Comment on "Quantum chaotic system in the generalized Husimi representation"

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In a recent paper [K. Życzkowski, Phys. Rev. A **35**, 3546 (1987)] the generalized Husimi distribution was used to investigate the quantum kicked rotator. It was shown that in the classically chaotic region the Husimi distribution displayed a "rippled irregular shape." It was suggested that such behavior could be considered as a qualitative criterion for quantum chaos. In this Comment it is suggested that such behavior is not necessarily associated with quantum chaos. The rippled irregular features may be due to quantum interference effects between superposed states.

A number of authors[1-7] have recently suggested that the Husimi distribution may provide a useful tool for comparing quantum and classical dynamics. Such a tool would be especially useful in the context of quantum chaos. In Ref. 5 a particular type of Husimi distribution, called the Q function, was used to compare the classical and quantum dynamics of a nonlinear oscillator. One of the justifications for using Husimi distributions in this context is that they may be interpreted as true joint phase-space probability distributions for a special class of "least disturbing" simultaneous measurement of position and momentum.[8] Specific models for such measurements are given in Refs. 9 and 10.

In a recent interesting paper, Życzkowski[1] used a generalization of the Husimi distribution to discuss the dynamics of the quantized kicked rotator. The results of that work clearly demonstrate the considerable insight provided by the Husimi representation in this context. The Husimi distribution for an eigenstate of the evolution operator was computed and shown to be peaked on the secondary resonances of the classical standard map. In the classically chaotic region the peaks are quite well defined but the region between the peaks is filled with a number of ridges and valleys or "ripples." Życzkowski suggests that these ripples may be considered as one of the qualitative criteria for quantum chaos. This claim is not easy to justify as the assumed initial state was not sufficiently semiclassical. In this Comment I would like to suggest another interpretation of such ripple features.

Consider a simple harmonic oscillator with energy eigenstates designated $|n\rangle$. Let $\{|\alpha\rangle\}$ be a particular class of oscillator minimum uncertainty states with

$$\alpha=\left[\frac{\omega}{2\hbar}\right]^{1/2}\langle\hat{q}\rangle+i(2\hbar\omega)^{-1/2}\langle\hat{p}\rangle\ , \tag{1}$$

$$\langle\Delta\hat{q}^{\,2}\rangle=\frac{\hbar}{2\omega}\ , \tag{2}$$

$$\langle\Delta\hat{p}^{\,2}\rangle=\frac{\hbar\omega}{2}\ , \tag{3}$$

$$\langle\Delta\hat{q}\Delta\hat{p}\rangle_s=0\ , \tag{4}$$

where $\Delta\hat{A}\equiv\hat{A}-\langle\hat{A}\rangle$ and $\langle\hat{A}\hat{B}\rangle_s=\langle\hat{A}\hat{B}+\hat{B}\hat{A}\rangle/2$. Thus $\{|\alpha\rangle\}$ are in fact the Glauber coherent states.[11] If $\hat{\rho}$ represents the state of an oscillator a Husimi distribution may be constructed as

$$Q(\alpha)=tr(\hat{\rho}|\alpha\rangle\langle\alpha|)\ . \tag{5}$$

This particular Husimi distribution is called the Q function in quantum optics.[12] It is normalized with respect to the measure $d^2\alpha/\pi$.

Consider now an oscillator state corresponding to a coherent superposition of two energy eigenstates,

$$\hat{\rho}=\tfrac{1}{2}(|n\rangle\langle n|+|m\rangle\langle m|+|n\rangle\langle m|+|m\rangle\langle n|)\ . \tag{6}$$

In polar coordinates (r,θ) where $\alpha=re^{i\theta}$ the Q function for this state is

$$Q(r,\theta)=\tfrac{1}{2}\{P_n(r)+P_m(r)+2[P_n(r)P_m(r)]^{1/2}\cos[(n-m)\theta]\}\ , \tag{7}$$

where

$$P_n(r)=\frac{r^{2n}}{n!}e^{-r^2}\ . \tag{8}$$

The function $P_n(r)$ is sharply peaked at $r^2=n$ when $n\gg 1$ and may be considered as a phase-space probability density peaked on the classical tori corresponding to the quantum state $|n\rangle$, and distributed uniformly in phase. If we had chosen the state to be an incoherent mixture of the two eigenstates, i.e.,

$$\hat{\rho}=\tfrac{1}{2}(|n\rangle\langle n|+|m\rangle\langle m|)\ , \tag{9}$$

the Q function would be

$$Q_m(r,\theta)=\tfrac{1}{2}[P_n(r)+P_m(r)]\ . \tag{10}$$

Thus it is clear that the "interference term" in Eq. (7) is due to the quantum coherence between the superposed states $|n\rangle$ and $|m\rangle$.

In Fig. 1 the Q function in Eq. (7) is plotted for $n-m=8$. The interference fringes between the tori at $r^2=n$ and $r^2=m$ are clearly visible. This example illustrates that ripples in the Q function need not necessarily have any connection with chaotic dynamics but rather

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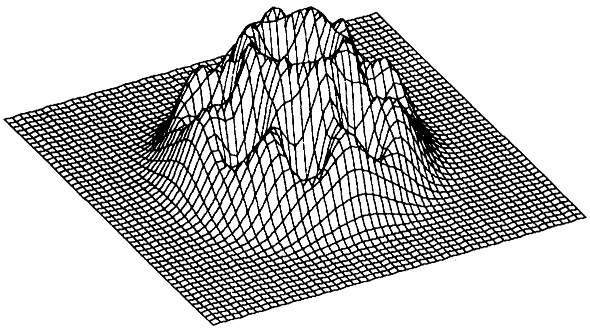

FIG. 1. Plot of the Q function against real and imaginary parts of α, for an initial superposition of two energy eigenstates, $|n\rangle$ and $|m\rangle$, with $n=2$, $m=10$. The coordinate on the left is $\mathrm{Im}(\alpha)$ and the coordinate at the bottom is $\mathrm{Re}(\alpha)$. The origin is in the center and the range for both coordinates is $-4.0 \leq \mathrm{Im}(\alpha), \mathrm{Re}(\alpha) \leq 4.0$.

reflect an underlying quantum coherence between states mixed by the dynamics. Similar behavior is evident in the model of Ref. 6.

Returning now to the study of Życzkowski, an alternative interpretation of the ripples between the peaks on the classical resonances may be given as follows. The single eigenstate of the evolution operator used by Życzkowski corresponds to two classical motions on the secondary resonances of the classical map.[13] Thus one might expect this state to be some form of superposition state of these two classical motions. The fact that the Q function is doubled peaked is evidence of such a superposition. The fact that ripples occur between the peaks indicates that this is a quantum superposition and not a classical mixture. The ripples in the Q function reflect the underlying quantum coherence between these states. In the example discussed above we considered the Q function for a superposition of two classical motions with different energy. The example of Życzkowski corresponds to a superposition of two classical motions with the same quasienergy. The interpretation of the ripples in the Q function is the same, however,

It is known that quantum recurrence phenomena arise due to the fact that unitary evolution preserves quantum coherence between superposed states.[14] In as much as the Husimi distribution reveals quantum coherences in the kicked rotator evolution operator eigenstates it provides indirect evidence for quasiperiodic behavior and the nonoccurrence of chaotic behavior in the quantum dynamics. Further investigation of nonlinear dynamical problems using the Husimi distribution will determine whether this interpretation of the ripple features found by Życzkowski is tenable.

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