Low-contrast bandgaps of a planar parabolic spiral lattice

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We show that a planar aperiodic lattice, mimicking the appearance of a sunflower, supports photonic bandgaps for weak dielectric contrast. The pattern’s high orientational order and spatially uniform modal pitch yields an isotropic Fourier space. A 2D structure of cylinders (\( \varepsilon = 2 \)) in air possesses a wide 21% TM bandgap, versus 5.6% for a sixfold lattice or 14% for a 12-fold fractal tiling. The isotropic gap frequencies imply flat bands, and thus application in nonlinear optics and low threshold lasers, where a reduced group velocity in all directions may be desired. © 2009 Optical Society of America

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Photonic crystals are engineered materials with a periodic variation of the permittivity along one or more directions \([1]\). Many of their potential applications require a complete photonic bandgap (PBG), which is more readily achieved in lattices with high rotational symmetry. Aperiodic crystals, or quasi-crystals, may possess arbitrarily high local or statistical symmetry, suggesting wider isotropic gaps in weakly modulated \((\Delta \varepsilon < 3)\) structures \([2,3]\).

A quasi-crystal “essentially discrete” Fourier space \([4]\) shows long-range order in the absence of translational symmetry. Consequently, PBGs have been demonstrated in 8- \([5]\), 10- \([6]\), and 12-fold \([6,7]\) structures. Like the crystal lattice, PBG formation in quasi-crystals is best viewed in the Fourier domain; \(k\) vectors with the strongest Fourier coefficients can be related to the deepest modulation of the density of modes by a generalized Bragg condition \([8]\). Intuitively, the ideal Fourier space for 2D band gaps is a circle, i.e., a “Bragg ring.” One such pattern with this property is the infinite pinwheel tiling \([9]\). However, the circular symmetry is not preserved in real (finite) samples.

Photonic quasi-crystals are often modeled using periodic “approximants” \([8]\). In the infinite limit, these approximants recover the properties of the true quasi-periodic parent lattice. PBGs in weakly modulated quasi-crystals are often excluded, even though the approximants may be too small to retain elements of long-range order present in the lattice. In particular, fractal patterns \([7,10]\) may exhibit extremely long-range order owing to their inherent self-similarity.

In this Letter, we analyze a point set mimicking the head of a sunflower, previously studied in both planar \([11]\) and fiber geometries \([12]\). Mathematically, this pattern (herein “the sunflower”) is a form of Fermat’s spiral representing an optimal packing of points evolving about a polar origin. The number of visible spirals, or parastichies, in each direction appear as consecutive numbers in the Fibonacci series, the ratio of which approximates the golden ratio \(\tau\). Formally, an \(N\)-point pattern is expressed by

\[
x(n) = A \cos(n \Psi) \sqrt{n},
\]

\[
y(n) = A \sin(n \Psi) \sqrt{n},
\]

where \(\{n \in 0:N\}\), \(A\) is a scaling factor and \(\Psi = 2\pi/\rho^2\) is the golden angle \([13]\). This remarkably simple definition contrasts sharply with conventional quasi-crystals, typically generated by matching rules, projection, substitution, or multigrids.

Figure 1 shows the scanning electron microscopy (SEM) image, Delaunay triangulation, and both calculated and experimental diffraction patterns of a 500 point sunflower fabricated by e-beam lithography. The pattern has a strong modal nearest-neighbor pitch \(a = 2.2 \mu m\), derived from Delaunay triangulation [Fig. 1(b)] of the SEM image \((r/a = 0.245)\). Expressing the real space density distribution \(\rho(r)\) as a finite sum of plane waves,

\[
\rho(r) = \sum_{n=1}^{N} \frac{e^{i \Psi n}}{\sqrt{n}},
\]

Fig. 1. (Color online) (a) SEM image of a 500-point sunflower. (b) Delaunay triangulation of (a), highlighting the parastichies. (c) Calculated and (d) experimental diffraction patterns of the fabricated sample. The faint fourfold symmetry in (d) is an artefact due to the test patterns’ square tiling.

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\[ \rho(\mathbf{r}) = \sum_n f_n(\mathbf{k}_n) \exp(i\mathbf{k}_n \cdot \mathbf{r}), \]

where \( f_n \) are the Fourier coefficients and \( \mathbf{k}_n \) are the reciprocal lattice vectors indexed by \( n \), yields a dense diffraction pattern [Fig. 1(c)] with relatively weak low \( k \) components. The experimental diffraction pattern [Fig. 1(d)] is in excellent agreement. Its key feature is a circular band with an average radius of 0.45 \( \mu \)m\(^{-1}\), corresponding to the sunflower’s modal pitch, and a spiral structure resembling the parastichies of the real space lattice. Indeed, the number of counterpropagating spirals—68 clockwise, 75 counter-clockwise—is equal to the sum of the first ten Fibonacci numbers. The diffraction pattern’s largely continuous nature also suggests that, like the pinwheel [4], the sunflower is not strictly a quasi-crystal.

The sunflower’s peak \( f_n \) is more than 1 order of magnitude lower than an equivalent sixfold lattice. Assuming that the total amplitude in the Fourier domain remains fixed, the peak \( f_n \) is clearly reduced as the number of plane waves required to resolve the lattice increases. Since gap widths are, to first order, proportional to the amplitude of the strongest Fourier components [2], a PBG open on the sunflower’s pseudo-Brillouin zone could be expected to suffer. However, the broad range of \( \mathbf{k} \), may support a wide gap as a superposition of several overlapping gaps.

Like other curvilinear lattices [14], the sunflower’s local morphology varies between distorted four- or sixfold symmetry. For a square 50a\(^2\) section centered at a radial offset \( r_0 = 100a \) [Fig. 2(a)], the isotropic ring disintegrates into six elements, while the modal pitch remains stable. To recover the sunflower’s isotropic ring we require increasingly larger domains as \( r_0 \to \infty \), such that a sufficient amount of spiral curvature is retained [Fig. 2(b)]. In a weakly modulated sunflower this condition is satisfied by long-range interations, and thus any PBG will be formed on the Bragg ring. Conversely, a PBG in a strongly modulated sunflower will depend on the symmetry of the local environment.

To determine potential low-contrast PBGs, we used a commercial finite-difference time-domain code (RSof’s FullWAVE) to simulate a square section of the sunflower, centered on the origin. The structure consists of infinitely extended dielectric rods (\( \varepsilon = 2 \)) in air, for which TM (\( E \mid \)rods) gaps are favored. The radius–pitch ratio (\( r/a \)) is 0.3, equivalent to a dielectric fill factor of 0.2961 and chosen to coincide with the gap-maximizing parameters for a sixfold lattice.

Three alternative structures were studied with the same parameters: a sixfold hexagonal; a 12-fold Archimedean-like tiling as per Hiett et al. [15]; and a 12-fold Stampfli-inflated tiling, as per Zoorob et al. [7]. In each case, the major \( E \)-field component radiating from a nonideal (Gaussian) point source was recorded every 0.5° at a radial distance of 80a. The source location was offset from the origin to avoid selectively exciting modes with certain symmetries.

Figure 3 shows transmission spectra for all four structures as a function of incident angle. Gap–midgap ratios (\( \Delta \omega / \omega_0 \)) are measured between points on the upper and lower gap edges, where the intensity is −5 dB relative to the source. Most notably, the sunflower [Fig. 3(d)] possesses the widest complete TM gap, with a ratio of 21%. In comparison, the hexagonal lattice [Fig. 3(a)] has a 5.6% gap, the dodecagonal [Fig. 3(b)] a 1.9% upper gap, and the Stampfli [Fig. 3(c)] a relatively wide 14% gap. The hexagonal and dodecagonal figures agree well with calculations by plane-wave expansion (5.3% and 2%, respectively), thus validating our method.

The periodically changing gap frequencies in Fig. 3(a) highlight the sixfold symmetry of the lattice and limit the complete TM gap. Similar behavior in Fig. 3(b) arises from the underlying sixfold tiling of the locally 12-fold lattice, evident in Fig. 3(f). The reduced

![Fig. 2. Calculated diffraction patterns of the sunflower for (a) a 50a\(^2\) section and (b) a 100a\(^2\) section, both offset \( r_0 = 100a \) along the positive \( x \) axis. Darker shades indicate greater magnitudes, and the dc component is suppressed to improve contrast.](image1)

![Fig. 3. Transmitted intensity (\( \propto |E_z|^2 \), logscale) as a function of in-plane angle and normalized frequency (\( \omega a / 2 \pi c = a / \lambda \)) for (a) hexagonal, (b) dodecagonal, (c) Stampfli-inflated, and (d) sunflower lattices. Darker shades indicate lower transmission. (e)–(h) Corresponding diffraction patterns showing only the strongest \( f_n \). Symbol size is proportional to \( |f_n| \).](image2)
gap width for the dodecagonal lattice is attributed to the distribution of scattering amplitude over more Bragg peaks without improved isotropy. In contrast, the Stampfli tiling exhibits true 12-fold statistical symmetry (Fig. 3(g)) and thus supports a wider low-contrast gap than the sixfold lattice. However, with its broad, dense, and isotropic Fourier space (Fig. 3(h)), the sunflower appears to be optimal for a wide, low-contrast gap. Indeed, to the best of our knowledge, the sunflower’s TM gap is the widest known for $\Delta \varepsilon = 2$, by improving more than 10% on published figures [2], and furthering the possibility of low-index PBG materials. Briefly considering the TE case (not shown), we find that while the sunflower does not support a gap for the same lattice parameters, the Stampfli lattice does. Moreover, this gap coincides with the TM gap to yield an approximately 4.6% complete and absolute gap, which warrants further investigation.

The sunflower’s low-contrast PBG depends on long-range interaction. The field is therefore delocalized, implying less sensitivity to local perturbation and lower fabrication tolerances in real-world devices. Furthermore, gap frequencies in low-contrast crystals are less sensitive to systematic error in the dielectric fill factor. To clarify the lattice dimensions required for a deep PBG, we studied the evolution of the sunflower’s TM gap in radial increments of $16a$. From $r = 16 - 32a$, there is no definitive bandgap, although the density of modes maintains a weakly depleted region. From $r = 48a$, there is a clear gap populated by resonant cavity states, while for $r \geq 64a$ a defect-free gap emerges.

As expected from the locally low symmetry, a high-contrast ($\Delta \varepsilon = 12$) sunflower supports a primary PBG comparable in both width and position to those of the three alternative lattices. Gap formation in the sunflower thus has two origins. In the high $\Delta \varepsilon$ regime, the strong scattering leads to gaps based on Bragg reflection from the locally distorted four- or sixfold ordering. These gaps, which correlate with those of more conventional structures, can also be viewed in terms of the Mie resonance condition for single rods [3]. For low $\Delta \varepsilon$, the long-range interaction permits sampling of the sunflower’s unique spiral structure. Feedback therefore occurs isotropically from the Bragg ring, which incorporates many spatial frequencies to further broaden the gap. In contrast, Bragg reflection from periodic crystals acts on a sparse pure-point Fourier space [3], thereby restricting the overlap of low-contrast spectral gaps.

The sunflower’s isotropic gap frequencies also suggest the existence of very low group velocity ($v_g$) modes across all $k_{||}$. A reduced $v_g$ increases light-matter interaction and can enhance nonlinear effects and signal gain in active media, permitting either smaller devices or reduced operating power [16]. In addition, the sunflower’s asymmetry, and thus lack of degenerate modes [13], indicates that the gap edges may support strong single-moded resonances with potential for band edge lasing.

In conclusion, the sunflower displays a much wider TM gap for $\Delta \varepsilon = 2$ than previously studied structures, as a result of long-range interaction and an isotropic Fourier space. This pattern lends itself to applications in low-index PBG materials and nonlinear optics involving slow light.

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