Effects of Laminar Boundary Layer on a Model Broad-Crested Weir

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Research Report No. CE28
September, 1981
EFFECTS OF LAMINAR BOUNDARY LAYER
ON A MODEL BROAD-CRESTED WEIR

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RESEARCH REPORT NO. CE 28
Department of Civil Engineering
University of Queensland
September, 1981

Synopsis

This report describes an approximate method for the analysis of flow with a laminar boundary layer over a model broad-crested weir. The method has been used to demonstrate the effects of the laminar boundary layer and the results have shown that these effects are significant. If the energy is calculated from depth and discharge measurements, large energy losses appear to occur at low flow rates in laboratory experiments. Losses real and apparent, are explained and the methods presented may be used to estimate the correct energy values.

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INTRODUCTION

1. Flow Over a Model Broad-crested Weir

The flow over a model broad-crested weir (Figure 1) is often studied in undergraduate laboratory classes because

(i) prototypes of these weirs are commonly used as flow measuring devices,

(ii) subcritical, critical and supercritical flow can be observed, and

(iii) the experiment affords an opportunity for the demonstration and application of the Specific Energy concept.

If \( q \) is the two-dimensional discharge and \( y \) the depth, provided the velocity is uniform over a cross-section, the specific energy, \( E \), is given by

\[
E = y + \frac{q^2}{2gy^2} \quad (1a)
\]

\[
= y + \frac{V^2}{2g} \quad (1b)
\]

in which \( g \) is the acceleration due to gravity and \( V \) is the velocity.

If there are no losses and the bed level upstream and downstream of the weir are the same (see Figure 1),

\[
E_1 = E_2 \quad (2)
\]

\[
E_2 = E_1 - w \quad (3)
\]

in which \( w \) is the level of the weir sill relative to the upstream bed level.
2.

Flow is assumed to occur at critical depth over the weir. Hence,

\[ y_2 = y_c \]  

(4)

\[ E_2 = E_c = 1.5 y_c \]  

(5)

in which \( y_c = \left( \frac{q^2}{g} \right)^{1/2} \) is the critical depth and \( E_c \) is the critical or minimum specific energy for the discharge \( q \). If the bed is smooth and there are no discontinuities in the slope along MNPR (Figure 1), the assumption of no energy losses between sections 1 and 3 would generally be made, e.g. in the calculation of depth at section 3 given the discharge and depth at section 1.

The broad-crested weir experiment is done in our laboratory in a channel 250 mm wide constructed with glass sides and perspex bottom. The model weir has the profile shown in Figure 1 and detailed in Table 1. The sill is 64 mm above the channel bed which is horizontal. Discharge is measured with a weighing tank and depths are measured by pointer gauges with verniers marked in gradations of 0.2 mm. The specific energy at a cross-section is determined by Equation la. Over the range of flows used, the measured depths at station 3 lie between 3 and 30 mm. Some difficulty occurs in the measurement of depth because of the presence of capillary waves, and at the low flows the experimental error is relatively large. Consequently, there is a wide scatter in the experimental results for low flow rates but, in general, the results indicate that there are substantial losses at low flows and that the relative losses decrease as the flow rate increases. Some experimental results obtained by the writer are plotted in Figure 2. In Figure 2 and in some later figures, \( y_c \) is chosen for the independent variable rather than \( q \) because \( y_c \) gives a more convenient scale. The effects of a change of ± 0.1 mm in the measured depths at station 3 have been computed for some values and are shown by the vertical bars through these values and by the plot of percentage change in \( E_2/E_1 \).
Conversely, the value of $y$ must be insensitive to the value for $E$ used in supercritical regions and the neglect of any losses when the depth is calculated is justifiable. Problems arise when the specific energy in the supercritical region is required and it is not surprising that students' results have a wide variation, especially for the lower flow rates. However,
there is a general trend apparent and an investigation was
done to determine if the general trend could be explained by
the presence of a laminar boundary layer.

1.2 Objective of the Investigation

The objective of the study described in this report
was the development of an approximate analysis to demonstrate
the effects of the laminar boundary layer and to estimate the
magnitude of these effects for the laboratory experiment.

This objective has been achieved. An approximate
method for the analysis of the effects of the laminar boundary
layer which gives results consistent with the observed pattern
has been developed. However, because of the assumptions and
approximations used in the analysis, the method cannot be
expected to yield an accurate, quantitative result.

The effects of the laminar boundary layer are often
important in hydraulic model studies at small scales, e.g. in
model testing of dam spillways. The method described in this
paper may be readily adapted and used to obtain an estimate
of these effects. Furthermore, the work done indicates that
the general method used is satisfactory and could form the
basis for the development of an accurate method for quantitative
assessments.

2. METHOD OF SOLUTION

The solution involves the calculation of the variation
of depth and velocity along the weir sill, M - N, and down the
slope, N - R, (Figure 1) and the calculation of the values of
momentum thickness, \( \delta \), displacement thickness, \( \delta^* \), and the
shear stress along the bed, \( \tau_w \), from a boundary layer analysis.
Since the velocity depends on \( \tau_w \) and the boundary layer analysis
on the velocity, an iterative approach is necessary.

2.1 Calculation of Depth and Velocity

For the purposes of an approximate analysis the flow
over the weir sill is assumed to occur at critical depth and
critical velocity. A preliminary estimate based on the results
for a laminar boundary layer on a smooth flat plate showed
that the energy losses along the sill are very small and they
have been neglected in the analysis. The following method was
developed for the calculation of depth and velocity down the
slope, N - P, and along the horizontal bed, P - R (Figure 1)
for a known discharge.

The profile is described by a number of points at
which \( x \), the horizontal distance, and \( z \), the height of the bed
above datum are specified. In the analysis, values for \( \tau_w \) are
also specified at these points.

If \( i \) and \( i + 1 \) are two points on the slope and the
depth, \( y_i \), is known at \( i \), the depth, \( y_{i+1} \), at \( i + 1 \) is found
from a momentum analysis. The control volume used is shown in
Figure 4.

![FIGURE 4: Control volume for momentum analysis](image-url)
The following assumptions are made:

(i) the depth, $y$, is measured normal to the bed and the velocity is uniform over this depth,

(ii) the pressure distribution is hydrostatic and, therefore, the pressure force per unit width acting at the end of the control volume is (1)

$$ P_i = \rho \frac{g}{2} y_i^2 \cos \beta $$

(7)

(iii) the mean shear stress at the wall is the average of the two values for points $i$ and $i + 1$,

(iv) the volume of water in the control volume of unit width is $0.5 (y_i + y_{i+1}) \Delta S$, and

(v) $\Delta S$, the distance along the bed between points $i$ and $i + 1$, is made equal to the length of the straight line segment joining the two points and, $\beta$, the slope is made equal to the slope of this segment. ($\beta$ is treated as positive.)

The momentum analysis for unit width parallel to the bed gives

$$ P_i + W \sin \beta - \tau_w \Delta S - P_{i+1} = \rho \frac{g}{2} (V_{i+1} - V_i) $$

(8)

in which $W$ is the weight of water in the control volume and $\rho$ is the density of water.

Define a momentum function, $M$, by

$$ M = \frac{y^3 \cos \beta}{2} + \frac{q^2}{2g} $$

(9).

Equation 8 may be rearranged as

$$ M_{i+1} = M_i - S \left( \frac{\tau_w}{pg} - \left( \frac{y_i + y_{i+1}}{2} \right) \sin \beta \right) $$

(10).

9.

If $y_i$ is known, Equation 10 may be solved for $y_{i+1}$.

A convenient, iterative method is as follows:-

Assume a value for $y_{i+1}$, substitute it into the right hand side of Equation 10 and calculate a value for $M_{i+1}$.

Solve Equation 9 for this value of $M_{i+1}$ to obtain the next estimate for $y_{i+1}$. Continue the iterations until satisfactory convergence is obtained.

The depth at the top of the slope is estimated and the method is used to calculate the depth and velocity at the other points. Because of the local curvature of the free surface at the top of the slope the depth will be less than the critical depth, $y_c$, and a value of $0.9 y_c$ has been adopted.

The acceleration, $dV/dS$ is also required for the boundary layer analysis. Once $V$ has been found for each point, $dV/dS$ is computed by the use of the three-point Lagrangian interpolation formulas (2).

2.2 Boundary Layer Calculations

A simple, easy to use method for laminar boundary analysis was sought. Schlichting (3) describes a simple technique based on the von-Karman Pohlhausen method with an approximation suggested by Walz which reduces the laminar boundary layer analysis to a simple quadrature. However, the one parameter, Pohlhausen velocity profiles are unsatisfactoyr for rapidly accelerating flow and are not suited to the present application. Walz (4) claims that Hartree profiles are better and shows that the same techniques may be used with these profiles. Therefore, Hartree profiles are used but the results show that these are also unsatisfactory for precise calculations in this application.

The simplified boundary layer analysis is

(i) determine $\beta$ from
\[
\frac{\partial V}{\partial S} = \int_{S_0}^{S} \frac{V^b}{v^a} \, ds
\]

in which \(V = V(S)\) is the velocity external to the boundary layer, \(v = \nu/\rho\) is the kinematic viscosity (\(\nu\) is the viscosity) and \(S = 0\) where \(\partial V/\partial S = 0\), \(a = 0.441\) and \(b = 4.165\) for Hartree profiles (4).

The integral may be evaluated by numerical quadrature. In the present study \(S = 0\) was assumed at the upstream end of the sill, the contribution to the integral from the flow along the sill is \(V S_k\) (where \(k\) is the length of the sill) and the trapezoidal rule was used to evaluate the integral down the slope.

(i) calculate the profile shape parameter, \(\Gamma\),

\[
\Gamma = \frac{\partial}{\partial V} \frac{\partial V}{\partial S}
\]

\(\partial V/\partial S\) is found from the velocity variation previously computed (Section 2.1).

(ii) determine the parameter \(\alpha = \alpha(\Gamma)\) and hence the wall shear stress, \(\tau_w\),

\[
\tau_w = \frac{\nu V \alpha}{\theta}
\]

\(\alpha\) and \(\Gamma\) values are tabulated in Walz (4). Intermediate values may be found by linear interpolation.

For accelerating flows, \(\Gamma\) varies between 0.0 and 0.082 and \(\alpha\) varies between 0.2212 and 0.3922. In the present study, values for \(\Gamma\) considerably larger than the allowable maximum were calculated at the top of the slope. Where this occurred, the value of 0.3922 was used for \(\alpha\).

Because a simple, approximate analysis is required, the estimate for \(\alpha\) given above has been adopted. The approximation does, however, mean that the results obtained are open to question and should not be used for other than preliminary estimates.

Since the displacement thickness is required only along the section PR (Figure 1) where \(\partial V/\partial S\) is close to zero, a value of \(\delta*/\theta = 2.6\), from the results for a flat plate, is used for the calculation of \(\delta*\) from \(\theta\) at points along PR.

### Table 1: Weir profile

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<tr>
<th>(x) (m)</th>
<th>(z) (m)</th>
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</tr>
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<tbody>
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<td>0.0000</td>
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</tr>
<tr>
<td>0.520</td>
<td>0.0000</td>
<td>point R</td>
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#### 2.3 Solution Procedure

For any given two-dimensional discharge, \(q\), the critical depth, \(y_c\), and the critical velocity, \(V_c\), are calculated and assumed to apply along the sill. The contribution to the integral in Equation 11 from the boundary layer along the sill is evaluated. The flow and boundary layer growth down the slope and along the bed downstream of the slope are calculated by an iterative procedure as follows:

(i) assume \(\tau_w\) values for all specified points \((\tau_w = 0\) is an acceptable assumption);  

(ii) calculate depths, velocities and accelerations (Section 2.1);  

(iv) determine \(\delta*\)
(iii) with values for $V$ and $dV/dS$ from (ii) calculate $\theta$ and $\tau_w$ at the specified points (Section 2.2).

Repeat (ii) and (iii) until satisfactory convergence is obtained. About four iterations are usually sufficient.

The calculations may be done manually with the use of graphs and an electronic calculator but are tedious and time consuming. The method can be readily programmed for a computer and this is recommended if a number of cases are to be studied.

3. RESULTS

The method was applied to the model broad-crested weir 64 mm high with the profile given in Table 1.

As a check of the method for velocities (Section 2.1) an analysis was carried out for $q = 3.132 \times 10^{-2} \text{m}^3/\text{s}$ ($y_c = 0.01 \text{ m}$) with $\tau_w = 0$ and $y = 0.01 \text{ m}$ at the top of the slope. The specific energy calculated for flow between P and R (Figure 1) from the computed depth and velocity was 79.6 mm which was less than 1% different from the exact value of 79.0 mm.

Complete analyses were done for flows over the weir corresponding to critical depth values of 10, 15, 20, 40 and 60 mm. The Reynolds No. based on $\theta$ was calculated at point P for all flows and was below the minimum at which the boundary layer becomes turbulent (2). Initial values of $\tau_w = 1.0 \text{ N/m}^2$ were assumed for all runs. A typical result for the variation of $\tau_w$ down the slope is shown in Figure 5. In each analysis the depth and the velocity at point P were used to calculate the specific energy, $E_b'$, at P from Equation 1. $E_b'$ is an estimate of the specific energy at P when the energy losses from the boundary shear are taken into account and is less than the value for no losses, $E_o'$. The laminar boundary layer also causes the external flow and, hence the free surface, to be displaced a distance equal to the boundary layer displacement thickness, $\delta^*$. Therefore, a depth, $y_a$, equal to the calculated depth plus the displacement thickness is an estimate of the depth that would be measured by a pointer gauge. The apparent specific energy, $E_a$, is the specific energy calculated from Equation 1a when $y_a$ is used. $E_b'/E_o$ and $E_a'/E_o$ are plotted as functions of the critical depth, $y_c'$, in Figure 6. The experimental values for $E_b'/E_o$ are also plotted in Figure 6.

The plots of $E_b'/E_o$ and $E_a'/E_o$ are consistent with the pattern observed in the laboratory. The experimental results obtained by the author for this report lie within the two curves and it appears that the two effects of the boundary...
The results indicate that the effects of the laminar boundary layer are significant and that, even if the depth could be accurately measured, there are large apparent losses of energy at low flow rates because of the boundary layer displacement thickness. If the experiment is used to demonstrate the specific energy concept and the assumption of no losses is made, a sufficiently high flow rate should be chosen such that the apparent losses are acceptable. The results suggest that, in the experiment studied, agreement between specific energy theory and experiment to within 10% should be possible for $y_c > 35$ mm, i.e. $q > 2.05 \times 10^{-3}$ m$^3$/s.

The problems that make the boundary layer analysis suspect are associated with the large values of acceleration, $dV/dS$, at the top of the slope. A more accurate analysis of the flow at the top of the slope was carried out for $y_c = 0.02$ m. The finite element technique for free-surface, gravity affected flows described by Isaacs (5) was used. The region of analysis was $-0.10 < x < 0.18$ m. The mesh is shown in Figure 7.

**FIGURE 6**: $E_k/E_0$ and $E_a/E_0$ as functions of $y_c$

**FIGURE 7**: Finite element mesh
FIGURE 8: Velocities from finite element and momentum analyses ($y_c = 0.02$ m)

FIGURE 9: Accelerations from finite element and momentum analyses ($y_c = 0.02$ m)
The finite elements used were cubic triangular elements with the stream function and the velocity components as nodal parameters. In the finite element analysis, critical conditions were assumed 0.10 m upstream of the start of the slope.

The velocity variations found in the finite element and momentum analyses (both for $T_w = 0$) are shown in Figure 8. The agreement between the results from the two analyses is good. The results indicate that the choice of $y = 0.9 y_c$ at the top of the slope for the momentum analysis is reasonable. The accelerations computed from the velocities are shown in Figure 9 and both analyses yield comparable results.

It appears that the problems in the proposed method are not caused by the method adopted for the calculation of $w$ depth and velocity. Therefore, it is concluded that the approximate method used for the boundary layer calculations is not satisfactory and for reliable results one of the advanced methods of laminar boundary layer analysis is required.

5. CONCLUSION

An approximate method for the analysis of the flow and the growth of the laminar boundary layer over a broad-crested weir model has been presented. Analyses done with the method indicate that laminar boundary layer effects are significant, especially at low flow rates. As the flow rate increases the effects decrease.

A comparison of the results from the analysis and of experimental results shows that the analytical results correctly predict the trend and that the method could be used for preliminary estimates. Because of the approximations used in the method the quantitative values are not accurate enough for precise values. The study has shown that research into methods for laminar boundary analysis in regions of rapidly accelerating flow would be of considerable benefit in the assessment of model studies in hydraulics.
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