Quantum limits to all-optical phase shifts in a Kerr nonlinear medium

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We consider two copropagating fields in a nonlinear Kerr medium, each with a particular phase and intensity. The Kerr medium possesses an intensity-dependent refractive index and the phase shift of each field thus depends on the intensities of the fields. Classically it is possible to induce an arbitrary phase shift of one field (the signal field) by either increasing the intensity of the other field (the control field) or by increasing the interaction length. We show that if the intensity of the control field is low, the phase shift on the signal is limited by the discrete nature of the photon-number distribution in the control field and cannot be increased simply by increasing the interaction length. In general the maximum phase shift of the signal field is ϕ if the control field possesses ϕ photons. This limit arises as a consequence of quantum recurrence effects.

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I. INTRODUCTION

Suppose that two single-mode electromagnetic fields interact in a nonlinear Kerr medium which possesses an intensity-dependent refractive index. Each of the two fields, referred to as the control field and the signal field, possesses an intensity and phase used to characterize the fields. The intensity of the control field and the interaction time between the two fields can be adjusted to produce the desired phase shift on the signal field, if we treat the system classically (Fig. 1). The two fields could be distinguished by either frequency or polarization.

In this treatment we ignore the self-induced intensity-dependent phase shift [1, 2] in order to be able to concentrate on the mutual phase shift in the medium. This assumption, although difficult to realize in practice, allows us to emphasize the basic limits to phase shifting in Kerr media in the clearest fashion. Classically one expects that arbitrary phase shifts are possible in Kerr media. The question arises as to whether arbitrary phase shifts are possible in a quantum treatment. This concern is very important if we consider schemes for designing all-optical switches [3] and all-optical logic gates [4] where optically induced phase shifts of π in Kerr media are important. We demonstrate that, although phase shifts are not limited in classical electrodynamics, the phase shift is limited in a quantum treatment.

In classical electrodynamics we can conceive of fields with an intensity and phase and with no noise on these variables. Given that the two fields copropagate through the Kerr medium, the control field c induces a phase shift on the signal field a given by

$$\phi = \chi I_c , \tag{1}$$

where I_c is the intensity of the control field in dimensionless units of photon number and

$$\chi = 2\chi^{(3)} \left(\frac{\hbar\omega^2}{2\epsilon_0^2 V} \right) \frac{L}{c} , \tag{2}$$

where V is the interaction volume, ω is the frequency of the light, L is the interaction length, and c is the speed of light in the medium. Given a small control-field intensity and a small nonlinear susceptibility, an arbitrary phase shift ϕ can be arranged by increasing the interaction length L. (Of course dissipative effects become more significant as the interaction length increases, but we ignore these difficulties in order to concentrate on a more fundamental limitation.)

A classical field with no intensity and no phase noise is an extreme idealization and cannot apply to quantum fields which are intrinsically noisy. Therefore let us consider a situation in which the control field contains some intensity noise. In this situation the phase of the signal field undergoes a spreading, even as it is shifted, due to the distribution of intensities in the control field. In an extreme situation the spreading in phase might come to dominate and render a phase shift of ϕ impossible. For example, a phase shift of π might induce a diffusion of the phase over 2π. This phase-diffusion-limited phase shift is a classical and a quantum limit to phase shifts in a Kerr medium: it is a quantum limit insofar as quantum fields of practical use (such as coherent fields) have intrinsic intensity noise. It is also a classical limit because the phase spreading arises for any field with intensity noise. However, there also exists a purely quantum limit which is described below.

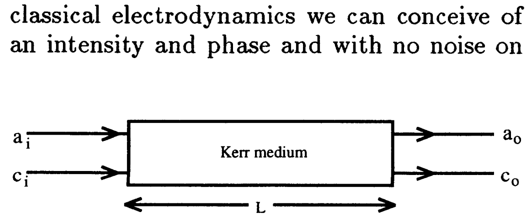

FIG. 1. A signal field a and a control field c interact via a nonlinear Kerr medium of length L.

II. THE NONLINEAR OPTICAL KERR MODEL

For simplicity we treat cw fields and ignore noise sources that would arise such as guided-acoustic-wave Brillouin scattering and stimulated Raman scattering in optical fibers, for example. The assumptions of no external or medium-induced noise and no loss are very limiting assumptions, but are necessary to elucidate the fundamental limits to all-optical switching. Classical coherent fields can possess a well-defined intensity and phase, but complementarity prevents this in the quantum field. The quantum counterpart to the classical coherent field is the coherent state which is a quadrature-phase minimum-uncertainty state [5]. The coherent state is used as an input state to discuss the basic quantum limits to phase shifting and we demonstrate how the classical behavior is obtained in the appropriate high-intensity limit.

Let $\hat{a}_i$ ($\hat{a}_o$) denote the annihilation operator for the input (output) *signal-field mode* and $\hat{c}_i$ ($\hat{c}_o$) denote the annihilation operator for the input (output) *control-field mode.* Given the model interaction Hamiltonian

$$\hat{H}_I = \hbar\tilde{\chi}\hat{n}_a\hat{n}_c, \tag{3}$$

with $\tilde{\chi} = \chi c/L$, it is apparent that the photon number in each mode is a constant of motion. The input and output operators are thus related by

$$\hat{a}_o = \exp\left(-i\chi\hat{n}_c\right)\hat{a}_i,\ \hat{c}_o = \exp\left(-i\chi\hat{n}_a\right)\hat{c}_i\ , \tag{4}$$

where $\hat{n}_a \equiv \hat{a}_i^\dagger\hat{a}_i = \hat{a}_o^\dagger\hat{a}_o$ are the (constant) photon-number operators for signal- and control-field modes, respectively. One application of this model involves measuring the phase shift of the probe field which allows a quantum nondemolition (QND) measurement of the signal-field photon number [6]. There is a related class of QND measurements based on a more general Kerr interaction than that used here [2]. In these cases the interaction is written in terms of the quadrature-phase amplitudes for both the signal and probe. The coupling is such that one of the quadrature amplitudes of the probe carries QND information on a quadrature amplitude of the signal.

The classical equation which corresponds to Eq. (4) is

$$\alpha_o = \exp\left(-i\chi I_c\right)\alpha_i\ , \tag{5}$$

where α_i (α_o) is the complex amplitude of the signal input (output) field. The (dimensionless) intensity of the control field I_c is given in terms of the input control-field amplitude γ_c by

$$I_c = |\gamma_i|^2. \tag{6}$$

In order to arrange a phase shift of ϕ we require that Eq. (1) holds. Even if I_c is very small and the nonlinearity is also very small, we can satisfy (1) simply by increasing the interaction length in the nonlinear medium which thereby increases χ.

If there are intensity fluctuations in the control field we must average over values for I_c in Eq. (5). For example, if the control-field amplitude possesses the Gaussian distribution

$$P_c(\gamma,\gamma^*) = \frac{1}{2\pi\Delta}\exp\left(-\frac{|\gamma-\gamma_i|^2}{2\Delta}\right)\ , \tag{7}$$

where Δ is the variance in the real and imaginary parts of the control field complex amplitude and γ_i is the initial mean amplitude. The mean amplitude for the signal field is then calculated to be

$$\begin{aligned}\langle\alpha_o\rangle &= \langle\alpha_i\rangle\frac{1}{\pi}\int d^2\gamma\, P_c(\gamma,\gamma^*)\exp(-i\chi|\gamma|^2)\\ &= \frac{\langle\alpha_i\rangle\exp(-i\chi I_c)}{1+2i\chi\Delta}\exp\left(-2\frac{\chi^2\Delta I_c}{1+2i\chi\Delta}\right).\end{aligned} \tag{8}$$

Apart from a small additional phase shift, the effect of intensity fluctuations is to cause a "decay" of the mean amplitude. This decay is due to the spreading in the phase of the signal-output amplitude which arises from the intensity fluctuations in the control field. Nonetheless, there exists a discernible phase shift of χI_c in the expression (8).

In quantum optics the signal field and probe field are specified as coherent states rather than as a complex amplitude probability distribution. For a given coherent state one can *define* an amplitude probability density and then subject the density to an evolution according to the classical equations of motion. However, the classical evolution of the probability density can be quite different from the amplitude probability density which is determined by the quantum evolution of the initial coherent state for nonlinear systems with fields of low intensity [7].

In order to compute the corresponding quantum result we assume that the initial state of each field is in the single-mode coherent state which is represented in the Fock number state basis as

$$\langle n|\alpha\rangle = \exp(-|\alpha|^2/2)\frac{\alpha^n}{\sqrt{n!}}. \tag{9}$$

The signal field is initially in the coherent state $|\alpha_i\rangle_a$ and the control field is in the initial state $|\gamma_i\rangle_c$. Thus the mean amplitude of the output state is

$$\begin{aligned}\langle\hat{a}_i\rangle &= {}_c\langle\gamma_i|_a\langle\alpha_i|\exp(i\hat{H}_i/\hbar)\hat{a}_i\exp(-i\hat{H}_i/\hbar)|\alpha_i\rangle_a|\gamma_i\rangle_c\\ &= e^{-\eta-i\phi}\,\alpha_i\end{aligned} \tag{10}$$

for

$$\eta = \tfrac{1}{2}I_c\sin^2\left(\frac{\chi}{2}\right),\ \phi = I_c\sin\chi. \tag{11}$$

There are two interesting features in this result. First there is a decay η of the mean amplitude. This is the quantum analog of the classical phase spreading which is induced by the intensity fluctuations in the control field. For small χ, we observe that $\eta \propto \chi^2|\gamma_i|^2$ as we observed for the classical Gaussian distribution (7). The most interesting feature, however, is the phase-shift term

$$\phi = |\gamma_i|^2\sin\chi. \tag{12}$$

For small χ this reduces to the classical result $\phi = |\gamma_i|^2\chi$ as observed in (1). An arbitrary phase shift can be seen to be impossible from expression (12). As $\sin\chi$ is bounded between 1 and -1, the phase shift ϕ is bounded between

$|\gamma_i|^2$ and $-|\gamma_i|^2$ regardless of how long the interaction length is. The periodicity of the trigonometric function means that there exists a characterisic interaction length in the nonlinear medium beyond which the dynamics recurs. The origin of the recurrence, as we show, arises from the discreteness of the photon-number distribution in the quantized control field and is therefore distantly related to quantum recurrence effects in related quantum systems [8].

III. RECURRENCE EFFECTS

Given a coherent state input $|\alpha\rangle_a|\gamma\rangle_c$, the output state is given by the density operator

$$\hat{\rho}_o = \exp(-i\chi\hat{n}_a\hat{n}_c)|\alpha\rangle_a|\gamma\rangle_c\langle\gamma|_a\langle\alpha|\exp(i\chi\hat{n}_a\hat{n}_c). \tag{13}$$

The reduced density operator for the signal mode is obtained by tracing over the control mode and the result is

$$\hat{\rho}_o^{(a)} = \sum_{n=0}^{\infty} P_\gamma(n)|\alpha e^{-i\chi n}\rangle_a\langle\alpha e^{-i\chi n}| , \tag{14}$$

where

$$P_\gamma(n) = \exp(-|\gamma|^2)|\gamma|^2/n! \tag{15}$$

is the Poisson distributed photon-number distribution in the control field with mean $|\gamma|^2$. Equation (14) represents a classical mixture of coherent states, each of which is rotated through an angle χn with respect to the original state. Each coherent state can be represented as a point in the complex plane with an associated error circle which represents fluctuations in the amplitude. The resulting state is thus represented in the plane as a set of (possibly overlapping) circles of differing weight. This "phase-space" representation can be made more definite by calculating contours of the Q function. We return to this point at the end of this section.

The dominant element in the mixed state in Eq. (14) occurs where the Poisson distribution is peaked at $n = \bar{n} = |\gamma|^2$. Thus the center of the resulting distribution is localized at the classically expected phase shift $\chi|\gamma|^2$. The width of the photon-number distribution in the control field determines the resulting spread in the phase as expected. However, because of the discrete sum over integers which appears in (14), very interesting results arise for special values of χ.

Let us consider the case where $\chi = \pi$. The sum in eqn (14) then breaks into two sums, one over even integers and the other over odd integers. The resulting state is

$$\begin{aligned}\hat{\rho}_a &= \left(\sum_{n=0}^{\infty} P_\gamma(2n)\right)|\alpha\rangle_a\langle\alpha| \\ &\quad + \left(\sum_{n=0}^{\infty} P_\gamma(2n+1)\right)|-\alpha\rangle_a\langle-\alpha| \\ &= \tfrac{1}{2}(1+e^{-2|\gamma|^2})|\alpha\rangle_a\langle\alpha| + \tfrac{1}{2}(1-e^{-2|\gamma|^2})|-\alpha\rangle_a\langle-\alpha|.\end{aligned} \tag{16}$$

This is an incoherent mixture of two coherent states which are π out of phase. We could also consider the case that $\chi = \pi/2$, which is a mixture of four coherent states $|\alpha\rangle$, $|i\alpha\rangle$, $|-\alpha\rangle$, and $|-i\alpha\rangle$ which are separated in phase by $\pi/2$. The unusual behavior for large values of χ is manifested as mixtures of very distinct coherent states and leads directly to the phase-shift limitation which is discussed above. For example, suppose that we have four photons in the control field $|\gamma|^2 = 4$ and we are seeking

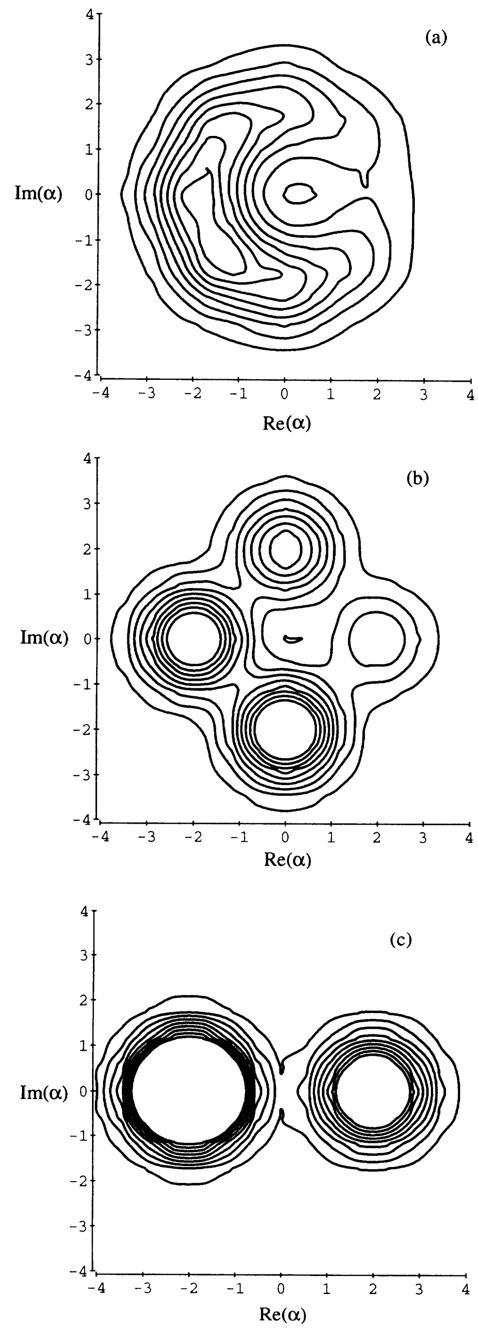

FIG. 2. Contours of the Q function for the reduced signal-field state. The signal field is initially in a coherent state with a mean of four photons and we set $|\gamma|^2\chi = \pi$ with (a) $\chi = \pi/4$, (b) $\chi = \pi/2$, and (c) $\chi = \pi$.

a classical phase shift of π. Classically this would be achieved if we could arrange for $\chi = \pi/4$. At this value of χ the mixture of very distinct coherent states has not formed. Alternately, if we let $|\gamma|^2 = 1$, then $\chi = \pi$ and the ouput state is given by (16) with an almost equal weight on the coherent states $|\alpha\rangle$ and $|-\alpha\rangle$. Thus for one photon a phase shift of π cannot be discerned.

The formation of these special mixtures of coherent states can be pictured most easily by plotting contours of the Q function [9]. The Q function is a real, positive phase-space distribution for the signal defined by

$$Q_a(\alpha) = {}_a\langle\alpha|\hat{\rho}_o^{(a)}|\alpha\rangle_a. \tag{17}$$

In Fig. 2 we plot contours of the Q function in the plane of real and imaginary α for various values of γ and χ.

In Fig. 2(a) we consider the case that $\gamma = 2$ and $\chi = \pi/4$. The classical equation would predict an exact phase shift of π for the signal field in this case. In addition we expect a spread in the phase of the signal due to the Poissonian spread of intensity in the control field. Both of these features are evident in Fig. 2(a). In Figs. 2(b) and 2(c) we consider the cases for which $\chi = \pi/2, \chi = \pi$, respectively, and fix $|\gamma|^2\chi = \pi$. The figures clearly reveal the formation of a mixture of four coherent states at $\chi = \pi/2$ and a mixture of two coherent states at $\chi = \pi$. Classically such states never form under these conditions: the contours of the phase-space density would simply continue to spread in phase. This phase-shift limit arises solely from the discreteness of the control-field energy levels and is a manifestation of the quantum field.

IV. THE EXPERIMENTAL SIGNATURE OF THE PHASE-SHIFT LIMIT

The phase-shift limit is one aspect of a quantum recurrence phenomenon. As a quantum effect, a direct experimental observation of the graininess of the control-field energy levels would provide further evidence of the quantized nature of the electromagnetic field. The graininess of the field has been verified by observations of quantum recurrences in the Jaynes-Cumming interaction between a two-level atom and a single mode of the electromagnetic field [8, 10]. For this atom-field interaction the recurrences arise in the terms for the collapse and revival sequence of the oscillating photon-number inversion. A similar collapse and revival sequence occurs in this model for the mean amplitude. For example, from Eq. (10) the real part of the mean amplitude $X = \mathrm{Re}\langle\hat{a}\rangle$ obeys the equation

$$X_o = X_i \exp\left(-\frac{1}{2} I_c \sin^2 \frac{\chi}{2}\right) \cos(I_c \sin\chi). \tag{18}$$

If X is plotted against χ (which is proportional to the interaction length), then X undergoes a decay but revives to the initial value at $\chi = 2\pi$. The revivals are due to the graininess of the control field which produces a mixture of distinct coherent states of the signal field.

The mixture of coherent states at $\chi = \pi/2$ and π are characterized by phase-sensitive noise in the complex amplitude plane. Homodyne detection is one technique for detecting phase-sensitive noise. Much effort has been invested into accurate homodyne detection schemes for optical fields primarily in connection with the efforts to detect squeezed states of light [11]. In the present case the objective is to observe a bimodal distribution of the distribution of the real quadrature-phase amplitude measurements which would result at $\chi = \pi$ for a control field with a mean photon number of 1.

Classically, in the absence of phase spreading, the resulting distribution would possess a single peak with a mean at a point π shifted from the initial amplitude. The inclusion of phase spreading causes the distribution to be broadened but the distribution remains single peaked. A homodyne detection which reveals two amplitude peaks π out of phase, as expected for the reduced signal state (16), is evidence of the graininess of the control field.

The detection of bimodal distributions will be difficult. Current Kerr nonlinearities are quite small: for example, to produce $\chi \approx 1$ in a glass fiber at $\omega \approx 5 \times 10^{14}$ rad/sec would require a fiber of length ~ 1000 km [12]. However, the experiment might be possible if media with large third-order susceptibilies are available. A high nonlinearity allows a short interaction time which minimizes the effects of dissipation. Dissipation is expected to reduce the effects of the graininess of the control field. The effects of dissipation require further attention.

Given that a mixture of two coherent states is produced a sensitive detector should be adequate to detect the bimodal distribution. Unlike the superpositions of coherent states which have been discussed in the literature [13] we seek here to detect an *incoherent mixture* of two coherent states which makes the signal-output state more robust than the coherent-superposition states.

V. CONCLUSIONS

Fundamental limits to all-optical phase shifts which can be obtained in Kerr media have been identified. The limits arise for the case that the intensity of the control field is low. One limit is related to the intensity fluctuations in the control field and can be understood in a classical context: the fluctuations in the control-field couple as phase diffusion in the signal field. The second limit establishes a maximum phase shift in the signal field and is purely quantum mechanical: the limited phase shift is a consequence of the discreteness of the energy distribution in the control field.

Essentially we have demonstrated that the discreteness of the energy levels in a coherent control field limits the phase shift which can be induced on a signal field. The phase shift of the amplitude of the signal field is bounded between $-\phi$ and ϕ, where ϕ is the mean number of photons in the control field. Of course one can consider very nonclassical states of the electromagnetic radiation to attempt to bypass these limits, but here we have expressed the limits in terms of the quantum analog to classical coherent states. The limit to the phase shift has repercussions in all-optical switching devices where phase shifts of π can be required.

The classical limit is obtained in the limit of an intense control field (large mean photon number) and small nonlinearities. In the classical limit arbitrary phase shifts can be induced and the phase spreading can be made arbitrarily small. In the near-classical regime the intensity fluctuations are very small compared to the intensity of the field and the graininess of the photon field becomes very fine and does not limit the phase shift of the signal field.

Dissipative effects have not been considered here but are expected to degrade the purely quantum features. Dissipation is avoided by employing a medium with a large nonlinearity which allows a small interaction length. With current technology the fundamental limit to phase shifting is not expected to be noticeable but is an important and surprising feature of Kerr media. Finally, the mixtures of distinct coherent states of the signal field provides evidence of the graininess of the electromagnetic field and merits further investigation.

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