PHYSICAL REVIEW A **78**, 052106 (2008)

Phase-space-region operators and the Wigner function: Geometric constructions and tomography

Demosthenes Ellinas[1] and Anthony J. Bracken[2,*]

[1]*Technical University of Crete, Department of Sciences, Division of Mathematics, GR 731 00 Chania, Crete, Greece*

[2]*Collegium Budapest Szentharomsag, ut. 2 1014 Budapest, Hungary*

(Received 26 August 2008; published 7 November 2008)

Quasiprobability measures on a canonical phase space give rise through the action of Weyl's quantization map to operator-valued measures and, in particular, to region operators. Spectral properties, transformations, and general construction methods of such operators are investigated. Geometric trace-increasing maps of density operators are introduced for the construction of region operators associated with one-dimensional domains, as well as with two-dimensional shapes (segments, canonical polygons, lattices, etc.). Operational methods are developed that implement such maps in terms of unitary operations by introducing extensions of the original quantum system with ancillary spaces (qubits). Tomographic methods of reconstruction of the Wigner function based on the radon transform technique are derived by the construction methods for region operators. A Hamiltonian realization of the region operator associated with the radon transform is provided, together with physical interpretations.

DOI: 10.1103/PhysRevA.78.052106 PACS number(s): 03.65.Fd, 03.67.−a

I. INTRODUCTION

The pioneering works of Weyl [1], von Neumann [2], Wigner [3], Groenewold [4], Moyal [5], and Berezin [6] laid the foundations for what is today called the phase-space formulation of quantum mechanics [7–9]. This formulation has proved increasingly useful for, e.g., quantum optics [10,11], quantum tomography [12], and research into the foundations of quantum theory itself [13–16]. Ongoing studies continue to reveal the richness of the underlying mathematical structures [17–26].

It is now well understood that the map $\mathcal{W}$ defined implicitly by Wigner [3], from operators on Hilbert space (such as quantum observables) to functions on phase space—we refer to this as the Weyl-Wigner transform—has as its inverse Weyl's quantization map [1] from functions on phase space (such as classical observables) to operators on Hilbert space. Both can be defined concisely in terms of a suitable kernel [27,28], which on the one hand is an operator on Hilbert space and on the other is a function on phase space. While $\mathcal{W}$ carries a density operator on Hilbert space into a Wigner function on phase space, the inverse map $\mathcal{W}^{-1}$ carries a Liouville density on phase space into a Groenewold operator [20] on Hilbert space. Just as the Wigner function is a quasiprobability function in general, so the Groenewold operator is a quasidensity operator in general: neither quantity is always positive definite [29].

Weyl's quantization map $\mathcal{W}^{-1}$ can be applied to any reasonably smooth function on phase space and so, in particular, to the characteristic function of any subregion of the space itself. The Hermitian operator obtained in such a special case is a general operator-valued probability measure (OVM) [26], and we usually call it a *region operator.* Up to normalization, a region operator is a special case of a Groenewold operator, corresponding to a Liouville density that is constant on the region in question. In previous studies some of the mathematical properties of Groenewold and region operators have been investigated [18,21,22,29].

At least two important points relevant to our work here are worth emphasizing in the theory of region operators: (i) As OVMs they can, combined with density operators, pure or mixed, that described the state of a quantum system, provide occupation quasiprobabilities (or *quasiprobability masses*) for various domains of phase space by means of integrals of the Wigner function over those domains. Due to the lack of positivity of those OVMs, for some states, they give negative-valued probabilities over any domain or, for some particular states, they can give positive probabilities only over some certain regions of phase space. Also there are states which provide always positive quasiprobability mass over any region, and so they feature classical characteristics; examples are the coherent states with Gaussian Wigner functions. (ii) The integrals over various phase-space domains of the OVMs or region operator provide a new kind of observables that can be considered as formalizing questions about the state of a quantum system in phase space, the latter being now conceptualized as the even space of a random experiment; therefore, region operators correspond to quantum mechanical measurements. As such their operational construction is important for their applicability in real situations where we request no trivial phase-space information about a quantum system.

Our aim here is to further develop strategies for the construction of region operators and to relate their properties more closely to applications in physics. To this end we discuss region operators associated with straight lines in the phase plane and their relevance to quantum tomography (Secs. II–IV for the case of tomography with generalized quasiprobability functions) and go on to construct potentially important assemblages of such lines (Secs. V and VI). The

*On leave from, Centre for Mathematical Physics, Department of Mathematics, University of Queensland, Brisbane 4072 Australia.

final section summarizes our results and discusses some of the prospects of the theory of region operators.

II. REGION OPERATORS AND QUANTUM TOMOGRAPHY

Let $\hat{Q}$ and $\hat{P}$ denote a pair of dimensionless Hermitian coordinate and momentum operators that strongly satisfy the canonical commutation relation in a Hilbert space $\mathcal{H}$, and let $|x\rangle$ and $|k\rangle$ denote their generalized eigenvectors, for $-\infty < x < \infty$, $-\infty < k < \infty$. Let q and p denote a corresponding pair of dimensionless phase space variables, and set

$$\hat{A} = \frac{1}{\sqrt{2}}(\hat{Q} + i\hat{P}), \quad \hat{A}^\dagger = \frac{1}{\sqrt{2}}(\hat{Q} - i\hat{P}),$$

$$\alpha = \frac{1}{\sqrt{2}}(q + ip), \quad \alpha^* = \frac{1}{\sqrt{2}}(q - ip). \tag{1}$$

Introduce also the number operator and its eigenvectors

$$\hat{N} = \hat{A}^\dagger \hat{A}, \quad \hat{N}|n\rangle = n|n\rangle \tag{2}$$

and the parity operator

$$\hat{\Pi} = (-1)^{\hat{N}} = \int |-x\rangle\langle x|dx = \int |-k\rangle\langle k|dk. \tag{3}$$

Now consider the Hermitian Wigner operator (elsewhere called the Stratonovich-Weyl kernel [28])

$$\hat{\Delta}(q,p)\ [=\hat{\Delta}(\alpha)] = \hat{D}(q,p)\hat{\Pi}\hat{D}(q,p)^\dagger = \hat{D}(2q,2p)\hat{\Pi}, \tag{4}$$

where the unitary displacement operator

$$\hat{D}(q,p)\ [=\hat{D}(\alpha)] = \exp(ip\hat{Q} - iq\hat{P}) = \exp(\alpha\hat{A}^\dagger - \alpha^*\hat{A}) \tag{5}$$

satisfies

$$\hat{D}(q,p)\hat{Q}\hat{D}(q,p)^\dagger = \hat{Q} - q, \quad \hat{D}(q,p)\hat{P}\hat{D}(q,p)^\dagger = \hat{P} - p,$$

$$\hat{D}(q,p)|x\rangle = e^{ip(x+q/2)}|x+q\rangle, \quad \hat{D}(q,p)|k\rangle = e^{-iq(k+p/2)}|k+p\rangle. \tag{6}$$

As (4) shows, the Wigner operator $\hat{\Delta}(q,p)$ is a displaced parity operator, reflecting $\hat{P}$ and $\hat{Q}$ about the point (q,p) in phase space:

$$\hat{\Delta}(q,p)\hat{Q}\hat{\Delta}(q,p) = 2q - \hat{Q}, \quad \hat{\Delta}(q,p)\hat{P}\hat{\Delta}(q,p) = 2p - \hat{P},$$

$$\hat{\Delta}(q,p)|x\rangle = e^{2ip(q-x)}|2q - x\rangle, \quad \hat{\Delta}(q,p)|k\rangle = e^{-2iq(p-k)}|2p - k\rangle,$$

$$\langle x|\hat{\Delta}(q,p)|y\rangle = \delta(x + y - 2q)e^{ip(x-y)}. \tag{7}$$

The Wigner function corresponding to a given state vector $|\Psi\rangle$ is simply expressed in terms of the Wigner operator as

$$W(q,p) \equiv \frac{1}{\pi}\int \Psi(x-y)\Psi(x+y)^* e^{2ipy}dy = \frac{1}{\pi}\langle\Psi|\hat{\Delta}(q,p)|\Psi\rangle \tag{8}$$

and can also be written in terms of the density operator $\hat{\rho} = |\Psi\rangle\langle\Psi|$ as

$$W(q,p) = \frac{1}{\pi}\mathrm{Tr}[\hat{\Delta}(q,p)\hat{\rho}]\ [=\mathcal{W}(\hat{\rho}/2\pi)(q,p), \text{ say}]. \tag{9}$$

More generally, the Weyl-Wigner transform of an arbitrary operator $\hat{F}$ is defined as

$$F(q,p) = \mathcal{W}(\hat{F})(q,p) = 2\mathrm{Tr}[\hat{\Delta}(q,p)\hat{F}]. \tag{10}$$

Conversely, the quantization of an arbitrary function F is given by the inverse Weyl-Wigner transform (equal to Weyl's quantization map) as

$$\hat{F} = \mathcal{W}^{-1}(F) = \frac{1}{\pi}\int \hat{\Delta}(q,p)F(q,p)dqdp. \tag{11}$$

Important orthogonality and completeness properties of the Wigner operator underlying these relations are

$$\mathrm{Tr}[\hat{\Delta}(q,p)\hat{\Delta}(q',p')] = \frac{\pi}{2}\delta(q-q')\delta(p-p'),$$

$$\int \hat{\Delta}(q,p)\mathrm{Tr}[\hat{\Delta}(q,p)\hat{F}]dqdp = \frac{\pi}{2}\hat{F}. \tag{12}$$

Now consider, for $-\infty < u < \infty$ and $0 \leq \theta < \pi$, the operators

$$\hat{R}(u,\theta) = \frac{1}{\pi}\int \delta(u - q\cos\theta - p\sin\theta)\hat{\Delta}(q,p)dqdp$$
$$= \frac{1}{\pi}\int \hat{\Delta}(q(u,v,\theta),p(u,v,\theta))dv, \tag{13}$$

with $q(u,v,\theta) = u\cos\theta + v\sin\theta$, $p(u,v,\theta) = u\sin\theta - v\cos\theta$, and $-\infty < v < \infty$. Here the δ function is the characteristic (generalized) function of the straight line in the q-p plane that has perpendicular displacement u from the origin and whose normal makes an angle θ, measured in the counterclockwise sense, with the q axis. We refer to $\hat{R}(u,\theta)$ as the region operator corresponding to the line (or region) in phase space. The quasiprobability density on that line for a state vector $|\Psi\rangle$ is

$$R(u,\theta) = \langle\Psi|\hat{R}(u,\theta)|\Psi\rangle = \int W(q(u,v,\theta),p(u,v,\theta))dv. \tag{14}$$

We recognize this as the radon transform of the Wigner function, as typically involved in the tomographic reconstruction of the latter [12,30–32], and consequently refer to the family of line region operators $\hat{R}(u,\theta)$, as *radon operators* in what follows. The radon transform of the Wigner function can also be expressed in terms of the density operator as

$$R(u,\theta)=\mathrm{Tr}[\hat{\rho}\hat{R}(u,\theta)]=\frac{1}{\pi}\mathrm{Tr}\left(\hat{\rho}\int\hat{\Delta}(q(u,v,\theta),p(u,v,\theta))dv\right)=\frac{1}{\pi}\mathrm{Tr}\left(\hat{\Pi}\int\hat{D}(q(u,v,\theta),p(u,v,\theta))^{\dagger}\hat{\rho}\hat{D}(q(u,v,\theta),p(u,v,\theta))dv\right)$$
$$\equiv\mathrm{Tr}[\hat{\Pi}\varepsilon_{u,\theta}(\hat{\rho})]\equiv\mathrm{Tr}[\varepsilon^{*}_{u,\theta}(\hat{\Pi})\hat{\rho}]. \qquad (15)$$

In the last line we have introduced the positive map $\varepsilon_{u,\theta}$ and its dual $\varepsilon^{*}_{u,\theta}$ [33], which arise naturally here—viz.,

$$\varepsilon_{u,\theta}(\hat{\rho})=\frac{1}{\pi}\int\hat{D}(q(u,v,\theta),p(u,v,\theta))^{\dagger}\hat{\rho}\hat{D}(q(u,v,\theta),p(u,v,\theta))dv,$$

$$\varepsilon^{*}_{u,\theta}(\hat{\Pi})=\hat{R}(u,\theta)=\frac{1}{\pi}\int\hat{D}(q(u,v,\theta),p(u,v,\theta))\hat{\Pi}\hat{D}(q(u,v,\theta),p(u,v,\theta))^{\dagger}dv. \qquad (16)$$

Note that $\hat{\Pi}/\pi\equiv\hat{\Delta}(0,0)/\pi$ is itself a region operator, corresponding to the single point (0,0) in the q-p plane, and (16) shows how the radon operator for an arbitrary straight line can be obtained from the region operator for the point at the origin with the use of the map $\varepsilon^{*}_{u,\theta}(\hat{\Pi})$. More generally, the region operator corresponding to the point (q_0,p_0) with characteristic (generalized) function $\delta(q-q_0)\delta(p-p_0)$ is given by a multiple of the Wigner operator evaluated at that point—viz.,

$$\frac{1}{\pi}\int\delta(q-q_0)\delta(p-p_0)\hat{\Delta}(q,p)dqdp=\frac{1}{\pi}\hat{\Delta}(q_0,p_0). \qquad (17)$$

As special cases of radon operators we consider two straight lines parallel to the axes with parameters ($u=q_0$, $\theta=0$) and ($u=p_0$, $\theta=\pi/2$), which cut the q axis at q_0 and the p axis at p_0, respectively. The corresponding Radon operators, expressed as ε^*-mapped parity operators (i.e., point region operators), are

$$\varepsilon^{*}_{q_0,0}(\hat{\Pi})=\frac{1}{\pi}\int e^{ip\hat{Q}}(e^{-iq_0\hat{P}}\hat{\Pi}e^{iq_0\hat{P}})e^{-ip\hat{Q}}dp=\frac{1}{\pi}\int\hat{D}(q_0,p)\hat{\Pi}\hat{D}(q_0,p)^{\dagger}dp=|q_0\rangle\langle q_0|, \qquad (18)$$

$$\varepsilon^{*}_{p_0,\pi/2}(\hat{\Pi})=\frac{1}{\pi}\int e^{-iq\hat{P}}(e^{ip_0\hat{Q}}\hat{\Pi}e^{-ip_0\hat{Q}})e^{iq\hat{P}}dq=\frac{1}{\pi}\int\hat{D}(q,p_0)\hat{\Pi}\hat{D}(q,p_0)^{\dagger}dq=|p_0\rangle\langle p_0|. \qquad (19)$$

We see that these particular radon operators are position or momentum (generalized) projection operators, a result that has been obtained previously for the respective region operators, by means of the geometrically motivated trace increasing positive maps [25].

To obtain a corresponding result for the radon operator $\hat{R}(u,\theta)$, we note that

$$e^{i\phi\hat{N}}\hat{A}e^{-i\phi\hat{N}}=e^{-i\phi}\hat{A},\quad e^{i\phi\hat{N}}\hat{A}^{\dagger}e^{-i\phi\hat{N}}=e^{i\phi}\hat{A}^{\dagger}\Rightarrow e^{i\phi\hat{N}}\hat{Q}e^{-i\phi\hat{N}}=\cos\phi\hat{Q}+\sin\phi\hat{P},\quad e^{i\phi\hat{N}}\hat{P}e^{-i\phi\hat{N}}=\cos\phi\hat{P}-\sin\phi\hat{Q}\Rightarrow e^{i\phi\hat{N}}\hat{\Delta}(q,p)e^{-i\phi\hat{N}}=\hat{\Delta}(q\cos\phi-p\sin\phi,p\cos\phi+q\sin\phi)\Rightarrow e^{i\phi\hat{N}}\hat{R}(u,\theta)e^{-i\phi\hat{N}}=\hat{R}(u,\theta+\phi). \qquad (20)$$

It follows that

$$\varepsilon^{*}_{u,\theta}(\Pi)\ [=\hat{R}(u,\theta)]=e^{i\theta\hat{N}}\varepsilon^{*}_{u=p_0,\pi/2}(\Pi)e^{-i\theta\hat{N}}=|u,\theta\rangle\langle u,\theta|, \qquad (21)$$

where

$$(\cos\theta\hat{P}-\sin\theta\hat{Q})|u,\theta\rangle=u|u,\theta\rangle,$$

so that every radon operator is a (generalized) projection operator and, in particular, is non-negative. From (14), the radon transform of the Wigner function is therefore also a non-negative function of u and θ. As is well known [34], the quasiprobability density on any straight line in the q-p-plane is in fact a true probability density, and we see that $R(u,\theta)\delta u$ is the probability that the measured value of $(\cos\theta\hat{P}-\sin\theta\hat{Q})$ lies in the interval $(u,u+\delta u)$.

There are several equivalent inversion formulas for the radon transform: in particular [35],

$$W(q,p)=\frac{1}{\pi}\lim_{a\to 0}\int R(u+q\cos\theta+p\sin\theta,\theta)G_a(u)du\,d\theta, \qquad (22)$$

where

$$G_a(u)=\begin{cases}\dfrac{1}{\pi a^2}, & |u|\leqslant a,\\[2ex] \dfrac{1}{\pi a^2}\left(1-\dfrac{1}{\sqrt{1-a^2/u^2}}\right), & |u|>a.\end{cases} \qquad (23)$$

[It is notable that the same inversion formula applies in the quantum and classical cases. In both cases the radon transform data are non-negative, but in the quantum case, the inverse (i.e., the Wigner function) is not in general non-negative. Evidently the radon transform data in the classical case satisfy constraints that ensure non-negativity of the inverse, constraints that are not satisfied in the quantum case. This is a question worthy of further investigation.]

The expression for the Wigner operator in terms of radon operators, corresponding to (22), is

$$\hat{\Delta}(q,p) = \lim_{a\to 0} \int \hat{R}(u+q\cos\theta+p\sin\theta,\theta)G_a(u)dud\theta$$
$$= \lim_{a\to 0} \int G_a(u)|u+q\cos\theta+p\sin\theta,\theta\rangle\langle u+q\cos\theta$$
$$+p\sin\theta,\theta|dud\theta. \qquad (24)$$

III. DYNAMICAL CONSTRUCTION OF RADON OPERATORS

We have seen in (16) how the radon operator for the straight line with parameters (u,θ) can be constructed from the region operator for the point (0, 0) in the phase plane, with the use of the map $\varepsilon^*_{u,\theta}$. There is another way of expressing radon operators that suggests a dynamical interpretation, and this in turn suggests a dynamical approach to the basic problem of quantum tomography—the problem of reconstruction of the Wigner function—because the Wigner operator can be constructed from the radon operators as in (24). To this end we rewrite (16) as

$$\hat{R}(u,\theta) = \frac{1}{\pi}\int e^{-i(v\sin\theta\hat{P}+v\cos\theta\hat{Q})}$$
$$\times\hat{\Delta}(u\cos\theta,u\sin\theta)e^{i(v\sin\theta\hat{P}+v\cos\theta\hat{Q})}dv$$
$$= \frac{1}{\pi}\int e^{i\hat{K}t}\hat{\Delta}(u\cos\theta,u\sin\theta)e^{-i\hat{K}t}d(\Omega t)$$
$$= \varepsilon_{\hat{K}}(\hat{\Delta}(u\cos\theta,u\sin\theta)), \qquad (25)$$

say, where we have changed the measure of integration from v to Ωt and introduced the operator

$$\widehat{K(\theta)} = \Omega(\cos\theta\hat{Q}+\sin\theta\hat{P}) = \frac{\Omega}{\sqrt{2}}(\hat{A}e^{-i\theta}+\hat{A}^\dagger e^{i\theta}) \qquad (26)$$

and the associated map $\varepsilon_{K(\hat{\theta})}$. While (16) shows how the radon operator for the straight line with parameters (u,θ) can be obtained by the action of the map $\varepsilon^*_{u,\theta}$ on $\hat{\Pi}$, which is (up to a factor $1/\pi$) the region operator for the point at the origin in the phase plane, (25) shows how that the radon operator can be obtained alternatively from the region operator for one point on the line itself: namely, the point nearest the origin, with coordinates $(u\cos\theta,u\sin\theta)$, through the action of the map $\varepsilon_{\hat{K}}$. For the case of a straight line through the origin in the phase plane, we have $u=0$ and the two maps coincide,

$$\hat{R}(0,\theta) = \varepsilon^*_{0,\theta}(\hat{\Pi}) = \varepsilon_{\widehat{K(\theta)}}(\hat{\Pi}). \qquad (27)$$

Note that (25) implies that

$$R(u,\theta) = \mathrm{Tr}[\varepsilon_{\hat{K}}(\hat{\Delta}(u,\theta))\hat{\rho}]. \qquad (28)$$

The result (25) suggests the physical identification of t with time and Ω with the Rabi frequency of the cavity QED interaction between a two-level atom and a one-mode quantized electromagnetic (EM) field. As we shall see, the effective Hamiltonian of such an interaction is

$$\hat{H}(\theta) = \sigma_3\otimes\widehat{K(\theta)} = \frac{\Omega}{\sqrt{2}}\sigma_3\otimes(\hat{A}e^{-i\theta}+\hat{A}^\dagger e^{i\theta}), \qquad (29)$$

and the reduced dynamics of an observable, such as the operator that at time $t=0$ equals the region operator for the point $(u\cos\theta,u\sin\theta)$, is

$$\frac{1}{\pi}\hat{\Delta}(u\cos\theta,u\sin\theta)(t) = \frac{1}{\pi}\mathrm{Tr}_A(e^{it\hat{H}(\theta)}|0\rangle\langle 0|\otimes\hat{\Pi}e^{-it\hat{H}(\theta)})$$
$$= \frac{1}{\pi}\hat{\Delta}(u\cos\theta+\Omega t\sin\theta,u\sin\theta$$
$$-\Omega t\cos\theta). \qquad (30)$$

In this equation the initial atomic state has been chosen to be the ground state $|0\rangle$, and the subsequent tracing of the atomic system has produced a time-evolved point region operator: namely, the region operator corresponding to the point with coordinates $(u\cos\theta+\Omega t\sin\theta, u\sin\theta-\Omega t\cos\theta)$. This point moves linearly in time along the line with parameters (u,θ), from one end at $t=-\infty$, through the point with coordinates $(u\cos\theta,u\sin\theta)$ at $t=0$, and on to the other end at $t=\infty$. According to (25), the map $\varepsilon_{\hat{K}}$ and hence the radon operator for the line are then obtained by integration of this point region operator along the whole line, with respect to the measure Ωt.

The physical relevance of radon region operators and their construction from point region operators by trace-increasing maps follows from the fact that the Hamiltonian $\hat{H}(\theta)$ is obtained in the so-called strong-classical-field Jaynes-Cummings model of microwave cavity quantum electrodynamics (cQED) experiments. This indicates a dynamical basis for the theory of radon operators, another subject worthy of further investigation.

To see the connection with the cQED model, we note that in the rotating-wave approximation the Hamiltonian of that model is

$$\hat{H} = \frac{\omega_0}{2}\sigma_3\otimes\mathbf{1}+\omega\mathbf{1}\otimes a^\dagger a+g(\sigma_- e^{-i\theta+i\omega_L t}+\sigma_+ e^{i\theta-i\omega_L t})\otimes\mathbf{1}$$
$$+\sqrt{2}\Omega(\sigma_-\otimes\hat{A}^\dagger+\sigma_+\otimes\hat{A}), \qquad (31)$$

where Ω is the atom-cavity interaction strength, ω the single-mode cavity field frequency, and ω_0 the atomic transition frequency, while g and θ are the amplitude and phase of a classical field with frequency ω_L. In the case of resonance where the atom, cavity, and driving-classical-field frequencies are all equal and in the strong-classical-field approximation where $g\gg\Omega$, the effective Hamiltonian becomes $\hat{H}(\theta)$ as in (29). (See also [36–43].)

IV. REGION OPERATORS FOR TOMOGRAPHY OF GENERALIZED QUASI-PROBABILITY FUNCTIONS

The region operators that have initially been introduced in association with integrals of Wigner function, when seen also as operator-valued measures defined on phase space [26], motivates the question of having similar OVMs—i.e., region operators—associated with other than the Wigner, phase-space quasiprobability functions, such as the P and Q functions. In this section we will give a brief partial answer to this more general question, in relation to the tomography problem along the lines of the two previous sections, by employing the so-called s parametrization of generalized quasiprobability densities [34].

From the power series expansion of the s-parametrized quasiprobability phase-space function $F(\alpha;s)$ of [44,45],

$$F(\alpha;s)=\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(1+s)^n}{(1-s)^{n+1}}(-1)^n\langle n|D(\alpha)^\dagger\hat{\rho}D(\alpha)|n\rangle, \quad (32)$$

we get

$$F(\alpha;s)=\frac{2}{\pi}\mathrm{Tr}[\hat{\rho}\hat{F}(\alpha;s)]\equiv\frac{2}{\pi}\mathrm{Tr}[\hat{\rho}D(\alpha)\hat{\Pi}(s)D(\alpha)^\dagger], \quad (33)$$

where the s-parametrized parity operator is displaced from $\hat{\Pi}$ by the coherent-state generating operators:

$$\hat{\Pi}(s)=(1+s)^{\hat{N}}\hat{\Pi}(1-s)^{-(\hat{N}+1)}. \quad (34)$$

Special cases are the Glauber-Sudarshan P function [46–48] $F(\alpha;1)\equiv P(\alpha)$, the Wigner function $F(\alpha;0)\equiv W(\alpha)$, and the positive-$Q$ function $F(\alpha;-1)\equiv Q(\alpha)$.

Remarks

(i) The "point operator" corresponding to the origin in phase-space plane is now $\hat{\Pi}(s)/\pi$.

(ii) The tomographic construction of the Wigner function in terms of the radon region operator outlined above can now be extended to the construction of any s-parametrized quasiposability phase-space function. In particular, the s-parametrized radon region operator is given by [cf. (15) and (16)]

$$\begin{aligned}\hat{R}(u,\theta;s)&=\varepsilon^*_{u,\theta}(\hat{\Pi}(s))=\frac{1}{\pi}\int\delta(u-q\cos\theta-p\sin\theta)\\&\quad\times\hat{D}(q,p)\hat{\Pi}(s)\hat{D}(q,p)^\dagger dqdp\\&=\frac{1}{\pi}\int\delta(u-q\cos\theta-p\sin\theta)\hat{\Delta}(q,p;s)dqdp\\&=\frac{1}{\pi}\int\hat{\Delta}(q(u,v,\theta),p(u,v,\theta);s)dv,\end{aligned} \quad (35)$$

with $q(u,v,\theta)$ and $p(u,v,\theta)$ as before. Now $\hat{\Delta}(q,p;s)/\pi$ appearing here is the point region operator corresponding to the point (q,p).

The generalization $\varepsilon_{\hat{K}}$ of this completely positive trace-increasing map, operating on $\hat{\Delta}(u\cos\theta,u\sin\theta;s)$, would also lead to an interpretation in terms of the Jaynes-Cummings model Hamiltonian operator, as indicated for the case $s=0$ in the last section.

V. REGION OPERATORS FOR CANONICAL POLYGONS

In this section we construct region operators for canonical polygons centered on the origin in phase space (to be precise, region operators $\hat{K}_m$ for canonical 2^m-gons, $m=2,3,\ldots$) by operating, with a single unitary operator in an extended space, on the region operator $\hat{K}_\nabla$ of an isosceles triangle. The triangle has angles $\pi(2^{m-1}-1)/2^m$, $\pi(2^{m-1}-1)/2^m$, and $\varphi=\pi/2^{m-1}$, the third angle being subtended at the origin in phase space.

To this end we first introduce m ancillary copies of the two-dimensional (qubit) Hilbert space $\mathcal{H}_2$ with orthonormal basis $\{|0\rangle,|1\rangle\}$ and now work in the tensor product space

$$\mathcal{H}_2^{\otimes m}\otimes\mathcal{H}, \quad (36)$$

where $\mathcal{H}$ is the 'canonical' space introduced in Sec. II, spanned by the eigenstates $|0\rangle,|1\rangle,|2\rangle,\ldots$ of the number operator. We think of the ancillary spaces as "control spaces" and the space $\mathcal{H}$ as a "target space." Then we introduce $\hat{P}_r=|r\rangle\langle r|$, the projection operator onto the basis vector $|r\rangle$ in $\mathcal{H}_2$ for $r=0,1$, and also the rotation operator $\hat{U}(\varphi)=e^{i\varphi\hat{N}}$, acting on $\mathcal{H}$, and we define the unitary operator

$$\hat{V}(\phi,m)\quad:\quad\mathcal{H}_2^{\otimes m}\otimes\mathcal{H}\to\mathcal{H}_2^{\otimes m}\otimes\mathcal{H},$$

$$\begin{aligned}\hat{V}(\phi,m)=&\sum_{r_1,r_2,\ldots,r_m=0,1}\hat{P}_{r_1}\otimes\hat{P}_{r_2}\otimes\ \ldots\ \otimes\hat{P}_{r_m}\\&\otimes\hat{U}(\phi)^{r_1 2^{m-1}+\cdots+r_m 2^0}.\end{aligned} \quad (37)$$

Suppose that the canonical system is in the state $\hat{\rho}$ and that the system of ancillaries is in the state $\hat{\rho}_A^{\otimes m}$, where $\hat{\rho}_A$ is some density matrix acting on the space H_2. By means of the unitary operator $\hat{V}(\varphi,m)$ we introduce the completely positive trace-preserving and -increasing map

$$\hat{\rho}\to\varepsilon_{\phi,m}(\hat{\rho})=\mathrm{Tr}_A[\hat{V}(\phi,m)(\hat{\rho}_A^{\otimes m}\otimes\hat{\rho})\hat{V}^\dagger(\phi,m)]. \quad (38)$$

The quasiprobability mass on the isosceles triangle in the phase space of the canonical system, in the transformed state $\varepsilon_{\varphi,m}(\hat{\rho})$, is $\mathrm{Tr}[\varepsilon_{\varphi m}(\hat{\rho})\hat{K}_\nabla)$. But this equals the quasiprobability mass over the resulting canonical 2^m-gon when the system is in the state $\hat{\rho}$, because it follows by duality that

$$\mathrm{Tr}[\varepsilon_{\phi m}(\hat{\rho})\hat{K}_\nabla]=\mathrm{Tr}[\hat{\rho}\varepsilon^*_{\phi,m}(\hat{K}_\nabla)]=\mathrm{Tr}(\hat{\rho}\hat{K}_m). \quad (39)$$

Explicitly, the action of the dual map reads

$$\begin{aligned}\varepsilon^*_{\phi,m}(\hat{K}_\nabla)&\equiv\hat{K}_m=\mathrm{Tr}_A[\hat{V}^\dagger(\phi,m)(1^{\otimes m}\otimes\hat{K}_\nabla)\hat{V}(\phi,m)]\\&=\sum_{r_1,r_2,\ldots,r_m=0,1}e^{-i(r_1 2^{m-1}+\cdots+r_m 2^0)\phi\hat{N}}\hat{K}_\nabla e^{i(r_1 2^{m-1}+\cdots+r_m 2^0)\phi\hat{N}}\\&=\sum_{s=0}^{2^m-1}e^{-is\phi\hat{N}}\hat{K}_\nabla e^{is\phi\hat{N}}.\end{aligned} \quad (40)$$

Remarks

(i) Given that the trace of a region operator equals the area of its support [25,26], taking the trace of the canonical

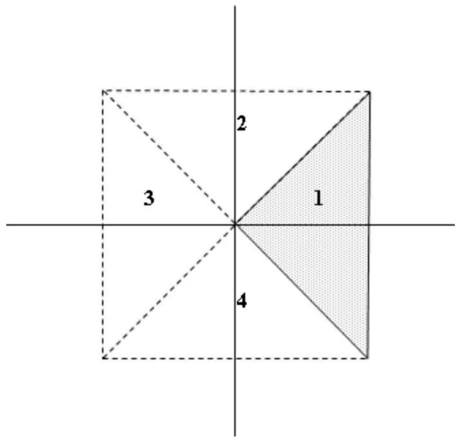

FIG. 1. Graphical depiction of the construction of a region operator with support on a square centered at the origin, starting from the corresponding operator for a orthogonal triangle marked by 1, by the action of a single positive trace- (area-) increasing map on it as described in the text. The positive map generates three more triangles, marked by 2, 3, and 4 in the figure, the union of which with triangle 1 makes up the square.

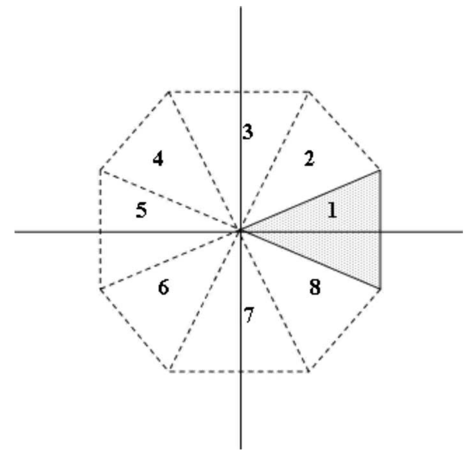

FIG. 2. The same as in Fig. 1, for the case of constructing a canonical octagon starting with an isosceles triangle marked by 1 in the figure. The action of the corresponding positive trace- (area-) increasing map, on the region operator of the initial triangle, generates seven new ones with support on the triangles marked by 2,…,8 in the figure. The union of all these triangles with triangle 1 produces the region operator of the canonical octagon.

2^m-gon region operator obtained in last equation by means of the trace-increasing map introduced earlier, gives m times the area of the initial triangle which in fact equals the area of the resulting canonical 2^m-gon.

(ii) Given an initial isosceles triangle with angle $\varphi=\pi/2^{m-1}$ at the origin (see Fig. 1), then rotations about the origin through angles $0,\varphi,2\varphi,3\varphi,\ldots,(2^m-1)\varphi$ added together produce a canonical 2^m-gon. Two examples are the construction a square from an orthogonal triangle, with angles $\pi/4$, $\pi/4$, and $\varphi=\pi/2$ and with $m=2$ (Fig. 1) and the construction of a canonical octagon from an isosceles triangle with angles $3\pi/8$, $3\pi/8$, and $\varphi=\pi/4$ and with $m=3$ (Fig. 2).

VI. LATTICE REGION OPERATORS

In this section we construct region operators for finite and infinite lattices, specifically region operators $\hat{K}_{SL}$ for square lattices, by starting with the region operators of finite straight-line segments and operating on them by a single unitary operator in an extended space.

We reintroduce the momentum operator $\hat{P}$ acting on $\mathcal{H}$ and define the translation operator $\hat{T}(a)=e^{ia\hat{P}}$, for the step variable $a\in\mathbb{R}$. Then we introduce m ancillary spaces as in the preceding section, and with $\hat{P}_r$ as before, we define the unitary operator

$$\hat{V}(\phi,m) \quad : \quad \mathcal{H}_2^{\otimes m}\otimes\mathcal{H}\rightarrow\mathcal{H}_2^{\otimes m}\otimes\mathcal{H},$$

$$\hat{V}(a,m)=\sum_{r_1,r_2,\ldots,r_m=0,1}\hat{P}_{r_1}\otimes\hat{P}_{r_2}\otimes\cdots\otimes\hat{P}_{r_m}\otimes\hat{T}(a)^{r_1 2^{m-1}+\cdots+r_m 2^0}. \tag{41}$$

Next we introduce the (generalized) region operator for the straight-line segment lying along the p axis, from $p=-L/2$ to $p=L/2$, obtained from the point region operator at the origin (the parity operator $\hat{\Pi}/\pi$) as [26]

$$\varepsilon_L^{*\hat{Q}}(\hat{\Pi})=\hat{K}_L^{\hat{Q}}=\int_{-L/2}^{L/2}\hat{D}(0,p)\hat{\Pi}\hat{D}(0,p)^\dagger dp=\frac{\sin(L\hat{Q})}{\hat{Q}}\hat{\Pi}. \tag{42}$$

As in Sec. V, we now suppose that the canonical system is in the state $\hat{\rho}$ and that the ancillary system is in the state $\hat{\rho}_A^{\otimes m}$, where $\hat{\rho}_A$ is some density matrix on the space $\mathcal{H}_2$. Using the unitary operator $\hat{V}(a,m)$ this time, we introduce the positive trace-increasing map

$$\hat{\rho}\rightarrow\varepsilon_{a,m}(\hat{\rho})=\mathrm{Tr}_A[V(a,m)(\hat{\rho}_A^{\otimes m}\otimes\hat{\rho})\hat{V}(a,m)^\dagger]. \tag{43}$$

The quasiprobability mass for the canonical system in the transformed state $\varepsilon_{a,m}(\hat{\rho})$ on a line segment of length L is $\mathrm{Tr}[\varepsilon_{a,m}(\hat{\rho})\hat{K}_L^{\hat{Q}}]$ [25,26]. This equals the quasiprobability mass when the system is in state $\hat{\rho}$, on the set of 2^m parallel straight-line segments with distance a between them, running from $(sa,-L/2)$ to $(sa,L/2)$ in the phase plane, for $s=0,1,\ldots,2^m-1$, because it follows by duality that we can also write

$$\mathrm{Tr}[\varepsilon_{a,m}(\hat{\rho})\hat{K}_L^{\hat{Q}}]=\mathrm{Tr}[\hat{\rho}\varepsilon_{a,m}^*(\hat{K}_L^{\hat{Q}})]=\mathrm{Tr}(\hat{\rho}\hat{K}_\parallel), \tag{44}$$

where $\hat{K}_\parallel$ is the region operator for the set of parallel segments. Explicitly, the action of the dual map reads

$$\varepsilon^*_{a,m}(\hat{K}_L^{\hat{Q}}) = \mathrm{Tr}_A[\hat{V}^\dagger(a,m)(1^{\otimes m}\otimes \hat{K}_L^{\hat{Q}})V(a,m)]$$
$$= \sum_{r_1,\ldots,r_m=0,1} e^{-i(r_1 2^{m-1}+\cdots+r_m 2^0)a\hat{P}}\hat{K}_L^{\hat{Q}} e^{i(r_1 2^{m-1}+\cdots+r_m 2^0)a\hat{P}}$$
$$= \sum_{s=0}^{2^m-1}\frac{\sin(L[\hat{Q}-sa])}{\hat{Q}-sa}\hat{\Delta}(-sa,0) = \hat{K}_\parallel. \qquad (45)$$

Having constructed $\hat{K}_\parallel$, we proceed to construct the operator for a square lattice. This is obtained by choosing $L=(2^m-1)a=Ma$ (say), rotating the set of parallel segments through $\pi/2$ clockwise about the point $(L/2,0)$ in the phase plane and then superposing them over the original ones [see Figs. 3(a)–3(c)]. This last step of rotation and superposition is realized by a further map

$$\varepsilon^*_{RS}:\hat{K}_\parallel \to \hat{K}_{SL},$$

$$\hat{K}_{SL} = \hat{K}_\parallel + \hat{U}_{L/2}\left(\frac{\pi}{2}\right)^\dagger \hat{K}_\parallel \hat{U}_{L/2}\left(\frac{\pi}{2}\right). \qquad (46)$$

Here $\hat{U}_{L/2}(\varphi)$ is the rotation operator about the point $(L/2,0)$, generated by the translated number operator $\hat{N}_{L/2}$, which is defined like $\hat{N}$ in (1) and (2), but with $\hat{Q}$ replaced by $\hat{Q}-L/2$—i.e.,

$$\hat{U}_{L/2}(\phi) = e^{i\phi\hat{N}_{L/2}}, \quad \hat{N}_{L/2} = \hat{N} - \frac{1}{2}L\hat{Q} + \frac{1}{8}L^2. \qquad (47)$$

This map is also implemented unitarily in an extended space by the operator

$$\hat{V}_{\pi/2} = \hat{P}_1\otimes 1 + \hat{P}_0\otimes \hat{U}_{L/2}\left(\frac{\pi}{2}\right)$$
$$= \mathrm{diag}\left(1,\hat{U}_{L/2}\left(\frac{\pi}{2}\right)\right) = e^{(i\pi/4)(1-\sigma_3)\otimes\hat{N}_{L/2}},$$

$$\hat{V}_{\pi/2}:\mathcal{H}_2\otimes\mathcal{H}\to\mathcal{H}_2\otimes\mathcal{H}, \qquad (48)$$

with $\hat{P}_r$ as in Sec. V. Explicitly,

$$\hat{K}_{SL} \equiv \varepsilon^*_{RS}(\hat{K}_\parallel) = \mathrm{Tr}_A[\hat{V}^\dagger_{\pi/2}(\mathbf{1}\otimes\hat{K}_\parallel)\hat{V}_{\pi/2}]. \qquad (49)$$

Dually from this last map we also have

$$\varepsilon_{RS}(\hat{\rho}) = \hat{\rho} + \hat{U}_{L/2}\left(\frac{\pi}{2}\right)\hat{\rho}\hat{U}_{L/2}\left(\frac{\pi}{2}\right)^\dagger. \qquad (50)$$

The final formula for the square-lattice-region operator is, from (46),

$$\hat{K}_{SL} = \sum_{s=0}^{M}\frac{\sin(L[\hat{Q}-sa])}{\hat{Q}-sa}\hat{\Delta}(-sa,0) + \sum_{s=-M/2}^{M/2}\frac{\sin(L[\hat{P}-sa])}{\hat{P}-sa}\hat{\Delta}(0,-sa), \quad M=2^m-1. \qquad (51)$$

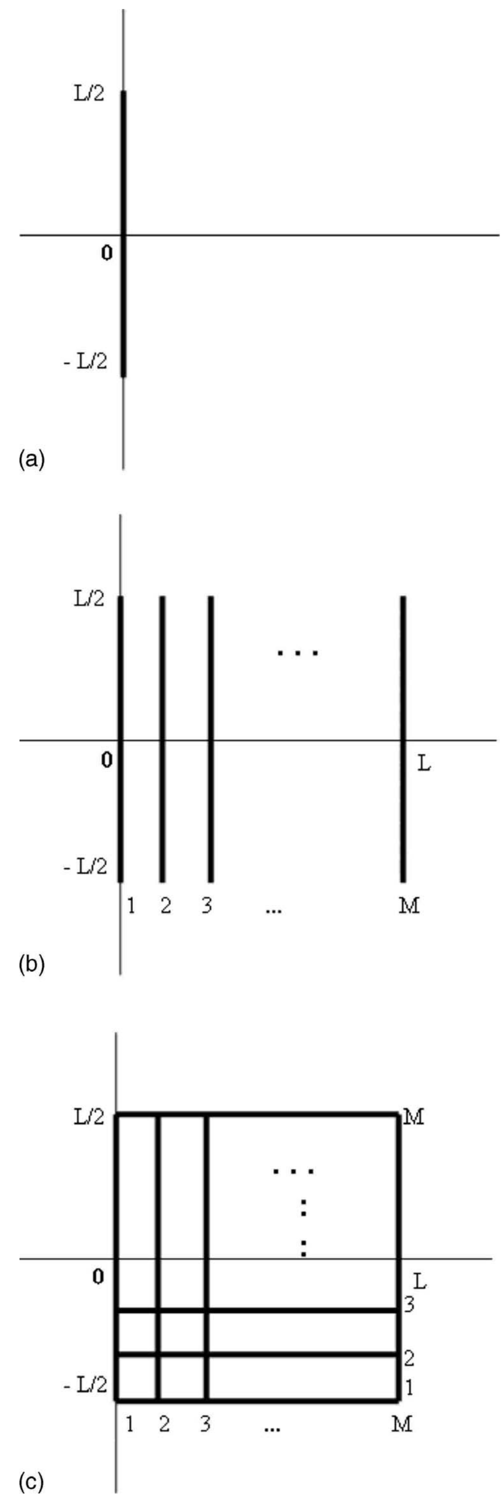

FIG. 3. Construction of a region operator defined over a square lattice with m horizontal and vertical parallel-line segments of length L. The construction begins with the region operator with support on the line-segment extended symmetrically with respect to the origin along the vertical axis (a). The first positive map (see text) acts on the line segment region operator to produce $m-1$ new ones parallel to it, spaced equally by $\frac{L}{m}$ (b). The second positive map (see text) operates on the region operator of previous steps to superimpose on it a $\frac{\pi}{2}$-rotated copy of the initial (b), producing the final region operator of the square lattice.

By means of (4) and (5), the square-lattice-region operator becomes

$$\hat{K}_{SL} = \sum_{s=0}^{M} \frac{\sin(L[\hat{Q} - sa])}{\hat{Q} - sa} e^{i2sa\hat{P}} \hat{\Pi} + \sum_{s=-M/2}^{M/2} \frac{\sin(L[\hat{P} - sa])}{\hat{P} - sa} e^{-i2sa\hat{Q}} \hat{\Pi}, \quad M = 2^m - 1. \tag{52}$$

Let us summarize the action of the above maps and their duals: the region operator for the square lattice is obtained from the region operator at the point of the origin—namely, the parity operator—by the action of the composite map

$$\hat{K}_{SL} = \varepsilon^*_{RS} \circ \varepsilon^*_{a,m} \circ \varepsilon^{*\hat{Q}}_{L}(\hat{\Pi}), \tag{53}$$

as shown in Figs. 3(a)–3(c). The quasiprobability mass on the square lattice in phase space, for the Wigner function corresponding to a state $\hat{\rho}$, is then

$$\mathrm{Tr}[\hat{\rho}\varepsilon^*_{RS} \circ \varepsilon^*_{a,m} \circ \varepsilon^{*\hat{Q}}_{L}(\hat{\Pi})] = \mathrm{Tr}[\varepsilon^{\hat{Q}}_{L} \circ \varepsilon_{a,m} \circ \varepsilon_{RS}(\hat{\rho})\hat{\Pi}]. \tag{54}$$

For $L=3a$, with $m=2$ qubits, the associated Kraus generators [39] are $\{1, \hat{T}(a)=e^{ia\hat{P}}, \hat{T}(2a)=e^{i2a\hat{P}}, \hat{T}(3a)=e^{i3a\hat{P}}\}$.

The region operator for an infinite square lattice with spacing a can be obtained from (51), (53), and (54) by letting $L, M \to \infty$ with a fixed.

VII. DISCUSSION

This work extends the theory of region operators for the simplest case of a phase-space plane. Region operators where seen to result from quantization, by Weyl's quantization rule, of characteristic functions of regions in phase space. They stand as operator-valued measures for these regions and, given a state density operator of a quantum system, they can assign to these regions quasiprobabilities (masses), by means of integrals of systems' Wigner function over those regions. The problem of operational construction of region operators is addressed here along lines of previous works, by means of positive trace-increasing maps operating on some initial region operator (e.g., the operator of the point at the origin) or dually on the density operator. As such maps can be lifted to unitary operators in extended space—i.e., spaces that include together with the state space of original system some additional auxiliary spaces—the construction problem can be cast into a search for appropriate Hamiltonian operator acting on an extended Hilbert space. This has been achieved for region operators of phase-space lines in terms of the problem of tomographic construction of Wigner function and other generalized quasidensities by means of the radon transform. Along similar lines region operators for canonical polygons and square lattices have been constructed. These constructions demand space extensions that in physical terms are translated to a number of auxiliary qubit systems. A number of important points can been raised at this stage of development of the theory of region operators; some of them, especially those that can be answered based on results of this work, should be mentioned: construction of Hamiltonian models in extended spaces that generate region operators for a variety of interesting domains, interrelation of Groenewold operators and region operators and their associated problems of classicality and positivity; to these and related problems, we aim to return elsewhere.

ACKNOWLEDGMENTS

This work was supported by EPEAEK research program Pythagoras II, funded by Greek Ministry of Education and partially by EU.

[1] H. Weyl, Z. Phys. **46**, 1 (1927). See also H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover, New York, 1931).
[2] J. von Neumann, Math. Ann. **104**, 570 (1931).
[3] E. P. Wigner, Phys. Rev. **40**, 749 (1932).
[4] H. Groenewold, Physica (Amsterdam) **12**, 405 (1946).
[5] J. Moyal, Proc. Cambridge Philos. Soc. **45**, 99 (1949).
[6] F. A. Berezin, Sov. Phys. Usp. **23**, 763 (1980).
[7] F. E. Schroeck, *Quantum Mechanics in Phase Space* (Kluwer, Boston, 1996).
[8] *Quantum Mechanics in Phase Space: An Overview with Selected Papers*, edited by C. K. Zachos, D. B. Fairlie, and T. L. Curtright (World Scientific, Singapore, 2005).
[9] D. A. Dubin, M. A. Hennings, and T. B. Smith, *Mathematical Aspects of Weyl Quantization and Phase* (World Scientific, Singapore, 2000).
[10] W. P. Schleich and M. G. Raymer, J. Mod. Opt. **44**, 12 (1997).
[11] W. P. Schleich, *Quantum Optics in Phase Space* (Wiley-VCH, New York, 2001).
[12] U. Leonhardt, *Measuring the Quantum State of Light* (Cambridge University Press, Cambridge, England, 1997).
[13] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, Ann. Phys. (N.Y.) **111**, 61 (1978).
[14] M. Kontsevich, Lett. Math. Phys. **66**, 157 (2003).
[15] C. Zachos, Int. J. Mod. Phys. A **17**, 297 (2002).
[16] A. J. Bracken, Rep. Math. Phys. **57**, 17 (2006).
[17] T. Curtright, D. Fairlie, and C. Zachos, Phys. Rev. D **58**, 025002 (1998).
[18] A. J. Bracken, H.-D. Doebner, and J. G. Wood, Phys. Rev. Lett. **83**, 3758 (1999).
[19] A. J. Bracken, G. Cassinelli, and J. G. Wood, J. Phys. A **36**, 1033 (2003).
[20] A. J. Bracken, J. Phys. A **36**, L329 (2003).
[21] A. J. Bracken, D. Ellinas, and J. G. Wood, Acta Phys. Hung. B **20**, 121 (2004); arXiv:quant-ph/0304110.
[22] A. J. Bracken, D. Ellinas, and J. G. Wood, J. Phys. A **36**, L297 (2003).
[23] J. G. Wood and A. J. Bracken, J. Math. Phys. **46**, 042103 (2005).

[24] A. J. Bracken and J. G. Wood, Phys. Rev. A **73**, 012104 (2006).
[25] D. Ellinas and I. Tsohantjis, Rep. Math. Phys. **57**, 69 (2006); arXiv:quant-ph/0510140
[26] D. Ellinas, in *Foundations of Probability and Physics 4*, edited by G. Adenier, C. A. Fuchs, and A. Khrennikov, AIP Conf. Proc. No. 889 (AIP, Melville, NY, 2006), pp. 289–293.
[27] R. L. Stratonovich, Sov. Phys. JETP **4**, 891 (1957).
[28] C. Brif and A. Mann, Phys. Rev. A **59**, 971 (1999).
[29] A. J. Bracken and J. G. Wood, Europhys. Lett. **68**, 1 (2004).
[30] S. R. Deans, *The Radon Transform and some its Applications* (Wiley, New York, 1983).
[31] V. G. Romanov, *Integral Geometry and Inverse Problems for Hyperbolic Equations* (Springer-Verlag, Berlin, 1974).
[32] S. Helgason, *The Radon Transform* (Birkhaüser, Boston, 1980).
[33] By means of the cyclic property of the trace, the expectation value of an observable $\hat{X}$—viz., $\langle \hat{X}, \varepsilon(\hat{\rho})\rangle = \mathrm{Tr}[\hat{X}\varepsilon(\hat{\rho})]$—on a state $\hat{\rho}$, mapped by a *CP* map ε as $\varepsilon(\hat{\rho}) = \Sigma_i \hat{A}_i \hat{\rho} \hat{A}_i^\dagger$, serves to define the dual map $\varepsilon^*(\hat{X}) = \Sigma_i \hat{A}_i^\dagger \hat{X} \hat{A}_i$ that satisfies the equation $\langle \hat{X}, \varepsilon(\hat{\rho})\rangle = \mathrm{Tr}[\hat{\rho}\varepsilon^*(\hat{X})]$.
[34] M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, Phys. Rep. **106**, 121 (1984).
[35] Y. Nievergelt, SIAM Rev. **28**, 79 (1986).
[36] E. Solano, G. S. Agarwal, and H. Walther, Phys. Rev. Lett. **90**, 027903 (2003).
[37] H. Mabuchi and H. M. Wiseman, Phys. Rev. Lett. **81**, 4620 (1998).
[38] X. B. Zou, K. Pahlke, and W. Mathis, Phys. Rev. A **69**, 015802 (2004).
[39] *States, Effects and Operations: Fundamental Notions of Quantum Theory*, edited by K. Kraus, A. Böhm, J. D. Dollard, and W. H. Wootters, Lecture Notes in Physics, Vol. 190 (Springer, Berlin, 1983).
[40] M. S. Kim, G. Antesberger, C. T. Bodendorf, and H. Walther, Phys. Rev. A **58**, R65 (1998).
[41] L. G. Lutterbach and L. Davidovich, Phys. Rev. Lett. **78**, 2547 (1997).
[42] P. Bertet, A. Auffeves, P. Maioli, S. Osnaghi, T. Meunier, M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. **89**, 200402 (2002).
[43] P. J. Bardroff, M. T. Fontenelle, and S. Stenholm, Phys. Rev. A **59**, R950 (1999).
[44] H. Moya-Cessa and P. L. Knight, Phys. Rev. A **48**, 2479 (1993).
[45] A. Royer, Phys. Rev. A **15**, 449 (1977).
[46] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[47] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[48] E. C. G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963).