Canad. Math. Bull. Vol. (), pp. 1–13
http://dx.doi.org/10.4153/CMB-2016-007-4

Generalized Goldberg Formula

Antonio De Nicola and Ivan Yudin

Abstract. In this paper we prove a useful formula for the graded commutator of the Hodge codifferential with the left wedge multiplication by a fixed p-form acting on the de Rham algebra of a Riemannian manifold. Our formula generalizes a formula stated by Samuel I. Goldberg for the case of 1-forms. As first examples of application we obtain new identities on locally conformally Kähler manifolds and quasi-Sasakian manifolds. Moreover, we prove that under suitable conditions a certain subalgebra of differential forms in a compact manifold is quasi-isomorphic as a CDGA to the full de Rham algebra.

1 Introduction

Since the beginnings of differential geometry the importance of formulae that relate various differential objects on a manifold has been apparent. Let us mention among others the Bianchi identities, Weitzenböck formulae, and Frölicher–Nijenhuis calculus. It should be noted that all the above results can be obtained by elementary, although long and tedious, computations. Their importance lies in the psychological and practical plane, as they permit us to work with the quantities in question without undergoing error-prone calculations, thus forming a swiss-army-knife kit for a differential geometer. In this article we prove a formula that we hope will deserve a place in the kit.

Let (M, g) be a Riemannian manifold. As usual, $\Omega^*(M)$ denotes the de Rham algebra of differential forms on M and $\delta\colon \Omega^*(M) \to \Omega^{*-1}(M)$ denotes the Hodge codifferential. Given a k-form ω, we denote by ϵ_ω the operator on $\Omega^*(M)$ defined by $\epsilon_\omega\theta = \omega \wedge \theta$, for every $\theta \in \Omega^l(M)$. In Theorem 3.2, we prove the following expression for the graded commutator of δ with ϵ_ω in terms of Frölicher–Nijenhuis operators (to be defined later)

$$[\delta, \epsilon_\omega] = \epsilon_{\delta\omega} - \mathcal{L}_{\omega^\#} - (-1)^k i_{\omega^\diamond}. \tag{1.1}$$

Here, $\omega^\# \in \Omega^{k-1}(M, TM)$ denotes the vector valued form obtained from $\omega \in \Omega^k(M)$ by metric contraction on the last coordinate, and $\omega^\diamond \in \Omega^k(M, TM)$ is a vector valued k-form defined in Section 3.

Received by the editors September 18, 2015; revised January 8, 2016.
Published electronically 0, 0.
Research partially supported by CMUC – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC, and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020 (A.D.N. and I.Y.), by MICINN (Spain) grants MTM2011-15725-E, MTM2012-34478 (A.D.N.), and by the exploratory research project in the frame of Programa Investigador FCT IF/00016/2013 (I.Y.).
AMS subject classification: **53C25**, 53D35.
Keywords: graded commutator, Hodge codifferential, Hodge laplacian, de Rham cohomology, locally conformal Kaehler manifold, quasi-Sasakian manifold.

Let ξ be a vector field and η its metric dual 1-form. In Corollary 3.3 we show that in this case Formula (1.1) takes the form

$$\{\delta, \epsilon_\eta\} + \mathcal{L}_\xi = \epsilon_{\delta\eta} + i_{(\mathcal{L}_\xi g)^\#}, \tag{1.2}$$

where the curly bracket denotes the anticommutator. Equation (1.2) was stated by Goldberg in [9] and [10, p. 109]. In both cases, Goldberg refrained from explicitly proving this result. Nevertheless, he proved a partial case of (1.2) on [10, pp. 110–111] under the condition that ξ generates a flow of conformal transformations. The absence of a published proof may be one of the reasons that equation (1.2) is not widely known.

Let us give a simple example of use of (1.1). Let (M, g, J) be a Kähler manifold and let $\Omega(X, Y) = g(X, JY)$ be its fundamental 2-form. Then $\Omega^\# = J$ is parallel and Ω is closed and coclosed. One gets easily that the associated vector valued 2-form $\Omega^\diamond$ vanishes (see equation (3.9)). Thus, (1.1) becomes

$$[\delta, \epsilon_\Omega] + \mathcal{L}_J = 0. \tag{1.3}$$

Upon complexification of $\Omega^*(M)$, we can write $d = \partial + \bar\partial$ with

$$\partial \colon \Omega^{p,q}(M) \longrightarrow \Omega^{p+1,q}(M), \quad \bar\partial \colon \Omega^{p,q}(M) \longrightarrow \Omega^{p,q+1}(M).$$

Since $i_J\beta = (p - q)i\beta$ for all $\beta \in \Omega^{p,q}(M)$, we get that

$$\mathcal{L}_J\beta = [i_J, d]\beta = [i_J, \partial + \bar\partial]\beta = -i(\bar\partial - \partial)\beta.$$

Thus, $[\delta, \epsilon_\Omega] - d^c = 0$, where $d^c = i(\bar\partial - \partial)$. This is of course a well-known formula in Kähler geometry, but usually it takes several pages of local computations to prove it.

Note: Equation numbers that are not referenced may have been removed.

In Theorem 3.4 we show the importance of the condition

$$[\delta, \epsilon_\omega] + \mathcal{L}_{\omega^\#} = 0 \tag{1.4}$$

for a p-form ω. Namely, we prove that if (1.4) holds for all $\omega \in S$, where S is a subset of the de Rham algebra $\Omega^*(M)$ of a Riemannian manifold (M, g), then the subalgebra

$$\Omega^*_{\mathcal{L}_{S^\#}}(M) := \left\{ \beta \mid \mathcal{L}_{\omega^\#}\beta = 0,\ \omega \in S \right\}$$

of $\Omega^*(M)$ is quasi-isomorphic to $\Omega^*(M)$ as a commutative differential graded algebra (CDGA), with the quasi-isomorphism given by the embedding. Then the cohomology ring of $\Omega^*_{\mathcal{L}_{S^\#}}(M)$ is isomorphic to the de Rham cohomology ring of M. Note that in the case where M is Kähler manifold, the above-mentioned quasi-isomorphism is the first step in the proof of formality of Kähler manifolds given in [4].

Employing our formula, in Theorem 3.5 we give a complete characterization of all forms ω that satisfy the condition (1.4). Namely, we prove that a p-form ω on a Riemannian manifold (M, g) satisfies (1.4) if and only if one of the following cases holds:

(a) $p = 1$ and $\omega^\#$ is a Killing vector field;
(b) $p \geq 2$ and ω is parallel.

In Section 4 we consider the case of locally conformal Kähler manifolds. By applying formula (1.1), we get the following result, which in a sense generalizes equation (1.3). Let (M, J, g) be a locally conformal Kähler manifold with fundamental

2-form Ω, Lee 1-form θ, and anti-Lee 1-form η. Then for any p-form β we have

$$[\delta, \epsilon_\Omega]\beta = (p-n)\eta \wedge \beta - \mathcal{L}_J\beta + \Omega \wedge i_{\theta^\#}\beta.$$

Finally, in Section 5 we show how our formula works in the context of quasi-Sasakian manifolds. In Theorem 5.1 we prove the following result. Let (M, ϕ, ξ, η, g) be a quasi-Sasakian manifold and let $A := -\phi \circ \nabla\xi$. Then

$$[\delta, \epsilon_\Phi] = -\operatorname{tr}(A)\epsilon_\eta - \mathcal{L}_\phi + 2\epsilon_\eta i_A. \tag{1.5}$$

The special case of formula (1.5) for Sasakian manifolds was first proved by Fujitani [8] by complicated computation in local coordinates. This formula was crucial for the proof of the main result in our recent article [3] on the hard Lefschetz theorem for Sasakian manifolds. We hope that (1.5) will allow us to obtain a suitable generalization of the hard Lefschetz theorem for quasi-Sasakian manifold.

2 Preliminaries

In this section we remind the reader of some notions and results of Frölicher–Nijenhuis calculus [6,7], which will be used later.

A *commutative differential graded algebra* (A, d) (CDGA for short) is a graded algebra $A = \bigoplus_{k\ge 0} A_k$ over $\mathbb{R}$ such that for all $x \in A_k$ and $y \in A_l$ we have

$$xy = (-1)^{kl} yx,$$

together with a differential d of degree one such that $d(xy) = d(x)y + (-1)^k x d(y)$ and $d^2 = 0$. Let M be a smooth manifold of dimension n. Then the direct sum

$$\Omega^*(M) := \bigoplus_{k=1}^{n} \Omega^k(M)$$

is a CDGA with the multiplication given by the wedge product $\wedge$ and the differential given by the exterior derivative $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$.

Let (A, d) be a CDGA. We say that a linear operator $D\colon A \to A$ is a *derivation of degree p* if $D(A_k) \subset A_{k+p}$ for all k, and

$$D(xy) = D(x)y + (-1)^{kp} x D(y)$$

for all $x \in A_k$ and $y \in A_l$.

We write $\Omega^k(M, TM)$ for the space of skew-symmetric TM-valued k-forms on M. Denote by Σ_m the permutation group on $\{1, \dots, m\}$. For k and s such that $k + s = m$, let $\mathrm{Sh}_{k,s}$ be the subset of (k, s)-shuffles in Σ_m. Thus for $\sigma \in \mathrm{Sh}_{k,s}$, we have

$$\sigma(1) < \sigma(2) < \cdots < \sigma(k), \qquad \sigma(k+1) < \cdots < \sigma(k+s).$$

Let $\phi \in \Omega^p(M, TM)$. We define the operator i_ϕ of degree $p-1$ on $\Omega^*(M)$ by

$$(i_\phi\omega)(Y_1, \dots, Y_{p+k-1}) = \sum_{\sigma \in \mathrm{Sh}_{p,k-1}} (-1)^\sigma \omega\big(\phi(Y_{\sigma(1)}, \dots, Y_{\sigma(p)}), Y_{\sigma(p+1)}, \dots, Y_{\sigma(p+k-1)}\big),$$

where $\omega \in \Omega^k(M)$. The Lie derivative $\mathcal{L}_\phi$ is an operator of degree p on $\Omega^*(M)$ defined as the graded commutator $[i_\phi, d]$.

We now recall the fundamental theorem of Frölicher–Nijenhuis calculus.

Theorem 2.1 ([6]) *Let $D\colon \Omega^*(M) \to \Omega^*(M)$ be a derivation of degree p. Then there are unique $\phi \in \Omega^p(M, TM)$ and $\psi \in \Omega^{p+1}(M, TM)$ such that $D = \mathcal{L}_\phi + i_\psi$.*

As a consequence of the above theorem, we get the following:

(a) If a TM-valued p-form ϕ is different from 0, then $i_\phi \neq 0$.
(b) If $D\colon \Omega^*(M) \to \Omega^*(M)$ is a derivation such that $[D, d] = 0$, then there is a unique $\phi \in \Omega^p(M, TM)$ such that $D = \mathcal{L}_\phi$.

For a k-form $\omega \in \Omega^k(M)$ and TM-valued p-form ϕ, we define the TM-valued $(p+k)$-form $\omega\wedge\phi$ by

$$(\omega\wedge\phi)(Y_1, \dots, Y_{p+k}) = \sum_{\sigma\in \mathrm{Sh}_{k,p}} (-1)^\sigma \omega(Y_{\sigma(1)}, \dots, Y_{\sigma(k)})\phi(Y_{\sigma(k+1)}, \dots, Y_{\sigma(k+p)}).$$

Following [7], we will define the *contraction* (sometimes called *trace*) operator

$$\mathrm{C}\colon \Omega^p(M, TM) \longrightarrow \Omega^{p-1}(M)$$

as follows. Every $\phi \in \Omega^p(M, TM)$ can be written locally as a finite sum $\sum_{i\in I} \omega_i \wedge X_i$, where X_i are vector fields and $\omega_i \in \Omega^p(M)$. Then

$$\mathrm{C}(\phi) := \sum_{i\in I} i_{X_i}\omega_i.$$

One can check that $\mathrm{C}(\phi)$ does not depend on the choice of the local presentation for ϕ. We will use the following property [7, eq. (2.12)]:

$$\mathrm{C}(\omega \wedge \phi) = (-1)^k \omega\wedge \mathrm{C}(\phi) + (-1)^{(k+1)p} i_\phi \omega \tag{2.1}$$

for any $\omega \in \Omega^k(M)$ and $\phi \in \Omega^p(M, TM)$. Given $\omega \in \Omega^k(M)$, we define

$$\begin{aligned} \epsilon_\omega\colon \Omega^p(M, TM) &\longrightarrow \Omega^{p+k}(M, TM) \\ \phi &\longmapsto \omega\wedge\phi. \end{aligned}$$

For an operator $A\colon \Omega^*(M) \to \Omega^*(M)$ and $\omega \in \Omega^*(M)$, we abbreviate the composition $\epsilon_\omega \circ A$ by $\omega \wedge A$. It is easy to check that $\omega \wedge i_\phi = i_{\omega\wedge\phi}$.

We will need the following fact.

Proposition 2.2 *Let M be a smooth manifold, $\omega \in \Omega^k(M)$, and $\phi \in \Omega^p(M, TM)$. Then*

$$\omega \wedge \mathcal{L}_\phi = \mathcal{L}_{\omega\wedge\phi} - (-1)^{p+k} i_{(d\omega)\wedge\phi}.$$

Proof The computation

$$\mathcal{L}_{\omega\wedge\phi} = [i_{\omega\wedge\phi}, d] = [\omega \wedge i_\phi, d] = (-1)^{k+p}(d\omega) \wedge i_\phi + \omega \wedge \mathcal{L}_\phi$$

proves the claim. ∎

3 Generalized Goldberg Formula

In this section we prove the main result of the article. Let M be a smooth manifold equipped with a Riemannian metric g and let ∇ denote the corresponding Levi–Civita connection. Using ∇, we can define the map

$$d^\nabla\colon \Omega^p(M,TM) \to \Omega^{p+1}(M,TM)$$

similarly to the standard exterior derivative, as follows

$$\begin{aligned} d^\nabla\phi(Y_1,\dots,Y_{p+1}) = &\sum_{s=1}^{p+1}(-1)^{s-1}\nabla_{Y_s}\big(\phi(Y_1,\dots,\widehat{Y}_s,\dots,Y_{p+1})\big)\\ &+\sum_{s<t}(-1)^{s+t}\phi\big([Y_s,Y_t],Y_1,\dots,\widehat{Y}_s,\dots,\widehat{Y}_t,\dots,Y_{p+1}\big). \end{aligned}$$

Since for the Levi–Civita connection we have $[Y,Z]=\nabla_Y Z-\nabla_Z Y$, one can easily check that

$$(d^\nabla\phi)(Y_1,\dots,Y_{p+1}) = \sum_{s=1}^{p+1}(-1)^{s+1}(\nabla_{Y_s}\phi)(Y_1,\dots,\widehat{Y}_s,\dots,Y_{p+1}). \tag{3.1}$$

Moreover, note that d^∇ is related to the Riemann curvature by the formula

$$(d^\nabla)^2\phi(Y_1,\dots,Y_{p+2}) = \sum_{\sigma\in \mathrm{Sh}_{2,p}}(-1)^\sigma R(Y_{\sigma(1)},Y_{\sigma(2)})\big(\phi(Y_{\sigma(3)},\dots,Y_{\sigma(p+2)})\big).$$

For $\omega\in\Omega^k(M)$ and $\phi\in\Omega^p(M,TM)$, we have

$$d^\nabla(\omega\wedge\phi) = (d\omega)\wedge\phi + (-1)^k\omega\wedge(d^\nabla\phi).$$

Note that for any vector field $X\in\Omega^0(M,TM)$, we get

$$d^\nabla X(Y) = \nabla_Y X.$$

Hence, $d^\nabla X=\nabla X$. Thus, we can think about ∇-parallel vector fields as a generalization of harmonic functions. For any k-form ω and any vector field X, we get

$$\mathcal{L}_X\omega = \nabla_X\omega + i_{\nabla X}\omega.$$

In other words $\nabla_X = \mathcal{L}_X - i_{d^\nabla X}$. This equation suggests the following generalization of the covariant derivative. Namely, for $\phi\in\Omega^p(M,TM)$, we define

$$\nabla_\phi := \mathcal{L}_\phi - (-1)^p i_{d^\nabla\phi}. \tag{3.2}$$

We get

$$\begin{aligned} \omega\wedge\nabla_\phi &= \omega\wedge\mathcal{L}_\phi - \omega\wedge i_{d^\nabla\phi} = \mathcal{L}_{\omega\wedge\phi} - (-1)^{p+k}i_{(d\omega)\wedge\phi} - (-1)^p i_{\omega\wedge d^\nabla\phi}\\ &= \mathcal{L}_{\omega\wedge\phi} - (-1)^{p+k}i_{d\omega\wedge\phi+(-1)^k\omega\wedge d^\nabla\phi} = \mathcal{L}_{\omega\wedge\phi} - (-1)^{p+k}i_{d^\nabla(\omega\wedge\phi)} \end{aligned}$$

that is, $\omega\wedge\nabla_\phi = \nabla_{\omega\wedge\phi}$. This equation is a generalization of the property $f\nabla_X = \nabla_{fX}$ for the usual covariant derivative, where $f\in C^\infty(M)$ and $X\in\Omega^0(M,TM)$.

The Hodge codifferential is abstractly defined as the Hodge dual of the operator d on Ω. It is well known that given a local orthonormal frame $X_1,\dots,X_n$ on $U\subset M$, the following local expression for the codifferential holds

$$\delta = -\sum_{t=1}^n i_{X_t}\circ\nabla_{X_t}.$$

Since both i_{X_t} and ∇_{X_t} are derivations of $\Omega^*(U)$, we see that δ is a differential operator of order 2 on $\Omega^*(U)$, and thus also on $\Omega^*(M)$.

Let $\omega \in \Omega^p(M)$. Then $[\delta, \epsilon_\omega]$ is a differential operator of order 1 and of degree $p-1$ on $\Omega^*(M)$. Thus, it can be expressed in a unique way as a sum $\epsilon_\alpha + \nabla_\phi + i_\psi$ for suitable $(p-1)$-form α, TM-valued $(p-1)$-form ϕ, and TM-valued $(p+1)$-form ψ. Our aim is to identify α, ϕ, and ψ for a given ω.

For $\omega \in \Omega^p(M)$, we define $\omega^\# \in \Omega^{p-1}(M, TM)$ and $\omega^\nabla \in \Omega^p(M, TM)$ by

$$\omega^\# = \sum_{t=1}^n (i_{X_t}\omega)\wedge X_t \qquad \omega^\nabla = \sum_{t=1}^n (\nabla_{X_t}\omega)\wedge X_t. \tag{3.3}$$

It is easy to see that $\omega^\#$ and ω^∇ do not depend on the choice of the orthonormal frame $X_1, \dots, X_n$. Therefore, $\omega^\#$ and ω^∇ are well defined. By applying the contraction operator C to (3.3), we get

$$\mathrm{C}(\omega^\#) = \sum_{t=1}^n i^2_{X_t}\omega = 0, \tag{3.4}$$

$$\mathrm{C}(\omega^\nabla) = \sum_{t=1}^n i_{X_t}\nabla_{X_t}\omega = -\delta\omega. \tag{3.5}$$

Proposition 3.1 *For any $\omega \in \Omega^p(M)$, we have $d^\nabla(\omega^\#) + (d\omega)^\# = \omega^\nabla$.*

Proof Let $X_1, \dots, X_n$ be an orthonormal frame on an open set U in M. By definition of ω^∇ and the Leibniz rule for d^∇, we get

$$d^\nabla(\omega^\#) = \sum_{t=1}^n d(i_{X_t}\omega)\wedge X_t + (-1)^{p-1}\sum_{t=1}^n i_{X_t}\omega\wedge\nabla X_t. \tag{3.6}$$

Further,

$$(d\omega)^\# = \sum_{t=1}^n i_{X_t}(d\omega)\wedge X_t. \tag{3.7}$$

Note that for every $1 \le t \le n$, we have

$$d(i_{X_t}\omega) + i_{X_t}(d\omega) = \mathcal{L}_{X_t}\omega = \nabla_{X_t}\omega + i_{\nabla X_t}\omega.$$

Therefore, summing (3.6) with (3.7), we get

$$\begin{aligned} d^\nabla(\omega^\#) + (d\omega)^\# &= \sum_{t=1}^n \nabla_{X_t}\omega\wedge X_t + \sum_{t=1}^n i_{\nabla X_t}\omega\wedge X_t + (-1)^{p-1}\sum_{t=1}^n i_{X_t}\omega\wedge\nabla X_t \\ &= \omega^\nabla + \sum_{t=1}^n i_{\nabla X_t}\omega\wedge X_t + (-1)^{p-1}\sum_{t=1}^n i_{X_t}\omega\wedge\nabla X_t. \end{aligned}$$

Let us denote the expression

$$\sum_{t=1}^n i_{\nabla X_t}\omega\wedge X_t + (-1)^{p-1}\sum_{t=1}^n i_{X_t}\omega\wedge\nabla X_t$$

by T. Since $T = d^\nabla(\omega^\#) + (d\omega)^\# - \omega^\nabla$, we see that T does not depend on the choice of the orthonormal basis $X_1, \dots, X_n$ and that T is a tensor on M. Let $x \in M$. Then there is an local orthonormal frame $X_1, \dots, X_n$ on an open neighbourhood of x such that $(\nabla X_t)_x = 0$ for every $1 \le t \le n$. Computing T_x with respect to this basis, we see that $T_x = 0$. Since x is an arbitrary point of M, we see that $T \equiv 0$. ∎

Let us define for every $\omega \in \Omega^p(M)$ the TM-valued form

$$\omega^\diamond = d^\nabla(\omega^\#) + \omega^\nabla.$$

Note that by Proposition 3.1 we can write it in two other ways:

$$\omega^\diamond = 2d^\nabla(\omega^\#) + (d\omega)^\#, \tag{3.8}$$

$$\omega^\diamond = 2\omega^\nabla - (d\omega)^\#. \tag{3.9}$$

Now (3.4) and (3.5) give the following expression for $\delta\omega$ in terms of $\omega^\diamond$:

$$\delta\omega = -\tfrac{1}{2}\,\mathrm{C}(\omega^\diamond). \tag{3.10}$$

We can now prove the announced formula, (1.1), for the commutator of the codifferential with the left wedge multiplication by a k-form.

Note: "(1.1)" inserted.

Theorem 3.2 *Let $\omega \in \Omega^p(M)$. Then*

$$[\delta, \epsilon_\omega] = \epsilon_{\delta\omega} - \nabla_{\omega^\#} - (-1)^p i_{\omega^\nabla}, \tag{3.11}$$

or, using the Lie derivative instead of the covariant derivative,

$$[\delta, \epsilon_\omega] = \epsilon_{\delta\omega} - \mathcal{L}_{\omega^\#} - (-1)^p i_{\omega^\diamond}. \tag{3.12}$$

Proof Let X be a vector field and $\omega \in \Omega^p(M)$. Then

$$\begin{aligned}
[i_X \circ \nabla_X, \epsilon_\omega] &= [i_X, \epsilon_\omega] \circ \nabla_X + i_X \circ [\nabla_X, \epsilon_\omega] \\
&= \epsilon_{i_X\omega} \nabla_X + i_X \epsilon_{\nabla_X\omega} \\
&= \epsilon_{i_X\omega} \nabla_X + [i_X, \epsilon_{\nabla_X\omega}] + (-1)^p \epsilon_{\nabla_X\omega} i_X \\
&= \nabla_{i_X\omega\wedge X} + \epsilon_{i_X\nabla_X\omega} + (-1)^p \epsilon_{\nabla_X\omega} i_X \\
&= \epsilon_{i_X\nabla_X\omega} + \nabla_{i_X\omega\wedge X} + (-1)^p i_{\nabla_X\omega\wedge X}.
\end{aligned}$$

Now (3.11) follows by substituting X_t instead of X and summing up over t.

Since $\omega^\# \in \Omega^{p-1}(M, TM)$, from (3.2) we get

$$\nabla_{\omega^\#} = \mathcal{L}_{\omega^\#} - (-1)^{p-1} i_{d^\nabla(\omega^\#)} = \mathcal{L}_{\omega^\#} + (-1)^p i_{d^\nabla(\omega^\#)}.$$

Therefore,

$$[\delta, \epsilon_\omega] = \epsilon_{\delta\omega} - \mathcal{L}_{\omega^\#} - (-1)^p \big(i_{d^\nabla(\omega^\#)} + i_{\omega^\nabla} \big). \qquad \blacksquare$$

As a corollary we can get [9, Formula (4)] in Goldberg's article.

Corollary 3.3 *Let ξ be a vector field on a Riemannian manifold M and let η be its metric dual 1-form. Then $\eta^\diamond = (\mathcal{L}_\xi g)^\#$; that is,*

$$\{\delta, \epsilon_\eta\} + \mathcal{L}_\xi = \epsilon_{\delta\eta} + i_{(\mathcal{L}_\xi g)^\#}, \tag{3.13}$$

where $\{\cdot,\cdot\}$ denotes the anti-commutator of operators and $(\mathcal{L}_\xi g)^\#$ is the metric contraction of the $(0,2)$-tensor $\mathcal{L}_\xi g$.

Proof We have to check that $d^\nabla\eta^\# + \eta^\nabla = (\mathcal{L}_\xi g)^\#$. Since $\eta^\# = \xi$, we have for any vector field Y,

$$(d^\nabla\eta^\#)(Y) = (d^\nabla\xi)(Y) = \nabla_Y\xi = \sum_{t=1}^n g(X_t, \nabla_Y\xi)X_t, \tag{3.14}$$

where $X_1, \dots, X_n$ is a local orthonormal frame on M. Further,

$$\eta^{\nabla}(Y) = \sum_{t=1}^{n} (\nabla_{X_t}\eta)(Y)X_t = \sum_{t=1}^{n} g(\nabla_{X_t}\xi, Y)X_t. \tag{3.15}$$

It is well known that

$$(\mathcal{L}_{\xi} g)(Y,Z) = g(\nabla_Y \xi, Z) + g(\xi, \nabla_Z \xi)$$

for any vector fields ξ, Y, and Z. Therefore, adding (3.14) and (3.15), we get

$$(d^{\nabla}\xi + \eta^{\nabla})(Y) = \sum_{t=1}^{n} (\mathcal{L}_{\xi} g)(X_t, Y)X_t = (\mathcal{L}_{\xi} g)^{\#}(Y). \qquad \blacksquare$$

Let S be a set of differential forms on M. We will denote by $S^{\#}$ the set of vector valued forms $\omega^{\#}$, where $\omega \in S$. Further, we write $\Omega^{*}_{\mathcal{L}_{S^{\#}}}(M)$ for the intersection of the kernels of operators $\mathcal{L}_{\omega^{\#}}$, $\omega \in S$.

Recall that a morphism of CDGAs is a morphism of algebras that preserves the degree and commutes with the differentials. Let $f\colon (A,d) \to (B,d)$ be a morphism of CDGAs. For every $k \geq 0$, the map f induces a morphism between the k-th cohomologies

$$H^k(f)\colon H^k(A) \longrightarrow H^k(B).$$

If all the morphisms $H^k(f)$ are isomorphisms, then f is called a *quasi-isomorphism* of CDGAs.

We have the following theorem that generalizes several known facts.

Theorem 3.4 *Let (M,g) be a compact Riemannian manifold. Suppose $S \subset \Omega^{*}(M)$ is such that $[\delta, \epsilon_{\omega}] + \mathcal{L}_{\omega^{\#}} = 0$ for all $\omega \in S$. Then the inclusion*

$$j\colon \Omega^{*}_{\mathcal{L}_{S^{\#}}}(M) \hookrightarrow \Omega^{*}(M)$$

is a quasi-isomorphism of CDGAs.

Proof Let $\omega \in S$. Since $[\delta, \epsilon_{\omega}] + \mathcal{L}_{\omega^{\#}} = 0$ and $\delta^2 = 0$, we get that

$$[\delta, \mathcal{L}_{\omega^{\#}}] = -\big[\delta, [\delta, \epsilon_{\omega}]\big] = 0.$$

Since the Hodge Laplacian Δ is the graded commutator of d and δ, we have also that $[\Delta, \mathcal{L}_{\omega^{\#}}] = 0$.

Let β be a harmonic p-form. We are going to show that $\beta \in \Omega^{p}_{\mathcal{L}_{S^{\#}}}(M)$. This will imply by Hodge theory that j induces a surjection in cohomology. Since $[\Delta, \mathcal{L}_{\omega^{\#}}] = 0$ for all $\omega \in S$, we get immediately that $\Delta(\mathcal{L}_{\omega^{\#}}\beta) = 0$, *i.e.*, $\mathcal{L}_{\omega^{\#}}\beta$ is harmonic. But, since β is closed, we have that $\mathcal{L}_{\omega^{\#}}\beta = d i_{\omega^{\#}}\beta$ is an exact form. Thus, by Hodge theory, $\mathcal{L}_{\omega^{\#}}\beta = 0$.

It is left to show that j induces an injection in cohomology. Let $\beta \in \Omega^{p}_{\mathcal{L}_{S^{\#}}}(M)$ such that $[\beta] = 0$ in $H^p(M)$. Then $\beta = dG\delta\beta$, where G is the Green operator for Δ. We are going to show that $G\delta\beta \in \Omega^{p}_{\mathcal{L}_{S^{\#}}}(M)$. For this, it is enough to prove that $\mathcal{L}_{\omega^{\#}}G = G\mathcal{L}_{\omega^{\#}}$ for every $\omega \in S$. In fact, then

$$\mathcal{L}_{\omega^{\#}}G\delta\beta = G\delta\mathcal{L}_{\omega^{\#}}\beta = 0, \quad \forall \omega \in S.$$

We have

$$I - G\Delta = \Pi_\Delta, \qquad I - \Delta G = \Pi_\Delta, \tag{3.16}$$

where Π_Δ is the orthogonal projection on the set of harmonic forms. Now we multiply the equation $\mathcal{L}_{\omega^\#}\Delta = \Delta\mathcal{L}_{\omega^\#}$ by G on the left and right-hand sides. We get

$$G\mathcal{L}_{\omega^\#}\Delta G = G\Delta\mathcal{L}_{\omega^\#}G.$$

Applying (3.16), we obtain

$$G\mathcal{L}_{\omega^\#} - G\mathcal{L}_{\omega^\#}\Pi_\Delta = \mathcal{L}_{\omega^\#}G - \Pi_\Delta\mathcal{L}_{\omega^\#}G.$$

As we saw above, $\mathcal{L}_{\omega^\#}$ annihilates harmonic forms, hence $\mathcal{L}_{\omega^\#}\Pi_\Delta = 0$. To finish the proof it is enough to check that $\Pi_\Delta\mathcal{L}_{\omega^\#} = 0$. Let $\alpha \in \Omega^k(M)$. By Hodge theory, we can write α as $\alpha_\delta + \alpha_\Delta + \alpha_d$, where α_δ is in the image of δ, α_d is in the image of d, and α_Δ is harmonic. Note that $\mathcal{L}_{\omega^\#}\alpha_\Delta = 0$. Further, $\mathcal{L}_{\omega^\#}\alpha_d = \pm d i_{\omega^\#}\alpha_d$, where the sign depends on the degree of ω. In particular, $\mathcal{L}_{\omega^\#}\alpha_d$ is exact, and therefore $\Pi_\Delta\mathcal{L}_{\omega^\#}\alpha_d = 0$. Finally, since $[\delta, \epsilon_\omega] + \mathcal{L}_{\omega^\#} = 0$, we get

$$\mathcal{L}_{\omega^\#}\alpha_\delta = -[\delta, \epsilon_\omega]\alpha_\delta = -\delta(\omega \wedge \alpha_\delta).$$

Hence, $\mathcal{L}_{\omega^\#}\alpha_\delta$ is a coexact form, and thus $\Pi_\Delta\mathcal{L}_{\omega^\#}\alpha_\delta = 0$. ■

The previous theorem shows the importance of the property $[\delta, \omega] + \mathcal{L}_{\omega^\#} = 0$ for a differential form ω. In the following theorem we characterize all the forms with this property.

Theorem 3.5 *Let (M, g) be a Riemannian manifold and ω a p-form on M, with $p \geq 1$. Then $[\delta, \epsilon_\omega] + \mathcal{L}_{\omega^\#} = 0$ if and only if one of the following conditions holds:*

(i) *$p = 1$ and $\omega^\#$ is a Killing vector field;*
(ii) *$p \geq 2$ and ω is parallel.*

Proof Let us first consider the case $p = 1$. Suppose $\xi = \omega^\#$ is Killing. Then $\mathcal{L}_\xi g = 0$. By Corollary 3.3, we have $\omega^\diamond = (\mathcal{L}_\xi g)^\# = 0$. Applying (3.10), we get $\delta\omega = -\frac{1}{2}\,\mathrm{C}(\omega^\diamond) = 0$. By (3.13), we obtain that $\{\delta, \epsilon_\omega\} + \mathcal{L}_\xi = 0$.

Now, suppose that $\{\delta, \epsilon_\omega\} + \mathcal{L}_\xi = 0$. Then from (3.13), we have

$$\epsilon_{\delta\omega} + i_{(\mathcal{L}_\xi g)^\#} = 0. \tag{3.17}$$

Applying (3.17) to the constant function with the value 1, we get $\delta\omega = 0$. Thus $i_{(\mathcal{L}_\xi g)^\#} = 0$. By Theorem 2.1, we have $\mathcal{L}_\xi g = 0$, and thus ξ is a Killing vector field.

Now suppose $p \geq 2$ and $\nabla\omega = 0$. Then, by looking at defining formulae one readily sees that $\delta\omega = 0$, $d\omega = 0$, and $\omega^\nabla = 0$. Thus, by (3.12) we get that $[\delta, \epsilon_\omega] + \mathcal{L}_{\omega^\#} = 0$.

Finally, suppose that $[\delta, \epsilon_\omega] + \mathcal{L}_{\omega^\#} = 0$. Then by (3.12), we have

$$\epsilon_{\delta\omega} - (-1)^p i_{\omega^\diamond} = 0. \tag{3.18}$$

Applying (3.18) to the constant function 1, we get that $\delta\omega = 0$. Therefore, $i_{\omega^\diamond} = 0$ and, by Theorem 2.1, we have $\omega^\diamond = 0$. Using (3.9) and (3.3), we obtain

$$0 = \omega^\diamond = \sum_{t=1}^{n} 2\nabla_{X_t}\omega\wedge X_t - \sum_{t=1}^{n} i_{X_t}\omega\wedge X_t = \sum_{t=1}^{n}(2\nabla_{X_t}\omega - i_{X_t}d\omega)\wedge X_t,$$

where $X_1, \dots, X_n$ is a local orthonormal frame on M. Since $X_1, \dots, X_n$ are linearly independent at every point, we obtain that $2\nabla_{X_t}\omega = i_{X_t}d\omega$ for all t. But this implies that

$$2\nabla_Z\omega = i_Z d\omega \tag{3.19}$$

for every vector field Z.

Let $Y_0, \dots, Y_p$ be vector fields. Then by using (3.19) we get

$$\begin{aligned}
2(d\omega)(Y_0, \dots, Y_p) &= \sum_{s=0}^{p}(-1)^s(2\nabla_{Y_s}\omega)(Y_0, \dots, \widehat{Y_s}, \dots, Y_p)\\
&= \sum_{s=0}^{p}(-1)^s(i_{Y_s}d\omega)(Y_0, \dots, \widehat{Y_s}, \dots, Y_p)\\
&= \sum_{s=0}^{p}(d\omega)(Y_0, \dots, Y_p) = (p+1)d\omega(Y_0, \dots, Y_p).
\end{aligned}$$

Since $p \neq 1$, we obtain $d\omega = 0$. Now (3.19) implies $\nabla\omega = 0$. ■

4 Locally Conformal Kähler Manifolds

In this section, we show how Theorem 3.2 works in the context of locally conformal Kähler manifolds.

Let (M^{2n+2}, g) be a Riemannian manifold and let J be a complex structure on M. Then (M, J, g) is called *Hermitian* if $g(JX, JY) = g(X, Y)$ for all vector fields X, Y on M. For an Hermitian manifold (M, J, g), we define its *fundamental* 2*-form* Ω by $\Omega(X, Y) = g(X, JY)$. Thus, $\Omega^\# = J$. An Hermitian manifold (M, J, g) is called *locally conformal Kähler* (l.c.K.) if there exists a 1-form θ (called the *Lee form*) such that $d\Omega = \theta \wedge \Omega$. We are going to apply Theorem 3.2 to $\omega = \Omega$. For this we have to compute $\Omega^\diamond$ and $\delta\Omega$. We define $\eta = i_J\theta$. It is proved in [5, Corollary 1.1] that

$$(\nabla_X J)Y = \tfrac{1}{2}\big(\eta(Y)X - \theta(Y)JX - g(X, Y)\eta^\# - \Omega(X, Y)\theta^\#\big).$$

Thus,

$$\begin{aligned}
d^\nabla J(X, Y) &= (\nabla_X J)Y - (\nabla_Y J)X\\
&= \tfrac{1}{2}\big(\eta(Y)X - \theta(Y)JX - \eta(X)Y + \theta(X)JY - 2\Omega(X, Y)\theta^\#\big)\\
&= \tfrac{1}{2}(-(\eta\wedge \mathrm{Id})(X, Y) + (\theta\wedge J)(X, Y)) - (\Omega\wedge\theta^\#)(X, Y).
\end{aligned}$$

Hence, we get

$$d^\nabla J = \tfrac{1}{2}(\theta\wedge J - \eta\wedge \mathrm{Id}) - \Omega\wedge\theta^\#.$$

Using the definition of #, it is easy to check that

$$(d\Omega)^\# = (\theta \wedge \Omega)^\# = \Omega\wedge\theta^\# - \theta\wedge\Omega^\# = \Omega\wedge\theta^\# - \theta\wedge J. \tag{4.1}$$

Thus, by (3.8)

$$\Omega^\diamond = 2d^\nabla J + (d\Omega)^\# = -\eta\wedge \mathrm{Id} - \Omega\wedge\theta^\#. \tag{4.2}$$

Moreover, due to (3.4), by contracting (4.1) we get

$$\mathrm{C}(\Omega\wedge\theta^\#) = \mathrm{C}(\theta\wedge J)$$

Hence by (3.10), we obtain from (4.2)

$$\delta\Omega = -\tfrac{1}{2}\,\mathrm{C}(\Omega^{\diamond}) = \tfrac{1}{2}\big(\mathrm{C}(\eta\wedge\mathrm{Id}) + \mathrm{C}(\Omega\wedge\theta^{\#})\big) = \tfrac{1}{2}\big(\mathrm{C}(\eta\wedge\mathrm{Id}) + \mathrm{C}(\theta\wedge J)\big).$$

Using (2.1), we have

$$\begin{aligned}\mathrm{C}(\eta\wedge\mathrm{Id}) &= -\,\mathrm{C}(\mathrm{Id})\eta + i_{\mathrm{Id}}\eta = -(2n+2)\eta + \eta = -(2n+1)\eta,\\ \mathrm{C}(\theta\wedge J) &= -\,\mathrm{C}(J)\theta + i_J\theta = \eta.\end{aligned}$$

Therefore,

$$\delta\Omega = \tfrac{1}{2}\big(\eta - (2n+1)\eta\big) = -n\eta.$$

Applying Theorem 3.2, we get the following formula that in a sense generalizes equation (1.3), which holds for Kähler manifolds.

Theorem 4.1 *Let (M, J, g) be a locally conformal Kähler manifold. Let Ω be the fundamental 2-form, θ the Lee 1-form, and $\eta = i_J\theta$. Then, for any p-form β, we have*

$$[\delta, \epsilon_\Omega]\beta = (p-n)\eta\wedge\beta - \mathcal{L}_J\beta + \Omega\wedge i_{\theta^{\#}}\beta.$$

5 Quasi-Sasakian Manifolds

In this section we will show how Theorem 3.2 can be used to get useful formulae for commutators on quasi-Sasakian manifolds.

Recall that an *almost contact metric* structure on a manifold M^{2n+1} is a quadruple (ϕ, ξ, η, g), where ϕ is an endomorphism of TM, ξ is a vector field, η is a 1-form, and g is a Riemannian metric such that

$$\begin{aligned}\phi^2 &= -\mathrm{Id} + \eta\otimes\xi, & \eta(\xi) &= 1,\\ g(\phi X, Y) &= -g(X, \phi Y), & \eta(X) &= g(X, \xi),\end{aligned}$$

for any vector fields X and Y. As a consequence, one easily gets that $\phi(\xi) = 0$ and $\eta\circ\phi = 0$. We define an almost complex structure J on $M\times\mathbb{R}$ by

$$J\Big(X, f\frac{d}{dt}\Big) = \Big(\phi X - f\xi, \eta(X)\frac{d}{dt}\Big),$$

where f is a smooth function on $M\times\mathbb{R}$. If J is integrable, the almost contact metric structure (ϕ, ξ, η, g) on M is called *normal*. We define a 2-form Φ by

$$\Phi(X, Y) = g(X, \phi Y), \text{ for any } X, Y \in \mathfrak{X}(M).$$

A normal almost contact metric structure (ϕ, ξ, η, g) on M is called *quasi-Sasakian* if Φ is closed.

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a quasi-Sasakian manifold. We define $A := -\phi\circ\nabla\xi$. We are going to apply Theorem 3.2 to $\omega = \Phi$. For this we have to compute $\Phi^{\#}$, $\Phi^{\diamond}$, and $\delta\Phi$. From the definition of Φ, we have that $\Phi^{\#} = \phi$. Since Φ is closed, from (3.8), we get $\Phi^{\diamond} = 2d^{\nabla}\phi$. In [11] it was shown that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \qquad g(AX, Y) = g(X, AY).$$

Thus, by (3.1), we have

$$(d^{\nabla}\phi)(X,Y) = (\nabla_X\phi)(Y) - (\nabla_Y\phi)(X)$$
$$= \eta(Y)AX - g(AX,Y)\xi - \eta(X)AY + g(X,AY)\xi = -(\eta\wedge A)(X,Y).$$

Therefore,

$$\Phi^{\diamond} = -2\eta\wedge A. \tag{5.1}$$

Further, by (3.10), we have

$$\delta\Phi = -\tfrac{1}{2}\,\mathrm{C}(\Phi^{\diamond}) = \mathrm{C}(\eta\wedge A). \tag{5.2}$$

By (2.1), we have

$$\mathrm{C}(\eta\wedge A) = -\eta\wedge\mathrm{C}(A) + i_A\eta = -\,\mathrm{C}(A)\eta + i_A\eta. \tag{5.3}$$

Since $A = -\phi\circ\nabla\xi$ and $\eta\circ\phi = 0$, combining (5.2) and (5.3), we finally get $\delta\Phi = -\,\mathrm{C}(A)\eta$. Thus, by Theorem 3.2 and (5.1), we have

$$[\delta,\epsilon_\Phi] = -\epsilon_{\mathrm{C}(A)\eta} - \mathcal{L}_\phi + i_{2\eta\wedge A}.$$

Since A is an endomorphism of TM, we actually have $\mathrm{C}(A) = \mathrm{tr}(A)$. Hence, we have proved the following result.

Theorem 5.1 *Let (M,ϕ,ξ,η,g) be a quasi-Sasakian manifold. Then*

$$[\delta,\epsilon_\Phi] = -\,\mathrm{tr}(A)\epsilon_\eta - \mathcal{L}_\phi + 2\epsilon_\eta i_A.$$

The most important examples of quasi-Sasakian manifolds are co-Kähler manifolds (see [2]) and Sasakian manifolds (see [1]). For every co-Kähler manifold, one has $\nabla\xi = 0$, and thus $A = 0$. Therefore, in co-Kähler case, we get $[\delta,\epsilon_\Phi] = -\mathcal{L}_\phi$, which could also have been achieved by using the fact that ϕ is parallel on a co-Kähler manifold and Theorem 3.5.

For Sasakian manifolds, one has $\nabla\xi = -\phi$, and thus $A = \phi^2 = -\mathrm{Id} + \eta\wedge\xi$. Therefore $\mathrm{tr}\,A = -2n$ in this case. Applying Theorem 5.1, we get

$$[\delta,\epsilon_\phi] = 2n\epsilon_\eta - \mathcal{L}_\Phi + 2\epsilon_\eta(-i_{\mathrm{Id}} + \epsilon_\eta i_\xi) = 2n\epsilon_\eta - \mathcal{L}_\phi - 2\epsilon_\eta i_{\mathrm{Id}}. \tag{5.4}$$

Formula (5.4) was first proved by Fujitani in [8] by complicated computation in local coordinates. This formula was crucial for some proofs in our recent article [3] on the hard Lefschetz theorem for Sasakian manifolds. We hope that Theorem 5.1 will permit us to find a suitable generalization of the Hard Lefschetz Theorem for quasi-Sasakian manifold.

References

[1] C. P. Boyer and K. Galicki, *Sasakian geometry*. Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008.

[2] B. Cappelletti-Montano, A. De Nicola, and I. Yudin, *A survey on cosymplectic geometry*. Rev. Math. Phys. 25(2013), no. 10, 1343002, 55. http://dx.doi.org/10.1142/S0129055X13430022 http://dx.doi.org/10.1142/S0129055X13430022

[3] ———, *Hard Lefschetz theorem for Sasakian manifolds*. J. Differential Geom. **101**(2015), no. 1, 47–66.

[4] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds*. Invent. Math. **29**(1975), no. 3, 245–274. http://dx.doi.org/10.1007/BF01389853

[5] S. Dragomir and L. Ornea, *Locally conformal Kähler geometry*. Progress in Mathematics, 155, Birkhäuser Boston, Inc., Boston, MA, 1998. http://dx.doi.org/10.1007/978-1-4612-2026-8

[6] A. Frölicher and A. Nijenhuis, *Theory of vector-valued differential forms. I. Derivations of the graded ring of differential forms*. Nederl. Akad. Wetensch. Proc. Ser. A. **59**; Indag. Math. **18**(1956), 338–359. http://dx.doi.org/10.1016/S1385-7258(56)50046-7

[7] ———, *Some new cohomology invariants for complex manifolds. I. II*. Nederl. Akad. Wetensch. Proc. Ser. A. **59**; Indag. Math. **18**(1956), 540–552, 553–564.

[8] T. Fujitani, *Complex-valued differential forms on normal contact Riemannian manifolds*. Tôhoku Math. J. (2) **18**(1966), 349–361. http://dx.doi.org/10.2748/tmj/1178243376

[9] S. I. Goldberg, *Conformal transformations of Kaehler manifolds*. Bull. Amer. Math. Soc. **66**(1960), 54–58. http://dx.doi.org/10.1090/S0002-9904-1960-10390-4

[10] ———, *Curvature and homology*. Pure and Applied Mathematics, 11, Academic Press, New York-London, 1962.

[11] S. Kanemaki, *Quasi-Sasakian manifolds*. Tôhoku Math. J. **29**(1977), no. 2, 227–233. http://dx.doi.org/10.2748/tmj/1178240654

CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
e-mail: antondenicola@gmail.com yudin@mat.uc.pt