q_s(t) - \tilde{q}_s(t)\Big|$$
$$\leq \sqrt{3}C_A\ e^{-\delta t}\Big(|p_0 - \tilde{p}_0| + \eta|q_{d0} - q_{d0}| + \eta|q_{s0} - \tilde{q}_{s0}|\Big).$$

Remark 6.2. *Consider the case where* m_s *and* m_d *are very small as compared to* bdm_p. *Since* $\delta \to (b+d)/(bdm_p)$ *as* $m_s, m_d \to 0+$, *we have* $\delta \approx (b+d)/(bdm_p)$.

Remark 6.3. *The formula (33) gives also a general explicit continuous dependence for the solution of the initial value problem (28)-(29) on the parameters* a, b, c, d, m_d, m_p, m_s, *and on the initial value* (p_0, q_{d0}, q_{s0}).

7. A nonhomogeneus linear case

We return to consider the average firm and consumer behavior, i.e., equations (25)-(27), (19), and (21) with $a_2 > 0$ and $b_2 < 0$. We are solving a homogeneous linear equation

$$y'(t) - Ay(t) = \mathbf{f}(t),\ t \in [0, \infty[, \tag{36}$$

where A is given by its inverse with the replacements $b = a_2/n^2$ and $d = -b_2/m^2$ in (35) and $\mathbf{f}(t) = \big(0, f_2(t), f_3(t)\big)$,

$$f_2(t) = \frac{\eta}{mm_d}\Big(b_0 + b_1 t + \frac{b_3 b_4}{m} t\Big),\ f_3(t) = -\frac{\eta}{nm_s}(a_0 + a_1 t).$$

We recall the variation of constant formula for the solution of (36) with the initial condition $y(0) = y_0 \in R^3$:

$$y(t) = \mathrm{e}^{At}\, y_0 + \int_0^t \mathrm{e}^{A(t-s)}\, \mathbf{f}(s)\, ds,\ t \in [0, \infty[, \tag{37}$$

By integrating by parts this gives

$$\begin{aligned} y(t) = {} & \mathrm{e}^{At}\left(y_0 + A^{-1}\mathbf{f}(0) + A^{-2}\mathbf{f}'(0)\right) \\ & -A^{-1}\big(\mathbf{f}(0) + A^{-1}\mathbf{f}'(0)\big) - A^{-1}\mathbf{f}'(0)t \text{ for all } t \in [0, \infty[. \end{aligned}$$

Due to Lemmas 6.2 and 6.3, the first term decays exponentially, as $t \to \infty$. Hence, there exist affine functions[10] $p_\infty, q_{d\infty}, q_{s\infty} \colon [0, \infty[\to R$ such that

$$p(t) - p_\infty(t) \to 0, q_d(t) - q_{d\infty}(t) \to 0, q_s(t) - q_{s\infty}(t) \to 0$$

exponentially, as $t \to \infty$. This behavior can also be seen in Figures 1-4.

8. A Smooth Nonlinear Autonomous Case

Next we consider the case in which the marginal costs and the willingness to pay are given by

$$\frac{\partial C}{\partial q_s}(t, q_s) = g(q_s), \quad h(t, q_d) = h(q_d), \tag{38}$$

where $g, h \colon R \to R$. If $g'(0) > 0$ and $h'(0) < 0$, the matrix $A(t)$ is the same constant as in the previous section. But $\mathbf{f}$ is different;

$$\mathbf{f}(t, \mathbf{x}) = \Big(0, \eta\big(h(x_2) - h'(0)x_2\big)/m_d, \eta\big(g'(0)x_3 - g(x_3)/m_s\Big). \tag{39}$$

Define

$$Q = \big\{x \in R \mid h(x) = g(x),\, h'(x) = h'(0),\, g'(x) = g'(0)\big\}. \tag{40}$$

Theorem 8.1. *Let $g, h \colon R \to R$ be locally Lipschitzian with $g'(0) > 0$ and $h'(0) < 0$, and let there be $q_\infty \in Q$ at which g' and h' are continuous. Then the solution (p, q_d, q_s) of (28)-(29) satisfies for some positive constant δ_{q_∞}:*

$$\begin{aligned} & \big|p(t) - h(q_\infty)\big| + \eta\big|q_d(t) - q_\infty\big| + \eta\big|q_s(t) - q_\infty\big| \to 0, \\ & \qquad \text{as } |p_0 - h(q_\infty)| + \eta|q_{d0} - q_\infty| + \eta|q_{s0} - q_\infty| \to 0 \text{ uniformly in } t; \\ & \big|p(t) - h(q_\infty)\big| + \eta\big|q_d(t) - q_\infty\big| + \eta\big|q_s(t) - q_\infty\big| \to 0, \\ & \qquad\quad \text{as } t \to \infty \text{ if } |p_0 - h(q_\infty)| + \eta|q_{d0} - q_\infty| + \eta|q_{s0} - q_\infty| \le \delta_{q_\infty}. \end{aligned}$$

Proof. The proof follows the lines of [16, pp. 33–34, 75]. **q.e.d.**

Remark 8.1. *Depending on the nonlinear behavior of g and h it may happen that the solution of (28)-(29) does not converge at all, as $t \to \infty$, if $\big(p(0), q_d(0), q_s(0)\big)$ is not close enough to $\big(h(q_\infty), q_\infty, q_\infty\big)$.*

[10]That is, $p_\infty(t) = p_1 + p_2 t$, where $p_1, p_2 \in R$ are constants, etc.

9. A Monotone Nonlinear Autonomous Case

Next, we consider by different methods a class of nonlinear autonomous cases: we assume very little on the smoothness of the marginal costs and the willingness to pay, but we assume them to be monotone. Indeed, we do not assume them to be continuous, single-valued or everywhere defined. For example, constraints like $0 \le q_s \le q_{max}$ are included. Our assumptions allow us to prove the stabilization of the solution, as $t \to \infty$, for any initial conditions. We can also show that the dynamic model of [17] with only one differential equation is a limit case of our theory, as the inertial masses of supply and demand tend to zero, that is, the dynamic equations for q_s, q_d, tend to stationary equations.

Let us recall some notions of nonlinear analysis. For further details the reader may refer e.g. [4]. Let H be a real Hilbert space with the inner product $(\cdot\,,\,\cdot)_H$ and the norm $\|\cdot\|_H$. We denote the interior of the set $C \subset H$ by Int C. A set $B \subset H \times H$ is an *operator* H, its *domain* is $D(B) = \{x \in H \mid (x, y) \in B$, for some $y \in H\}$, its *value at* $x \in H$ is $Bx = \{y \mid (x, y) \in B\}$, and its *inverse* is $B^{-1} = \{(y, x) \mid (x, y) \in B\}$. An operator $B \subset H \times H$ is *monotone*, if $(y_2 - y_1, x_2 - x_1)_H \ge 0$, for each (x_1, y_1), $(x_2, y_2) \in B$. A monotone operator $A \subset H \times H$ is *maximal monotone operator* if it is not contained by any other monotone operator $B \subset H \times H$. An operator $B \subset H \times H$ is *strongly monotone*, if there is $\mu > 0$ such that

$$(y_2 - y_1, x_2 - x_1)_H \ge \mu\|x_1 - x_2\|_H^2, \text{ for each } (x_1, y_1),\ (x_2, y_2) \in B.$$

Let $T > 0$, $k = 1, 2, \dots$, and $r \in [1, \infty[$. By $L^r(0, T)$ we denote the space of Lebesgue measurable functions $u\colon [0, T] \to R$, for which $\int_0^T |u(t)|^r\,dt < \infty$. By $C\big([0, T]; R^k\big)$ we denote the space of continuous functions $[0, T] \to R^k$, etc. The Sobolev space $W^{k,r}(0, T)$ is given by

$$\begin{aligned} W^{1,r}(0, T) = \{u\colon [0, T] \to R \mid u(t) = u(0) + \int_0^t v(\tau)\,d\tau, \\ \text{for each } \ t \in [0, T] \text{ and for some } \ v \in L^r(0, T)\}. \end{aligned}$$

Theorem 9.1. *Let $m_p, m_d, m_s > 0$, $p_0, q_{d0}, q_{s0}, q_\infty \in R$ and $g, -h \subset R \times R$ be maximal monotone operators such that $g(q_\infty) \cap h(q_\infty) \neq \emptyset$. If the problem*

$$\begin{aligned} m_p p'(t) = q_d(t) - q_s(t),\ m_d q_d'(t) \in h(q_d(t)) - p(t), \\ m_s q_s'(t) \in p(t) - g(q_s(t)),\ \textit{for a.e. } t \in]0, \infty[, \end{aligned} \tag{41}$$

$$p(0) = p_0, \quad q_d(0) = q_{d0}, \quad q_s(0) = q_{s0}, \tag{42}$$

has a solution[11] $(p, \eta q_d, \eta q_s) \in C\big([0,\infty[; R^3\big)$ *which satisfies* $p', q_d', q_s' \in L^2(0,k)$*, for each* $k > 0$*, then*

$$m_p(p(t)-p_\infty)^2 + m_d(q_d(t)-q_\infty)^2 + m_s(q_s(t)-q_\infty)^2 \\ \le m_p(p_0-p_\infty)^2 + m_d(q_{d0}-q_\infty)^2 + m_s(q_{s0}-q_\infty)^2, \quad (43)$$

for each $t \ge 0$ *and* $p_\infty \in g(q_\infty) \cap h(q_\infty)$*. If, in addition,* g *and* $-h$ *are strongly monotone, then for some* $p_\infty \in g(q_\infty) \cap h(q_\infty)$*,*

$$|p(t)-p_\infty| + \eta|q_d(t)-q_\infty| + \eta|q_s(t)-q_\infty| \to 0, \;\; \text{as } t \to \infty. \quad (44)$$

Proof. By (41) and the monotonicity of g and $-h$,

$$\frac{d}{dt}\Big(m_p\big(p(t)-p_\infty\big)^2 + m_d\big(q_d(t)-q_\infty\big)^2 + m_s\big(q_s(t)-q_\infty\big)^2\Big) \le 0$$

for a.a. $t \in]0,T[$. This implies (43). The rest of proof is a straightforward application of the standard techniques of the theory of nonlinear differential equations with maximal monotone operators (the chain rule, the demiclosedness of maximal monotone operators, etc.), see e.g. [4]. We just mention that the key idea is to consider the functions $p_n(t) = p(n+t)$, $q_{dn}(t) = q_d(n+t)$ and $q_{sn}(t) = q_s(n+t)$, $n = 1, 2, \dots$, in the space $L^2(0,1)$. **q.e.d.**

Theorem 9.2. *Let* $g, -h \subset R \times R$ *be strongly monotone maximal monotone operators and let* $T, m_p, M_0 > 0$*,* $p_0 \in R$*,* $q_{d0} \in \operatorname{Int} D(h)$*, and* $q_{s0} \in \operatorname{Int} D(g)$ *be fixed. Let the problem (41)–(42) have a solution* $(p, \eta q_d, \eta q_s)$ *on* $[0,T]$*,* p*,* q_d*,* $q_s \in W^{1,2}(0,T)$*, for each* $m_d, m_s \in]0, M_0]$*. Then there are* $q_{d\infty}$*,* $q_{s\infty} \in W^{1,2}(0,T)$*,* $p_\infty \in W^{2,2}(0,T)$ *and a constant* $M > 0$ *which satisfy, as* $m_d, m_s \to 0+$*,*

$$m_p p_\infty'(t) = q_{d\infty}(t) - q_{s\infty}(t), \;\; \text{for a.e. } t \in]0,T[, \quad (45)$$

$$p_\infty(0) = p_0, \quad (46)$$

$$p_\infty(t) \in h(q_{d\infty}(t)) \cap g(q_{s\infty}(t)), \;\; \text{for each } t \in [0,T], \quad (47)$$

$$p(t) \to p_\infty(t) \;\;\text{uniformly on } [0,T], \quad (48)$$

$$q_d(t) \to q_{d\infty}(t),\; q_s(t) \to q_{s\infty}(t), \;\; \text{for a.e. } t \in]0,T[, \quad (49)$$

$$m_d q_d'(t),\;\; m_s q_s'(t) \to 0, \;\; \text{for a.e. } t \in]0,T[, \quad (50)$$

$$|p(t)-p_\infty(t)| + \|m_d q_d'\|_{L^2(0,t)} + \|m_s q_s'\|_{L^2(0,t)} \\ +\eta\|q_d - q_{d\infty}\|_{L^2(0,t)} + \eta\|q_s - q_{s\infty}\|_{L^2(0,t)} \\ \le M(\sqrt{m_d} + \sqrt{m_s}), \;\; \text{for each } t \in [0,T]. \quad (51)$$

Proof. We just give the idea of the proof. The same techniques as in the proof of Theorem 9.1 are applied. Observe that the uniform convergence of q_d and q_s is not stated. The reason is that the problem

[11] To be a solution means that q_d, q_s, and p are differentiable a.e. on $]0,T[$, continuous on $[0,T]$, $q_d(t) \in D(g)$, $q_s(t) \in D(f)$ for a.a $t \in]0,T[$, and they satisfy (41)-(42).

(41)-(42) is of form $(\mathcal{A}y)' + \mathcal{B}y \ni 0$ where $\mathcal{A}$ and $\mathcal{B}$ are maximal monotone operators in R^3, not of the form $y' + \mathcal{C}y \ni 0$ with monotone $\mathcal{C}$. **q.e.d.**

Remark 9.1. *By the approach used in* [9] *it can be proved that these convergence properties as well as the continuous dependence of the solution of (41) on the initial conditions hold also in the case of time dependent but bounded g and h. The similar results can likely be realized in the case of unbounded f and g. See* [1].

Remark 9.2. *Assume that g and $-h$ are maximal monotone and strongly monotone. Then g^{-1}, $-h^{-1}$, and $L\colon = g^{-1} - h^{-1}$ are Lipschitzian. Then (45) reads as $m_p p' + Lp = 0$, and thus (45)-(47) has a unique solution, for any initial condition $p_0 \in R$. Observe that (45)-(47) is indeed the dynamical model of* [17].

Let us finish this section by a theorem on the limit, as $m_p \to 0+$. The proof is elementary.

Theorem 9.3. *Let $T > 0$, $p_0 \in R$, and $p \in W^{1,2}(0,T)$ be a solution of (45). Assume that L above is strongly monotone with a constant $\mu > 0$ and Lipschitzian. Then there is $\hat{p} \in R$ such that $L\hat{p} = 0$ and*

$$|p(t) - \hat{p}| \le |p_0 - \hat{p}| e^{-\mu t/m_p}, \text{ for each } t \in [0,T].$$

Remark 9.3. *If g and $-h$ are monotone and Lipschitzian, then L is strongly monotone. In our linear autonomous case above all the conditions of Theorems 9.1, 9.2 and 9.3 are satisfied. Moreover,*

$$Lp = \frac{b+d}{bd}p - \frac{ad+bc}{bd}, \ \mu = \frac{b+d}{bd} \text{ and } \hat{p} = \frac{ad+bc}{bd}.$$

Remark 9.4. *If $p_0 = \hat{p}$ where $L\hat{p} = 0$ and L is monotone, then the equilibrium condition $L\hat{p} = 0$ is a special case of the dynamical model (45)-(47). Under the conditions of Theorem 9.3, $L\hat{p} = 0$ is the limit case $m_p \to 0$ of (45)-(47). These relations are part of the Correspondence Principle described by Samuelson* [18].

10. Conclusions

We presented a model of the behavior of an industry in perfect competition. A new feature in the modeling is the consumers' active role in the evolution of the industry. We dynamized the consumer behavior so that a consumer adjusts his consumption by comparing his willingness to pay for one unit and the unit price. Firms adjust their production flows by comparing the market price and their marginal costs, and price adjusts at the market according to excess demand. The modeling was executed on the basis of the agents' average behavior.

One aim in our modeling was to show a link in between market behavior and economic growth. The model separates the industry level economic growth from business cycles, which occur due to the adjustment of economic agents. Growth, on the other hand, may occur due to increases in technology or consumers' wealth, due to a positive change in consumers' preferences concerning this good or increasing returns to scale. We showed that the proposed system is stable (i.e. its solution is asymptotically stable and continuously depends on the data) under certain monotonicity or continuity assumptions.

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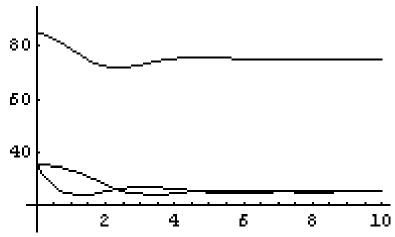
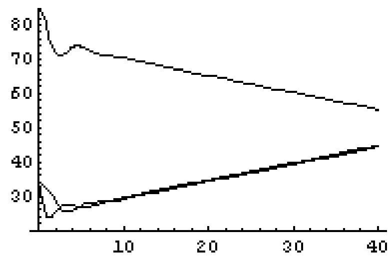
Figure 1



$a_0 / n = 50,\ a_1 / n = 0,\ a_2 / n^2 = 1,\ b_0 / m = 100,\ b_1 / m = 0$

$b_2 / m^2 = -1,\ b_3 / m^2 = 0,\ m_d = m_s = m_p = 1.$

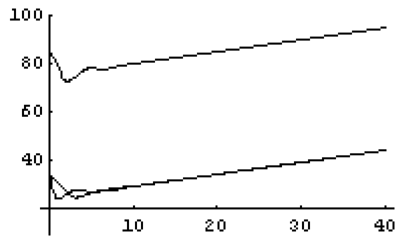
Figure 2



$a_0 / n = 50,\ a_1 / n = -1,\ a_2 / n^2 = 1,\ b_0 / m = 100,\ b_1 / m = 0$

$b_2 / m^2 = -1,\ b_3 / m^2 = 0,\ m_d = m_s = m_p = 1.$

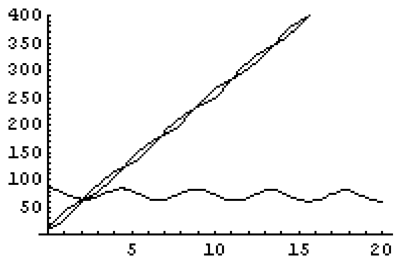
Figure 3



$a_0 / n = 50,\ a_1 / n = 0,\ a_2 / n^2 = 1,\ b_0 / m = 100,\ b_1 / m = 1,$

$b_2 / m^2 = -1,\ b_3 / m^2 = 0,\ m_d = m_s = m_p = 1.$

Figure 4



$a_0 / n = 50,\ a_1 / n = 0,\ a_2 / n^2 = -0.01,\ b_0 / m = 100,$

$b_1 / m = 0,\ b_2 / m^2 = -0.01,\ b_3 / m^2 = 0,\ m_d = m_s = m_p = 1.$