UNIVERSITÀ DEGLI STUDI DI TRENTO
DOTTORATO DI RICERCA IN MATEMATICA

XI CICLO

Gianluca Occhetta

Extremal rays of smooth projective varieties

Relatore
Prof. Marco Andreatta

3 Dicembre 1999

Sottrarre è un modo di aggiungere

Contents

Introduction

In two fundamental papers [Mor79] and [Mor82] Mori studied smooth threefolds with non numerically effective canonical bundle, i.e. threefolds on which there exists a curve C such that $K_X.C < 0$. We will call, for shortness, these curves negative curves.

His main results were the so called cone theorem and contraction theorem; the first asserts that in the cone of effective curves (more precisely the closure of the cone generated by effective curves in the $\mathbb{R}$ vector space of 1-cycles modulo numerical equivalence) the negative curves stay in a part of this cone which is polyhedral, and whose rays are generated by rational curves. The second states that each ray (or even a face) of this polyedral part of the cone can be contracted; that is given a ray R_i there is a proper surjective map $\varphi_i : X \to Z_i$ onto a normal variety Z_i, with connected fibers and such that the anticanonical divisor of X is φ_i-ample and φ_i contracts all the curves in the ray. Such a map is often called a *Fano-Mori* elementary contractions, or an *extremal contraction*.

In order to prove these results he used very much the theory of deformation of curves on a projective variety; in particular he developed a technique, which is now called "bend-and-break" technique, which produces rational negative curves and which can be roughly summarized as follows. If a curve is "sufficiently" negative then it has non trivial deformations, even with a fixed point; then in the degenerate part of such a deformation the curve "breaks" with at least one rational component. With this technique, in constructing the extremal contractions, he also proved that the fibers of these contractions are covered by rational curves.

As a final output he gave a complete description of Fano-Mori contractions of smooth threefolds (and of smooth surfaces, revising the classical theory in this new language): this is considered as the first step of the so called *Minimal Model Program* in dimension three.

The cone theorem and the contraction theorem were generalized in higher dimen-

sion by Kawamata [Ka84a] [Ka84b], assuming also the existence of some mild singularities. He used a different approach with cohomological methods which overcomes the difficulty of generalizing deformation theory of curves to singular varieties. In particular he proved that a manifold of any dimension, with mild singularities, whose canonical bundle is not numerically effective, admits a Fano-Mori contraction.
It is important to mention that, by Kawamata contraction theorem, a Fano-Mori contraction of a smooth variety is given by the linear system associated to a multiple of a divisor $K_X + rL$, with L a φ-ample line bundle on X and r a positive integer; we will say that the contraction is *supported* by $K_X + rL$.

A natural problem is then to describe these contractions in any dimension, and it is reasonable to start assuming that the variety X, on which they are defined, is smooth. The theory of deformation of curves can be again used, studying families of rational curves on the fibers of these contractions; Ionescu [Io86] and Wiśniewski [Wiś89] introduced the length of a ray, namely the natural number $l(R_i) = min\{-K_X.C \mid [C] \in R_i\}$ which is directly related to the dimension of the space of deformations of the curves in the ray, and proved a bound relating the dimensions of the fibers and of the exceptional locus of the contraction of the ray to the dimension of the ambient variety and the length.
In this thesis we will consider Fano-Mori contractions of higher dimensional varieties supported by $K_X + rL$ for which the inequality of Ionescu and Wiśniewski is an equality or is "almost" an equality (for a precise definition see III.2.1) and such that $l(R) = r$, i.e. such that the contracted curve generating the extremal ray are lines with respect to L; contractions of this kind appear in the so called *adjunction theory* [BS95]. Then we try to generalize the results to contractions supported by $K_X + det\mathcal{E}$, with $\mathcal{E}$ ample vector bundle on X; contraction of this kind arise naturally considering pairs $(X, \mathcal{E})$ such that $\mathcal{E}$ has a section whose zero locus is a variety with non nef canonical bundle, which are studied in the last part of the thesis.

Going into details: in the first chapter the terminology and the fundamental results of the theory of extremal rays and extremal contractions, and of the adjunction theory of polarized varieties are recalled, together with some basic facts about ample vector bundles and Albanese varieties.
Chapter II is dedicated to the study of rational curves whose deformations can not break, hence the name of *unbreakable* rational curves. A fundamental example of an unbreakable rational curve is an extremal rational curve, i.e. a rational curve whose numerical class generates an extremal ray of the ambient variety and is minimal.
In the first section we recall the definitions of families of rational curves and we show how to construct a family with good properties starting with an unbreakable rational curve (see [Mor79], [BS95], [Wiś89] and [Ko96]). The basic facts about

"bend-and-break" techniques are summarized in Appendix A. Then unbreakable families are used to give a criterion for a good contraction to be elementary, and to study Fano manifolds of large pseudoindex, as a first step to extend Wiśniewski's classification [Wiś90a],[Wiś91b].
In the third Chapter we use deformation arguments to prove the existence of "transverse" rational curves in the fibers of Fano-Mori contractions, then we show how, under assumptions which ensure that the family of rational curves in a fiber is "large" enough, we can have a quite precise description of the fiber itself. This description is then used to study small contractions supported by divisors of the form $K_X + (n - 2d)L$, generalizing some results of Zhang [Zh95].
The aim of Chapter IV is to generalize some results on contractions supported by divisors of the form $K_X + rL$ to contractions supported by divisors of the form $K_X + det\mathcal{E}$, with $\mathcal{E}$ ample vector bundle of rank r on X. In particular we will consider contractions whose fibers are covered by "large" families of rational curves, extending some results of the preceding chapter.
In general, we have to strengthen our hypothesis on the fibers, allowing only some mild singularities, but, in some special cases, we are able to prove that the fibers have to be smooth, generalizing a theorem of Andreatta, Ballico and Wiśniewski on small contractions [ABW93].
As a byproduct we obtain a characterization of projective spaces, quadrics and $\mathbb{P}$-bundles over curves by means of families of rational curves with bounded degree with respect to the determinant bundle of an ample vector bundle: if X is a log terminal variety with a sufficiently large family of this kind, then X is smooth and is one of the above varieties.
Chapter V deals with Fano-Mori contractions on a smooth projective variety X arising as extensions of Fano-Mori contractions defined on zero loci of sections of an ample vector bundle. The existence of an ample section Z with non nef canonical bundle carries the existence on X of negative curves whose deformations must meet Z, providing a lifting property for the contraction. These results, contained in [AO99], generalize classical ones by L. Bădescu [Băd82a], [Băd81], [Băd82b] and A.J. Sommese ([Som76] and chapter 5 of [BS95], in particular Theorem 5.2.1 which should be compared with our results in section 3) and more recent ones by A. Lanteri and H. Maeda [LM95],[LM96],[LM97], who were the first ones to study the problem of special sections of ample vector bundles.

Acknowledgements

I would like to thank Antonio Lanteri, who introduced me to the study of Algebraic Geometry, Edoardo Ballico, Massimiliano Mella and Jaroslaw Wiśniewski, always willing to answer my questions and to listen to my problems, Lucian Bădescu and Paltin Ionescu, for a pleasant stay at the Mathematical Institute of the Romanian Academy in november 1997, Andrew Sommese, with whom I discussed part of this work, during my visit at the Notre Dame University, and, finally, Marco Andreatta, for his patience, support and advice during the whole Ph.D. program in Trento; as Shel Shapiro sang: "Lights turn on, thanks to what you say".

There are many people which are not mathematicians, but helped me a lot during these years and deserve my thanks: first of all my parents and my sister for their constant support and encouragement; they held me up with their affection and spurred me with their criticism.

Then I would like to remember all the friends who shared with me these years, either accompanying and sustaining or reproving and correcting me when necessary, associating in my gratefulness those of the eleventh hour to those who bore me for many years.

Among them I cannot help mentioning Mauro *Frit*, Marco *The Gentle* and all his family, Luca, Oriana, Checco, Ivan, Mariella, Laura *The Queen*, Anna K., Lucia, Giacomo, Giancarlo, Chiara, Andrea *Barbas* & Chiara and Davide, to whom I ideally pass the baton.

Chapter I

Background material

I.1 Cone of curves

I.1.1 Preliminaries

Let X be a normal complex projective variety of dimension n. We will use the following notations:

$Z_{n-1}(X) :=$ the group of Weil divisors on X.

$Z_1(X) :=$ the free abelian group generated by irreducible reduced curves on X.

$Div(X) :=$ the group of Cartier divisors on X.

$Pic(X) :=$ the group of line bundles on X.

Let C be a complete curve on X and $\nu : \widetilde{C} \to C$ its normalization. For every $D \in Pic(X)$ we define an intersection number $D.C := deg_{\widetilde{C}} f^*D$; this intersection product induces a bilinear form

$$(.) : Pic(X) \times Z_1(X) \to \mathbb{Z}$$

which induces on both groups an equivalence relation, which we will call *numerical equivalence* and we will denote by $\equiv$. We now define

$$N_1(X) := \frac{Z_1(X)}{\equiv} \otimes \mathbb{R} \qquad N^1(X) := \frac{Pic(X)}{\equiv} \otimes \mathbb{R}.$$

This two groups are canonically dual via (.); by the Neron-Severi theorem we have $\dim N_1(X) = \dim N^1(X) < \infty$; we will denote by $\rho(X)$ the dimension of these two groups, and call it the *Picard number* of X. Inside $N_1(X)$ we consider

$\overline{NE}(X) :=$ the closed convex cone generated by irreducible and reduced curves.
$\overline{NE}_D(X) := \{C \in \overline{NE}(X) \mid D.C \geq 0\}$ with $D \in N^1(X)$.

With these definitions we can state the Kleiman ampleness criterion:

Theorem I.1.1.1 *$H \in Pic(X)$ is ample if and only if its numerical class $[H] \in N^1(X)$ is positive on $\overline{NE}(X)\backslash\{0\}$.*

I.1.2 Singularities

Let X be a normal variety and $X_0 \subset X$ its smooth locus. In X_0 is possible to define a notion of regular differential in a point $p \in X_0$ and the sheaf $\Omega^n_{X_0}$; given the immersion $j : X_0 \hookrightarrow X$ we define $\omega_X = j_*(\Omega^n_{X_0})$; it is possible to show that ω_X admits an immersion in $\mathcal{K}(X)$, the total quotient sheaf of $\mathcal{O}_X$; fixing that immersion it is possible to associate to ω_X a Weil divisor that we will denote by K_X. This construction can be repeated for $j_*(\Omega^n_{X_0})^{\otimes m}$, for every $m \in \mathbb{Z}$; in this way for every $m \in \mathbb{Z}$ is possible to define a Weil divisor mK_X.

Definition I.1.2.1 Let X be a normal variety; X is called $\mathbb{Q}$-Gorenstein if there exists an integer $r \in \mathbb{N}$ such that $rK_X \in Div(X)$; X is called *Gorenstein* if $K_X \in Div(X)$.

Definition I.1.2.2 A normal variety X is said to have *terminal singularities* (respectively *canonical singularities, log terminal singularities*), if the following conditions are satisfied:

1. X is $\mathbb{Q}$-Gorenstein.
2. There exists a resolution of singularities $f : Y \to X$ such that $K_Y = f^*K_X + \sum a_i E_i$, with $a_i \in \mathbb{Q}$ and $a_i > 0$ (respectively $a_i \geq 0$, $a_i > -1$), where the E_i are exceptional divisors for f.

I.1.3 Extremal rays and contractions

Let now X be a normal $\mathbb{Q}$-Gorenstein variety; the following theorems will describe the structure of the cone $\overline{NE}(X)$.

Theorem I.1.3.1 *(Cone theorem): Let X be a log terminal variety; then*

$$\overline{NE}(X) = \overline{NE}_{K_X}(X) + \sum R_j,$$

where $R_j = \mathbb{R}^+[C_j]$ for some irreducible reduced rational curve $C_j \in Z_1(X)$ such that $K_X.C_j < 0$. Moreover, the R_j are discrete in the space $\{Z \in N_1(X) \mid K_X.Z < \epsilon\}$ for every $\epsilon > 0$.

Definition I.1.3.2 The R_j are called *extremal rays* of X.

Theorem I.1.3.3 *(Contraction theorem) Let X be a log terminal variety. Let $H \in Div(X)$ a nef divisor such that $F := H^{\perp} \cap \overline{NE}(X) \backslash \{0\}$ is entirely contained in $\{Z \in N_1(X) \mid K_X.Z < 0\}$. Then there exists a projective morphism $\varphi : X \to Y$ onto a normal projective variety, which is characterized by the following properties:*

1. *For any irreducible curve C on X, $\varphi(C)$ is a point if and only if $(H.C) = 0$.*
2. *φ has only connected fibers.*
3. *$H = \varphi^* A$ for some ample Cartier divisor $A \in Div(Y)$.*

The divisor H is called a good supporting divisor *for the contraction φ.*

Theorem I.1.3.4 *(Rationality theorem) Let X be a log terminal variety and $H \in Div(X)$ an ample divisor; if K_X is not nef then*

$$r := max\{t \in \mathbb{R} \mid H + tK_X \text{ is nef}\}$$

is a rational number; moreover $r \geq 1/(n+1)$.

I.2 Fano-Mori contractions

Definition I.2.1 A *contraction* is a proper map $\varphi : X \to Z$ of normal irreducible varieties with connected fibers. The map φ is *birational* or otherwise dim $Z <$ dim X; in the latter case we say that φ is of *fiber type*. The exceptional locus $E(\varphi)$ of a birational contraction φ is equal to the smallest subset of X such that φ is an isomorphism on $X \backslash E(\varphi)$.

Suppose from now on that X is smooth.

Definition I.2.2 A contraction φ of a manifold X is called a *Fano-Mori contraction* if the anticanonical divisor $-K_X$ is φ-ample; we say that φ is *elementary* if $Pic(X)/\varphi^*(PicZ) \simeq \mathbb{Z}$. An elementary contraction is called *small* if its exceptional locus has codimension ≥ 2.

Example-Definition I.2.3 If Z is a point, X is a Fano variety, i.e. a variety s.t. $-K_X$ is ample. Suppose that X is smooth; we call *index* of X the largest positive integer $i =: i(X)$ s.t. $-K_X = i(X)L$ for some ample line bundle L on X; A Fano manifold X is called a *del Pezzo* manifold if there exists an ample line bundle L on X s.t. $-K_X = (\dim X - 1)L$.

Remark I.2.4 Let A be an ample divisor on Z and $H =: \varphi^* A$, i.e. H is a good supporting divisor for φ; then $H = K_X + rL$, where r is a positive integer and L is a φ-ample line bundle.

Definition I.2.5 Let φ be a Fano-Mori contraction of X and let $E = E(\varphi)$ be the exceptional locus of φ (if φ is of fiber type then $E := X$); let S be an irreducible component of a (non trivial) fiber F. We define the positive integer l as

$$l = min\{-K_X \cdot C : C \text{ is a rational curve in } S\}.$$

If φ is the contraction of a ray R, then $l(R) := l$ is called the *length of the ray.*

Remark I.2.6 We have ([Mor82]) that, if X has an extremal ray R then there exists a rational curve C such that $0 < -K_X.C \leq n+1$, i.e. the length of R is $\leq n+1$. A curve C which realizes the length is called an *extremal* curve.

In [Wiś89], the author studied manifolds having an extremal ray of length $n, n+1$.

Proposition I.2.7 *Let X be a smooth n-dimensional complex projective variety and R an extremal ray of X. Then*

1. *If R has length $n+1$ then $Pic(X) \simeq \mathbb{Z}$ and $-K_X$ is ample on X.*
2. *If R has length n then, either $Pic(X) \simeq \mathbb{Z}$ and $-K_X$ is ample on X or $\rho(X) = 2$ and there exists a morphism $contr_R : X \to B$ onto a smooth curve B whose general fiber is a smooth $(n-1)$-dimensional variety which satisfies conditions of (I.2.7.1).*

Now we go back to the setup of definition (I.2.5); we have the following very useful

Proposition I.2.8 *[Wiś91c] In the set-up of definition (I.2.5) the following formula holds*

$$dimS + dimE \geq dim\ X + l - 1.$$

In particular this implies that if φ is of fiber type then $l \leq (dimZ - dimW + 1)$ and if φ is birational then $l \leq (dimZ - dim(\varphi(E))$.
If a manifold admits different Fano-Mori contractions, then the dimensions of general fibers of different contractions are bounded by the following

Theorem I.2.9 *([Wiś91c, Theorem 2.2]) Let a manifold X of dimension n admit k different contractions (of different extremal rays). If by $m_i, i = 1, 2, \ldots, k$ we denote the dimensions of images of these contractions, then*

$$\sum_{i=1}^{k}(n - m_i) \leq n$$

I.3 Adjunction theory and special varieties

Definition I.3.1 A *complex polarized variety* is a pair (X, L) where X is an irreducible reduced projective scheme over $\mathbb{C}$ and L an ample line bundle on X.

Definition I.3.2 Let (X, L) be a polarized variety, and suppose that X has log terminal singularities and that K_X is not nef; then

$$\tau(X, L) = \min\{t \in \mathbb{Q} \mid (K_X + \tau L) \text{ is nef } \}$$

is called *nef value* of the pair (X, L). This minimum exists by the Rationality theorem (I.1.3.4).

It is possible to give a characterization of polarized varieties with high nef values; the next two theorems describe the cases $\tau = n + 1, n$.

Theorem I.3.3 *[Fuj87], [Ma90] Let X be a normal irreducible projective variety with log-terminal singularities. Let L be an ample line bundle on X. Then $K_X + nL$ is nef unless $(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.*

Theorem I.3.4 *[Fuj87], [BS95, (7.2.2)], [Me97, III 3.3]Let X be a normal irreducible projective variety with log-terminal singularities. Let L be an ample line bundle on X and suppose that $K_X + nL$ is nef but not ample. Then $(X, L) \simeq (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$ or $(X, L) \simeq (\mathbb{P}(\mathcal{E}), \xi_{\mathcal{E}})$ for an ample rank n vector bundle on a smooth curve.*

Now we recall the definitions of some special varieties.

Definition I.3.5 Let L be an ample line bundle on X. The pair (X, L) is called a *scroll* (respectively a *quadric fibration*, a *del Pezzo fibration*) over a normal variety Y of dimension $m < n$ if there exists a surjective morphism with connected fibers $\phi : X \to Y$ such that

$$K_X + (n - m + 1)L \approx p^*\mathcal{L}$$

(respectively $K_X+(n-m)L \approx p^*\mathcal{L}$, $K_X+(n-m-1)L \approx p^*\mathcal{L}$) for some ample line bundle $\mathcal{L}$ on Y. X is called a *classical scroll* or a $\mathbb{P}$-bundle (respectively *quadric bundle*) over a projective variety Y of dimension r if there exists a surjective morphism $\phi : X \to Y$ such that every fiber is isomorphic to $\mathbb{P}^{n-r}$ (respectively to a quadric in $\mathbb{P}^{n-r+1}$) and if there exists a vector bundle $\mathcal{E}$ of rank $n - r + 1$ (respectively of rank $n - r + 2$) on Y such that $X \simeq \mathbb{P}(\mathcal{E})$ (respectively exists an embedding of X over Y as a divisor of $\mathbb{P}(\mathcal{E})$ of relative degree 2).

Remark I.3.6 A scroll is a Fano-Mori contraction of fiber type such that the inequality in (I.2.8) is actually an equality, i.e. $l = (dimX - dimY + 1)$ and moreover if C is a rational curve such that $-K_X.C = l$ then it exists an ample line bundle L such that $L.C = 1$, i.e. C is a line with the respect to L. The

contrary is almost true in the sense that if φ is a Fano-Mori contraction with the above properties then it factors through a scroll, that is the face which is contracted by φ contains a sub-face whose contraction is a scroll.
Similarly a quadric (respectively a del Pezzo) fibration is a Fano-Mori contraction of fiber type such that $l = (dimX - dimY)$ (resp. $l = (dimX - dimY - 1)$) and moreover if C is a rational curve such that $-K_X.C = l$ then it exists an ample line bundle L such that $L.C = 1$, i.e. C is a line with the respect to L.

The following theorem, which we are going to use very often, gives a sufficient condition for a contraction to be a $\mathbb{P}^d$-bundle.

Theorem I.3.7 *[Fuj87], [BS95, Proposition 3.2.1] Let $p : X \to Y$ be a surjective equidimensional morphism onto a normal variety Y and let L be an ample line bundle on X such that $(F, L_F) \simeq (\mathbb{P}^r, \mathcal{O}(1))$ for the general fiber F of p. Then $p : X \to Y$ gives to (X, L) the structure of a $\mathbb{P}^d$-bundle.*

In particular, by the above theorem, an equidimensional scroll is a $\mathbb{P}^d$-bundle; in general, we have the following

Remark I.3.8 Let $\varphi : X \to Y$ be a scroll and let $\Sigma \subset Y$ be the set of points y such that $dim(\varphi^{-1}(y)) > k := dimX - dimY$ then $codim\Sigma \geq 3$, thus if $dimY \leq 2$ then the scroll is a $\mathbb{P}^k$-bundle ([Som86, Theorem 3.3]).

The following theorem gives a condition for a quadric fibration to be a classical quadric bundle

Theorem I.3.9 *[ABW93, Theorem B] Let (X, L) be a quadric fibration $\varphi : X \to Y$, with L an ample line bundle on X; assume that φ is an elementary contraction and that φ is equidimensional. Then $\mathcal{E} := p_*L$ is a locally free sheaf of rank dim $X-$ dim $Y + 2$ and L embeds X into $\mathbb{P}(\mathcal{E})$ as a divisor of relative degree 2, i.e. X is a classical quadric bundle.*

We conclude this section recalling the following

Theorem I.3.10 *([Laz83], [PS89]) Let X be a smooth variety of dimension n and suppose there exists a finite-to-one morphism $f : \mathbb{P}^n \to X$. Then $X \simeq \mathbb{P}^n$. If there exists a finite-to-one morphism $f : \mathbb{Q}^n \to X$ then $X \simeq \mathbb{P}^n$ or $X \simeq \mathbb{Q}^n$.*

I.4 Δ-genus and the Apollonius method

The results of this section are contained in [Fuj90]; let (X, L) be a polarized variety.

Definition I.4.1 The Δ-genus of (X, L) is defined by the formula

$$\Delta(X, L) = n + d(X, L) - h^0(X, L).$$

where $d(X, L) = L^n$ is called the *degree* of (X, L).

Definition I.4.2 Let $\chi(X, L) = \sum \chi_j t^{[j]}$ with $t^{[j]} = t(t+1)\dots(t+j-1)$ be the Hilbert polynomial of (X, L); we set $g(X, L) = 1 - \chi_{n-1}(X, L)$ and we call $g(X, L)$ the *sectional genus* of (X, L).

In the case X is Gorenstein, we have the sectional genus formula:

$$2g(X, L) - 2 = (K_X + (n-1))L)L^{n-1}.$$

Let us recall some general properties of the Δ-genus ([Fuj90]).

Theorem I.4.3 *Let (X, L) be a polarized variety. Then*

1. *$\Delta(X, L) > \dim Bs \mid L \mid$, assuming by convention that $\dim \emptyset = -1$. In particular $\Delta(X, L) \geq 0$;*
2. *If $\Delta(X, L) = 0$ then L is very ample.*

Theorem I.4.4 *Let (X, L) be an n-dimensional polarized variety. Assume that $\Delta(X, L) = 0$. Then either:*

1. *$(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ if $d(X, L) = 1$;*
2. *$(X, L) \simeq (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$ if $d(X, L) = 2$;*
3. *(X, L) is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$, $X \simeq \mathbb{P}(\mathcal{E})$ for a vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ which is a direct sum of line bundles of positive degrees;*
4. *$(X, L) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$; or*
5. *(X, L) is a generalized cone over a smooth submanifold $V \subset X$ with $\Delta(V, L_V) = 0$.*

Definition I.4.5 Let (X, L) be a polarized variety of dimension n, and let D be a member of $\mid L \mid$. Suppose that D is irreducible and reduced as a subscheme of X; in such a case D is called a *rung* of (X, L). The pair (D, L_D) is a polarized variety of dimension $n - 1$.

If (D, L_D) is a rung of (X, L), and r is the restriction map $r : H^0(X, L) \to H^0(D, L_D)$ we have $\Delta(X, L) - \Delta(D, L_D) = \dim \mathrm{Coker}(r) \leq h^1(X, \mathcal{O}_X)$. In particular

$$\Delta(D, L_D) \leq \Delta(X, L)$$

and the equality holds if and only if r is surjective. In this case D is said to be a *regular* rung.

Definition I.4.6 A sequence $X = X_n \supset X_{n-1} \supset \ldots \supset X_1$ of subvarieties of X is called a *ladder* of (X, L) if X_j is a rung of (X_{j+1}, L_{j+1}) for each $j \geq 1$. It is said to be *regular* if each rung X_j is regular.

An example of the use of a regular ladder is the following

Proposition I.4.7 *Let (X, L) be a polarized variety with $g(X, L) = 0$; suppose that (X, L) has a ladder such that each rung X_j is a normal variety. Then $\Delta(X, L) = 0$.*

Weakening the assumptions on L is possible to define a broader class of varieties.

Definition I.4.8 A *complex quasi-polarized variety* is a pair (X, L), with X an irreducible reduced projective scheme over $\mathbb{C}$ and L a nef and big line bundle on X.

Theorem I.4.9 $\Delta(X, L) \geq 0$ *for any quasi-polarized variety (X, L). Moreover, if $\Delta = 0$, there exists a variety W, a birational morphism $f : X \to W$ and a very ample line bundle H on W such that $L = f^*H$ and $\Delta(W, H) = 0$.*

Theorem I.4.10 *Let (X, L) be a normal quasi-polarized variety, $\pi : M \to X$ a desingularization of X and suppose that $h^n(X, -\pi^*tL) = 0$ for $1 \leq t \leq n$. Then there is a birational morphism $f : X \to \mathbb{P}^n$ such that $L = f^*\mathcal{O}_{\mathbb{P}^n}(1)$.*

Theorem I.4.11 *Let (X, L) be a normal quasi-polarized variety, $\pi : M \to X$ a desingularization of X and suppose that $h^n(M, -\pi^*tL) = 0$ for $1 \leq t \leq n-1$ and $h^n(M, -\pi^*nL) = 1$. Then either*

1. *$L^n = g(X, L) = 1$*
2. *There is a birational morphism $f : X \to \mathbb{Q}^n$ such that $L = f^*\mathcal{O}_{\mathbb{P}^n}(1)$ with $\mathbb{Q}^n$ a possibly singular quadric.*

I.4.1 Slicing techniques

In the study of Fano-Mori contractions is often very useful to apply a kind of inductive method, called Apollonius method, slicing the fibers.

I.4.1.1 Assume that $\varphi : X \to Z$ is a Fano-Mori contraction of a variety X with at most log terminal singularities onto a normal affine variety Z, L is an ample line bundle on X and φ is supported by $K_X + rL$.

Lemma I.4.1.2 *[AW93, Lemma] (Horizontal slicing) Suppose that $\varphi : X \to Z$ is as in (I.4.1.1). Let X' be a general divisor from the linear system $\mid L \mid$. Then, outside of the base point locus of $\mid L \mid$, the singularities of X' are not worst than those of X and any section of L on X' extends to X.*

Moreover, if we set $\varphi' := \varphi_{|X'}$ and $L' = L_{X'}$ then $K_{X'} + (r-1)L'$ is φ'-trivial. If $r \geq 1 + \epsilon(dim\ X - dim\ Z)$ then φ' is a contraction, i.e. has connected fibers.

Lemma I.4.1.3 *[AW93, Lemma] (Vertical slicing) Assume that $\varphi : X \to Z$ is as in (I.4.1.1); Let $X'' \subset X$ be a non trivial divisor defined by a global function $h \in H^0(X, K_X + rL) = H^0(X, \mathcal{O}_X)$; then, for a general choice of h, X'' has singularities not worst than those of X and any section of L on X' extends to X.*

These slicing arguments were used for the first time in [AW93] to prove the following

Theorem I.4.1.4 *[AW93,] (Relative base point freeness). Let $\varphi : X \to Z$ be a Fano-Mori contraction as in (I.4.1.1). Let F be a fiber of φ. Assume that dim $F \leq r + 1 - \epsilon(dim\ X - dim\ Z)$. Then $Bs \mid L \mid$ does not meet F. Moreover (after we restrict φ to a neighborhood of F, if necessary) there exists a closed embedding $X \to \mathbb{P}^N \times Z$ over Z such that $L \simeq \mathcal{O}_{\mathbb{P}^N}(1)$ (we say that L is φ-very ample).*

An example of application of the relative base point freeness, together with slicing is the following

Theorem I.4.1.5 *([AW97, Theorem 5.1]) Let $\varphi : X \to Z$ be a Fano-Mori contraction of a smooth variety and let $F = \varphi^{-1}(z)$ be a fiber. Assume that φ is supported by $K_X + rL$, with L a φ-ample line bundle on X.*

1. *If dim $F \leq r - 1$ then Z is smooth at z and φ is a projective bundle in a neighborhood of F.*

2. *If dim $F = r$ then, after possible shrinking of Z and restricting φ to a neighborhood of F, Z is smooth and*

   (a) if φ is birational then φ blows down a smooth divisor $E \subset X$ to a smooth codimension $r - 1$ subvariety $S \subset Z$.

   (b) if φ is of fiber type and dim Z = dim $X - r$ then φ is a quadric bundle.

   (c) if φ is of fiber type and dim Z = dim $X - r + 1$ then $r \leq$ dim $X/2$, $F \simeq \mathbb{P}^r$ and the general fiber is $\mathbb{P}^{r-1}$.

I.5 Ample vector bundles

Let $\mathcal{E}$ be a rank r vector bundle on a variety X, and let $S^t(\mathcal{E})$ denote the t-th symmetric power of $\mathcal{E}$ for $t \geq 0$, with the convention $S^0(\mathcal{E}) = \mathcal{O}_X$. Let $\mathcal{S} = \bigoplus_{t=0}^{\infty} S^t(\mathcal{E})$ be the symmetric algebra of $\mathcal{E}$; then $\mathbb{P}(\mathcal{E}) := \text{Proj}(\mathcal{S})$ is a $\mathbb{P}^d$-bundle on X, $\pi : \mathbb{P}(\mathcal{E}) \to X$.
There is an invertible line bundle $\xi_{\mathcal{E}}$, which, over $x \in \mathbb{P}(\mathcal{E})$ is the quotient space

of $(\pi\mathcal{E})_x$ corresponding to the one dimensional quotient of $\mathcal{E}_{\pi(x)}$ defining x. On $\mathbb{P}(\mathcal{E})$ we have $\xi_F \simeq \mathcal{O}_{\mathbb{P}^d}(1)$ for any fiber F of π; the line bundle $\xi_{\mathcal{E}}$ is called the *tautological line bundle* on $\mathbb{P}(\mathcal{E})$. We have $\pi^*\xi_{\mathcal{E}} \simeq \mathcal{E}$ and $\pi_*(t\xi_{\mathcal{E}}) = S^t(\mathcal{E})$, for any integer $t \geq 0$. The canonical bundle of $\mathbb{P}(\mathcal{E})$ is given by the formula

$$K_{\mathbb{P}(\mathcal{E})} \equiv \pi^*(K_X + det\mathcal{E}) - r\xi_{\mathcal{E}}$$

Definition I.5.1 We say that a vector bundle $\mathcal{E}$ is *ample* if the tautological line bundle is ample on $\mathbb{P}(\mathcal{E})$.

Lemma I.5.2 *Let $\mathcal{E}$ be an ample vector bundle of rank r on a complex variety X. For any rational curve $C \subset X$ we have*

$$(det\mathcal{E}).C \geq r.$$

Moreover, if C is smooth and the equality holds, then $\mathcal{E}_C = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$.

Lemma I.5.3 *Let Y be a complex projective variety of dimension n, $\mathcal{E}$ an ample vector bundle on Y, s a global section of $\mathcal{E}$; denote with $V(s)$ the zero set of s. Then*

$$dimV(s) \geq n - r.$$

Proof. See [Ful84, Example 12.1.3].

Proposition I.5.4 *Let X, $\mathcal{E}$ and Z be as before. Let Y be a subvariety of X of dimension $\geq r$. Then $dim\ Z \cap Y \geq dim\ Y - r$.*

Proof. Consider $\mathcal{E}_Y$, the restriction of $\mathcal{E}$ to Y and s_Y, the restriction of s to $\Gamma(Y, \mathcal{E}_Y)$. Applying lemma (I.5.3) to Y and s_Y we get

$$\dim\ (Z \cap Y) = \dim\ V(s_Y) \geq \dim\ Y - r$$

Now we recall some results about special vector bundles on projective spaces and quadrics.

Proposition I.5.5 *([OSS, Theorem 3.2.1]) If $\mathcal{E}$ is a rank r vector bundle on $\mathbb{P}^n$ and $x \in \mathbb{P}^n$ is a point such that $\mathcal{E}_{|l} = \oplus^r \mathcal{O}_{\mathbb{P}^1}$ for every line l through x then $\mathcal{E}$ is the trivial bundle $\oplus^r \mathcal{O}_{\mathbb{P}^n}$.*

Proposition I.5.6 *([Wiś89, Lemma 3.6.1]) If $\mathcal{E}$ is a rank r vector bundle on $\mathbb{Q}^n$ and the restriction of $\mathcal{E}$ to every line of $\mathbb{Q}^n$ is $\oplus^r \mathcal{O}_{\mathbb{P}^1}$ then the bundle $\mathcal{E}$ is the trivial bundle $\oplus^r \mathcal{O}_{\mathbb{Q}^n}$.*

Proposition I.5.7 *([Wiś90b, Proposition 1.9]) If $\mathcal{E}$ is a uniform vector bundle on $\mathbb{P}^n$ of rank r, with splitting type $(a, \ldots, a, a+1)$ then $\mathcal{E}$ is either decomposable into a sum of line bundles or (if $r \geq n$) isomorphic to $T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(a-1) \oplus \mathcal{O}_{\mathbb{P}^n}(a)^{\oplus(r-n)}$.*

I.6 The Albanese variety

In this last section we recall the definition of the Albanese variety of a smooth projective variety, together with some basic properties that we are going to use in Chapter III.

Theorem I.6.1 *Let X be a smooth complex projective variety. There exists an abelian variety $Alb(X)$ and a holomorphic map $\alpha : X \to Alb(X)$ with the following universal property: for every complex torus T and every holomorphic map $F : X \to T$, there exists a unique holomorphic map $\tilde{f} : Alb(X) \to T$ such that $\tilde{f} \circ \alpha = f$. This abelian variety is called the Albanese variety of X, and we have*

$$Alb(X) \simeq \frac{H^0(X, \Omega^1)^*}{H_1(X, \mathbb{Z})}$$

Proposition I.6.2 *[Uen75, Proposition 9.19] Assume that the image of the Albanese mapping $\alpha : X \to Alb(X)$ is a curve C. Then C is a nonsingular curve of genus $g = dim\ Alb(X)$ and the fibers of $\alpha : X \to C$ are connected.*

The last proposition will be used in the study of varieties with large families of rational curves, because of the following well-known

Proposition I.6.3 *[LB92, Proposition 9.5] Every rational map $f : \mathbb{P}^n \to A$, from a projective space to an abelian variety A is constant.*

Chapter II

Families of rational curves

This chapter is dedicated to the study of rational curves whose deformations can not break, hence the name of *unbreakable* rational curves.
A fundamental example of an unbreakable rational curve is an extremal rational curve, i.e. a curve whose numerical class generates an extremal ray of the ambient variety; we will study families arising in this way in chapter IV.
In the first section we recall the definitions of families of rational curves and we show how to construct a family with good properties starting with an unbreakable rational curve; in the second we use unbreakable families to give a criterion for a good contraction to be elementary, while the last section is dedicated to the study of Fano manifolds of large pseudoindex. The main references for this chapter are [Mor79], [BS95], [Wiś89] and [Ko96].

II.1 Families of unbreakable rational curves

Definition II.1.1 A *family of rational curves* on a quasi-projective variety X is a quadruple $\mathcal{V} = (V, T, p, q)$, consisting of a reduced variety T, called the parameter space, and a reduced variety $V \subset T \times X$, with the maps $p : V \to X$, $q : V \to T$ induced by the product projections, such that q is proper, algebraic, with all fibers generically reduced with reduction being irreducible rational curves.

$$\begin{array}{ccc} V & \xrightarrow{q} & T \\ {\scriptstyle p}\downarrow & & \\ X & & \end{array}$$

Such a family is called *irreducible* (respectively *compact*, *connected*) if T is irreducible (respectively compact, connected).
If any two distinct fibers of q are mapped to distinct curves of X by p, we say

that the family is a *Chow family of rational curves*; a family $\mathcal{V}$ of rational curves is *maximal* if there is no family $\mathcal{V}'$ containing $\mathcal{V}$ and not equal to it.

Given any family of rational curves, $\mathcal{V} = (V, T, p, q)$ it follows, since the fibers of q are irreducible, generically reduced curves, that there is an holomorphic map from T to the Chow variety of X. If T is compact, then the image in the Chow variety of X will be a variety, and in this case we can replace our family by a Chow family which contains all the curves in the family.

We define the *locus of the family*, $E(\mathcal{V})$ to be $p(V)$, and we say that the family is a *covering family* if $E(\mathcal{V})$ is dense in X; the dimension, $\dim V - \dim E(\mathcal{V})$, of the generic fiber of p is denoted by δ and we refer to it as to the *dimension of the family.*
Moreover we define $V_x := p^{-1}(x)$ and $T_x := q(V_x)$; T_x is the subvariety of T parametrizing curves of the family which contain x; we denote its dimension by δ_x. Note that the fiber $F_x := p^{-1}(x)$ over a point $x \in E(\mathcal{V})$ maps homeomorphically under q to T_x; so, by upper semicontinuity of dimensions of the fibers of p we have that

$$\delta_x \geq \delta \qquad \text{for all } \ x \in E(\mathcal{V}).$$

Definition II.1.2 An irreducible and reduced curve $C \subset X$ is *unbreakable* if the integer homology class $[C] \in H_2(X, \mathbb{Z})$ associated to C cannot be written as a nontrivial sum of the integer homology classes of effective curves.

Example II.1.3 If L is an ample line bundle on X, and C is a curve such that $L.C = 1$, then C is unbreakable.

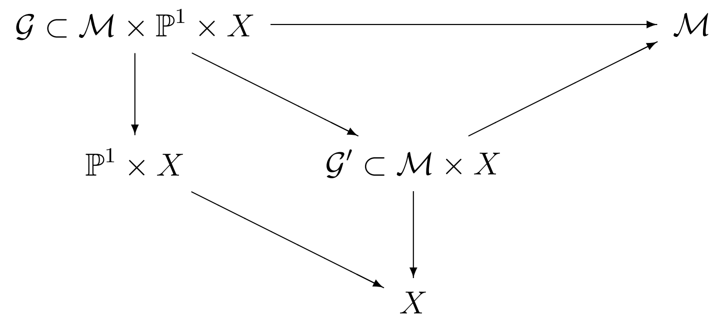
Example II.1.4 If $\mathcal{E}$ is an ample vector bundle of rank r on X, and C is a curve such that $det\mathcal{E}.C < 2r$, then C is unbreakable.

Example II.1.5 If C is an extremal rational curve on X, then C is unbreakable.

Proof. Let R be the extremal ray to which C belongs; if we could write $[C] = a[C_1] + b[C_2]$, since C is extremal, both C_1 and C_2 would have to belong to R, but, among curves belonging to R, C has the minimum intersection number with the anticanonical bundle, and we would have a contradiction. □

II.1.6 We show how to construct a maximal compact Chow family of rational curves, starting with an unbreakable rational curve (cfr. [BS95, 6.2.1] and [Wiś89, Appendix]).

Let C be an unbreakable rational curve on a variety X; let $\nu : \mathbb{P}^1 \to X$ be the normalization of C. Consider the graph $G(\nu) \subset \mathbb{P}^1 \times X$. Using the Hilbert scheme of $\mathbb{P}^1 \times X$ we obtain a connected universal flat family of rational curves



where $\mathcal{M} = \mathrm{Hilb}_{G(\Phi)}(\mathbb{P}^1 \times X)$. Let $\mathcal{G}'$ be the image of the family $\mathcal{G}$ under the projection on $\mathcal{M} \times X$. The fibers of $\mathcal{G}' \to \mathcal{M}$ are irreducible and generically reduced because C is unbreakable and moreover we have the following

Lemma II.1.7 *The morphism $\mathcal{M} \to Chow_X$ has a closed image.*

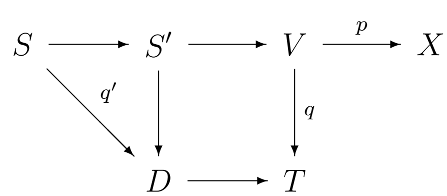
Proof. For a proof see [Wiś89, Lemma A4]; we only stress that the unbreakability assumption is essential for this result.

By the lemma we can replace the family with the associated Chow family, completing the construction.

The following result is a key fact in the study of unbreakable families.

Lemma II.1.8 (Non breaking lemma) *Let X be a smooth variety of dimension n and $\mathcal{V} = (V, T, p, q)$ an irreducible maximal compact Chow family of unbreakable rational curves, and fix $x \in E(\mathcal{V})$; given a connected variety $B \subset T$, consider $\widetilde{B} := q^{-1}(B)$ and let $f : \widetilde{B} \to Z$ be a morphism which takes $\widetilde{B} \cap p^{-1}(x)$ to a point. Either f is a constant map, or it is finite-to-one on $\widetilde{B} \backslash \widetilde{B} \cap p^{-1}(x)$.*

Proof. Suppose, by contradiction that exists an irreducible curve $D' \subset \widetilde{B}$ which is mapped to a point $y \neq x$ by p. Let $D \to q(D')$ be the normalization of $q(D') \subset T$, let $S' \to D$ be the morphism obtained by base change from $q : V \to T$ and $S \to S'$ the normalization of S', with induced morphisms $q' : S \to D$, $p' : S \to X$.



We recall a general well known fact:

Lemma II.1.9 *[Wiś89, Lemma 1.14] Let S be a normal surface, with a morphism $\pi : S \to C$ onto a smooth curve C. If every fiber of π is an irreducible generically reduced variety of dimension 1 whose normalization is $\mathbb{P}^1$, then $\pi : S \to C$ is a $\mathbb{P}^1$-bundle.*

By the lemma $q' : S \to D$ is a $\mathbb{P}^1$-bundle. The universal property of the Chow scheme yields that $p'(S)$ is two dimensional, otherwise the curve $q(D')$ parametrizes only one 1-cycle $p'(S)$.
On the surface S there are two contractible flat sections $p'^{-1}(x)$ and $p'^{-1}(y)$, and, applying lemma (A.1.2) we get a contradiction. □

Corollary II.1.10 *Let x be a point of $E(\mathcal{V})$ and T_x the subvariety of T parametrizing curves of the family containing x; then $\delta_x = dim\ T_x \leq n-1$.*

Proof. Apply lemma (II.1.8), taking $B = T_x$ and $f = p$, recalling that q has one dimensional fibers.

Corollary II.1.11 *Let x be a point of $E(\mathcal{V})$ and let $E_x = p(q^{-1}(T_x))$ be the locus of X covered by curves in the family passing through x; then every morphism $h : E_x \to Z$ is either finite to one or takes E_x to a point.*

Proof. Apply lemma (II.1.8), taking $B = T_x$ and $f = h \circ p$. □

II.1.1 Anticanonical degree of families of rational curves

Let C be a general member of a family of rational curves on a n-dimensional variety X; we now recall how to give a lower bound to the anticanonical degree of C, in terms of δ.

Let X be a normal complex projective variety. Fix a smooth point $x \in X$. Let S be a connected component of $Hom(\mathbb{P}^1, X, i)_{red}$, where $i : \{0\} \to \{x\}$.
Let $\Phi : S \times \mathbb{P}^1 \to X$ be the evaluation map and let $t = \dim \text{Im } \Phi$. Let S_0 be an irreducible component of Reg$S$ such that $t = \dim \text{Im } (\Phi_{|S_0 \times \mathbb{P}^1})$.

Proposition II.1.1.1 *[KS98, Proposition 3.1] For a general $s \in S_0$*

$$\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(t-1)} \subset (\nu_s^* \Omega_X^1)^V.$$

Now take S' to be a connected component of $Hom(\mathbb{P}^1, X)_{red}$, $\Phi' : S' \times \mathbb{P}^1 \to X$ the evaluation map and $t' = \dim \text{Im } \Phi'$. Let S'_0 be an irreducible component of RegS' such that $t' = \dim \text{Im } (\Phi_{|S_0 \times \mathbb{P}^1})$.

Proposition II.1.1.2 *[KS98, Proposition 3.3] For a general $s \in S'_0$*

$$\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}{}^{\oplus(t'-1)} \subset (\nu_s^* \Omega_X^1)^V.$$

Using these two propositions, in the case of a covering family of rational curves, we have:

Corollary II.1.1.3 *Let $\mathcal{V} = (V, T, p, q)$ be a family of rational curves on X, such that $E(\mathcal{V})$ is a dense subset of X and let δ be the dimension of the family at a general point of X. Then, for a general curve C in the family, denoted by $\nu : \mathbb{P}^1 \to C$ the normalization of C, we have*

$$(\nu^*\Omega^1_X)^\vee = \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$$

with $\quad a_1 \geq 2,\ a_i \geq 1 \quad$ *for* $i = 2, \ldots, \delta+1 \quad ,\ a_i \geq 0 \quad$ *for* $i = \delta+2, \ldots, n.$

Proof. We can apply proposition (II.1.1.1) with $t = \delta + 1$ and proposition (II.1.1.2) with $t' = n$. □

Lemma II.1.1.4 *[KS98, Lemma 3.4] Let $C \subset X$ be an irreducible curve on X, and $\nu : \tilde{C} \to X$ the composite of the normalization of C and the closed immersion $C \to X$. Assume that $C \not\subset$ Sing X, and assume that X is $\mathbb{Q}$-Gorenstein. Then*

$$c_1((\nu^*\Omega^1_X)^\vee) \leq (-K_X.C).$$

Corollary II.1.1.5 *Let X be a $\mathbb{Q}$-Gorenstein variety, such that exists a family of rational curve which covers X and whose dimension at a general point is δ. Then, if C is a general curve in the family, we have*

$$-K_X.C \geq \delta + 2.$$

Considering now a family of unbreakable rational curves on a smooth variety, we can be more precise in relating the anticanonical degree of curves to the dimension of the family.

Proposition II.1.1.6 *Let X be a smooth variety of dimension n and let $\mathcal{V} = (V, T, p, q)$ a maximal connected family of unbreakable rational curves. Then*

$$dim\ V \geq -K_X.C + n - 2$$

Proof. Let C be a curve in the family and $\Phi : \mathbb{P}^1 \to C$ its normalization. Applying (A.1.8) to $G(\Phi)$, the graph of C we find that, for any component H_Φ of the Hilbert scheme containing $G(\Phi)$ we have

$$\dim H_\Phi \geq -K_X.C + n.$$

Any two graphs $G(\Phi_1)$, $G(\Phi_2)$, for generically one-to-one maps $\Phi_i : \mathbb{P}^1 \to X$, going to the same C, satisfy $\Phi_1 = \Phi_2 \circ g$ for some $g \in \text{Aut}\ (\mathbb{P}^1)$; let T_Φ be the irreducible component of T corresponding to H_Φ; we have

$$\dim V \geq \dim T_\Phi + 1 \geq \dim H_\Phi - \dim \text{Aut}\ (\mathbb{P}^1) + 1 \geq -K_X.C + n - 2$$

and the result is proved. □

Corollary II.1.1.7 *Let X be a smooth variety of dimension n and $\mathcal{V} = (V, T, p, q)$ a maximal covering family of unbreakable rational curves. Then*

$$dim\ V = -K_X.C + n - 2$$

and

$$\delta = -K_X.C - 2$$

Proof. Combine proposition (II.1.1.6) with corollary (II.1.1.5), recalling that $\dim V = \delta + \dim E(\mathcal{V})$. □

II.2 Unbreakable families and elementary contractions

Definition II.2.1 We define the *pseudoindex* of a Fano manifold X as

$$min\{-K_X.C, \quad \text{where } C \text{ is a rational curve on } X\}$$

Proposition II.2.2 *[ABW91] Let $\pi : X \to W$ be a contraction of a face of a smooth n-fold such that $dim X > dim W = m$. Suppose that for every rational curve C in a general fiber of π we have $-K_X \cdot C \geq (n+1)/2$. Then π is an elementary contraction except if*

a) $-K_X \cdot C = (n+2)/2$ for some rational curve C on X, W is a point, X is a Fano manifold of pseudoindex $(n+2)/2$ and $\rho(X) = 2$

b) $-K_X \cdot C = (n+1)/2$ for some rational curve C, and $dim W \leq 1$

Proof. Suppose that $-K_X.C \geq (n+2)/2$ for every rational curve in X. By (A.1.10), through a general point of a general fiber there is a rational curve l such that

$$-K_X.l = -K_F.l \leq \dim F + 1 = n - m + 1$$

By our assumption we also have that $-K_X.C > (n - m + 1)/2$ for every rational curve, so the curve l can not break. Let (V, T, p, q) be the non breaking family of rational curves of X, obtained as deformations of l; the dimension of this family at every point is greater or equal than $-K_X.l - 2$. Suppose that π is not an elementary contraction; therefore there is a contraction $\varphi = \text{contr}_R$ where $R = \mathbb{R}_+[l_2]$ is an extremal ray contracted by π and not containing l. Let F be a fiber of φ; then $\dim F \geq l(R) - 1$ (I.2.8); let T_p be the locus of curves from the family T which pass through a given point p; we have $\dim T_p \geq -K_X.l - 1$. But, by the non-breaking lemma (II.1.8), we must have

$$\dim F + \dim T_p \leq \dim X$$

if $T_p \cap F \neq \emptyset$ and for $p \notin F$, that is

$$-K_X.l - K_X.l_2 \leq n+2$$

and so we have a contradiction if $K_X.C > (n+2)/2$ for every curve and equality everywhere if $K_X.C = (n+2)/2$ in particular, in this last case we have that $\dim F = -K_X.l - 1 = n - \dim T - 1$.
Then, by [BSW90, Lemma 1.4.5] we have that $NE(X) = NE(F) + \mathbb{R}_+[l]$. Since F is a positive dimensional fiber of an elementary contraction, we conclude that $NE(F) = \mathbb{R}_+$ and thus that $\rho(X) = 2$. Thus $\rho(Y) = 0$, i.e. Y is a point; therefore X is a Fano manifold of pseudoindex $(n+2)/2$.

If $-K_X.C = (n+1)/2$ then a general fiber of π, G, is a Fano manifold of pseudoindex $\geq (n+1)/2$. If $m \geq 2$ we have that the pseudoindex of G is greater than $(\dim G)/2+1$; therefore $\rho(G) = 1$ (see [Wiś90a]). This implies in particular that, if π is not an elementary contraction, then $\varphi = \mathrm{cont}_R$, as above, is birational and therefore that $\dim F \geq -K_X.l_2$ (I.2.8); since $\rho(X) \geq 3$ it follows as above that

$$-K_X.l - K_X.l_2 > n,$$

and thus we arrive at the contradiction with $-K_X.C \leq (n+1)/2$ using again [BSW90, Lemma 1.4.5]. □

Corollary II.2.3 *Let $\pi : X \to W$ a fiber type contraction supported by $K_X + rL$ and suppose that $r \geq (n+1)/2$. Then π is an elementary contraction except if*

a) $r = (n+2)/2$, $X \simeq \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$.

b) $r = (n+1)/2$, dim$W = 1$ and the general fiber of π is $\mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$.

c) $r = (n+1)/2$, X is $\mathbb{P}^{r-1} \times \mathbb{Q}^r$, $\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r-1)})$ or $\mathbb{P}_{\mathbb{P}_r}(T\mathbb{P}^r)$.

Proof. In case a) X is a Fano manifold of Picard number two and index $(n+2)/2$ and so is $\mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$ by a result of Wiśniewski [Wiś90a, Theorem B]; in case b) the same theorem leads to the same conclusion for the general fiber.
In case c) X is a Fano manifold with $b_2 \geq 2$ and index $(n+1)/2$; these are classified by Wiśniewski [Wiś91b]. □

II.3 Unbreakable families and Fano manifolds of large pseudoindex

Wiśniewski used families of unbreakable rational curves to classify Fano manifolds of large index ([Wiś90a], [Wiś91b]); in this section we try to extend his classification to the case of some Fano manifolds of large pseudoindex.

Proposition II.3.1 *Let X be a smooth projective variety of dimension $n \geq 4$, $\mathcal{E}$ an ample vector bundle of rank r on X such that $K_X + det\mathcal{E} = \mathcal{O}_X$ and $n = 2r-1$. Suppose that $b_2(X) \geq 2$. Then $b_2(X) = 2$ and*

i) X has a projective bundle structure on $\mathbb{P}^r$ or $\mathbb{Q}^r$.

ii) X has two $\mathbb{P}^{r-1}$-bundle structures on two smooth Fano manifolds of Picard number 1.

Proof. Consider the contraction of X to a point, which is supported by the divisor $K_X + det\mathcal{E}$; by proposition (II.2.2) either $b_2(X) = 1$ or X is a Fano manifold of pseudoindex r and Picard number 2.
The cone $NE(X)$ is thus generated by two extremal rays R_1 and R_2, both of length $\geq r$; let $\varphi_1 : X \to Y_1$ and $\varphi_2 : X \to Y_2$ be the extremal contractions associated to these rays, and let F_1 and F_2 be fibers of these contractions; by the inequality (I.2.8)

$$n + 1 = 2r \leq l(R_1) + l(R_2) \leq (\dim F_1 + \epsilon_1) + (\dim F_2 + \epsilon_2) \leq n + \epsilon_1 + \epsilon_2 \quad (*)$$

with $\epsilon_i = 1$ if R_i is nef and $\epsilon_i = 0$ if R_i is not nef; in particular we have $l(R_1) = l(R_2) = r$ and $r - 1 \leq \dim F_i \leq r$.

Claim The contractions φ_i are equidimensional.

Proof. By the preceding remarks, if φ_i is not equidimensional, then its generic fiber has dimension $r-1$ (and so, having length r, φ_i is of fiber type and the generic fiber, by adjunction is $\mathbb{P}^{r-1}$), and the jumping fibers have dimension r; since the contraction is elementary the image in Y_i of the jumping fibers has codimension ≥ 3.
Let m be this codimension, take $r - m$ hyperplane sections A_j of Y_i and let $X' = \varphi_i^{-1}(\cap A_j)$; the morphism $\varphi_{i|X'} : X' \to \cap A_j$ is again a good contraction, with general fiber $\mathbb{P}^r$ and some isolated jumping fibers of dimension r; we can apply the following

Lemma II.3.2 *[AM97, Lemma 3.3] Let X be a smooth complex variety and $(W, 0)$ an analytic germ such that $W \backslash \{0\} \simeq \Delta^m \backslash \{0\}$. Suppose that there exists an holomorphic map $\pi : X \to W$ with $-K_X$ π-ample with $F \simeq \mathbb{P}^s$ for all fibers F of π such that $F \neq F_0$, and that codim$F_0 \geq 2$. Then there exists a π-ample line bundle L on X such that $L_F \simeq \mathcal{O}_{|F}(1)$.*

So we can suppose that the contraction φ_i is supported by a divisor of the form $K_X + rL$; by (I.4.1.5), φ_i is equidimensional; this implies that there are no jumping fibers for φ_i and the claim is proved. □

Now we go back to $(*)$, which tells us that one of the two extremal contractions of X, say φ_1, is of fiber type, with fibers of dimension $r - 1$. This contraction is supported by $K_X + det\mathcal{E}'$, with $\mathcal{E}' = \mathcal{E} \otimes \varphi_1^* A_1$, with A_1 ample on Y_1. So we can apply [AM97][Theorem 3.2] and obtain that Y_1 is smooth and φ_1 is a $\mathbb{P}^{r-1}$-bundle $X = \mathbb{P}(\mathcal{F}) \to Y_1$.

Case a) The contraction φ_2 is birational.

In this case $\varphi_2 : X \to Y_2$ is a birational, equidimensional contraction with r-dimensional fibers and it is supported by $K_X + det\mathcal{E}''$, with $\mathcal{E}'' = \mathcal{E} \otimes \varphi_2^* A_2$, with A_2 ample on Y_2, and so, by [AM97, Theorem 3.1] Y_2 is smooth and φ_2 is the blow up of a smooth subvariety of Y_2. The morphism φ_1 restricted to a fiber of φ_2 (which is a $\mathbb{P}^r$) is finite-to-one, and so, by (I.3.10) $Y_1 \simeq \mathbb{P}^r$.

Case b) The contraction φ_2 is of fiber type with r-dimensional fibers.

In this case the general fiber of φ_2 is, by adjunction a projective space $\mathbb{P}^r$ or a smooth quadric $\mathbb{Q}^r$; the restriction of φ_1 to one of these fibers is finite to one and so Y_1 is $\mathbb{P}^r$ or $\mathbb{Q}^r$ by (I.3.10).

Case c) The contraction φ_2 is of fiber type with $(r-1)$-dimensional fibers.

In this case the contraction φ_2 is supported by $K_X + det\mathcal{E}''$, with $\mathcal{E}'' = \mathcal{E} \otimes \varphi_2^* A_2$, with A_2 ample on Y_2, so we can apply [AM97][Theorem 3.2] and obtain that Y_2 is smooth and φ_2 is a $\mathbb{P}^{r-1}$-bundle $X = \mathbb{P}(\mathcal{F}) \to Y_2$. □

If X is as in case i) of the preceding proposition, then we can give a detailed description of all possible cases:

Proposition II.3.3 *Let X, $\mathcal{E}$ be as in proposition (II.3.1,i)); Then $(X, \mathcal{E})$ is one of the following:*

1. $(\mathbb{P}(\mathcal{O}_{\mathbb{P}^r}(1) \oplus \mathcal{O}_{\mathbb{P}^r}{}^{\oplus(r-1)}), \mathcal{O}(1,1)^{\oplus r})$
2. $(\mathbb{P}(T\mathbb{P}^r(-1)), \mathcal{O}(1,1)^{\oplus r})$
3. $(\mathbb{P}^{r-1} \times \mathbb{Q}^r, \mathcal{O}(1,1)^{\oplus r})$
4. $(\mathbb{P}^{r-1} \times \mathbb{P}^r, \mathcal{O}(1,2) \oplus \mathcal{O}(1,1)^{\oplus r-1})$

Proof. Up to twist $\mathcal{F}$ with a power of $\mathcal{O}(1)$ we can assume that $0 < c_1(\mathcal{F}) \leq r$.

Claim: The line bundle $\xi_{\mathcal{F}}$ is nef.

Proof. Consider the vector bundle $\mathcal{F}(1)$; we have

$$r\xi_{\mathcal{F}(1)} = -K_X + \varphi_1^*(K_{Y_1} + c_1(\mathcal{F}(1))).$$

Since $c_1(\mathcal{F}(1)) \geq r+1$ the line bundle $r\xi_{\mathcal{F}(1)}$ is ample, being the sum of an ample line bundle and a nef one. In particular $\mathcal{F}(1)$ is an ample vector bundle, and so, for every line in Y_1

$$\mathcal{F}(1)_{|l} \simeq \oplus_1^r \mathcal{O}_{\mathbb{P}^1}(a_i)$$

with all the a_i positive; so, on every line $\mathcal{F}$ splits as a sum of line bundles of nonnegative degree; in particular the restriction of $\xi_{\mathcal{F}}$ to $Z =: \mathbb{P}_l(\mathcal{F})$ is nef. Observe that Z is a section of the rank $r-1$ vector bundle $\varphi_i^*(\oplus^{r-1}\mathcal{O}(1))$ and apply theorem (V.1.3) to get the nefness of $\xi_{\mathcal{F}}$.

Case a) $0 < c_1(\mathcal{F}) < r$.

Take a line l in $Y = \mathbb{P}^r, \mathbb{Q}^r$ and consider the restriction of $\mathcal{F}$ to it. We have

$$\mathcal{F}_{|l} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$$

with $a_2, \ldots, a_r \geq 0$; take a section C, corresponding to the surjection $\mathcal{F}_{|l} \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0$; we have $\xi_{\mathcal{F}}.C = 0$, and so

$$r \leq -K_X.C = r\xi_{\mathcal{F}}.C - \varphi^*(K_Y + det\mathcal{F}).C = r + \epsilon - c_1(F)$$

with $\epsilon = 1$ if $Y \simeq \mathbb{P}^r$ and $\epsilon = 0$ if $Y \simeq \mathbb{Q}^r$. The only possibility is $\epsilon = 1 \quad (Y \simeq \mathbb{P}^r)$ and $c_1(\mathcal{F}) = 1$, that is, $\mathcal{F}$ is a uniform rank r vector bundle on $\mathbb{P}^r$ with splitting type $(1, 0, \ldots, 0)$; by (I.5.7) $\mathcal{F}$ is $\mathcal{O}_{\mathbb{P}^r}(1) \oplus \mathcal{O}_{\mathbb{P}^r}{}^{\oplus(r-1)}$ or $T\mathbb{P}^r(-1)$.

Case b) $c_1(\mathcal{F}) = r$.

In this case either $\mathcal{F}$ is uniform of splitting type $(1, 1, \ldots, 1)$ or there is a line l such that

$$\mathcal{F}_{|l} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r).$$

In the first case, by (I.5.5) and (I.5.6) we have $\mathcal{F} \simeq \oplus^r \mathcal{O}(1)$, while, in the second, arguing as in case a) we find a contradiction. □

Corollary II.3.4 *Let $\pi : X \to W$ a fiber type contraction supported by $K_X + det\mathcal{E}$ with $\mathcal{E}$ ample vector bundle of rank r on X and suppose that $r \geq (n+1)/2$. Then π is an elementary contraction except if*

a) $r = (n+2)/2$, $(X, \mathcal{E}) \simeq (\mathbb{P}^{r-1} \times \mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1} \times \mathbb{P}^{r-1}}(1,1))$.

b) $r = (n+1)/2$, *$dimW = 1$ and the general fiber of π is* $\mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$.

c) $r = (n+1)/2$

- *i)* $(X, \mathcal{E}) \simeq (\mathbb{P}^{r-1} \times \mathbb{Q}^r, \mathcal{O}(1,1)^{\oplus r})$.
- *ii)* $(X, \mathcal{E}) \simeq (\mathbb{P}^{r-1} \times \mathbb{P}^r, \mathcal{O}(1,2) \oplus \mathcal{O}(1,1)^{\oplus r-1})$.
- *iii)* $(X, \mathcal{E}) \simeq (\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r-1)}), \mathcal{O}(1,1)^{\oplus r})$.
- *iv)* *X has two $\mathbb{P}^{r-1}$-bundle structures on two smooth Fano manifolds of Picard number 1.*

Proof. Case a) follows from [AM97, Lemma 5.3], in case b) the same result leads to the same conclusion for the general fiber, while case c) follows from (II.3.1).

In case ii) of proposition (II.3.1) it is possible to show, using again unbreakable families of rational curves, that Y_1 and Y_2 have an extremal ray of length $r+1$. This fact motivates the following

Conjecture II.3.5 *Let X be a smooth projective variety of dimension n, $\mathcal{E}$ an ample vector bundle of rank r on X such that $K_X + det\mathcal{E} = \mathcal{O}_X$ and $n = 2r-1$. Suppose that $b_2(X) \geq 2$. Then $(X, \mathcal{E})$ is one of the following:*

1. $(\mathbb{P}^{r-1} \times \mathbb{Q}^r, \mathcal{O}(1,1)^{\oplus r})$.
2. $(\mathbb{P}^{r-1} \times \mathbb{P}^r, \mathcal{O}(1,2) \oplus \mathcal{O}(1,1)^{\oplus r-1})$.
3. $(\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r-1)}), \mathcal{O}(1,1)^{\oplus r})$.
4. $(\mathbb{P}_{\mathbb{P}^r}(T\mathbb{P}^r), \mathcal{O}(1,1)^{\oplus r})$.

II.3.1 Addendum

Conjecture (II.3.5) has been proved; a complete proof will soon appear elsewhere; we give here only a quick sketch:

Step 1 The image under φ_1 of a line in the fiber of φ_2 is unbreakable.

Step 2 Call C one of this curves and consider the family of its deformations; we deduce that $-K_{Y_1}.C \geq r+1 \Rightarrow -K_{Y_1}.C = r+1$.

Step 3 Let $\nu : \mathbb{P}^1 \to C$ be the normalization of C and consider the restriction of $\mathcal{F}$ to C and the pull back of $\mathcal{F}$ to $\mathbb{P}^1$; using the unbreakability of the fibers of φ_2 we get that $\nu^*\mathcal{F} \simeq \mathcal{O}(a)^{r-1} \oplus \mathcal{O}(a+1)$.

Step 4 The map $\mathbb{P}_{\mathbb{P}^1}(\nu^*\mathcal{F}) \to Y_2$ is the blow down of a smooth divisor to a smooth codimension 2 subvariety of $Y_2 \Rightarrow Y_2 \simeq \mathbb{P}^r$.

Chapter III

Rational curves and elementary contractions

A fundamental feature of a Fano-Mori contraction is the existence of rational curves in its fibers. In this chapter we show how, under assumptions which ensure that the family of rational curves in a fiber is "large" enough, we can have a quite precise description of the fiber itself.
The description obtained in this way is then used to study small contractions supported by $K_X + (n-2d)L$.

III.1 Deformations of curves and fibers of good contractions

III.1.1 Let X be a smooth complex projective variety, L an ample line bundle on X and let $\varphi : X \to Z$ be a Fano-Mori contraction supported by $K_X + rL$.

The idea of the next proposition is taken from [KMM92, Theorem 2.1], in view of the following

Remark III.1.2 Let X, L and φ be as in (III.1.1); let F be a fiber of φ; then the pointed deformations inside X of the curves lying in F must remain inside F.

Proposition III.1.3 *Let X, L and φ be as in (III.1.1), let F be a fiber of φ. Then, if $\pi : F \to Y$ is a non constant morphism to a projective variety Y, there exists, for any point $y \in \pi(F)$, a rational curve on F which meets $\pi^{-1}(y)$, but is not contracted by π.*

Proof. Since π is not constant, there exists a curve $i : B \to F$ not contracted by π which meets $\pi^{-1}(y)$. Let b be a point on B such that $\pi(i(b)) = y$; we may assume that B is irrational.

Assume that the ground field has positive characteristic; fix an ample divisor H on Y; since $-K_{X|F}$ is ample on F there exists a positive rational number α such that $-K_{X|F} - \alpha\pi^*H$ is ample on F; since $\pi \circ i$ is not constant, we may assume, by composing i with sufficiently many Frobenius morphisms, that

$$\alpha H \pi_*(i_*B) \geq \dim X \cdot g(B). \qquad \text{(III.1.4)}$$

and, in particular

$$-K_X.i_*B > \dim X \cdot g(B)$$

This implies ((A.1.6) and (A.1.4)) that there exist a curve $i' : B \to X$ and a connected non-zero effective rational 1-cycle Z on X, passing through $i(b)$ such that

$$i_*B \equiv i'_*B + Z.$$

Note that, by remark (III.1.2), $i'(B)$ and Z are contained in F. If Z is not contained in a fiber of π, then we are done; otherwise the inequality (III.1.4) is still valid for i' and moreover, since Z is non-zero

$$0 < -K_X.i'_*B < -K_X.i_*B.$$

Now we can repeat the same construction; at a certain point we have to stop, otherwise the $-K_X$ degrees of the curves would form an infinite decreasing sequence of positive integers. The characteristic zero case is treated by reduction to positive characteristic (see Appendix A). □

Corollary III.1.5 *Let X, L and φ be as in (III.1.1) and let F be a fiber of φ. Then F can not be a $\mathbb{P}$-bundle on a curve of positive genus.*

III.2 Family of rational curves and fibers of good contractions

III.2.1 Let X, L and $\varphi : X \to Z$ be as in (III.1.1) and suppose moreover that φ is elementary.

In this situation, if R is the extremal ray contracted by φ we have $l(R) \geq r$, and, by proposition (I.2.8) we have

$$(*) \qquad \dim F + \dim E \geq n + l(R) - 1 \geq n + r - 1.$$

In [ABW93] the authors investigated the case in which this two inequalities are equalities, and proved the following result:

Proposition III.2.2 *([ABW93, 1.1]) Let X, L and φ be as in (III.1.1), and assume that the inequalities in (*) are equalities for an irreducible component F' of a fiber of φ. Let F denote the normalization of F' and denote again by L the pullback of L to F; let $s = \dim F'$. Then $(F, L) \simeq (\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(1))$.*

Our aim is to go one step further in this direction, studying the situation in which (*) is nearly an equality. To be more precise we are interested in the case in which

$$(**) \qquad \dim\, F + \dim\, E = n + l(R) = n + r.$$

First of all we show that, with these assumptions, there exists on the fibers of $\varphi : X \to Z$ a "large" family of unbreakable rational curves.

Remark III.2.3 The assumption $l(R) = r$ is crucial; assuming only $\dim\, F + \dim\, E = n + r$ we still obtain a family of unbreakable rational curves on the fibers, but we have no way to prove the existence of a line, which is essential for the description of F.

Lemma III.2.4 *(cfr. proof of Theorem (1.1),[Wiś91c]) Let X, L and φ be as in (III.1.1); let F be an irreducible component of a fiber of φ such that $(**)$ is satisfied and let $s = \dim\, F$. Then there exists a family $\mathcal{V}' = (V', T', p', q')$ of unbreakable rational curves which covers E, such that, for a general $x \in F$ we have $\delta_x \geq s - 2$.*

Proof. Let $x \in F$ be a general point of F and let $C_0 \subset F$ be an extremal rational curve containing x; in particular we have $-K_X.C_0 = l(R) = r$. As in (II.1.6), deforming the curve C_0, we obtain an unbreakable (see example (II.1.5)) covering family of rational curves $\mathcal{V}' = (V', T', p', q')$. We have

$$\dim\, T' \geq \dim\, X - K_X.C_0 - 3 = n + r - 3$$

and, as the fibers of q are one dimensional

$$\dim\, V' \geq n + r - 2$$

The image of p is contained in E, and so, recalling that $\delta_x = \dim\, p^{-1}(x)$

$$\dim\, E \geq \dim V' - \delta_x \geq n + r - 2 - \delta_x$$

By $(**)$ we have $\dim\, E = n + r - s$, and so

$$\delta_x \geq s - 2. \qquad \square$$

In the following proposition we show how the existence of this family of rational curves leads us to the description of the fibers of φ.

Proposition III.2.5 *Let X, L and φ be as in (III.2.1), and assume that $(**)$ is satisfied for an irreducible component F' of a fiber of φ. Let F be the normalization of F' and denote again by L the pullback of L to F; let $s = \dim F$. Then $(F, L) \simeq (\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(1))$ or $(F, L) \simeq (\mathbb{Q}^s, \mathcal{O}_{\mathbb{Q}^s}(1))$, possibly singular, or F has a desingularization which is a $\mathbb{P}^{s-1}$-bundle over a smooth curve B.*

Proof. Consider the family $\mathcal{V}' = (V', T', p', q')$ constructed in lemma (III.2.4); and observe that the curves in this family are lines with respect to L.
Next we take F to be the normalization of F'; normalizing also the family and its graph we obtain a new unbreakable family $\mathcal{V} = (V, T, p, q)$ on F. Let $\Phi : M \to F$ be a desingularization of F, and $L_M = \Phi^* L$, and let m the inverse image of e, a smooth point of F. The curves in the family T_e, since they are not entirely contained in the singular locus of F, can be lifted to curves on M passing through the point m.
We claim that there is a family of these lifts-up of degree one with respect to L_M which covers a dense subset of M.
Indeed we know that a neighborhood of m is covered by these curves, so, if it were not such a family, the neighborhood would be contained in a countable sum of nowhere dense subsets of M, obtained by deforming each of the lifted-up curves, obtaining a contradiction.
By lemma (III.2.4) the dimension of the family at a general point is greater or equal than $s - 2$ and the family covers a dense subset of M, so, by (II.1.1.3), if $\nu : \mathbb{P}^1 \to C \subset M$ is the normalization of a general curve in the family we have

$$\nu^*(TM) \simeq \oplus \mathcal{O}_{\mathbb{P}^1}(a_i)$$

with $a_1 \geq 2, \quad a_i \geq 1 \text{ for} i = 2, \dots s-1, \quad a_s \geq 0$ and so

$$(K_M + tL_M).C < 0 \qquad \text{if } t < s.$$

So, since the family cover a dense subset of M, we have

$$h^s(M, -tL_M) = h^0(M, K_M + tL_M) = 0 \qquad \text{for } t < s$$

and

$$h^i(M, \mathcal{O}_M) \simeq h^0(M, \wedge^i TM^*) = 0 \qquad \text{for } i \geq 2$$

Case I: $h^1(M, \mathcal{O}_M) = 0$

By the Kawamata-Viehweg vanishing theorem we have

$$h^i(M, -tL_M) = 0 \qquad \text{for } i < s \ , \ t > 0$$

Combining all these vanishings, we see that, for $\chi(t) = \chi(M, tL_M)$, the Hilbert polynomial of L_M, we have $\chi(M, tL_M) = 0$ for $t = -1, -2, \dots, -s+1$ and

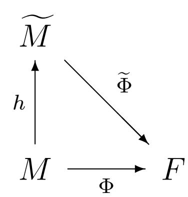
$\chi(M, \mathcal{O}_M) = 1$, so the polynomial is of the form

$$\chi(M, tL_M) = \frac{d}{s!}\left(\prod_{k=1}^{s-1}(t+k)\right)\left(t+\frac{s}{d}\right)$$

This implies that $h^0(M, L_M) = \chi(1) = s + d$ and then, computing the Δ-genus of the quasi-polarized variety (M, L_M)

$$\Delta(M, L_M) = \dim M + \deg L_M - h^0(M, L_M) = s + d - (s + d) = 0$$

Using [Fuj89, Theorem 1.1] we get a commutative diagram



where $\widetilde{M}$ is a smooth variety, $L_{\widetilde{M}}$ is a very ample line bundle on $\widetilde{M}$ such that $(\widetilde{M}, L_{\widetilde{M}})$ has Δ-genus zero, and so $\widetilde{M} \simeq \mathbb{P}^s, \mathbb{Q}^s$ or $\widetilde{M}$ is a $\mathbb{P}^{s-1}$-bundle on a rational curve. In the first two cases we can apply (I.4.10) or (I.4.11), respectively and, since $L_{\widetilde{M}}$ is very ample, obtain $(F, L) \simeq (\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(1))$ or $(F, L) \simeq (\mathbb{Q}^s, \mathcal{O}_{\mathbb{Q}^s}(1))$ (possibly singular), respectively.

Case II: $h^1(M, \mathcal{O}_M) \neq 0$

In this case we consider the Albanese map of M, $\alpha : M \to Alb(M)$; since rational curves in the family cover a dense subset of M and the dimension of the family at a general point is $s-1$, the image of the Albanese map of M, $\alpha(M) \subset Alb(M)$ must be a curve B (see I.6.3 and I.6.2).

Claim Let f be the generic fiber of α. Then the normalization of $\Phi(f)$ is $\mathbb{P}^{s-1}$. Using as in the preceding case vanishing theorems and Hilbert polynomials we

can show that, if f is a generic fiber of α then f is a quasi polarized varieties of Δ-genus 0; by [Fuj89, Theorem 1.1] the restriction of L to f is spanned and defines a birational map h to a polarized variety of Δ-genus 0; by [Fuj89, Theorem 2.2] this variety is $\mathbb{P}^{s-1}$ and $L_f = h^*(\mathcal{O}_{\mathbb{P}^{s-1}}(1))$. This implies that the normalization of $\Phi(f)$ is $\mathbb{P}^{s-1}$.

Consider the map $\widetilde{\Phi} : M \to F \times B$ induced by Φ and the image of M, $\widetilde{\Phi}(M)$

$$\begin{array}{ccc} M & \xrightarrow{\widetilde{\Phi}} & F \times B \supset \widetilde{\Phi}(M) \\ \Big\downarrow{\scriptstyle \alpha} & \swarrow{\scriptstyle p_2} & \\ B & & \end{array}$$

Let $\widetilde{M}$ be the normalization of $\widetilde{\Phi}(M)$ and $\tilde{\alpha} : \widetilde{M} \to B$ the composite of the normalization morphism and the projection on B.
We have a commutative diagram

$$\begin{array}{ccc} \widetilde{M} & & \\ \Big\uparrow{\scriptstyle \widetilde{\widetilde{\Phi}}} & \searrow{\scriptstyle \tilde{p}_1} & \\ M & \xrightarrow[\Phi]{} & F \end{array} .$$

The generic fiber of $\tilde{\alpha}$ is a normal variety, and, by the claim is $\mathbb{P}^{s-1}$; the line bundle $L_{\widetilde{M}} = \tilde{p}_1^* L$ is ample on the fibers of $\tilde{\alpha}$, so, up to tensor it with $\tilde{\alpha}^* H$, with H ample on B, we can assume that it is ample. So we can apply (I.3.7) and obtain that $(\widetilde{M}, L_{\widetilde{M}})$ is a $\mathbb{P}^{s-1}$-bundle over B. □

In the rest of the section we show how the result of the proposition can be improved under some additional assumptions (on the dimension of the fibers, on their singularities, etc.). Then, in the following sections we apply these results to the description of the global structure of some Fano Mori contractions.

Remark III.2.6 In the set up of the previous proposition, if we assume that F has dimension $\leq r+1$ then case II of the proof does not occur, i.e. $\Delta(F, L) = 0$

Proof. Let $X_k = \cap_1^k H_i$, with $H_i \in \mid L \mid$ generic divisors and $k = \dim F' - 1$. Let $F'_k = F' \cap X_k$. By (I.4.1.2) and (I.4.1.4), X_k is smooth and the restriction of φ to X_k is a contraction supported by K_{X_k}. Being a one dimensional fiber of a crepant birational contraction of a smooth variety, $F'_k \simeq \mathbb{P}^1$ ([AW98, Proposition 5.6.1]). Let $\Phi : M \to F'$ be a resolution of F; since $L_{|F}$ is spanned, we also have embedded resolutions $\Phi_k : M_k \to F'_k$ Suppose that (M, L_M) is a $\mathbb{P}^{s-1}$-bundle over a curve B; the curve $M_k \simeq \mathbb{P}^1$ is given by the intersection of r generic members of $\mid L_M \mid$, and so dominates the base curve, which has to be rational. □

Remark III.2.7 Actually, it can be shown ([AW97, Theorem 1.10, iii)]) that, in the hypothesis of remark (III.2.6) the fiber is normal, so that the result of proposition (III.2.5) holds for the fiber itself.

Remark III.2.8 If we assume that F has rational singularities, we can prove that $\Delta(F, L_F) = 0$.

In fact we can consider the Albanese mapping of F, which is defined in this case. (see [BS95, 2.4]) and obtain that, in case II of the proof, (F, L) is a $\mathbb{P}^{s-1}$-bundle over a smooth curve B, but, by corollary (III.1.5), B must be rational.

Proposition III.2.9 *Let X, L and φ be as in (III.1.1), and assume that $(**)$ is satisfied for an irreducible component F of a fiber of φ which has at worst log terminal singularities. Let $s = dim\ F$. Then $(F, L) \simeq (\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(1))$ or $(F, L) \simeq (\mathbb{Q}^s, \mathcal{O}_{\mathbb{Q}^s}(1))$, possibly singular or (F, L) is a $\mathbb{P}^{s-1}$-bundle on a rational curve.*

Proof. Let $x \in F$ be a general point of F and $C_0 \subset F$ be an extremal rational curve containing x and consider the deformations of the curve C_0. By the minimality of the intersection $-K_X.C_0$ the family $\mathcal{V} = (V, T, p, q)$ obtained in this way is unbreakable, and the curves in this family are lines with respect to the ample line bundle L.
By lemma (III.2.4) the dimension of the family at a general point is greater or equal than $s - 2$, and by lemma (II.1.1.5)

$$-K_F.C \geq s.$$

By theorem (I.3.3) either $(F, L_F) \simeq (\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(1))$ or $K_F + sL$ is nef.
In the second case by (I.3.4) F is a possibly singular quadric or a $\mathbb{P}^{s-1}$-bundle on a smooth curve B.
In the last case the curve B is rational, by (III.1.5). □

Proposition III.2.10 *Let X, L and φ be as in (III.2.1); let F be a fiber of φ and suppose that either $r \geq 3$ or F is an isolated fiber (i.e., if E is the irreducible component of the exceptional locus of φ containing F then $E = F$) which verifies (**). Then F can not be a rational scroll.*

Proof. Let s be the dimension of F and suppose that F is $\mathbb{P}^{s-1}$-bundle on a rational curve. Take a section $C \subset F$ such that the intersection number with L is minimal and let $d = L.C$; in $N_1(X)$; since the contraction is elementary we have $C \equiv dC_0$.
Let c be a point of C and consider the pointed deformations of C inside X; by remark (III.1.3) these deformations are contained in F. We claim that these deformations of C cannot be reducible; by contradiction suppose that, for a deformation of C, $\widetilde{C}$, we have

$$\widetilde{C} = \sum_1^k C_i.$$

Let f be a fiber of π, the bundle projection; since $\widetilde{C}.f = 1$ exactly one of the C_i must be horizontal, and the others contained in the fibers, but, by the minimality of $L.\widetilde{C}$ among horizontal sections, this is impossible.
Suppose now that $r \geq 3$; denoting by D_c the locus of deformations of C passing

through c, we have, by (II.1.1.7)

$$\dim D_c = -K_X.C - 1 \geq rd - 1,$$

and, by lemma (II.1.8) we have

$$s + 1 \leq rd - 1 + s - 1 \leq \dim D_c + \dim f \leq s$$

a contradiction.

If F is an isolated fiber of φ we can consider the (not necessarily) pointed deformations of C in X, which have to stay inside F, and prove as above that they cannot be reducible. So we construct an unbreakable family $\mathcal{F} = (V_{\mathcal{F}}, T_{\mathcal{F}}, p_{\mathcal{F}}, q_{\mathcal{F}})$ of deformations of C. By (II.1.8) we have $\delta_{\mathcal{F}} = 0$ and

$$s \geq \dim E(\mathcal{F}) = \dim V(\mathcal{F}) \geq rd + n - 2 \geq 2s - 2$$

The only possibility is that $s = 2, \quad d = 1, \quad n = 3, \quad r = 1$ and φ is a divisorial contraction contracting a two-dimensional quadric to a point. $\square$

III.3 Small contractions

Using the description of the fibers obtained in the previous section we now study some special cases of good contractions.

Theorem III.3.1 *Let $K_X + (n-2d)L$, with $d \geq 2$ be a good supporting divisor of an elementary small contraction $\varphi : X \to W$. Let E' be an irreducible component of the exceptional locus of φ of dimension $n - d$ and E the normalization of E'. Then either $\varphi(E')$ is a point and E is isomorphic to $\mathbb{P}^{n-d}$, $\mathbb{Q}^{n-d}$, possibly singular, or has a desingularization which is a $\mathbb{P}^{n-d-1}$-bundle on a smooth curve or $\varphi(E')$ is a curve and E' is a $\mathbb{P}^{n-d}$-bundle on that curve.*

Proof. Let F' be an irreducible component of a fiber of φ. The inequality (I.2.8), combined with our hypothesis gives

$$2n - 2d \geq \dim E' + \dim F' \geq n + (n - 2d) - 1.$$

If $\dim F' = n-d$ then $E' = F'$ and we have two possible cases: if $l(R) = n-2d-1$ we have $n = 5$ and $d = 2$ and this is the case studied in ([Zh95, Theorem 1]), while if $l(R) = n - 2d$ we are in the hypothesis of proposition (III.2.5).
If $\dim F' = n - d - 1$ the image of E' is a curve, B. By vertical slicing (I.4.1.3) we obtain a smooth variety $X'' \subset X$ with a good contraction $\varphi_{|X''}$ supported by $K_{X''} + rL_{X''}$ such that the irreducible components of the exceptional locus of $\varphi_{|X''}$ are the intersection of X'' with the irreducible components of the exceptional locus of φ. In particular $(E' \cap X'', L_{E' \cap X''})$ is $(\mathbb{P}^{n-d-1}, \mathcal{O}(1))$, by (III.2.2). So we have $(F, L_F) \simeq (\mathbb{P}^{n-d-1}, \mathcal{O}(1))$ for a general fiber of φ; the contraction φ is equidimensional, and so, by (I.3.7), E' is a $\mathbb{P}^{n-d-1}$-bundle over B. $\square$

Corollary III.3.2 *Let $K_X + (n-2d)L$, with $d \geq 2$ be a good supporting divisor of an elementary small contraction $\varphi : X \to Z$. Let E be an irreducible log terminal components of the exceptional locus of φ of dimension $n-d$. Then E is isomorphic to $\mathbb{P}^{n-d}$, $\mathbb{Q}^{n-d}$ and φ contracts E to a point or or $\varphi(E)$ is a smooth curve B and E is a smooth $\mathbb{P}^{n-d-1}$-bundle over B.*

Proof. The result follows reasoning as in the proof of theorem (III.3.1), taking into account proposition (III.2.9) and proposition (III.2.10). □

All the cases listed in the corollary do happen.

Construction III.3.3 Let $(F, \mathcal{E})$ be a pair consisting of a smooth variety F and a numerically effective vector bundle $\mathcal{E}$ such that $-K_F - det\mathcal{E}$ is ample. Let $X := \mathbb{P}(\mathcal{E} \oplus \mathcal{O})$, let ξ denote the tautological line bundle on X, consider the section of the projective bundle $\pi : X \to F$ determined by the surjection $\mathcal{E} \oplus \mathcal{O} \longrightarrow \mathcal{O} \longrightarrow 0$, and call it again F; it is easy to check that $N^*_{F \setminus X} = \mathcal{E}$. Since ξ is nef and $\xi - K_{\mathbb{P}(\mathcal{E} \oplus \mathcal{O})}$ is ample, by the Contraction theorem it follows that $m\xi$ is base point free for $m >> 0$ and defines a Fano-Mori contraction $\varphi : X \to Z$ with $Z = \mathbb{P}(\oplus_{m \geq 0} H^0(S^m(\mathcal{E} \oplus \mathcal{O})))$. If $\mathcal{E}$ is ample the map φ is the contraction of F to a point.

Example III.3.4 To construct examples for the contractions described in the corollary consider as $(F, \mathcal{E})$ the pairs $(\mathbb{P}^{n-d} \mathcal{O}_{\mathbb{P}}(1)^{\oplus d})$ and $(\mathbb{Q}^{n-d}, \mathcal{O}_{\mathbb{Q}}(1)^{\oplus d})$ for the first two possibilities, while for the last consider the pair $(\mathbb{P}^{n-d-1}, \mathcal{O}_{\mathbb{P}}(1)^{\oplus d})$; the same construction gives us a good contraction of a $(n-1)$-dimensional variety $Y = \mathbb{P}(\mathcal{E} \oplus \mathcal{O})$ which contracts a $\mathbb{P}^{n-d-1}$ to a point. Taking $X = Y \times B$, with B a smooth curve, we have an example of a small contraction which contracts a $\mathbb{P}^{n-d-1}$-bundle on B to B.

Remark III.3.5 In all this examples the flip of φ exists.

Proof. Consider $\hat{X}$, the blow up of X along E; by [AW98, Proposition 5.4] the bundle $-\hat{F}$ is generated by global sections and defines a Fano-Mori contraction $\hat{\varphi} : \hat{X} \to \hat{Z}$. □

Chapter IV

Ample vector bundles, low degree rational curves and good contractions

The aim of this chapter is to generalize some results on good contractions supported by divisors of the form $K_X + rL$ to Fano-Mori contractions supported by divisors of the form $K_X + det\mathcal{E}$. In particular we will consider contractions whose fibers are covered by "large" families of rational curves, extending propositions (III.2.2) and (III.2.5). Unfortunately, in this case, it is not possible to use vanishing theorems and Hilbert polynomials as in the line bundle case, so, in general, we have to strengthen our hypothesis on the fibers, allowing only some mild singularities.
However, in some special cases, without assumptions on singularities, we are able to prove that the fibers have to be smooth, generalizing a theorem of Andreatta, Ballico and Wiśniewski on small contractions.
As a byproduct we obtain a characterization of projective spaces, quadrics and $\mathbb{P}$-bundles over curves by means of families of rational curves with bounded degree with respect to the determinant bundle of an ample vector bundle: if X is a log terminal variety with a sufficiently large family of this kind, then X is smooth and is one of the above varieties.

IV.1 Generalized adjunction and rational curves of low degree

IV.1.1 Let X be a smooth complex projective variety, $\mathcal{E}$ an ample vector bundle of rank r on X and let $\varphi : X \to Z$ be an elementary contraction supported by $K_X + det\mathcal{E}$.

By inequality (I.2.8) and (I.5.2) we have

$$\dim E + \dim F \geq n + l(R) - 1 \geq n + r - 1$$

We want to study the border cases for this inequality:

$$(*) \qquad \dim E + \dim F = n + r - 1$$

$$(**) \qquad \dim E + \dim F = n + r.$$

First of all we show how these hypothesis ensure the existence of a "large" family of rational curves in the fibers of φ. Let X, $\mathcal{E}$ and φ be as in (IV.1.1); let F be an irreducible component of a fiber of φ and let $s = \dim F$.

Lemma IV.1.2 *Suppose that $(*)$ is satisfied. Then there exists a covering family $\mathcal{V} = (V, T, p, q)$ of unbreakable rational curves on F, such that, for a general $x \in F$ we have $\delta_x \geq s - 1$.*

Lemma IV.1.3 *Suppose that $(**)$ is satisfied. Then there exists a covering family $\mathcal{V} = (V, T, p, q)$ of unbreakable rational curves on F, such that, for a general $x \in F$ we have $\delta_x \geq s - 2 + \epsilon$, with $\epsilon = l(R) - r$.*

Proof. Same as lemma (III.2.4). □

Next we show how the existence of a "large" family of rational curves of low degree with respect to $det\mathcal{E}$ allows us to classify X.

Theorem IV.1.4 *Let X be a log terminal complex projective variety of dimension n and $\mathcal{E}$ an ample vector bundle of rank r on X, such that there exists a covering family $\mathcal{V} = (V, T, p, q)$ of rational curves on X of dimension δ and degree d with respect to $det\mathcal{E}$.*

1. *If $d = r \leq n + 1$ and $\delta = n - 1$ then $(X, \mathcal{E}) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r})$.*
2. *If $d = r + 1 \leq n + 1$ and $\delta = n - 1$ then $(X, \mathcal{E}) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus (r-1)} \oplus \mathcal{O}_{\mathbb{P}^n}(2))$ or $(X, \mathcal{E}) \simeq (\mathbb{P}^n, T\mathbb{P}^n)$.*
3. *If $d = r \leq n$ and $\delta = n - 2$ then $(X, \mathcal{E}) \simeq (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus r})$ or X is a $\mathbb{P}^{n-1}$-bundle on a smooth curve and $\mathcal{E}_{|f} \simeq \mathcal{O}(1)^{\oplus r}$ for every fiber f of the bundle projection.*

In the proof we will need the following

Lemma IV.1.5 *Let $(X, \mathcal{E})$ be as in (IV.1.4) and suppose that there exists a covering family $\mathcal{V} = (V, T, p, q)$ of rational curves on X, of dimension δ and degree $d < 2r$ with respect to $det\mathcal{E}$. Then there exists a rational curve $\bar{C} \subset \mathbb{P}(\mathcal{E}) = Proj_X(\mathcal{E})$ such that $\xi_{\mathcal{E}}.\bar{C} = 1$ and $-K_{\mathbb{P}(\mathcal{E})}.\bar{C} \geq \delta + 2 - d + r$.*

Proof. The canonical bundle of $\mathbb{P}(\mathcal{E})$ is given by the formula

$$K_{\mathbb{P}(\mathcal{E})} = p^*(K_X + \det\mathcal{E}) - r\xi_{\mathcal{E}}$$

Let $C \subset X$ be a curve of the family $\mathcal{V}$, and $\nu : \mathbb{P}^1 \to X$ the normalization of C, and consider the fiber product

$$\begin{array}{ccc} W & \xrightarrow{\bar{\nu}} & \mathbb{P}(\mathcal{E}) \\ \downarrow & & \downarrow p \\ \mathbb{P}^1 & \xrightarrow{\nu} & X \end{array}$$

We have $\nu^*\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$, with $a_i \geq 1 \quad \forall i$. Let $\widetilde{C}$ be the section corresponding to the surjection $\nu^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0$ and $\overline{C} = \bar{\nu}(\widetilde{C})$. We have $\xi_{\mathcal{E}}.\overline{C} = 1$ and, recalling that $K_X.C \geq \delta + 2$ (II.1.1.5)

$$\begin{aligned} -K_{\mathbb{P}(\mathcal{E})}.\overline{C} &= \nu^*(-K_{\mathbb{P}(\mathcal{E})}).\widetilde{C} = \bar{\nu}^*(p^*(-K_X - \det\mathcal{E}) + r\xi_{\mathcal{E}}).\widetilde{C} = \\ &= (\pi \circ \nu)^*(-K_X).\widetilde{C} - (\pi \circ \nu)^*(\det\mathcal{E}).\widetilde{C} + r\bar{\nu}^*(\xi_{\mathcal{E}}).\widetilde{C} = \\ &= \delta + 2 - d + r. \end{aligned}$$

□

Proof of theorem (IV.1.4).

Step 1: Assume that X is smooth.

Proof of 1) Consider in $\mathbb{P}(\mathcal{E})$ the maximal Chow family $\mathcal{V} = (V, T, p, q)$ of deformations of the unbreakable rational curve $\bar{C} = \bar{\nu}(\widetilde{C})$, where $\bar{C}$ and $\widetilde{C}$ are as in lemma (IV.1.5).

Claim $\mathcal{V}$ is a covering family for $\mathbb{P}(\mathcal{E})$.

Proof of the claim Let x be a point of $\bar{C}$, P_x the fiber of p containing x, T_x be the subvariety of T parametrizing curves in the family containing x, and δ_x be the dimension of T_x. If $\delta_x \geq n$, then $\dim D_x \geq n + 1$ and

$$\dim(D_x \cap P_x) \geq 1$$

and so $p(D_x)$ is a point (cfr. II.1.8), and this is impossible. So $\delta_x \leq n - 1$. The dimension of V is $\geq 2n + r - 2$ by corollary (II.1.1.6) and so we have, for the dimension of the exceptional locus of $\mathcal{V}$

$$\dim E(\mathcal{V}) \geq \dim V - \delta_x \geq n + r - 1$$

and the claim is proven. □

By [BSW90, Lemma 1.4.5] we have

$$NE(\mathbb{P}(\mathcal{E})) = NE(P) + \mathbb{R}_+[\bar{C}]$$

and this implies that $\mathbb{P}(\mathcal{E})$ is a Fano variety and $\bar{C}$ is an extremal rational curve generating an extremal ray R_2.
By lemma (IV.1.5) the contraction $\varphi = contr_{R_2}$, has length

$$l(R_2) \geq n + 1.$$

So, if F is a fiber of the contraction φ

$$dim F \geq n.$$

On the other hand, by the proof of the claim

$$\dim F \leq n + r - 1 - (r - 2) - 1 = n$$

and the two inequalities together imply $\dim F = n$, $l(R_2) = n + 1$. Using the inequality (I.2.8)

$$\dim E(R_2) \geq -\dim F + \dim \mathbb{P}(\mathcal{E}) + l(R_2) - 1 = n + r - 1$$

we see that the contraction of R_2 is of fiber type. The general fiber is a smooth Fano variety such that

$$-K_F = (-K_{\mathbb{P}(\mathcal{E})})_{|F} = (n+1)(\xi_{\mathcal{E}})_{|F}$$

and so the general fiber is $\mathbb{P}^n$. By (I.3.7) we conclude that $\varphi : \mathbb{P}(\mathcal{E}) \to Y$ is a $\mathbb{P}^n-$bundle. The map p restricted to a fiber of φ is finite-to-one and we can apply theorem (I.3.10) and conclude $X \simeq \mathbb{P}^n$. Using (I.5.5) we have that $\mathcal{E}$ splits into the direct sum of r copies of $\mathcal{O}_{\mathbb{P}^n}(1)$.

Proof of 2) By corollary (II.1.1.7) X has an extremal ray of length $n+1$, and so, by (I.2.7) is a Fano variety of Picard number 1.
If $r = n$ then $K_X + det\mathcal{E} = \mathcal{O}_X$, and, by [YZ90], recalling that X has an extremal ray of length $n+1$, we have $(X, \mathcal{E}) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-1)})$ or $(X, \mathcal{E}) \simeq (\mathbb{P}^n, T\mathbb{P}^n)$, so we can suppose that $r < n$.
Let L be the ample generator of $Pic(X)$; we have $K_X + det\mathcal{E} = aL$ with $a < 0$, because, if C is a general curve in the family $aL.C \leq (r+1) - (n+1) < 0$. This implies that $\mathbb{P}(\mathcal{E})$ is a Fano variety of Picard number two and there is an extremal ray contraction $\varphi : X \to Y$, different from p.
Let τ be the nef value of $\xi_{\mathcal{E}}$ and $\bar{C}$ the curve on $\mathbb{P}(\mathcal{E})$ given by lemma (IV.1.5); since $K_{\mathbb{P}(\mathcal{E})} + \tau\xi_{\mathcal{E}}.\bar{C} \geq 0$ we must have $\tau \geq n$; this implies that

$$l(R_2) \geq n$$

Case a) The contraction φ is birational.

In this case, for a fiber of φ, we have

$$\dim F \geq l(R_2) \geq n$$

and again, by [BSW90, Lemma 1.4.5] we find that

$$\dim F \leq n + r - 1 - (r - 2) - 1 = n$$

and the two inequalities together imply $\dim F = n$, $l(R_2) = n$ and $R_2 = \mathbb{R}_+[\bar{C}]$. The general fiber of this contraction is $\mathbb{P}^n$ by [ABW93, Lemma 1.1]. The restriction of p to the general fiber is finite to one and surjective, and by (I.3.10) $X \simeq \mathbb{P}^n$.

Case b) The contraction φ is of fiber type.

In this case, for a fiber of φ, we have

$$\dim F \geq l(R_2) - 1 \geq n - 1,$$

and, as in case a) we have $\dim F \leq n$, so the fibers of φ can have dimension $= n, n-1$.

If the dimension of the general fiber is $n-1$, then $l(R_2) = n$, $\tau = n$ and $\dim Y = r$; so we can apply (I.4.1.5) to conclude that φ is equidimensional. In fact, if φ had a fiber of dimension n, then we would be in case (2c) of (I.4.1.5) and we would have $n \leq \frac{n+r-1}{2}$, a contradiction with our hypothesis $r \leq n$. For the generic fiber of φ we have

$$K_F = (K_{\mathbb{P}(\mathcal{E})})_{|F} = -n(\xi_{\mathcal{E}})_{|F}$$

and so $(F, (\xi_{\mathcal{E}})_{|F}) \simeq (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$, and, by (I.3.7) $\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{F})$ is a $\mathbb{P}^{n-1}$-bundle over Y. This implies that Y is smooth, $\mathbb{P}(\mathcal{E})$ being so, and in particular $\bar{C}$, being an extremal rational curve, hence a line in $\mathbb{P}^{n-1}$, is smooth, and so C is smooth, too. This implies that, in the construction of lemma (IV.1.5) $W = p^{-1}(C) = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(r-1)})$ is the blow up of $\mathbb{P}^r$ along a codimension 2 linear space. The restriction of φ to W is a morphism which contracts the exceptional divisor of the blow up, and such that K_W is $\varphi_|$ ample. So the connected part of the Stein factorization of $\varphi_|$ is a contraction on $\mathbb{P}^r$, and $Y \simeq \mathbb{P}^r$ by (I.3.10). An easy computation shows that $\xi_{\mathcal{F}} = \xi_{\mathcal{E}}$ is ample, and $c_1(\mathcal{F}) = n+1$ so that $\mathcal{F}$ is uniform of splitting type $(1, \ldots, 1, 2)$, and so, by proposition (I.5.7) $\mathcal{F} = \mathcal{O}_{\mathbb{P}^r}(2) \oplus \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus(n-1)}$ or $\mathcal{F} = T\mathbb{P}^r \oplus \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus(n-r)}$.The first case is impossible, since the second contraction of $\mathbb{P}(\mathcal{F})$ would be birational, while, in the second case the second contraction is onto $\mathbb{P}^n$.

If the dimension of the general fiber is n, then φ is a quadric bundle, by (I.4.1.5). The restriction of p to the general fiber, a smooth n-dimensional quadric, is finite-to-one and we can use theorem (I.3.10) to get $X \simeq \mathbb{P}^n$ or $X \simeq \mathbb{Q}^n$, but the second case is impossible, since X has an extremal ray of length $n+1$.
It remains to prove that $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(r-1)}$, but this follows from (I.5.7) since $r < n$.

Proof of 3) By corollary (II.1.1.7) X has an extremal ray of length n, and so, by (I.2.7) there are two possibilities for X:

i. X is a Fano variety and $Pic(X) \simeq \mathbb{Z}$.

ii. There exists an extremal contraction $contr_R : X \to B$ onto a smooth curve B, whose general fiber is a smooth $n-1$ manifold with an extremal ray of length n.

In case i), reasoning as in lemma (IV.1.5) and in proof of theorem 2), we can prove that $\mathbb{P}(\mathcal{E})$ is a Fano variety, and the "other contraction", $\varphi : \mathbb{P}(\mathcal{E}) \to Y$ has length $l(R_2) \geq n$.

Case a) The contraction φ is birational.

In this case, as in proof of 2), we have that the general fiber of φ is $\mathbb{P}^n$. The restriction of p to the general fiber is finite to one and surjective, and by (I.3.10) $X \simeq \mathbb{P}^n$, but this is absurd, since X has an extremal ray of length n.

Case b) The contraction φ is of fiber type.

As in proof of 2) we find that φ is equidimensional and either is a $\mathbb{P}^{n-1}$-bundle contraction on $\mathbb{P}^r$, or a quadric bundle.
In the first case $\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is $\mathcal{F} = \mathcal{O}_{\mathbb{P}^r}(2) \oplus \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus(n-1)}$ or $\mathcal{F} = T\mathbb{P}^r \oplus \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus(n-r)}$ and this is impossible, since in the first case the second contraction of $\mathbb{P}(\mathcal{F})$ would be birational, while, in the second case the second contraction is onto $\mathbb{P}^n$, but X has an extremal ray of length n.
If φ is a quadric bundle contraction, the restriction of p to the general fiber, a smooth n-dimensional quadric, is finite-to-one and we can use theorem (I.3.10) to get $X \simeq \mathbb{P}^n$ or $X \simeq \mathbb{Q}^n$, but the first case is impossible, since X has an extremal ray of length n.
It remains to prove that $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(r)}$, but this follows from (I.5.5).
In case ii), the general fiber of $contr_R : X \to B$ is a smooth $n-1$ manifold which satisfies the assumptions of 1) and so is a smooth projective space. By [Me97, proof of 4.1.2, claim], X is a $\mathbb{P}^{n-1}$-bundle on B.

Step 2: X is singular.

The idea is to show that, under the assumptions we made, $\mathbb{P}(\mathcal{E})$ and hence X has to be smooth. This idea, as well as the techniques, are borrowed from [Zh95b] and [Me98].

Proof of 1) In case $r = n+1$ we have that $K_X + det\mathcal{E}$ is nef but not ample and so $X \simeq \mathbb{P}^n$ by [Zh95b, Theorem 2], so we can suppose that $r \leq n$.
We consider $\mathbb{P}(\mathcal{E})$, which now is a log terminal variety of dimension $n+r-1$, such that $\mathrm{Sing}(\mathbb{P}(\mathcal{E})) = p^{-1}\mathrm{Sing}(X)$.
Take on X a curve in the family which is contained in the smooth locus of X; the construction of lemma (IV.1.5) gives us a curve $\bar{C} \subset \mathbb{P}(\mathcal{E})$ such that $-K_{\mathbb{P}(\mathcal{E})}.\bar{C} \geq n+1$ and $\xi_{\mathcal{E}}.\bar{C} = 1$.
Let τ be the nef value of $\xi_{\mathcal{E}}$, and $\varphi : \mathbb{P}(\mathcal{E}) \to Y$ the associated extremal contraction; Since $K_{\mathbb{P}(\mathcal{E})} + \tau\xi_{\mathcal{E}}$ is nef on $\mathbb{P}(\mathcal{E})$ we have $(K_{\mathbb{P}(\mathcal{E})} + \tau\xi_{\mathcal{E}}).\bar{C} \geq 0$, and so $\tau \geq n+1$; since $r \leq n$, φ does not contract curves in the fibers of p; since no curve can be contracted by two different extremal contractions, if F is a fiber of φ, we have $\dim F \leq n$.

Claim The contraction φ is of fiber type.

Proof of the claim. See [Zh95b, Claim II].

The bundle $K_F + (n+1)(\xi_{\mathcal{E}})_{|F} = K_{\mathbb{P}(\mathcal{E})} + \tau\xi_{\mathcal{E}}$ is nef but not ample, implying $F \simeq \mathbb{P}^n$. The contraction φ is thus equidimensional, the general fiber is $\mathbb{P}^n$ and the ample line bundle $\xi_{\mathcal{E}}$ restricts to $\mathcal{O}_{\mathbb{P}^n}(1)$ on the fibers. By (I.3.7) $\mathbb{P}(\mathcal{E})$ is a $\mathbb{P}^n$-bundle on Y. Take a smooth Zariski open subset $U \subset Y$; $\varphi^{-1}(U)$ must be smooth, and this implies that $\mathrm{Sing}(X) = \emptyset$.

Proof of 2) and 3) If $r = n$ we have that $K_X + det\mathcal{E}$ is nef but not ample, and the results follow from [Me98, Theorem 2], so we can suppose $r < n$.

Take a curve in the family contained in the smooth locus of X. As in lemma (IV.1.5) we find a curve $\bar{C}$ on $\mathbb{P}(\mathcal{E})$ such that $-K_{\mathbb{P}(\mathcal{E})}.\bar{C} \geq n$ and $\xi_{\mathcal{E}}.C = 1$.
Let τ be the nef value of $\xi_{\mathcal{E}}$; since $(K_{\mathbb{P}(\mathcal{E})} + \tau\xi_{\mathcal{E}}).\bar{C} \geq 0$ we have $\tau \geq n$, so the contraction φ associated to $K_{\mathbb{P}(\mathcal{E})} + \tau\xi_{\mathcal{E}}$ does not contract curves in the fibers of p, and this implies, for a fiber of p, that $\dim F \leq n$, but we also have $\dim F \geq n-1$, since $K_{\mathbb{P}(\mathcal{E})} + \tau\xi_{\mathcal{E}}$ is trivial on F. Proof of [Me97, Theorem 3.2] works in our case to give the smoothness of X. □

Now we apply theorem (IV.1.4) to the study of Fano-Mori contractions.

Theorem IV.1.6 *Let X , $\mathcal{E}$ and φ be as in (IV.1.1), with $r \geq 2$, and suppose that $(*)$ is satisfied for a log terminal irreducible component F of a fiber of φ; let $s = dim\ F$. Then $(F, \mathcal{E}_{|F}) \simeq (\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(1)^{\oplus r})$.*

Proof. By lemma (IV.1.2) we are in the hypothesis of theorem (IV.1.4.1), which gives us the result. □

Theorem IV.1.7 *Let X , $\mathcal{E}$ and φ be as in (IV.1.1), with $r \geq 2$, let F be a log terminal irreducible component of a fiber of φ such that (∗∗) is satisfied; let $s = dim\ F$. Then $(F, \mathcal{E}_{|F}) \simeq (\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(1)^{\oplus r})$ or $(F, \mathcal{E}_{|F}) \simeq (\mathbb{Q}^s, \mathcal{O}_{\mathbb{Q}^s}(1)^{\oplus r})$ or F is a $\mathbb{P}^{s-1}$-bundle over a smooth curve B and $\mathcal{E}_{|f} = \mathcal{O}_{\mathbb{P}^{s-1}}(1)^{\oplus r}$ for every fiber of the $\mathbb{P}$-bundle contraction.*

Proof. Let R be the extremal ray contracted by φ; our assumptions give us two possibilities for the length of R.
If $l(R) = r$, by lemma (IV.1.3) we are in the hypothesis of (IV.1.4.2)); if $l(R) = r + 1$, by the same lemma, we are in the hypothesis of (IV.1.4.3)). □

IV.2 A theorem on small contractions

In [ABW93] the authors proved the following

Theorem IV.2.1 *Let X be a smooth complex projective variety, $f : X \to W$ be an elementary small contraction whose good supporting divisor is $K_X + (n-3)L$. Then the exceptional locus E of f is a disjoint union of its irreducible components E_i such that $E_i \simeq \mathbb{P}^{n-2}$ and $N_{E_i/X} \simeq \mathcal{O}_{\mathbb{P}^{n-2}}(-1)^{\oplus 2}$. Moreover there exists a flip $f^+ : X^+ \to W$ of f from a smooth projective variety X^+.*

The aim of this section is to generalize this result to small contractions supported by divisors of the form $K_X + det\mathcal{E}$, with $\mathcal{E}$ ample vector bundle of rank $n-3$. We prove the following

Theorem IV.2.2 *Let X be a smooth complex projective variety, $\mathcal{E}$ an ample vector bundle of rank $n-3$ on X, $f : X \to W$ be an elementary small contraction whose good supporting divisor is $K_X + det\mathcal{E}$. Then the exceptional locus E of f is a disjoint union of its irreducible components E_i such that $E_i \simeq \mathbb{P}^{n-2}$. Moreover there exists a flip $f^+ : X^+ \to W$ of f from a smooth projective variety X^+.*

Proof. Let F_i be an irreducible component of a fiber of f, contained in E_i; we have, by (I.2.8)

$$2n - 4 \geq \dim\ E_i + \dim\ F_i \geq n + (n-3) - 1$$

which implies $E_i = F_i$; moreover, by (IV.1.6), if E_i is smooth, then $E_i \simeq \mathbb{P}^{n-2}$. So the theorem will follow once we prove that E_i is smooth.

Let $\mathbb{P}(\mathcal{E}) = Proj_X(\mathcal{E})$, $p : \mathbb{P}(\mathcal{E}) \to X$ the natural projection, $\xi_{\mathcal{E}}$ the tautological bundle of $\mathcal{E}$; p is an elementary contraction and

$$K_{\mathbb{P}(\mathcal{E})} = p^*(K_X + det\mathcal{E}) - r\xi_{\mathcal{E}}$$

Our construction yields that $\rho(\mathbb{P}(\mathcal{E})/W) = 2$, and that $-K_{\mathbb{P}(\mathcal{E})}$ is $(f \circ p)$-ample; hence there exists an extremal ray $R_2 \in \overline{NE}(\mathbb{P}(\mathcal{E})/W)$ not contracted by p, with $l(R_2) \geq n-3$, and an elementary contraction $\varphi : \mathbb{P}(\mathcal{E}) \to Z$, with a commutative diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}) & \xrightarrow{\varphi} & Z \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \psi} \\ X & \xrightarrow{f} & W \end{array}$$

Lemma IV.2.3 *Every fiber of φ, $F(\varphi)$, is contained in the inverse image of a fiber of f, $F(f)$. In particular the exceptional locus of φ, $E(\varphi)$, is contained in the inverse image of the exceptional locus of f, $E(f)$.*

Proof. The first part of the lemma is trivial, because of the commutativity of the diagram, while the second part follows since the image via p of a positive dimensional fiber of φ has positive dimensional image. □

This lemma, in particular, gives us

$$\dim E(\varphi) \leq \dim E(f) + n - 4$$

$$\dim F(\varphi) \leq \dim F(f).$$

Our assumptions imply, for the contraction f

$$2(n-2) \geq \dim E(f) + \dim F(f) \geq n + l(R) - 1 \geq 2n - 4$$

hence $\dim E(f) = \dim F(f) = n - 2$. Using inequality (I.2.8) for φ we have

$$2n - 8 = \dim E(f) + \dim F(f) + n - 4 \geq \dim E(\varphi) + \dim F(\varphi) \geq$$

$$\geq \dim \mathbb{P}(\mathcal{E}) + l(R_2) - 1 \geq (2n-4) + (n-3) - 1 = 2n - 8$$

so that all the inequalities are equalities; in particular we have $\dim E(\varphi) = 2n-6$ and $\dim F(\varphi) = n - 2$.

The contraction φ verifies the assumptions of (III.2.2) for every irreducible component of a fiber; so the normalization of every irreducible component of a fiber is $\mathbb{P}^{n-2}$. By theorem ([AW97, Theorem 1.11]) the line bundle L is φ-very ample, and so every irreducible component of a fiber is $\mathbb{P}^{n-2}$.

Claim For each component E_i of the exceptional locus there exists a fiber of φ, $\bar{F}$ with normal bundle $N_{\bar{F}/\mathbb{P}(\mathcal{E})} = \mathcal{O}^{\oplus(n-4)} \oplus \mathcal{O}(-1)^{\oplus 2}$.

Proof. Same as first part of the proof of [ABW90, theorem 1.3], which doesn't use the bound on r.

Now we consider the Hilbert scheme (over Z) of $n-2$-planes in X (over Z); since $h^1(N_{\bar{F}/\mathbb{P}(\mathcal{E})}) = 0$ the Hilbert scheme is smooth at the point $\bar{t}$ corresponding to $\bar{F}$; let T be the unique irreducible component containing $\bar{t}$; since $h^0(N_{\bar{F}/\mathbb{P}(\mathcal{E})}) = n-4$, the dimension of T is $n-4$.
For a general $t \in T$, since a small deformation of a decomposable bundle is trivial, we have $N_{F_t/\mathbb{P}(\mathcal{E})} = \mathcal{O}^{\oplus(n-4)} \oplus \mathcal{O}(-1)^{\oplus 2}$. Let $U \subset T$ be a Zariski open subset such that for every $t \in U$ we have $N_{F_t/\mathbb{P}(\mathcal{E})} = \mathcal{O}^{\oplus(n-4)} \oplus \mathcal{O}(-1)^{\oplus 2}$.
By the property of the Hilbert scheme, U is smooth and, if Π is the universal $(n-2)$-plane over U, we have that Π is a $\mathbb{P}^{n-2}$-bundle over U. There is moreover a natural "evaluation" map $h : \Pi \to X$; this map is one to one and it is an immersion at every point (see [ABW90, proof of theorem 1.1]).
Therefore $h(U) \subset E_i(\varphi)$ is smooth, and, since $\mathrm{Sing}(E_i(\varphi)) = p^{-1}\mathrm{Sing}(E_i(f))$ we have that $E_i(\varphi)$ and $E_i(f)$ are smooth, too. □

Chapter V

Ample vector bundles with sections vanishing on special varieties

A very classical and natural way of classifying complex projective manifolds X consists in slicing X with a number of general hyperplane sections obtaining in this way a complex manifold of smaller dimension which is likely classifiable. Then one should *ascend* the geometrical properties of this new manifold and obtain a complete description of X. To stress the classical flavor of this approach it is sometime called *Apollonius method* [Fuj90]. The hard part of the Apollonius method are the ascending properties; in this chapter we will consider this problem in a slightly more general set up.

Let $\mathcal{E}$ be an ample vector bundle of rank r on X such that there exists a section $s \in \Gamma(\mathcal{E})$ whose zero locus $Z = (s = 0)$ is a smooth submanifold of the expected dimension $dimX - r := n - r$.
Assume that Z is not minimal in the sense of Mori's theory, that is $-K_Z$ is not nef; thus Z has at least one extremal ray (I.1.3.1) and an associated extremal contraction (I.1.3.3). Our question will then be, under which condition this contraction can be *lifted* to the ambient variety, determining its structure; this general situation is studied in section 1; suppose that F_Z is an extremal face in $\overline{NE(Z)}$ with supporting divisor $K_Z + \tau H_Z$; a lifting property is proved under the assumption that

$$H_Z \quad \text{is the restriction of an ample line bundle } H \text{ on } X. \tag{V.0.1}$$

Next we discuss some special situations in which the assumption (V.0.1) can be avoided. In the rest of the chapter we consider some special cases, namely if Z is a scroll, a quadric bundle or a del Pezzo fibration; the results are described in theorems (V.2.1), (V.3.1), (V.4.1) and corollaries.
These results generalize classical ones by L. Bădescu (see [Băd82a], [Băd81], [Băd82b]) and A.J. Sommese (see [Som76] and chapter 5 of [BS95], in particular Theorem 5.2.1 which should be compared with our results in section 1) and

more recent ones by A. Lanteri and H. Maeda ([LM95], [LM96], [LM97]), who were the first ones to study the problem of special sections of ample vector bundles.

V.1 Lifting of contractions

V.1.1 Let $\mathcal{E}$ be an ample vector bundle of rank r on X such that there exists a section $s \in \Gamma(\mathcal{E})$ whose zero locus $Z = (s = 0)$ is a smooth submanifold of the expected dimension $dim X - r := n - r$. Note that, with this assumptions, the restriction of $\mathcal{E}$ to Z is the normal bundle $N_X Z$ [Ful84, Example 6.3.4].

The idea of this section is to investigate the relation between $\overline{NE(X)}$ and $\overline{NE(Z)}$; one basic result in this direction is the following Lefschetz type theorem proved by Sommese in [Som78] and with slightly weaker assumptions in [LM95].

Theorem V.1.2 *Let X, $\mathcal{E}$ and Z be as in V.1.1 and let $i : Z \hookrightarrow X$ be the embedding. Then*

(V.1.2.1) $H^i(i) : H^i(X, \mathbb{Z}) \to H^i(Z, \mathbb{Z})$ is an isomorphism for $i \leq dim\ Z - 1$
(V.1.2.2) $H^i(i)$ is injective and its cokernel is torsion free for $i = dim\ Z$
(V.1.2.3) $Pic(i) : Pic(X) \to Pic(Z)$ is an isomorphism for $dim Z \geq 3$.
(V.1.2.4) $Pic(i)$ is injective and its cokernel is torsion free for $dim Z = 2$.
(V.1.2.5) $\rho(X) = \rho(Z)$ for $dim Z \geq 3$.

Note that, although the Picard groups are isomorphic, in general the ample cone of X is properly contained in the ample cone of Z. However, in special cases, something can be said; the following proposition generalizes a result of J. A. Wiśniewski on divisors.

Proposition V.1.3 *Let X be a Fano manifold of dimension n, $\mathcal{E}$ an ample vector bundle of rank r on X and Z the zero locus of a section of $\mathcal{E}$, smooth and of the expected dimension. If $Pic(X) \cong Pic(Z)$ and X has no elementary extremal contractions with all fiber of dimension $\leq r$ then a line bundle on X is ample (nef) if and only if its restriction to Z is ample (nef). The assumption is satisfied for instance if all the extremal rays of X have length $l(R) \geq r+2$ or $l(R) \geq r+1$ if R is not nef.*

Proof. Observe that, since X is Fano, a line bundle $\mathcal{L}$ on X is ample (nef) if and only if it has positive (nonnegative) intersection with any extremal ray of X. So take a line bundle $\mathcal{L}_Z$ ample on Z; if we prove that every extremal ray of X contains the class of a curve lying on Z we can conclude that $\mathcal{L}$ is ample (nef) on X.

By our assumption for every extremal ray of X its associated contraction has a fiber F of dimension $\geq r+1$; thus

$$\text{dim} F + \dim Z \geq n+1$$

and therefore, by proposition (I.5.4) the intersection of Z and F contains a curve, which belongs to the ray R. Using (I.2.8) one shows immediately that the assumption on the length implies the lower bound on the fiber. □

The same idea allows us to prove the following

Theorem V.1.4 *(Lifting of contractions) Let X, $\mathcal{E}$ and Z be as in (V.1.1) and assume that Z is not minimal and that $dimZ \geq 2$. Let F_Z be an extremal face of Z and $D_Z = m(K_Z + \tau H_Z)$ a good supporting divisor of F_Z, where m is a positive integer. Assume that there exists an ample line bundle H on X which is the extension of H_Z. Then $D = K_X + det\mathcal{E} + \tau H$ is nef, but not ample; thus it defines an extremal face F_X of X. Moreover, if $\tau \geq 2$ and $dimZ \geq 3$, under the identification of $N_1(X)$ with $N_1(Z)$ we have $F_X = F_Z$ and the contraction of every ray spanning F_Z lifts.*

Proof. Suppose by contradiction that D is not nef. There exists a curve on X on which D is negative; therefore there exists an extremal ray $R = \mathbb{R}[C]$ on X such that $D.C < 0$ and $l(R) > r + \tau$.
Let F be a component of a non trivial fiber of the contraction of R, $\rho : X \to W$; if $dimF \geq (r+1)$ we would have that $dimF + dimZ \geq n+1$ so, in view of proposition (I.5.4), a curve of the ray R lays on Z and this is absurd, since $D_{|Z}$ is nef. On the other hand if $dimF \leq r$, using the inequality (I.2.8), we have that ρ is of fiber type and that $dimF = r$; in particular ρ is equidimensional. Since $D.C < 0$ it follows also that $-K_X.C = r+1$, that $det\mathcal{E}.C = r$ and that the general fiber is $\mathbb{P}^r$; in particular $-(K_X + det\mathcal{E}).C = 1$. By (I.3.7) X is a $\mathbb{P}^r$-bundle over W; moreover $dim(Z \cap F) = 0$, since otherwise we will have a contradiction as in the previous paragraph. Thus, since $\mathcal{E}_{|F} = \oplus^r \mathcal{O}_{\mathbb{P}^s}(1)$, Z is isomorphic to W and this is a contradiction with the Lefschetz theorem (V.1.2).
To prove the last claim, observe that every extremal ray R in the face F_X has length $l(R) \geq \tau + r$, so the general non trivial fiber of the contraction of R has dimension $\dim F \geq \tau + r - 1 \geq r+1$, so, in view of proposition (I.5.4) a curve of the ray R lays on Z. □

Proposition V.1.5 *The hypothesis on the ampleness of H is not necessary if $dimZ \geq 2$ and $Pic(Z) \cong \mathbb{Z}$ or, more generally, if $Pic(i) : Pic(X) \to Pic(Z)$ is an isomorphism and $\overline{NE(Z)} = \overline{NE(Z)}_{K_Z \geq 0} + R$, i.e. if Z has only one extremal ray. Example (V.2.10) shows that the hypothesis is necessary if Z has at least two extremal rays.*

Proof. There exists a line bundle L which is ample on X; the restriction of this line bundle to Z, L_Z is ample on Z, so, if K_Z is not nef there exist a rational number $\sigma > 0$ such that $K_Z + \sigma L_Z$ is nef but not ample and it defines an extremal face G_Z (I.3.2). But we are supposing that on Z there is only one extremal ray, thus $F_Z = G_Z$.

Remark V.1.6 Note that the proof of (V.1.5) actually shows there is always an extremal contraction on Z which can be lifted to X.

Lemma V.1.7 *If $\varphi : Z \to W$ is a $\mathbb{P}$-bundle contraction on a smooth minimal variety W then Z has only one extremal ray.*

Proof. Suppose that Z has another extremal ray, R_1; there exists a rational curve C_0 such that $-K_Z.C_0 > 0$ and $\varphi(C_0)$ is not a point. Let $C = \varphi(C_0)$, let $\nu : \mathbb{P}^1 \to C$ be the normalization of C and consider the fiber product

$$\begin{array}{ccc} Z \times_W \mathbb{P}^1 & \xrightarrow{\bar\nu} & Z \\ \downarrow{\scriptstyle \bar\varphi} & & \downarrow{\scriptstyle \varphi} \\ \mathbb{P}^1 & \xrightarrow{\nu} & W \end{array} \qquad \text{(V.1.8)}$$

$\bar\varphi : Z_C := Z \times_W \mathbb{P}^1 \to \mathbb{P}^1$ is a $\mathbb{P}$-bundle on $\mathbb{P}^1$ and so $\rho(Z_C) = 2$; the morphism $\bar\nu$ induces a map of spaces of cycles $N_1(Z_C) \to N_1(Z)$ which is an embedding. The Mori cone $NE(Z_C)$ is contained in the intersection $N_1(Z_C) \cap NE(Z)$ and so, since $N_1(Z_C)$ is a plane in $N_1(Z)$ and passes through two different extremal rays of Z, $NE(Z_C)$ is contained in the negative part of $NE(Z)$.
By [KMM92, Corollary 2.8] $-K_{Z_C/\mathbb{P}^1}$ is not ample, so there exist an horizontal curve C_1 on Z_C such that $-K_{Z_C/\mathbb{P}^1}.C_1 \leq 0$; noting that

$$-K_{Z_C/\mathbb{P}^1} = \bar\nu^*\varphi^*K_W - \bar\nu^*K_Z$$

we get

$$\bar\nu^*\varphi^*K_W.C_1 = -K_{Z_C/\mathbb{P}^1}.C_1 + \bar\nu^*K_Z.C_1 \leq K_Z.\bar\nu(C_1) < 0$$

and therefore $K_W.\varphi(\bar\nu(C_1)) < 0$, which contradicts the minimality of W.

Remark V.1.9 We found the idea of the proof of (V.1.7) in [SW90a] and [KMM92].

For the rest of this section we will be in the hypothesis of theorem (V.1.4) and we will denote by $\varphi : Z \to W$ the contraction of the face F_Z and by $\phi : X \to Y$ the contraction of F_X. Let also $m = dimW$.
By the adjunction formula $-K_Z = -(K_X + \det\mathcal{E})_Z$, so $-K_Z$ is ϕ-ample.

On Z we have thus two contractions, φ and ϕ_Z. Now we are going to investigate the relation between them. Clearly we have a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{\;i\;} & Z \\ \phi\downarrow & \swarrow_{\phi_Z} & \downarrow\varphi \\ Y & \xleftarrow{\;\pi\;} & W \end{array} \qquad \text{(V.1.10)}$$

Lemma V.1.11 $\phi_Z(Z) \supseteq \phi(E(\phi))$.

Proof. We reason as in the proof of (V.1.4): since ϕ is the contraction of a ray of length $l(R) \geq r + \tau$, a non trivial fiber of ϕ has dimension $\geq \tau + r - 1$ and thus it has nonempty intersection with Z.

Proposition V.1.12 *If the contraction φ is of fiber type then also ϕ is of fiber type.*

Proof. If φ is of fiber type the commutativity of the diagram (V.1.10) implies that also ϕ_Z is of fiber type, so Z is contained in the exceptional locus of ϕ, $E(\phi)$, and by lemma (V.1.11) $\phi_Z(Z) = \phi(E(\phi))$.
Suppose that ϕ is birational; in this case $E(\phi) \subsetneq X$ and

$$\dim\, \phi(E(\phi)) = \dim\, \phi_Z(Z) < \dim\, Z = n - r.$$

Y has dimension n, so it is possible to find a subvariety $Y' \subset Y$ of dimension r which has empty intersection with $\phi((E(\phi))$; away from $E(\phi)$, ϕ is an isomorphism, so $X' = \phi^{-1}(Y') \subset X$ is a subvariety of X of dimension r which has empty intersection with $E(\phi)$ and therefore with Z, but this is absurd by proposition (I.5.4). $\square$

Proposition V.1.13 *If φ is of fiber type or if $\tau \geq 2$, ϕ_Z has connected fibers. Moreover ϕ_Z factors as $\phi_Z = \sigma \circ \varphi$ where $\sigma : W \to \phi_Z(Z)$ is the normalization morphism. In particular, if φ is of fiber type then $\phi_Z = \varphi$.*

Proof. The fibers of ϕ_Z are of the form $Z \cap F$ with F fiber of ϕ. If φ is of fiber type, then the same is for ϕ (see V.1.12) whose fibers have thus dimension $\geq dimX - dimY = n - m$; so dim $Z \cap F \geq n - r - m \geq 1$. If $\tau \geq 2$, reasoning as in the proof of theorem (V.1.4) we again have dim $Z \cap F \geq 1$. So theorem (V.1.2.1) applies to F and $\mathcal{E}_F$ and gives $H^0(Z \cap F, \mathbb{Z}) \cong H^0(F, \mathbb{Z}) \cong \mathbb{Z}$. Using the Universal Coefficient Theorem we get $H_0(Z \cap F) \cong \mathbb{Z}$.
Let $\sigma : W' \to \phi_Z(Z)$ be the normalization of $\phi_Z(Z)$; by the universal property ϕ_Z factors through $\widetilde{\phi_Z} : Z \to W'$ and σ; note also that, since ϕ_Z has connected fibers, the same is true for $\widetilde{\phi_Z}$.
Let $C \subset Z$ be any curve contracted by ϕ_Z (and hence by $\widetilde{\phi_Z}$); thus $(K_X + det\mathcal{E} + \tau H)_Z.C = 0$ which is equivalent to $(K_Z + \tau H_Z).C = 0$, i.e. C is contracted by φ.

By the commutativity of the diagram every curve contracted by φ is contracted by ϕ_Z (and hence by $\widetilde{\phi_Z}$). Therefore φ and $\widetilde{\phi_Z}$ are two Fano Mori contractions which contract the same extremal face, so they are the same morphism.
To prove the last claim recall that, by lemma (V.1.11) and proposition (V.1.12) if φ is of fiber type then $\phi_Z(Z) = Y$ and hence $\phi_Z(Z)$ is normal.

Proposition V.1.14 *If the contraction φ is birational and $\tau \geq 2$, then also ϕ is birational.*

Proof. Suppose ϕ is of fiber type; reasoning again as in the proof of (V.1.4) we can prove that dim $Z \cap F \geq 1$ for the generic fiber of ϕ, so $\phi_Z = \varphi$ is of fiber type, a contradiction in view of proposition (V.1.13).

Remark V.1.15 If φ is birational and $\tau = 1$ then ϕ can be of fiber type (see case 3. of proposition (V.2.12).

V.2 Scrolls and $\mathbb{P}^d$-bundles

Theorem V.2.1 *Let X, $\mathcal{E}$ and Z be as in (V.1.1) with $dimZ \geq 2$. We assume that Z has a scroll contraction $\varphi : Z \to W$ with respect to an ample line bundle on Z, H_Z, which is the restriction of an ample line bundle H on X.*
Then X has a Fano-Mori contraction $\phi : X \to W$ which is of fiber type and with supporting divisor $D = K_X + det\mathcal{E} + (n - m - r + 1)H$. The general fiber of ϕ is isomorphic to $\mathbb{P}^{n-m}$ and $\mathcal{E}$ restricted to it is $\oplus^r \mathcal{O}_{\mathbb{P}}(1)$.
If φ is elementary or $dimX = n \geq 2m - 1 = 2dimW - 1$ (this is always the case if $dimW \leq 3$) then ϕ is elementary and it is a scroll contraction (i.e. it is supported by the divisor $K_X + (n - m + 1)H$) and moreover in the second case even φ had to be elementary.

Proof. The morphism φ is a contraction supported by $K_Z + (n-r-m+1)H_Z$, so, applying theorem (V.1.4), we get a contraction $\phi : X \to Y$, defined by an high multiple of $D = K_X + det\mathcal{E} + (n-m-r+1)H$; this contraction is of fiber type and $Y = W$ by proposition (V.1.13). Let F be a general fiber of ϕ; then F is a smooth Fano manifold of dimension $n - m$ such that $-K_F = (det\mathcal{E} + (n-m-r+1)H)_F$. Thus $F = \mathbb{P}^{n-m}$ and $\mathcal{E}$ restricted to it is $\oplus^r \mathcal{O}_{\mathbb{P}}(1)(see$[Pet90,]). Moreover, for any line in a general fiber $(det\mathcal{E} - rH).l = 0$.
Assume now that dim $X \geq 2$ dim $W - 1$; note that dim $X \geq$ dim $Z + 1 \geq$ dim $W + 2$, so the inequality holds for $dimW \leq 3$.
By the proposition (II.2.2) the contraction $\phi : X \to W$ is an elementary contraction and so

$$det\mathcal{E} = rH + \phi^* B$$

that is ϕ is supported by $K_X + (n - m + 1)H$; note that also $\varphi : Z \to W$ had to be elementary, by the last claim of theorem (V.1.4) if dim $Z \geq 3$ and by (II.2.2) if dim $Z = 2$. □

Corollary V.2.2 *Assume now that $Z = \mathbb{P}(\mathcal{F})$ for some vector bundle $\mathcal{F}$ on W, and its tautological bundle is the restriction of an ample line bundle on X; then $X = \mathbb{P}(\mathcal{G})$ for some vector bundle $\mathcal{G}$ on W which admits $\mathcal{F}$ as a quotient; in this case $\mathcal{E} = \xi_{\mathcal{G}} \otimes \phi^*\mathcal{I}^\vee$ where $\mathcal{I}$ fits into the exact sequence*

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow 0$$

and $\phi : X \to W$ is the $\mathbb{P}$-bundle contraction.

Proof. The theorem gives us a contraction $\phi : X \to W$; we claim that ϕ is equidimensional; in fact if it has any fiber of dimension $> n - m$ then, by proposition (I.5.4), even $Z \to W$ should have a fiber of dimension $> (n-m-r)$. Since $\varphi : Z \to W$ is elementary ϕ is a scroll with the respect to H. The first part of the corollary is proven by (I.3.7). The second part is a well known fact about vector bundles (see [Ful84, B.5.6.]).

Example V.2.3 Let $\mathcal{F}$ be an ample vector bundle on a smooth curve C of genus $g > 0$. If $\mathcal{F}$ is decomposable into a sum of r bundles $\mathcal{F}_i$, by [Fuj80, Corollary 4.20] each $\mathcal{F}_i$ fits into an exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{G}_i \longrightarrow \mathcal{F}_i \longrightarrow 0$$

with $\mathcal{G}_i$ ample; so we can construct an exact sequence

$$0 \longrightarrow \oplus^r \mathcal{O}_C \longrightarrow \mathcal{G} = \bigoplus_{i=1}^{r} \mathcal{G}_i \longrightarrow \mathcal{F} = \bigoplus_{i=1}^{r} \mathcal{F}_i \longrightarrow 0.$$

On $X = \mathbb{P}(\mathcal{G})$ the vector bundle $\mathcal{E} = \xi_{\mathcal{G}} \otimes p^*(\oplus^r \mathcal{O}_C) = \oplus^r \xi_{\mathcal{G}}$ is ample and has a section vanishing on $\mathbb{P}(\mathcal{F})$.

Corollary V.2.4 *Let X, $\mathcal{E}$ and Z be as in (V.1.1) with dim$Z \geq 2$. We assume that Z is a $\mathbb{P}$-bundle over a smooth variety W and also that W is minimal. Then X is a $\mathbb{P}$-bundle over W and $\mathcal{E}_{|F} = \oplus^r \mathcal{O}_{\mathbb{P}}(1)$ for every fiber F of $\phi : X \to W$.*

Proof. The assumption on the tautological bundle is not necessary in this case as noted in (V.1.7).

Remark V.2.5 In case $r = 1$ the last corollary shows that [BS92, Conjecture 5.5.1] is true if $b \geq 3$, X is smooth and B is minimal.

Corollary V.2.6 *Let X, $\mathcal{E}$ and Z be as in (V.1.1) with dim $Z \geq 2$. We assume that Z has a scroll contraction $\varphi : Z \to W$ with dim $W \leq 1$ (or equivalently that Z is a $\mathbb{P}$-bundle over a smooth variety W of dimension ≤ 1). Then X is a $\mathbb{P}$-bundle over W and $\mathcal{E}_{|F} = \oplus^r \mathcal{O}_{\mathbb{P}}(1)$ for every fiber F of $\phi : X \to W$ except possibly for $W = \mathbb{P}^1$ and $Z = \mathbb{P}(\oplus^{(n-r)} \mathcal{O}_{\mathbb{P}^1}) = \mathbb{P}^1 \times \mathbb{P}^{(n-r-1)}$ or $Z = \mathbb{P}(\oplus^{(n-r-1)} \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$.*

Proof. The corollary will follow if we prove that under the assumptions Z has only one extremal ray.
If $\dim W = 0$ then $Z = \mathbb{P}^{(n-r)}$ and thus Z has only one extremal ray. If $\dim W = 1$ then $\rho(Z) = 2$, thus Z has one extremal ray or it is Fano. But if Z is a Fano manifold then $0 = h^1(\mathcal{O}_Z) = g(W)$, thus $W = \mathbb{P}^1$. Therefore we can assume that $Z = \mathbb{P}(\mathcal{E})$ for a vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ with $rank(\mathcal{E}) = s = n - r$ and $0 \leq c_1(\mathcal{E}) \leq s - 1$. But since $-K_Z = s\xi + (2 - c_1(\mathcal{E}))H$, with ξ the tautological bundle and H the pull back of a point in $\mathbb{P}^1$, if $c_1(\mathcal{E}) \geq 2$ then ξ and thus $\mathcal{E}$ would be ample. This is in contradiction with $c_1(\mathcal{E}) \leq s-1$. Thus $0 \leq c_1(\mathcal{E}) \leq 1$ which gives our claim.

Remark V.2.7 Even if W is not minimal the $\mathbb{P}$-bundle contraction can be sometime lifted to X. For instance if Z is not Fano and $Pic(W) = \mathbb{Z}$ the same proof of the above corollary gives the lifting. In particular if $W = \mathbb{P}^m$ and $\mathcal{F}$ is a vector bundle with $0 \leq c_1\mathcal{F} \leq (m-1)$ which is not spanned by global section then by [SW90b], Z is not Fano and the $\mathbb{P}$-bundle contraction $Z = \mathbb{P}(\mathcal{F}) \to W$ can be lifted.

Remark V.2.8 The above theorem and corollaries extend the results of Lanteri and Maeda; their papers inspired and motivated ours. Summarizing we have a precise description of X when Z is a scroll or a $\mathbb{P}$-bundle satisfying assumptions (1.1) in the introduction (Theorem (V.2.1) and corollary (V.2.2)). We have a good result also if we drop the assumption but W is minimal. If W is not minimal the situation is much more complicated; cases that occur if $\dim W \leq 1$ are described in the rest of this section; other simple cases are described in remark (V.2.7).

Proposition V.2.9 *Suppose that* $Z = \mathbb{P}(\oplus^{(n-r)}\mathcal{O}_{\mathbb{P}^1}) = \mathbb{P}^1 \times \mathbb{P}^{n-r-1}$ *is the zero set of a section of an ample vector bundle* $\mathcal{E}$ *of rank* r *on a smooth* X *of dimension* n*, then*

1. X *is a* $\mathbb{P}^{n-1}$*-bundle over* $\mathbb{P}^1$ *and* $\mathcal{E}_F = \oplus^r\mathcal{O}_F(1)$ *for any fiber.*
2. X *is a* $\mathbb{P}^{r+1}$*-bundle over* $\mathbb{P}^{n-r-1}$ *and* $\mathcal{E}_F = \oplus^r\mathcal{O}_F(1)$*, for any fiber.*
3. $(X, \mathcal{E})$ *is* $(\mathbb{P}^n, \oplus^{n-3}\mathcal{O}_{\mathbb{P}}(1) \oplus \mathcal{O}_{\mathbb{P}}(2))$ *or* $(\mathbb{Q}^n, \oplus^{n-2}\mathcal{O}_{\mathbb{Q}}(1))$ *and* Z *is a smooth quadric surface.*

Proof. Let $Z = \mathbb{P}^1 \times \mathbb{P}^{(n-r-1)}$ and p_1, p_2 the projections on the two factors; $Pic(Z) \simeq \mathbb{Z} < p_1^*\mathcal{O}_{\mathbb{P}^1}(1) > \oplus\, \mathbb{Z} < p_2^*\mathcal{O}_{\mathbb{P}^{n-r-1}}(1) >=: \mathbb{Z} < L_1 > \oplus\, \mathbb{Z} < L_2 >$; p_1 and p_2 are two Fano-Mori contractions with supporting divisors $(aL_1, 0)$ and $(0, bL_2)$ respectively $(a, b > 0)$; on Z there is a third Fano-Mori contraction, p, which is the contraction of Z to a point.
Suppose that Z is the zero locus of a section of an ample vector bundle $\mathcal{E}$ on X; by remark (V.1.6) at least one of the extremal contractions of Z lifts to X.

Suppose that p_1 lifts; as in the proofs of theorem (V.2.1) and corollary (V.2.2) we obtain that X is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ and $\mathcal{E}_F = \oplus^r \mathcal{O}_F(1)$.

Suppose now that p_2 lifts; as above we get that X is a $\mathbb{P}^{r+1}$-bundle over $\mathbb{P}^{n-r-1}$ and $\mathcal{E}_F = \oplus^r \mathcal{O}_F(1)$.

Finally, suppose that p lifts. We start with the case in which $Pic(i) : Pic(X) \to Pic(Z)$ is an isomorphism and $\rho(X) = \rho(Z)$. We want to show that in this case both p_1 and p_2 would lift, but this is not possible, in view of theorem (I.2.9).
X is a Fano variety and $-K_X = det\mathcal{E} + H$; this implies that the two extremal rays R_1, R_2 of X have length $\geq r+1$; If $R_i \quad (i = 1, 2)$ is not nef, then the fibers of the associated contraction have dimension $\geq r+1$, and so, by proposition (I.5.4), there exists a curve in the ray belonging to Z. If $R_i \quad (i = 1, 2)$ is nef, then the fibers of the associated contraction P_i have dimension $\geq r$ by (I.2.8); if a fiber has dimension $\geq r+1$ then its intersection with Z contains at least a curve and we are done, so we can suppose that P_i is equidimensional and all the fibers have dimension r. Recalling that, for the general fiber

$$K_F = (K_X)_F = -(det\mathcal{E})_F - H_F$$

we have that $(F, H_F) \simeq (\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r})(1))$, so, by theorem (I.3.7), X is a $\mathbb{P}^r$-bundle on a normal variety W. This fact and the previous expression of K_F yield $(F, \mathcal{E}_F) \simeq (\mathbb{P}^r, \oplus^r \mathcal{O}_{\mathbb{P}^r}(1))$ for every fiber; this implies that $Z \cap F$ is a point or contains a curve; in the second case we are done, while the first is impossible, since Z would be isomorphic to W, which has different Picard number.

So suppose that $Pic(i) : Pic(X) \to Pic(Z)$ is not an isomorphism or $\rho(X) \neq \rho(Z)$; by (V.1.2) this is possible only for $\dim Z = 2$ and in this case we have $Pic(X) \simeq \mathbb{Z}$. By the proof of theorem (V.1.4) $K_X + det\mathcal{E} + 2H = \mathcal{O}_X$ for some ample line bundle H such that $2H_Z = -K_Z$. Note that there are curves C on Z such that $H.C = 1$, so H is the ample generator of $Pic(X)$; write $det\mathcal{E} = sH$; since the index of a Fano manifold is at most $n+1$ we must have $s = r, r+1$. So $(X, \mathcal{E})$ is either $(\mathbb{P}^n, \oplus^{n-3}\mathcal{O}_{\mathbb{P}}(1) \oplus \mathcal{O}_{\mathbb{P}}(2))$ or $(\mathbb{Q}^n, \oplus^{n-2}\mathcal{O}_{\mathbb{Q}}(1))$ and Z is a smooth quadric surface. □

Example V.2.10 The effectiveness of case 3. is clear; to see the effectiveness of case 1. consider the sequence

$$0 \longrightarrow \oplus^n \mathcal{O}_{\mathbb{P}^1} \longrightarrow \oplus^n (\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(s-a)) \longrightarrow \oplus^n \mathcal{O}_{\mathbb{P}^1}(s) \longrightarrow 0$$

which is exact in view of [Bǎd81, Remark 1, p.170] and choose a, s in such a way that $0 < a - s < a$; the construction in [Ful84, B.5.6] applies and gives $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ as a section of the ample vector bundle $\mathcal{E} = \oplus^n \xi_{\mathcal{G}}$ on $X = \mathbb{P}(\mathcal{G})$. By the discussion above, it is clear that in this example, the contraction p_2 cannot be lifted; p_2 is supported by $K_Z + H_Z = bL_2 \quad (b > 0)$; recalling that $K_Z = -2L_1 - (n-r)L_2$ we have that $H_Z = 2L_1 + (n-r+b)L_2$ is an ample line bundle on Z which cannot be the restriction on an ample line bundle on X.

Remark V.2.11 The effectiveness of case 2. looks uncertain; we note that for $r = 1$, i.e. in the case of ample divisors this is not possible by a result of Sommese [BS92, Theorem 5.2.1].

Proposition V.2.12 *Suppose that $Z = \mathbb{P}(\oplus^{(n-r-1)}\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ is the zero set of a section of an ample vector bundle $\mathcal{E}$ of rank r on a smooth X of dimension n, then*

1. *X is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ and $\mathcal{E}_F = \oplus^{n-1}\mathcal{O}_F(1)$ for any fiber.*
2. *X is a scroll over $\mathbb{P}^{n-r}$ and $\mathcal{E}_F = \oplus^r\mathcal{O}_F(1)$ for the general fiber.*
3. *X is both as in 1 and in 2 and it is $\mathbb{P}^1 \times \mathbb{P}^{n-1}$, $r = 1$ and $\mathcal{E} = \mathcal{O}(1,1)$.*

Proof. Let $Z = \mathbb{P}(\oplus^{(n-r-1)}\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) =$ blow-up of $\mathbb{P}^{(n-r)}$ along a linear subspace of codimension 2. The Mori cone of Z is two dimensional, and it is spanned by two extremal rays, R_1 and R_2; let $p_i \quad i = 1, 2$ be the extremal contraction associated to R_i; p_1 is the $\mathbb{P}^{n-r-1}$-bundle map on $\mathbb{P}^1$, while p_2 is the blow down to $\mathbb{P}^{n-r}$; moreover, let p be the contraction of Z to a point, associated to the extremal face generated by R_1 and R_2. By (V.1.6), there is an extremal contraction that lifts to X.

Suppose that p_1 lifts; as in the proof of (V.2.1) and corollary (V.2.2) we obtain that X is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ and $\mathcal{E}_F = \oplus^r\mathcal{O}_{\mathbb{P}^{n-1}}(1)$.

Suppose now that p_2 lifts to an extremal contraction P_2 on X.

Lemma V.2.13 *The contraction P_2 is of fiber type.*

Proof. Assume by contradiction that P_2 is birational.
Claim. P_2 is divisorial and it is the blow up of a smooth X' along a subvariety of codimension $r + 2$.
The contraction P_2 is supported by a divisor of the form $K_X + det\mathcal{E} + H$, and this implies dim $F \geq l(R) \geq r + 1$, so, reasoning as in the proof of (V.1.13), we can prove that P_{2Z} has connected fibers. The fact that P_{2Z} factors through p_2 and the normalization of $P_{2Z}(Z)$ yields that all the nontrivial fibers of P_{2Z} have dimension 1; this implies that all the nontrivial fibers of P_2 have dimension at most $r + 1$ by proposition (I.5.4); so we can conclude that the contraction P_2 is equidimensional and all the nontrivial fibers have dimension $r + 1$; by (I.2.8) we have

$$dimE(P_2) \geq l(R) - dimF + n - 1 \geq n - 1$$

so P_2 is divisorial and dim $P_2(E(P_2)) = (n-1)-(r+1) = n-r-2$. In view of $(V.1.2)$ the contraction P_2 is elementary, and so we can apply [AW93, Theorem 4.1 and Corollary 4.11], proving the claim.
The Picard group of Z is generated by the tautological line bundle ξ and f, a fiber of the projection on $\mathbb{P}^1$, but also by $E(p_2)$ and $p_2^*\mathcal{O}(1)$; we have $p_2^*\mathcal{O}(1) = \xi$ and $E(p_2) = \xi - f$. Using the isomorphism $Pic(i) : Pic(X) \to Pic(Z)$ and the fact that p_2 and P_2 are elementary contractions, we get that the restriction homomorphism $Pic(j) : Pic(X') \to Pic(\mathbb{P}^{n-r})$ is an isomorphism. So there exists an ample generator of $Pic(X')$, H' whose restriction to $\mathbb{P}^{n-r}$ is $\mathcal{O}_{\mathbb{P}}(1)$. $Pic(X)$ is thus generated by P_2^*H and $E(P_2)$. Write $K_{X'} = kH'$ for some $k \in \mathbb{Z}$. Let l be a line in f; we obtain

$$
\begin{aligned}
-(n-r) = K_Z.l = (K_X + det\mathcal{E})_Z.l = \\
= (P_2^*K_{X'} + (r+1)E(P_2) + det\mathcal{E})_Z.l = \\
= (kP_2^*H' + (r+1)E(P_2) + det\mathcal{E})_Z.l = \\
= ((k+r+1)\xi - (r+1)f + det\mathcal{E}_Z).l = \\
k + r + 1 + det\mathcal{E}_Z.l \geq k + 2r + 1
\end{aligned}
$$

and so

$$k \leq -n - r - 1$$

which is absurd, since the index of a Fano variety is not greater than $n+1$, and the lemma is proven.

On X we thus have a fiber type elementary contraction, supported by an high multiple of $K_X + det\mathcal{E} + H$ on a variety of dimension $n-r$ by (V.1.11). For the general fiber of P_2 we have

$$K_F + det\mathcal{E}_F + H_F = \mathcal{O}_F$$

and so $(F, \mathcal{E}_F) = (\mathbb{P}^r, \oplus^r \mathcal{O}_{\mathbb{P}}(1))$; therefore $Z \cap F$ is a point for the generic fiber, and thus P_{2Z} is generically one-to-one and therefore coincides with p_2 (see (V.1.13)). So the conclusion is that, if p_2 lifts, X is a scroll over $\mathbb{P}^{n-r}$.

As a final case suppose that p lifts; if $Pic(i) : Pic(X) \to Pic(Z)$ is an isomorphism and $\rho(X) = \rho(Z)$; as in the proof of proposition (V.2.9) we can prove that also p_1 and p_2 lift, so X is a Fano variety which has a $\mathbb{P}^{n-1}$-bundle contraction on $\mathbb{P}^1$ and a scroll contraction on $\mathbb{P}^{n-r}$. The only possibility is that $r = 1$ and $X = \mathbb{P}_{\mathbb{P}^1}(\oplus^r \mathcal{O}) = \mathbb{P}^1 \times \mathbb{P}^{n-1}$.
If $Pic(i)$ is not an isomorphism or $\rho(X) \neq \rho(Z)$, by theorem (V.1.2) $Pic(X) \simeq \mathbb{Z}$ and dim$Z = 2$, so $Z = \mathbb{F}_1$ and X is a Fano variety; this case is ruled out in [LM97, Section 2]. □

Example V.2.14 Cases 1. and 3. are effective; examples for the first case can be constructed as in example (V.2.10) starting with the exact sequence

$$0 \longrightarrow \oplus^n \mathcal{O}_{\mathbb{P}^1} \longrightarrow \oplus^{n-1}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(s-a)) \oplus \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(s+1-a) \longrightarrow \dots$$

$$\dots \longrightarrow \oplus^{n-1}\mathcal{O}_{\mathbb{P}^1}(s) \oplus \mathcal{O}_{\mathbb{P}^1}(s+1) \longrightarrow 0$$

while in case 3. easy computations show that a smooth Z in the linear system $|\mathcal{O}(1,1)|$ must be a Fano variety with a $\mathbb{P}$-bundle contraction on $\mathbb{P}^1$ and a birational contraction on $\mathbb{P}^{n-1}$.

Remark V.2.15 The example of case 3. shows that the assumption $\tau \geq 2$ in (V.1.14) is necessary.

Remark V.2.16 As in the case $Z = \mathbb{P}^1 \times \mathbb{P}^{n-r-1}$ the effectiveness of case 2. looks uncertain; we note that for $r = 1$, i.e. in the case of ample divisors this is not possible except for the trivial case $X = \mathbb{P}^{n-1} \times \mathbb{P}^1$, $\mathcal{E} = \mathcal{O}_{X=\mathbb{P}^{n-1}\times\mathbb{P}^1}(1,1)$. This is a very well known fact that descends by a result of Sommese [BS92, Theorem 5.2.1] and classical results of Bădescu, [Băd81], [Băd82b].

V.3 Quadric fibrations and quadric bundles

Theorem V.3.1 *Let X, $\mathcal{E}$ and Z be as in (V.1.1) with $dimZ \geq 3$. We assume that Z has a quadric fibration contraction $\varphi : Z \to W$ with respect to an ample line bundle on Z, H_Z, which is the restriction of an ample line bundle H on X. Then X has a Fano-Mori contraction $\phi : X \to W$ which is of fiber type and with supporting divisor $D = K_X + det\mathcal{E} + (n-m-r)H$ with $n = dim\ X$ and $m = dim\,W$. For the general fiber of ϕ, F we have either $(F, \mathcal{E}_F) \simeq (\mathbb{P}^{n-m}, \oplus^{r-1}\mathcal{O}_{\mathbb{P}}(1) \oplus \mathcal{O}_{\mathbb{P}}(2))$ or $(F, \mathcal{E}_F) \simeq (\mathbb{Q}^{n-m}, \oplus^r \mathcal{O}_{\mathbb{Q}}(1))$.*
If φ is elementary then also ϕ is elementary and it is either a scroll contraction or a quadric fibration contraction (i.e. it is supported by the divisor $K_X + (n-m+1)H$ or by the divisor $K_X + (n-m)H$). The last result holds replacing the assumption on φ with the strongest assumption $dim\ X \geq 2m+1 = 2dimW + 1$ (this is always the case if $dimW \geq 1$).

Proof. The morphism φ is a contraction supported by $K_Z + (n-r-m)H_Z$, so, applying theorem (V.1.4), we get a contraction $\phi : X \to Y$, defined by an high multiple of $D = K_X + det\mathcal{E} + (n-m-r)H$; this contraction is of fiber type and $Y = W$ by proposition (V.1.13). Let F be a general fiber of ϕ; then F is a smooth Fano manifold of dimension $n-m$ such that $-K_F = (det\mathcal{E} + (n-m-r)H)_F$. Thus, by [PSW92, Theorem 0.1] applied to the ample vector bundle $\mathcal{E}_1 = (\mathcal{E} \oplus^{n-m-r} H)_F$ either $F = \mathbb{P}^{n-m}$ and $\mathcal{E}$ restricted to it is $\oplus^{r-1}\mathcal{O}_{\mathbb{P}}(1) \oplus \mathcal{O}_{\mathbb{P}}(2)$ or $F = \mathbb{Q}^{n-m}$ and $\mathcal{E}$ restricted to it is $\oplus^r \mathcal{O}_{\mathbb{Q}}(1)$. Moreover, for any line in a general fiber

$$(det\mathcal{E} - (r+\epsilon)H).l = 0 \qquad \text{(V.3.2)}$$

with $\epsilon = 1, 0$. By theorem (V.1.2) $\rho(X/W) = \rho(Z/W)$ and so, if φ is elementary, also ϕ is so and, by (V.3.2) $det\mathcal{E} = (r+\epsilon)H + \phi^* B$, that is ϕ is supported by $K_X + (n-m+\epsilon)H$. Assume now that dim $X \geq 2$ dim $W+1$; by proposition (II.2.2) the contraction $\phi : X \to W$ is an elementary contraction and so, again by (V.1.2) also $\varphi : Z \to W$ had to be elementary. □

Corollary V.3.3 *Assume now that there exists a vector bundle $\mathcal{F}$ on W and an embedding of Z into $\mathbb{P}(\mathcal{F})$ as a divisor of relative degree 2; assume moreover that $K_Z + (n-r-m)(\xi_{\mathcal{F}})_Z$ is a good supporting divisor of a quadric bundle elementary contraction and $(\xi_{\mathcal{F}})_Z$ is the restriction of an ample line bundle on X. Then either there exists an ample vector bundle $\mathcal{G}$ of rank $n-m+1$ such that $X = \mathbb{P}(\mathcal{G})$ or there exists a vector bundle $\mathcal{G}$ of rank $n-m+2$ and an embedding of X into $\mathbb{P}(\mathcal{G})$ as a divisor of relative degree two.*

Proof. If Z is a quadric bundle then ϕ is equidimensional; in fact if it has any fiber of dimension $> n-m$ then, by proposition (I.5.4), even $Z \to W$ should have a fiber of dimension $> (n-m-r)$. Since $\varphi : Z \to W$ is elementary ϕ is a scroll or a quadric fibration with the respect to H. We conclude by (I.3.7) and (I.3.9).

V.4 Del Pezzo fibrations

Theorem V.4.1 *Let X, $\mathcal{E}$ and Z be as in (V.1.1) with $dimZ \geq 3$. We assume that Z has a del Pezzo fibration contraction $\varphi : Z \to W$ with respect to an ample line bundle on Z, H_Z, which is the restriction of an ample line bundle H on X. Then X has a Fano-Mori contraction $\phi : X \to W$ which is of fiber type and with supporting divisor $D = K_X + det\mathcal{E} + (n-m-r-1)H$, with $n = dim\ \ X$ and $m = dim\ W$. If $n - m \geq 5$, for the general fiber of ϕ, F, we have either $(F, \mathcal{E}_F) \simeq (\mathbb{P}^{n-m}, \oplus^{r-1}\mathcal{O}_{\mathbb{P}}(1) \oplus \mathcal{O}_{\mathbb{P}}(3))$, $(F, \mathcal{E}_F) \simeq (\mathbb{P}^{n-m}, \oplus^{r-2}\mathcal{O}_{\mathbb{P}}(1) \oplus^2 \mathcal{O}_{\mathbb{P}}(2))$, or $(F, \mathcal{E}_F) \simeq (\mathbb{Q}^{n-m}, \oplus^{r-1}\mathcal{O}_{\mathbb{Q}}(1) \oplus \mathcal{O}_{\mathbb{Q}}(2))$ or F is a del Pezzo manifold with $b_2 = 1$ and $\mathcal{E}_F \simeq \oplus^r \mathcal{O}_F(1)$, where $\mathcal{O}_F(1)$ is the ample generator of $Pic(F)$.*
If φ is elementary then ϕ is elementary and it is either a scroll contraction or a quadric fibration contraction or a del Pezzo fibration contraction (i.e. it is supported by the divisor $K_X + (n-m+\epsilon)H$ with $\epsilon = 1, 0$ or -1). The last result holds replacing the assumption on φ with the strongest assumption $dimX = n \geq 2m + 3 = 2dimW + 3$.

Proof The morphism φ is a contraction supported by $K_Z + (n-r-m-1)H_Z$, so, applying theorem (V.1.4), we get a contraction $\phi : X \to Y$, defined by an high multiple of $D = K_X + det\mathcal{E} + (n-m-r-1)H$; this contraction is of fiber type and $Y = W$ by proposition (V.1.13). Let F be a general fiber of ϕ; then F is a smooth Fano manifold of dimension $n-m$ such that $-K_F = (det\mathcal{E} + (n-m-r)H)_F$. Thus, applying [PSW92, Main theorem] to the ample vector bundle $\mathcal{E}_1 = (\mathcal{E} \oplus^{n-m-r-1} H)_F$ we get the description of F and $\mathcal{E}_F$. Moreover, for any line in a general fiber

$$(det\mathcal{E} - (r + \epsilon)H).l = 0 \tag{V.4.2}$$

with $\epsilon = 1, 0$ or -1.
By theorem (V.1.2) $\rho(X/W) = \rho(Z/W)$ and so, if φ is elementary, also ϕ is so, by (V.4.2) $det\mathcal{E} = (r+\epsilon)H+\phi^*B$, that is ϕ is supported either by $K_X+(n-m+\epsilon)H$. Assume now that dim $X \geq 2$ dim $W + 3$; by proposition (II.2.2) the contraction $\phi : X \to W$ is an elementary contraction and so, again by (V.1.2) also $\varphi : Z \to W$ had to be elementary. □

Remark V.4.3 If $n - m = 4$ the rank of $\mathcal{E}$ can be 1 or 2 and, according to [PSW92, proposition 7.4] the possibilities for the general fiber are those listed in the theorem plus

1. $(F, \mathcal{E}_F) \simeq (\mathbb{P}^2 \times \mathbb{P}^2, \oplus^2\mathcal{O}(1, 1))$.
2. $(F, \mathcal{E}_F) \simeq (\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(1, 1))$.
3. $(F, \mathcal{E}_F) \simeq (\mathbb{Q}^4, S(2))$.
4. F is a Fano 4-fold with $b_2 = 1$ and index 1.

Remark V.4.4 If $n-m = 3$ the rank of $\mathcal{E}$ must be 1 and, according to [PSW92, Theorem 0.4] the possibilities for the general fiber are those listed in the theorem plus

1. $(F, \mathcal{E}_F) \simeq (\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}(2, 1))$.
2. $(F, \mathcal{E}_F) \simeq (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 1, 1))$.
3. F is a del Pezzo manifold with $b_2 \geq 2$ and $\mathcal{E} = \mathcal{O}_F(1)$.

Corollary V.4.5 *Let X, $\mathcal{E}$ and Z be as in (V.1.1) with dim $Z \geq 3$ and let (Z, H_Z) be a del Pezzo manifold with $b_2 = 1$. then one of the following occurs*

1. *$X \simeq \mathbb{P}^n$ and $\mathcal{E}$ is either $\oplus^2\mathcal{O}_{\mathbb{P}^n}(2) \oplus^{r-2} \mathcal{O}_{\mathbb{P}^n}(1)$ or $\mathcal{O}_{\mathbb{P}^n}(3) \oplus^{r-1} \mathcal{O}_{\mathbb{P}^n}(1)$.*
2. *$X \simeq \mathbb{Q}^n$ and $\mathcal{E}$ is $\mathcal{O}_{\mathbb{Q}^n}(2) \oplus^{r-1} \mathcal{O}_{\mathbb{Q}^n}(1)$.*
3. *X is a del Pezzo manifold with $b_2 = 1$ and $\mathcal{E} \simeq \oplus^r\mathcal{O}_X(1)$ where $\mathcal{O}_X(1)$ is the ample generator of $Pic(X)$.*
4. *$X \simeq \mathbb{P}^n$, $\mathcal{E}$ is $\oplus^r\mathcal{O}_{\mathbb{P}^n}(1)$ and $(Z, H_Z) \simeq (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$.*

Proof. The hypothesis on H is not necessary in this case as noted in (V.1.5) and the cases in (V.4.3) and (V.4.4) are ruled out, because of the isomorphism $Pic(Z) \simeq Pic(X) \simeq \mathbb{Z}$ so, if H_Z is the generator of $Pic(Z)$ we are in the first three cases. The only case in which H_Z is not the generator of $Pic(Z)$ is when $(Z, H_Z) \simeq (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$, and the description of X follows from (V.2.4).

V.5 Some final considerations

Using the same arguments we can consider the case in which Z has an extremal contraction on W whose general fiber F is a Fano variety of index $\leq \dim F - 2$. However, in these cases it is very difficult to provide a good description of the vector bundle $\mathcal{E}$ and to construct non trivial examples. These difficulties already show up in the case in which W is a point and $Pic(Z) \simeq Z$; we recall that a line on Z in this case is a rational curve which is a line with respect to a the generator of $Pic(Z)$. The existence of a line on Z is proved if the index of Z is $\geq (n-2)$, by recent results of M. Mella. A line exists also if 2 index $(Z) > \dim\ Z + 1$. The following proposition summarize the simplest cases.

Proposition V.5.1 *Let $X, \mathcal{E}$ and Z be as in (V.1.1); we assume that Z is a Fano variety of dimension ≥ 2 with $Pic(Z) \simeq \mathbb{Z}$ and that Z has a line. Then X is a Fano variety with $Pic(X) \simeq \mathbb{Z}$ and coindex$(Z) \geq$ coindex (X).*

Proof. Let H be a generator of $Pic(X) = Pic(Z)$ and let s and τ be positive integers such that $det\mathcal{E} = sH$ and $-K_Z = \tau H_Z$. Therefore, by adjunction formula and (V.1.13), we have that

$$K_X + det\mathcal{E} + \tau H = K_X + (s+\tau)H = \mathcal{O}_X,$$

thus X is a Fano manifold. If C is a line of Z then $det\mathcal{E} \cdot C = sH \cdot C = s$; thus, by (I.5.2), $s \geq r$. In particular this gives $n + 1 \geq \text{index}(X) = s + \tau \geq r + \tau$.

Appendix A

Bend and break

The basic idea of "bend-and break" techniques, due to Mori ([Mor79]) is very simple: if a curve deforms on a smooth variety while passing through a fixed point, it must at some point break up, with at least one rational component.
The two main references for the appendix are [Ko96, Chapter II] and [CKM88, Lecture 1].

A.1 Finding rational curves

In this section we recall the main steps of Mori's construction and his fundamental results.

Definition A.1.1 Let S be a proper surface and $B \subset S$ a proper curve. We say that B is *contractible* in S if there is a surface S' and a dominant morphism $g : S \to S'$ such that $g(B)$ is zero dimensional.

Lemma A.1.2 (Bend and break) *[Mor79] let B be a smooth proper and irreducible curve over a field and S an irreducible, proper and normal surface. Let $p : S \to B$ be a morphism. Assume that there is an open subset $B^0 \subset B$, a smooth projective curve C and an isomorphism*

$$f : [p^{-1}(B^0) \xrightarrow{p} B^0] \simeq [C \times B^0 \xrightarrow{\pi} B^0],$$

where π is the second projection.
A section $s : B \to S$ is called flat if $s(B^0) = \{c\} \times B^0$ under the above isomorphism.

1. *If $g(C) \geq 1$ and there is a contractible flat section $s : B \to S$ then, for some $b \in B - B^0$, the fiber $p^{-1}(b)$ contains a rational curve intersecting $s(B)$.*

2. *If $g(C) = 0$ and there are two contractible flat sections $s_1, s_2 : B \to S$ then, for some $b \in B - B^0$ the fiber $p^{-1}(b)$ is either reducible or nonreduced.*

Corollary A.1.3 *Let C be an irreducible, proper and smooth curve and X a proper variety. Let $p_1, \dots, p_k \in C$ be k distinct points and $g : \{p_1, \dots, p_k\} \to X$ a morphism. Assume that there is a smooth irreducible curve B^0 and a morphism*

$$h^0 : C \times B^0 \to X \times B^0 \in Hom(C, X, g)(B^0)$$

such that $h^0(C \times b)$ is one dimensional for some $b \in B^0$.
Let $B \supset B^0$ be a smooth compactification of B^0. There is a unique normal compactification $S \supset C \times B^0$ such that h^0 extends to a finite morphism $h : S \to X \times B$. Let $p : S \to B$ and $p_X : X \times B \to X$ be the natural projections.

1. *If $g(C) \geq 1$ and $k \geq 1$ then, for some $b \in B - B^0$ the 1-cycle $h_*(p^{-1}(b))$ contains a rational curve D which passes through $g(p_1)$.*

2. *If $g(C) = 0$, dim Im $[C \times B^0 \xrightarrow{h^0} X \times B^0 \longrightarrow X] = 2$ and $k \geq 2$ then, for some $b \in B - B^0$, the 1-cycle $h_*(p^{-1}(b))$ is either reducible or nonreduced.*

Corollary A.1.4 *Let X be a projective variety, let $f : C \to X$ be a rational curve. If dim $_{[f]}Hom(C, X, f_{|c}) \geq 1$ then there exists a morphism $f' : C \to X$ and an connected non-zero effective rational 1-cycle Z on X through $f(c)$ such that*

$$f_*(C) \equiv f'_*(C) + Z.$$

Corollary A.1.5 *Let X be a projective variety, let $f : \mathbb{P}^1 \to X$ be a smooth curve, and let c be a point of C. If dim $_{[f]}Hom(C, X, f_{|0,\infty}) \geq 2$ the 1-cycle $f_*\mathbb{P}^1$ is numerically equivalent to a connected non-integral effective rational 1-cycle passing through $f(0)$ and $f(\infty)$.*

It is possible to bound from below the dimension of the spaces of morphisms of curves in terms of the genus of the deformed curve and of its intersection number with the anticanonical bundle:

Theorem A.1.6 *[Ko96, Theorem II.1.2] Let C be a proper algebraic curve without embedded points and $f : C \to X$ a morphism to a smooth variety X of dimension n. Then*

$$dim\ _{[f]}Hom(C, X) \geq -K_X.C + n\chi(\mathcal{O}_C),$$

and equality holds if $H^1(C, f^*TX) = 0$.

An analogous result bounds the dimension of the space of “pointed” morphisms of curves:

Theorem A.1.7 *[Ko96, Theorem II.1.7] Let C be a projective curve without embedded points and X a smooth variety. Let $B \subset C$ be a finite closed subscheme. Assume that C is smooth along B; let $g : B \to Y$ and $f \in$ Hom(C,X,g) be morphisms. Then*

$$dim\ Hom_{[f]}(C, X, g) \geq \chi(C, f^*T_X \otimes I_B).$$

The first result about spaces of morphism can be used to get information about the Hilbert scheme of curves:

Theorem A.1.8 *[Ko96, Theorem II.1.14] Let C a proper algebraic curve without embedded points and $h : C \to X$ a closed immersion to a smooth variety X of dimension n; suppose moreover that C is smoothable. Then*

$$dim\ _{[C]}Hilb(X) \geq -K_X.C + (n-3)\chi(\mathcal{O}_C).$$

Using these results Mori proved the following theorems:

Theorem A.1.9 *Let X be a smooth projective variety such that $-K_X$ is ample. Then, through every point $x \in X$ there is a rational curve D_x such that*

$$-K_X.D_x \leq dim\ X + 1.$$

Theorem A.1.10 *Let X be a smooth projective variety, and let H be an ample divisor on X. Assume that there is a curve $C \subset X$ such that $-K_X.C \geq \varepsilon(C.H)$ for some $\varepsilon > 0$. The there is a rational curve $E \subset X$ such that*

$$dim\ X + 1 \geq -K_X.E \geq \varepsilon(E.H).$$

It is interesting to note that the proof of these theorems, even in characteristic zero, requires to go through positive characteristic. The main point is that, in positive characteristic, a curve of every genus admits endomorphisms of high degree, so that it is possible to increase the dimension of Hom $_{[f]}(C, X, f_{|\{c\}})$ in order to satisfy the assumptions of (A.1.4), as we will see in the next section.

A.2 Reduction to characteristic p

A.2.1 Let C be a curve in a smooth projective manifold X such that $-K_X.C > 0$ and suppose that both C and X are defined by equations with integral coefficients:

$$h_1(X_0, \ldots, X_N), \ldots, h_r(X_0, \ldots, X_N) \quad \text{define} \quad X$$

$$c_1(X_0, \ldots, X_N), \ldots, c_s(X_0, \ldots, X_N) \quad \text{define} \quad C.$$

Let $\mathbb{F}(p)$ be the field with p elements and $\overline{\mathbb{F}(p)}$ its algebraic closure. The equations above define varieties X_p and C_p in the projective space $\mathbb{P}^N_{\overline{\mathbb{F}(p)}}$; these varieties are nonsingular, and dim $C_p = 1$ for almost all p.
Now we consider the *Frobenius morphism* $f_p : C_p \to C_p$, given by the mapping

$$(X_0, \ldots, X_N) \longrightarrow (X_0^p, \ldots, X_N^p).$$

This morphism has degree p; iterating it m times and composing with the inclusion $C_p \to X_p$ we obtain a morphism $f_p^m : C_p \to X_p$ whose "base-pointed" deformation space has dimension bounded below by

$$p^m(-K_{X_p}.C_p) - g(C_p)\text{dim } X.$$

By "generic flatness over Spec $\mathbb{Z}$", $-K_{X_p}.C_p$ and $g(C_p)$ are constant for almost all p; hence it is possible to pick an m such that the above expression is positive for almost all p.

A.2.2 In the general case, in which the coefficients of the h_i (defining X in $\mathbb{P}^N$) and the f_j (defining C in $\mathbb{P}^M$) are not integers, these coefficients generate a finitely generated ring $\mathcal{R}$ over $\mathbb{Z}$. Let $\mathcal{P}$ be any maximal ideal in $\mathcal{R}$; then $\mathcal{R}/\mathcal{P}$ is a finite field, and so it is isomorphic to $\mathbb{F}(p^k)$. In this case the Frobenius morphism is given by raising the homogeneous coordinates $(X_0, \ldots, X_n)$ of $\mathbb{P}^M_{\mathbb{F}(p^k)}$ to the p^k-th power. The rest of the argument proceeds as above, giving a positive dimensional "base-pointed" deformation space for C.

Then we can apply the following

Principle A.2.3 If a homogeneous system of algebraic equations with integral coefficients has a nontrivial solution in $\overline{\mathbb{F}(p)}$, for infinitely many p, then it has a nontrivial solution in any algebraically closed field. (The general case using $\mathcal{P}$ in Spec $\mathcal{R}$ is analogous.)

Proof. By elimination theory, the existence of a common solution to a system of equations is given by the vanishing of a series of determinants of matrices whose entries are polynomials (with integral coefficients) in the coefficients of the equations. A determinant vanishes if it vanishes mod p for an infinite number of primes p.

Bibliography

[ABW90] M. Andreatta, E. Ballico and J. A. Wiśniewski. Projective manifolds containing large linear subspaces. In *Lecture Notes in Mathematics*, **1515** Springer-Verlag, Berlin-New York, (1992), 1–11.

[ABW91] ———. On contractions of smooth algebraic varieties. *preprint UTM 344*, (1991).

[ABW93] ———. Two theorems on elementary contractions. *Math. Ann.*, **297** (1993), 191–198.

[AM97] M. Andreatta and M. Mella. Contractions on a manifold polarized by an ample vector bundle. *Trans. Amer. Math. Soc.*, **349** (1997), 4669–4683.

[AO99] M. Andreatta and G. Occhetta. Ample vector bundles with sections vanishing on special varieties. *Internat. J. Math.*,**10** (1999), 677–696.

[AW93] M. Andreatta and J. A. Wiśniewski. A note on nonvanishing and applications. *Duke Math. J.*, **72** (1993), 739–755.

[AW98] ———. On contractions of smooth varieties. *J. Algebraic Geometry*, **7** (1998), 253–312.

[AW97] ———. A view on contraction of higher dimensional varieties. In *Algebraic Geometry – Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, AMS, Providence, RI,1997, 153–183.

[Băd81] L. Bădescu. On ample divisors, II. In *Week of Algebraic Geometry, Proc. Bucharest*, volume 40 of *Texte Math.*, Teubner, Leipzig, 1981, 12–32.

[Băd82a] ———. On ample divisors. *Nagoya Math J.*, **86** (1982), 155–171.

[Băd82b] ———. The projective plane blown up at a point as an ample divisor. *Atti Accad. Ligure Sci. Lett.*, **38** (1982), 88–92. .

[BS92] M. C. Beltrametti and A. J. Sommese. On the adjunction theoretic classification of polarized varieties. *J. reine und angew. Math*, **427** (1992), 157–192.

[BS95] ———. *The adjunction theory of complex projective varieties*, volume 16 of *Exp. Math.* de Gruyter, Berlin, 1995.

[BSW90] M. C. Beltrametti, A. J. Sommese and J. A. Wiśniewski. Results on varieties with many lines and their application to adjunction theory. In *Complex algebraic varieties, Bayreuth 1990*, *Lecture Notes in Math.*, **1507** Springer-Verlag, 1992, 16–38.

[CKM88] H. Clemens, J. Kollar and S. Mori. *Higher dimensional complex geometry* volume 166 of Asterisque SMF, 1988.

[Fuj87] T. Fujita. On polarized manifolds whose adjoint bundles are not semipositive. In *Algebraic geometry, Sendai*, volume 10 of *Adv. Studies in Pure Math.*, Kinokuniya-North-Holland, 1987, 167–178.

[Fuj80] ———. Hyperplane section principle. *Journal of the Math. Soc. of Japan*, **32** (1980), 153–169.

[Fuj89] ———. Remarks on quasi-polarized varieties. *Nagoya Math. J.*, **115** (1989), 105–123.

[Fuj90] ———. *Classification theory of polarized varieties*, volume 155 of *London Math. Soc. Lecture Notes Series.* Cambridge Univ. Press., Cambridge, 1990.

[Ful84] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Math.* Springer Verlag, Berlin Heidelberg New York Tokio, 1984.

[Io86] P. Ionescu Generalized adjunction and applications. *Math. Proc. Camb. phil. Soc.*,**99** (1986), 457–472.

[Ka84a] Y. Kawamata On the finiteness of the generators of a pluricanonical ring for a threefold of general type. *Am. J. of Math.*, **106** (1984), 1503–1512.

[Ka84b] Y. Kawamata The cone of curves of algebraic varieties. *Ann. of Math.*, **119** (1984), 603–633.

[KMM87] Y. Kawamata, K. Matsuda, and K. Matsuki. Introduction to the minimal model program. In *Algebraic geometry, Sendai*, volume 10 of *Adv. Studies in Pure Math.*, Kinokuniya-North-Holland, 1987, 283–360.

[KMM92] J. Kollár, Y. Miyaoka, and S. Mori. Rational connectedness and boundedness of Fano manifolds. *J. Diff. Geom.*, **36** (1992), 765–779.

[Ko96] J. Kollar *Rational curves on algebraic varieties*, Springer Verlag, 1996.

[KS98] Y. Kachi, E. Sato Polarized varieties whose points are joined by rational curves of small degrees. *Preprint*

[LB92] H. Lange, C. Birkenhake Complex abelian varieties. In *Grundlehren der mathematischen Wissenschaften*, **302** Springer-Verlag, 1992.

[Laz83] R. Lazarsfeld Some applications of the theory of positive vector bundles. in *Complete intersections (Acireale, 1983)*, volume 1092 of *Lecture Notes in Math.*, Springer, Berlin-New York, 1984, 29–61.

[LM95] A. Lanteri and H. Maeda. Ample vector bundles with sections vanishing on projective spaces or quadrics. *Intern. J. Math.*, **6** (1995), 587–600.

[LM96] ———. Ample vector bundles characterizations of projective bundles and quadric fibrations over curves. In *Higher dimensional complex varieties (Trento, 1994)*, de Gruyert, 1996, 247–259.

[LM97] ———. Geometrically ruled surfaces as zero loci of ample vector bundles. *Forum Math.*, **9** (1997), 1–15.

[Ma90] H. Maeda Ramification divisors for branched coverings of $\mathbb{P}^n$. *Math. Ann.*, **288** (1990), 195–199.

[Me98] M. Mella. Vector bundles on log terminal varieties. *Proc. Amer. Math. Soc.*, **126** (1998), 2199–2204.

[Me97] ——————. Contrazioni estremali e geometria n-dimensionale Tesi di di Dottorato Universita' di Trento (1997)

[Mor79] S. Mori. Projective manifolds with ample tangent bundle. *Ann. Math.*, **110** (1979), 593–606.

[Mor82] ——————. Threefolds whose canonical bundles are not numerically effective. *Ann. Math.*, **116** (1982), 133–176.

[OSS] C. Okonek, M. Schneider and H. Spindler Vector bundles on complex projective spaces. In *Progress in Mathematics*, **3** . Birkhuser, Boston, Mass., 1980.

[PS89] K.H. Paranjape, V. Srinivas Self maps of homogeneous spaces. *Invent. Math.*, **98** (1989), 425–444..

[Pet90] T. Peternell. A characterization of $\mathbb{P}^n$ by vector bundles. *Math. Zeitschrift*, **205** (1990), 487–490.

[PSW92] T. Peternell, M. Szurek, and J. A. Wiśniewski. Fano manifolds and vector bundles. *Math. Ann.*, **294** (1992), 151–165.

[Som76] A. J. Sommese. On manifolds that cannot be ample divisors. *Math. Ann.*, **221** (1976), 55–72.

[Som78] ——————. Submanifolds of abelian varieties. *Math. Ann.*, **233** (1978), 229–256.

[Som86] ——————. On the adjunction theoretic structure of projective varieties. In *Complex Analysis and Algebraic Geometry, Proceedings Göttingen , ed. H.Grauert*, volume 1194 of *Lecture Notes in Math.*, Springer-Verlag, 1986, 175–213.

[SW90a] M. Szurek and J. A. Wiśnieswki. Fano bundles over $\mathbb{P}^3$ and $\mathbb{Q}^3$. *Pacific J. of Math.*, **141** (1990), 197–208.

[SW90b] ——————. On Fano manifolds which are $\mathbb{P}^k$-bundles over $\mathbb{P}^2$. *Nagoya Math. J.*, **120** (1990), 89–101.

[Uen75] K. Ueno. Classification theory of algebraic varieties and compact complex spaces. In *Lecture Notes in Mathematics*, **439** Springer-Verlag, Berlin-New York, 1975.

[Wiś89] J. A. Wiśniewski. Lenght of extremal rays and generalized adjunction. *Math.Z.*, **200** (1989), 409–427.

[Wiś90a] ——————. On a conjecture of Mukai. *Manuscripta Math.*, **68** (1990), 135–141.

[Wiś91a] ——————. On deformation of nef values. *Duke Math. J.*, **64** (1991), 325–332.

[Wiś91b] ——————. On Fano manifolds of large index. *Manuscripta Math.*, **70** (1991), 145–152.

[Wiś91c] ——————. On contractions of extremal rays of Fano manifolds. *J. reine und angew. Math.*, **417** (1991), 141–157.

[Wiś90b] ——————. Fano manifolds and quadric bundles. *Math.Z.*, **214** (1993), 261–271.

[YZ90] Y.G. Ye and Q. Zhang. On ample vector bundles whose adjunction bundles are not numerically effective. *Duke Math. J.*, **60** (1990), 671–687.

[Zh95] Q. Zhang. A remark on elementary contractions. *Math. Proc. Camb. Phil. Soc.*, **118** (1995), 183–188.

[Zh95b] ————. Ample vector bundles on singular varieties. *Math. Z.*, **220** (1995), 59–64.