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Single-electron measurements with a micromechanical resonator

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A mechanical electroscope based on a change in the resonant frequency of a cantilever one micron in size in the presence of charge has recently been fabricated. We derive the decoherence rate of a charge superposition during measurement with such a device using a master equation theory adapted from quantum optics. We also investigate the information produced by such a measurement, using a quantum trajectory approach. Such instruments could be used in mesoscopic electronic systems, and future solid-state quantum computers, so it is useful to know how they behave when used to measure quantum superpositions of charge.

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I. INTRODUCTION

As devices for processing and storing information become smaller, the demands on the readout technology become ever greater. This is especially true for proposed solid-state quantum computers that store information in various quantum degrees of freedom (qubits): in quantum dots [1], nuclear spin [2,3], superconducting islands [4], and persistent currents [5], to cite just a small sample.

Kane has proposed storing a qubit in the spin of a single phosphorous nucleus implanted in silicon. In his original readout scheme, this was coupled by the hyperfine interaction to the spin of the donor electron bound weakly to the nucleus. A surface gate would then draw the electron towards an adjacent ancilla donor, to which it might tunnel, producing a doubly charged D^- state. Under appropriate bias conditions, this transfer can only occur if the nuclear spin of the qubit is oriented opposite the ancilla.

A spin measurement is thus reduced to detecting the transfer of a single-electron charge to the ancilla. This can be done by a sensitive electroscope such as a single-electron transistor [6]. However, the techniques used for fabricating microelectronics have recently been adapted to build mechanical structures at micron and even nanometer scales [7], and mechanical electroscopes sensitive to small numbers of electrons have been constructed [8]. We will consider how effectively such devices might perform the measurements required for quantum information processing.

Classical treatments of measurement sensitivity assume that the observable being measured has a definite value, which influences the measuring instrument in a definite way. The only question is how much data we must gather to reliably distinguish this effect from other influences on the apparatus, which produce noise. Once we know the size of the effect we wish to distinguish, and the level of noise in the system, some elementary statistics tell us the integration time required for a reliable measurement.

This assumption does not hold when we measure an observable of a quantum system. If the system is in a superposition state, the observable will not have a definite value until some sort of measurement is carried out. Any interesting quantum information device will produce such superpositions. The process by which the superposition is reduced so that the observable has a certain value imposes a minimum level of noise in the measurement, which might be increased by the same sources of technical noise that affect measurements of classical systems.

In the proposed readout scheme for the Kane computer, a donor electron is induced to tunnel between two phosphorous nuclei, depending on the state of the nuclear spins. In general, the nuclei are in a superposition of a state that would permit tunneling, and one that would prevent it. After this tunneling has occured, the electron is left in a superpostion of two position states, each localized on one nucleus. It then interacts with the electroscope, and in general, with other degrees of freedom in the crystal lattice, with the result that we see it become localized on one nucleus or the other, so that the electroscope gives a definite signal that the charge is present or absent.

Note that we are not discussing an ensemble of quantum systems subject to a single measurement, but rather a single quantum system subject to a dynamical measurement process. In such a situation we need to be able to describe the instantaneous conditional state of the measured system as the measurement results accumulate. This is quite different from the usual situation that prevails in condensed-matter systems, where typically, a measurement is made on a large number of (almost) identical constituents undergoing quantum dynamics, and the measurement results are already an average over an ensemble. Fortunately, mathematical techniques (known as quantum trajectory methods) are available to describe the conditional dynamics of a single quantum system subject to measurement with added noise, and these methods have been applied with considerable success to experiments in quantum optics and ion traps [9]. Recently, such methods have been applied to mesoscopic electronic systems [10–12].

II. THE MECHANICAL ELECTROSCOPE

The operation of a micromechanical electroscope is shown schematically in Fig. 1. The active part is an electrode, mounted on a cantilever no longer than 1 μm, which is

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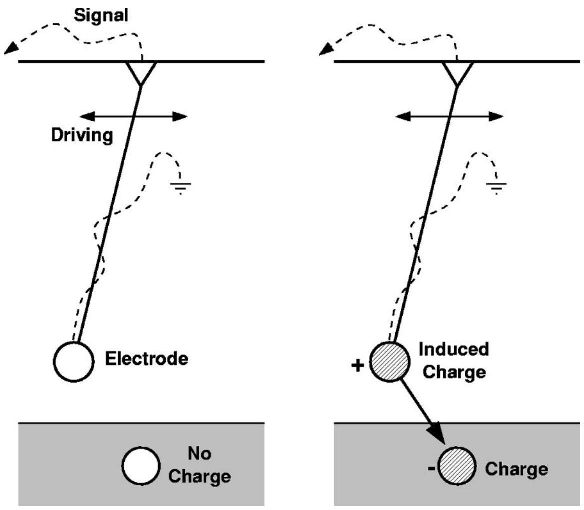

FIG. 1. Operation of the mechanical electroscope. A charge trapped near the surface of some material is coupled to a cantilever suspended above the surface, as explained in the text. The cantilever is driven at a rate E and damped by a combination of mechanical friction and reaction from the electronic readout loop at a rate γ. If an excess charge is present on the surface, the frequency of the pendulum is increased by $\delta\omega$. For simplicity, the figure shows a simple pendulum, but in practice, the cantilever would be a torsional pendulum, oscillating due to strain in the material.

set in motion near the charge to be measured. The electrode is held at constant potential, so that its motion with respect to the unknown charge induces a flow of charge between it and its voltage source. The induced charge gives the electroscope electric potential energy as well as elastic, and changes its resonant frequency. If we envision the electroscope being used to readout a qubit in a quantum computer, there will be two charge states we wish to distinguish. We will denote the difference between the resonant frequencies of the cantilever in these two states by $\delta\omega$; it is determined by geometry and the mutual capacitance between the electrode and the measured charge distribution.

We will assume the mechanical motion of the cantilever is elastic and treat it as a simple harmonic oscillator. Then its motion, including the capacitive coupling to the target charge, is described by a harmonic-oscillator Hamiltonian

$$H=\hbar(\omega_0+\delta\omega n_1)c^\dagger c, \tag{2.1}$$

where ω_0 is the resonant frequency of the cantilever in the absence of surface charge, and c the annihilation operator for its oscillation. The observable n_1 will be defined shortly.

During the readout of a Kane computer, a single-donor electron may occupy a bound state around either of two adjacent nuclei. We will denote these distinct spatial states by $|\psi\rangle$ and $|\phi\rangle$. Only one state (suppose $|\psi\rangle$) couples to the electroscope—this is how we can distinguish them.

During the readout, the surface gates will be configured to produce tunneling between the two nuclei, depending on the state of the nuclear-spin qubits. This entangles the charge states with the qubit states $|\uparrow\rangle$ and $|\downarrow\rangle$. We will denote the combined states by $|\mathbf{0}\rangle=|\uparrow\rangle\otimes|\phi\rangle$, and $|\mathbf{1}\rangle=|\downarrow\rangle\otimes|\psi\rangle$, according to the number of electrons interacting with the electroscope, which we will represent by the operator $n_1=|\mathbf{1}\rangle\langle\mathbf{1}|$. In general, the measured qubit will be in a superposition state, so the total state will take the form

$$|\Psi\rangle=a|\mathbf{0}\rangle+b|\mathbf{1}\rangle. \tag{2.2}$$

Table I gives numerical parameters for a cantilever electroscope fabricated in 1998. The frequency and operating temperature of this electroscope meant that themal noise completely dominated any quantum effects. Besides lowering the temperature, this could be changed by using a cantilever with a higher resonant frequency, and such devices have been fabricated. However, the sensitivity of the electroscope depends on the frequency changing significantly when change is present, and this might not be the case in higher-frequency cantilevers.

We note that the interaction Hamiltonian commutes with the number operator $\hat{n}_1$. Furthermore, in the absence of tunnelling, the free Hamiltonian for the charge state itself is proportional to the square of the charge (capacitive electrostatic energy) and itself commutes with the charge number operator. In the presence of the measurement, the number operator is thus a constant of motion. Such a measurement is known as a quantum nondemolition (QND) measurement [13]. Number eigenstates are not changed by the coupling to the apparatus, and moments of the number operator are constant in time. On the other hand, any state that is initially a coherent superposition in the number basis will be reduced to a mixture diagonal in this basis, a process known as decoherence. In an ideal quantum nondemolition measurement, the probability distribution for observed results at the conclusion of the measurement should accurately reflect the intrinsic probability distributions of the quantum nondemolition variable in the quantum state at the start of the measurement.

This model, where the electroscope performs a QND measurement of the coupled charge, is idealized. If such an electroscope was used to measure any interesting device, the motion of the cantilever would disrupt the distribution of the

TABLE I. Data for an electroscope fabricated by Cleland and Roukes [8].

Operating temperature T	4.2 K	$k_B T=3.6\times10^{-4}$ eV
Resonant frequency $\omega_0/2\pi$	2.6 MHz	$\hbar\omega_0=1.1\times10^{-8}$ eV
Torsional spring constant κ	1.1×10^{-10} Nm	
Amplitude $\theta_{\max}$	30 mrad	
Frequency shift per electron $\delta\nu$	0.1 Hz	
Quality factor ω_0/γ	6.5×10^3	

charge being measured. The nature and extent of this disruption would depend on the electrical properties of the system being measured; for the Kane computer, determining these is an unsolved problem in atomic physics. In general, back action (and interference from sources unrelated to the measurement) imposes a time limit on the measurement, after which the charge state will have been disrupted and the results will be meaningless. The results of this paper determine whether the electroscope can measure the charge with the necessary precision within that time.

To detect the change in resonant frequency, we must set the cantilever in motion with some driving mechanism. In the device described in Table I, this was supplied by driving an alternating current through a wire on the cantilever in the presence of a magnetic field. The current induced by the field in another wire was used to monitor the response of the cantilever to the driving.

However, the details of the driving are not important. As long as the cantilever is coupled weakly to the driving system and is not damped so strongly that its state changes significantly over the period of its vibration (in other words, it has high finesse), the effect can be described by a Hamiltonian. In the interaction picture, this takes the form $\hbar\mathcal{E}(\hat{c}+\hat{c}^\dagger)$, where $\mathcal{E}$ is the strength of the driving in units of frequency. If the finess of the cantilever is low, noise from the driving system affects its motion significantly, and the dynamics due to the driving cannot be approximated by a Hamiltonian.

The frequency shift could be detected in a number of ways. We could sweep the driving frequency and monitor the amplitude of the oscillations. Or else we could drive the oscillator at a constant frequency ω, and then detect the change in phase of the oscillation due to the shift in resonance frequency when a small charge is coupled; this is the method analyzed in this paper. We will assume that if the charge state is $|\mathbf{0}\rangle$, the cantilever will be driven on resonance; if it is $|\mathbf{1}\rangle$, the change $\delta\omega$ in its resonant frequency will cause its phase to differ from that of the driving force. The rate of change of the phase of the output current with frequency of the driving is greatest when the cantilever is driven near its resonant frequency.

We will measure time by the inverse damping rate γ^{-1}. Then, defining a dimensionless driving strength $E=\mathcal{E}/\gamma$ and a detuning $\Delta=\delta\omega/\gamma$, the Hamiltonian for the coherent driving, in the interaction picture, is

$$\hat{H}_D=\hbar E(\hat{c}+\hat{c}^\dagger)+\hbar\Delta n_1\hat{c}^\dagger\hat{c}. \tag{2.3}$$

In reality of course the mechanical oscillations of the cantilever will be subject to frictional damping, and accompanying mechanical noise. The rate of energy dissipation is specified by the quality factor Q which is the ratio of the resonance frequency to the width of the resonance. For linear response, this gives $Q=\omega_0/\gamma_M$, where γ_M is the decay rate of energy due to mechanical dissipation. Cleland and Roukes [14] have measured quality factors up to 2×10^4. With such quality factors and resonance frequencies approaching gigahertz, these devices are approaching low quality optical resonators. So we will treat the effect of mechanical damping with the master equation methods of quantum optics. These methods assume that the coupling of the resonator to the dissipative degrees of freedom is sufficiently weak [15,16]. Specifically, we assume that $\gamma_M\ll\omega_0,kT/\hbar$.

Under these assumptions, the coupling between the oscillator and the thermal mechanical reservoir is [17]

$$H_M=\sqrt{\gamma_M}[ca^\dagger(t)+c^\dagger a(t)], \tag{2.4}$$

where $a(t)$, $a^\dagger(t)$ are bosonic reservoir operators. The state of the reservoir will be taken to be that of a Planck thermal equilibrium density operator with temperature T_M.

We now consider in more detail the mechanism by which the small changes in resonance frequency induced by the proximity of a target charge are transduced. This may be done [8] by fabricating a wire loop on the mechanical oscillator and placing the whole apparatus in a strong magnetic field. As the mechanical oscillator moves, an induced voltage is set up in the loop and we may measure the induced current. When the current for the driving circuit is such as to drive the mechanical oscillator at its resonance frequency, the induction current is out of phase with the driving current. However, when a small target charge shifts the resonance frequency of the oscillator, the induced current shifts in phase with respect to the driving current. We can detect this phase shift by an electrical comparison of the driving current and induction current. This is essentially homodyne detection in which the driving current plays the role of a local oscillator. Unfortunately, this electrical transduction of the mechanical motion introduces another source of noise for the measurement.

The induction current is coupled into an external amplifier circuit that can be treated as a bosonic reservoir, with some nonzero noise temperature [18] T_E. The readout circuit variable coupled to the cantilever is the current operator $i(t)$ in the readout circuit. We will assume that the coupling is linear in the current and coordinate degree of freedom of the cantilever. Under standard assumptions, the interaction between the mechanical oscillator and the readout circuit is described by the interaction picture Hamiltonian,

$$H_R=i\sqrt{\gamma_E}[c^\dagger\Gamma(t)-c\Gamma^\dagger(t)], \tag{2.5}$$

where $\Gamma(t)=b(t)e^{i\omega_0 t}$ with the actual current in the circuit given by $i(t)=\sqrt{\hbar\omega_0/2Lz_0}(b(t)+b^\dagger(t))$, L being the inductance per unit length of the transmission line, and z_0 the quantization length. We will assume that the readout circuit reservoir is bosonic and also in thermal equilibrium at some temperature T_E.

Using the interaction Hamiltonians for the reservoir coupling [Eqs. (2.4) and (2.5)], we may obtain the Heisenberg equations of motion for the oscillator and reservoir variables. Using standard techniques [17], the reservoir variables may be eliminated to give a quantum Langevin stochastic differential equation describing the dynamics of the oscillator amplitude

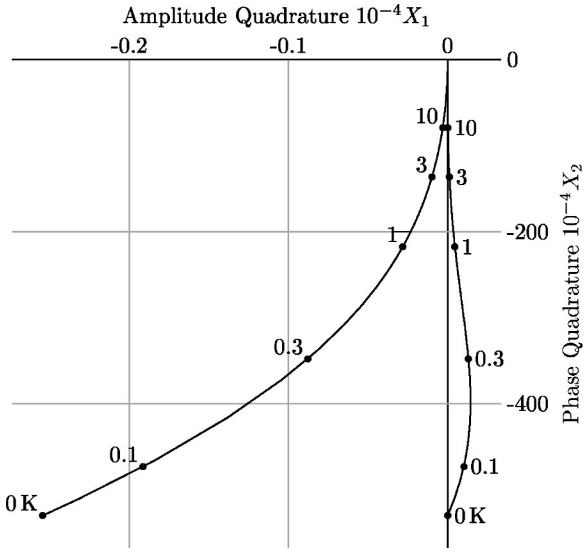

FIG. 2. The off-diagonal element Z of the density operator is a thermal state displaced by amplitudes α and β, which depend on the temperature [see Equation (3.13)]. When the cantilever is coupled to a hot bath, these coherent amplitudes decrease, and Z approaches a purely thermal state. The values these amplitudes take in the Cleland and Roukes electroscope at temperatures from absolute zero up to 10 K are plotted in the complex plane, in units of the ground-state fluctuations. The amplitudes of the diagonal elements A and B do not vary with temperature, but remain at the 0 K values.

$$\frac{da}{dt} = -i\,\delta\omega a - i\mathcal{E} - \frac{\gamma_M}{2}a - \frac{\gamma_E}{2}a + \sqrt{\gamma_M}a_{\rm in}(t) + \sqrt{\gamma_E}b_{\rm in}(t), \tag{2.6}$$

where $a_{\rm in}(t), b_{\rm in}(t)$ are the quantum noise sources for the mechanical and electrical reservoirs, respectively. These noise terms are defined by correlation functions, which are Fourier transforms of

$$\langle a_{\rm in}(t)\rangle = \langle b_{\rm in}(t)\rangle = 0, \tag{2.7}$$

$$\langle a_{\rm in}^\dagger(\omega) a_{\rm in}(\omega)'\rangle = \overline{n}(\omega, T_M)\,\delta(\omega - \omega'), \tag{2.8}$$

$$\langle a_{\rm in}(\omega) a_{\rm in}^\dagger(\omega)'\rangle = (\overline{n}(\omega, T_M) + 1)\,\delta(\omega - \omega'), \tag{2.9}$$

$$\langle b_{\rm in}^\dagger(\omega) b_{\rm in}(\omega)'\rangle = \overline{n}(\omega, T_E)\,\delta(\omega - \omega'), \tag{2.10}$$

$$\langle b_{\rm in}(\omega) b_{\rm in}^\dagger(\omega)'\rangle = [\overline{n}(\omega, T_E) + 1]\,\delta(\omega - \omega'), \tag{2.11}$$

where

$$\overline{n}(\omega, T) = \frac{1}{2}[\coth(\hbar\omega/2k_BT) - 1]. \tag{2.12}$$

Note the equation explicitly includes a friction term (proportional to γ_E) that arises form the electrical coupling to the readout circuit. The steady-state average amplitude $\alpha_n = \langle a(t)\rangle_{t\to\infty}$, is given by

$$\alpha_n = \frac{-2i\mathcal{E}}{(\gamma_M + \gamma_E) + 2i\,\delta\omega n_1}. \tag{2.13}$$

The actual measured quantity is the current in the readout circuit, that is to say the readout variable is an electrical bath variable, $b_{\rm out}$ at the output from the system interaction. The output amplitudes for both the mechanical and electrical baths are related to the input variables for these two baths and the amplitude of the mechanical oscillator by [13]

$$a_{\rm out}(t) = \sqrt{\gamma_M}a(t) - a_{\rm in}(t), \tag{2.14}$$

$$b_{\rm out}(t) = -i\sqrt{\gamma_E}a(t) - b_{\rm in}(t). \tag{2.15}$$

The average value of the electrical readout amplitude in the steady state is then found using equations Eqs. (2.6) and (2.15).

$$\langle b_{\rm out}\rangle = \sqrt{\gamma_E}\alpha_n, \tag{2.16}$$

where α_n is given in Eq. (2.13). We see that the steady-state amplitude of the cantilever, and hence, the output electrical signal undergoes a change in phase and amplitude, see Fig. 2. If we monitor the component in the imaginary direction (that is, in quadrature with the driving signal E) we will have maximum sensitivity to this change in phase. Furthermore, it is desirable to have E as large as possible so that small changes in phase translate into large changes in the quadrature.

We can now proceed to calculating the noise power spectrum for the measured current. The calculation is analogous to that for a double-sided cavity given in reference [13]. We now do not work in the rotating frame but return to the laboratory frame. The Fourier component of the output operator for the current is given by

$$b_{\rm out}(\omega) = \frac{[(\gamma_E - \gamma_M)/2 - i(\omega_0 - \omega) - i\,\delta\omega n_1]b_{\rm in}(\omega) - i\sqrt{\gamma_E}\mathcal{E}(\omega) + \sqrt{\gamma_E\gamma_M}a_{\rm in}(\omega)}{[(\gamma_E + \gamma_M)/2 + i(\omega_0 - \omega) + i\,\delta\omega n_1]}, \tag{2.17}$$

where $\mathcal{E}(\omega)$ is the Fourier component of the driving amplitude. If the driving is noiseless and monochromatic, $\mathcal{E}(\omega) = \mathcal{E}\delta(\omega - \omega_d)$. However, in reality there would be some noise in the driving amplitude derived from the electrical noise in the driving circuit. We will treat this as entirely classical.

Equations (2.13) and (2.16) suggest that the signal will appear in the quadrature of the current out of phase with the driving force, defined by

$$X_{2,\text{out}}(t) = i[b_{\text{out}}^{\dagger}(t) - b_{\text{out}}(t)], \tag{2.18}$$

with Fourier components $X_{2,\text{out}}(\omega)$. The measured power spectrum is then given by the correlation function,

$$S_{2,\text{out}}(\omega,\omega') = \langle X_{2,\text{out}}(\omega), X_{2,\text{out}}(\omega')\rangle. \tag{2.19}$$

Using the specified states for the electronic and mechanical noise operators, we find,

$$S_{2,\text{out}}(\omega,\omega') = [|\mathcal{B}(\omega)|^2\{2\overline{n}(\omega,T_E)+1\} + |\mathcal{A}(\omega)|^2\{2\overline{n}(\omega,T_M)+1\}]\delta(\omega-\omega'), \tag{2.20}$$

where

$$\mathcal{B}(\omega) = \frac{(\gamma_E-\gamma_M)/2 - i((\omega-\omega_0)+\delta\omega n_1)}{(\gamma_E+\gamma_M)/2 + i((\omega-\omega_0)+\delta\omega n_1)},$$

$$\mathcal{A}(\omega) = \frac{\sqrt{\gamma_E\gamma_M}}{(\gamma_E+\gamma_M/2) + i((\omega-\omega_0)+\delta\omega n_1)}.$$

To estimate the signal-to-noise ratio (SNR), we evaluate the spectrum at the driving frequency (that is to say, at the central Fourier component of the coherent driving);

$$S(\omega_0) = \frac{\{[(\gamma_E-\gamma_M)/2]^2 + (\delta\omega n_1)^2\}[2\overline{n}(\omega,T_E)+1] + \gamma_E\gamma_M[2\overline{n}(\omega,T_M)+1]}{[(\gamma_M+\gamma_E)/2]^2 + (\delta\omega n_1)^2}. \tag{2.21}$$

Equations (2.13), (2.16), and (2.18) show that the magnitude of the Fourier component of the mean signal at the driving frequency is given by

$$|\langle X_{2,\text{out}}(\omega_D)\rangle| = \frac{8\sqrt{\gamma_E}\mathcal{E}\delta\omega n_1}{(\gamma_M+\gamma_E)^2 + 4\delta\omega^2 n_1}. \tag{2.22}$$

The signal is a sharp peak at $\omega=\omega_d=\omega_0$, in which there is a noise power $S(\omega_0)$ per root Hertz. So the SNR per root Hertz is $|\langle X_{2,\text{out}}(\omega_D)\rangle|^2/S(\omega_0)$, or

$$\text{SNR} = \frac{16\gamma_E\mathcal{E}^2\delta\omega^2 n_1}{[(\gamma_M+\gamma_E)^2+4\delta\omega^2 n_1][(\gamma_E-\gamma_M)^2+4\delta\omega^2 n_1][2\overline{n}(\omega,T_E)+1] + \gamma_E\gamma_M[2\overline{n}(\omega,T_M)+1]}. \tag{2.23}$$

If the SNR required for the measurement is SNR_r, then we must average over noise for a time t such that $\text{SNR}_r = \text{SNR}/\sqrt{t}$. If we set $n_1=1$, so we are measuring the charge on one electron, the sensitivity is then $e\sqrt{t} = e\,\text{SNR}_r/\text{SNR}$.

III. UNCONDITIONAL DESCRIPTION OF THE MEASUREMENT

When we measure a quantum system, we bring an extremely large set of independent observables of our instrument and its environment into correlation with the measured system observable. The environment of the electroscope has two distinct components. First, there is the environment associated with the mechanical oscillator, which is responsible for mechanical damping and noise. Second, there is the environment associated with the electrical readout, which is responsible for Johnson-Nyquist noise in the electrical circuit, and ultimately provides the measured result. However, we are interested in what the measurement tells us about the system, not in the exact quantum state of the instrument and its environment. Useful instruments must operate independently of the detailed state of their environments.

There are two ways to describe the partial state of the charge and oscillator. First, we can ignore the results of the measurement and average over states of the environment completely. In this case, the evolution of the charge and oscillator is described by a master equation. Effectively we are averaging over the ensemble of partial states distinguished by different measurement records

Second, we can ask for the conditional states of the charge and oscillator, given a particular measurement record. Each member of the ensemble of partial states is associated with a distinct measurement record of the instrument. For it to be an effective measurement, observers must be able to distinguish the states of the instrument. In other words, the charge must end up correlated with some simple macroscopic quantity, like the current in a wire or the position of a pointer on a scale. It is then possible to ask for the particular partial state of the measured system that is correlated with a known pointer value. In other words, we need to be able to specify the conditional state of the system given a readout of the instrument variable that distinguishes different charge states. This is the conditional, or selective, description of the measured system. Of course, if we average over the readout variables, we must obtain the unconditional description of the system.

We begin with the unconditional description of the measurement. The dominant sources of excess noise that limit the quality of the measurement are the thermal mechanical noise and thermal electrical noise on the readout circuit. Under certain Markoff and rotating-wave assumptions [17,19], the explicit states of the mechanical and electrical reservoirs may be traced out. This leaves the following master equation for the density operator of the composite system of charge and cantilever,

$$\dot{\rho}(t) = -i[\mu(en_1)^2\hat{c}^\dagger\hat{c},\rho] - i[\mathcal{E}(\hat{c}+\hat{c}^\dagger),\rho] + \sum_{i=M,E}\gamma_i(\overline{n_i}+1)\mathcal{D}[c]\rho + \gamma_i\overline{n_i}\mathcal{D}[c^\dagger]\rho, \tag{3.1}$$

where the superoperator $\mathcal{D}$ is defined by

$$\mathcal{D}[c]\rho = c\rho c^\dagger - \frac{1}{2}(c^\dagger c\rho + \rho c^\dagger c). \tag{3.2}$$

This can be written in a more standard form

$$\dot{\rho} = -i[\mu(en_1)^2\hat{c}^\dagger\hat{c},\rho] - i[\mathcal{E}(\hat{c}+\hat{c}^\dagger),\rho] + \gamma(\overline{n}+1)\mathcal{D}[c]\rho + \gamma\overline{n}\mathcal{D}[c^\dagger]\rho, \tag{3.3}$$

where $\gamma \equiv \gamma_M + \gamma_E$, and $\overline{n} \equiv [\gamma_M\overline{n}(\omega,T_M) + \gamma_E\overline{n}(\omega,T_E)]/\gamma$.

We will begin solving this master equation by separating the dynamics of the cantilever and the charge. As before, we assume there is only one charge in the system, and consider the charge states $|\mathbf{0}\rangle$ and $|\mathbf{1}\rangle$. We can decompose ρ into a 2×2 matrix of cantilever operators

$$\rho = \hat{A}|\mathbf{0}\rangle\langle\mathbf{0}| + \hat{B}|\mathbf{1}\rangle\langle\mathbf{1}| + \hat{Z}|\mathbf{0}\rangle\langle\mathbf{1}| + \hat{Z}^\dagger|\mathbf{1}\rangle\langle\mathbf{0}|. \tag{3.4}$$

Since ρ is Hermititan, we need only three cantilever operators, $\hat{A}$, $\hat{B}$, and $\hat{Z}$. We can now decompose Eq. (3.3) into three independent equations involving only cantilever operators:

$$\frac{d\hat{A}}{dt} = -i[E(\hat{c}+\hat{c}^\dagger),\hat{A}] + (\overline{n}+1)\mathcal{D}[c]\hat{A} + \overline{n}\mathcal{D}[c^\dagger]A, \tag{3.5}$$

$$\frac{d\hat{B}}{dt} = -i[E(\hat{c}+\hat{c}^\dagger) + \Delta\hat{c}^\dagger\hat{c},\hat{B}] + (\overline{n}+1)\mathcal{D}[c]\hat{B} + \overline{n}\mathcal{D}[c^\dagger]\hat{B}, \tag{3.6}$$

$$\frac{d\hat{Z}}{dt} = -i[E(\hat{c}+\hat{c}^\dagger),\hat{Z}] + i\Delta\hat{Z}\hat{c}^\dagger\hat{c} + (\overline{n}+1)\mathcal{D}[c]\hat{Z} + \overline{n}\mathcal{D}[c^\dagger]\hat{Z}. \tag{3.7}$$

As before, we are now measuring time relative to the damping time $1/\gamma$.

If we measured the state of the charge by means other than the cantilever, the state of the cantilever immediately after the measurement would be $\hat{B}$ if the charge were present, or $\hat{A}$ if it were absent. Hence, $\hat{A}$ and $\hat{B}$ must be density operators, and Eqs. (3.5) and (3.6) have the form of master equations for a damped harmonic oscillator. Such equations, and their solutions, are familiar to quantum opticians. The stable solution is a displaced thermal state, which can be written

$$\rho = (1-e^{-\lambda(t)})D[\alpha(t)]e^{-\lambda(t)\hat{c}^\dagger\hat{c}}D^\dagger[\alpha(t)], \tag{3.8}$$

where $D(\alpha)$ is a displacement operator $\exp(\alpha\hat{c}^\dagger - \alpha^*\hat{c})$, and in the steady-state $\lambda = \hbar\omega_0/k_bT$. In the limit of low temperature $kT \ll \hbar\omega_0$, this becomes a coherent state $|\alpha\rangle\langle\alpha|$. In the steady state, the cantilever has as many thermal phonons as a resevoir mode with the same frequency, i.e., $e^{-\lambda} = \overline{n}/(\overline{n}+1)$. Its coherent amplitude α_0 reaches a balance with the driving and damping after a time around $2/\gamma$:

$$\alpha(t) = \alpha_0 e^{-\kappa t/2} - \frac{2iE}{\kappa}(1-e^{-\kappa t/2}), \tag{3.9}$$

$$\kappa = \begin{cases} 1 & n=0 \\ 1+2i\Delta & n=1 \end{cases}. \tag{3.10}$$

During measurement, the cantilever states $\hat{A}$ and $\hat{B}$ are displaced thermal states with distinct coherent amplitudes.

As the measurement proceeds, we expect the charge state to evolve from a coherent superposition of $|\mathbf{0}\rangle$ and $|\mathbf{1}\rangle$ to an incoherent mixture; in terms of our decomposition, we expect the off-diagonal term Z to decay with time. An operator of the form

$$Z = z(t)D(\alpha)\exp(-\lambda\hat{c}^\dagger\hat{c})D^\dagger(\beta), \tag{3.11}$$

where $z(t)$ is a (possibly complex) amplitude, solves Eq. (3.7) if α, β, λ, and z obey the following differential equations:

$$\frac{dl}{dt} = (\overline{n}+1)l^2 - (2\overline{n}+1+i\Delta)l + \overline{n}, \tag{3.12}$$

$$\frac{da}{dt} = \left[-i\Delta + (\overline{n}+1)l - \overline{n} - \frac{1}{2}\right]a - iE(1-l), \tag{3.13}$$

$$\frac{db}{dt} = \left[(\overline{n}+1)l - \overline{n} - \frac{1}{2}\right]b + iE(1-l), \tag{3.14}$$

$$\frac{dk}{dt} = -iE(a-b) + (\overline{n}+1)(l+ab-1) + 1. \tag{3.15}$$

Here, $l = \exp(-\lambda)$, $a = \alpha - l\beta$, $b = \beta^* - l\alpha^*$, and $k = \log z + l\alpha^*\beta - (1/2)(|\alpha|^2 + |\beta|^2)$.

In general, these equations can be solved numerically. However, there are some special cases where we can get interesting information analytically. First, we consider the zero-temperature limit, where the off-diagonal term $\hat{Z}$ is a projector $z|\alpha\rangle\langle\beta|$. The amplitudes α and β are the amplitudes of the diagonal terms given by Eq. (3.10), and z is a

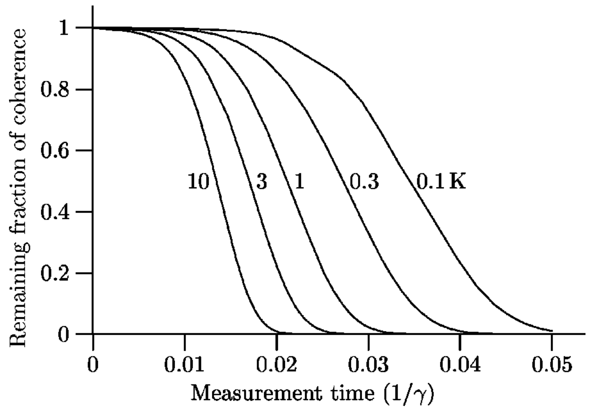

FIG. 3. The state of the cantilever takes some time to become entangled with the charge after they begin to interact, as the cantilever state moves towards its steady value. After this, the charge state decoheres rapidly. Here, the coherence between the two charge states is plotted as a function of time for an array of temperatures. The cantilever is initially in a thermal state at the appropriate temperature. Note the charge state has decohered long before the cantilever reaches its steady amplitude, which occurs after a time $1/\gamma$.

complex amplitude. Once α and β have reached their steady state, the trace of the off-diagonal term, decays exponentially with a rate $|\alpha-\beta|^2/2$.

If we assume the detuning Δ is small, and hence, $|\beta| \approx |\alpha| = 2E$, The difference between the steady-state amplitudes of Eq. (3.10) is

$$|\alpha-\beta|^2 = \frac{16E^2\Delta^2}{1+4\Delta^2} \approx 4|\alpha|^2\Delta^2. \tag{3.16}$$

Cleland and Roukes give enough information about their devices for us to calculate this explicitly [8]. Using the data in Table I, we can calculate α from the definition of the annihilation operator for a torsional pendulum

$$\alpha = \langle \hat{c} \rangle = \sqrt{\frac{\kappa}{2\hbar\omega_0}} \langle \theta_{\max} \rangle = 5.3\times 10^6. \tag{3.17}$$

The normalized detuning can be calculated from the frequency shift per electron and the measured quality factor

$$\Delta = \delta\omega/\gamma = \frac{2\pi\delta\nu Q}{\omega_0} = 2.4\times 10^{-4}. \tag{3.18}$$

The decoherence rate is then $3.2\times 10^6\gamma$ or $8.1\times 10^9\ \mathrm{s}^{-1}$.

As $\overline{n}$ increases from zero, the amplitudes α and β for the off-diagonal operator $\hat{Z}$ are reduced, as shown in Fig. 2. The initial decay of $z(t)$ is shown in Fig. 3, and the steady-decay rate, i.e., the limit of $|z'(t)/z(t)|$ when $t \gg 1/\gamma$, in Fig. 4. At low temperatures (below 130 mK), the increased thermal noise from the bath causes $\hat{Z}$ to decay more rapidly as the temperature of the bath is increased. Contrary to expectations, the steady decoherence rate of the charge superposition decreases as the bath temperature increases above 130 mK.

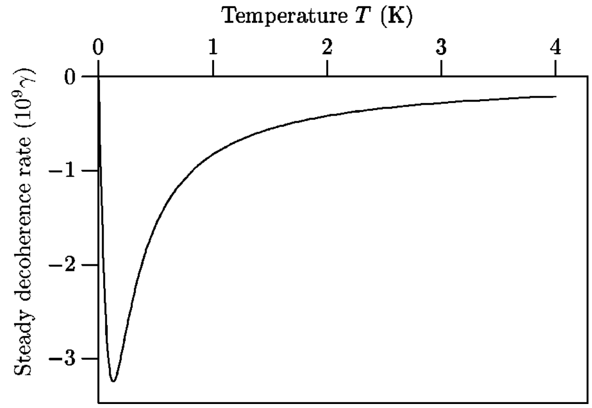

FIG. 4. When the measurement has been running for a time around $1/\gamma$, and the cantilever amplitudes have reached their steady state, any remaining coherence between the two charge states decays exponentially. Here, the rate of this decay is plotted as a function of temperature, for the device described in Table I. The maximum decay rate of $-3.2\times 10^9\gamma$ occurs at 130 mK. Beyond this point, the decay rate decreases with temperature, possibly because the increased thermal noise makes the coherent amplitude of the cantilever harder to distinguish.

The extra thermal noise increases the overlap between the oscillator states corresponding to the presence and absence of charge.

IV. CONDITIONAL DESCRIPTION

We now turn to the correlations between the charge and the reservoir system. These are important because we must be able to distinguish the results corresponding to different charges to make a measurement of the charge at all. They can be studied most simply using the quantum trajectory theory, which associates charge states with possible observed states of the apparatus [20].

We will assume we monitor the current in the electrical reservoir; this is equivalent to an optical homodyne measurement [21]. The inferred state of the charge as such a measurement proceeds is governed by a Wiener process, which is generated by a stochastic increment dW. The average of dW over the ensemble of possible measurement results is zero. Since the deviation of the Wiener process represented by dW increases proportional to $\sqrt{t}$, the average of $(dW)^2$ is dt. The simplest way to manipulate such differentials is to modify the chain rule, to give what is known as Ito calculus.

Given a particular measurement result, labeled by a Wiener increment dW, the evolution of the charge and cantilever is

$$d|\psi\rangle = \left[\frac{1}{i\hbar}\hat{H}dt - \frac{\gamma}{2}\left(c^\dagger c - 2\left\langle\frac{x}{2}\right\rangle c + \left\langle\frac{x}{2}\right\rangle^2\right)dt + \sqrt{\gamma}\left(c - \left\langle\frac{x}{2}\right\rangle\right)dW\right]|\psi\rangle. \tag{4.1}$$

When we insert the charge and cantilever Hamiltonian, and normalize time by the damping rate as before, this becomes

$$d|\psi\rangle = \left[-i\{E(c+c^\dagger) + \Delta n c^\dagger\} dt - \frac{1}{2}\left(c^\dagger c - 2\left\langle \frac{x}{2} \right\rangle c + \left\langle \frac{x}{2} \right\rangle^2 \right) dt + \left(c - \left\langle \frac{x}{2} \right\rangle \right) dW \right] |\psi\rangle. \tag{4.2}$$

When a particular function dW is selected from the Wiener ensemble, this can be solved to show the evolution of a pure state $|\psi\rangle$. These states form an ensemble with density operator ρ. Of course, ρ can be decomposed into many ensembles, so the evolution generated by Eq. (4.2) is not unique. The details are given in Carmichael [20].

Mixed states of the cantilever and charge must be written in the form of Eq. (3.4). However, pure states can always be written as

$$|\psi\rangle = |A\rangle \otimes |\mathbf{0}\rangle + |B\rangle \otimes |\mathbf{1}\rangle, \tag{4.3}$$

as before we will assume the state of the cantilever is initially coherent, so that

$$|\psi\rangle = p|\alpha 0\rangle + q|\beta_1|. \tag{4.4}$$

The differential of a scaled coherent state $q(t)|\beta(t)\rangle$ is

$$d(q|\beta\rangle) = \left(dq - \frac{1}{2} q d|\beta|^2 \right) |\beta\rangle + q\dot{\beta} dt c^\dagger |\beta\rangle. \tag{4.5}$$

Comparison with Eq. (4.2) gives Eqs. (3.9) for the evolution of α and β as before. Some Ito calculus manipulations show that

$$d|q|^2 = |pq|^2 (\langle x \rangle_\alpha - \langle x \rangle_\beta) dW, \tag{4.6}$$

where $\langle x \rangle_\alpha$ is the expectation value of the amplitude quadrature x in a coherent state $|\alpha\rangle$, which is just 2 Re α. The normalization of $|\psi\rangle$ requires that $d|p|^2 = -d|q|^2$.

We need to compare the gain in knowledge shown by this trajectory picture to the decay of coherence modeled by the master equation. The results of the measurement are the probabilities $|p|^2$ and $|q|^2$; the pure state that the observer will infer from these has a density operator

$$\rho = |p|^2 |0\rangle\langle 0| + |q|^2 |1\rangle\langle 1| + |pq| (|1\rangle\langle 0| + |0\rangle\langle 1|). \tag{4.7}$$

The off-diagonal terms in this have magnitude $|pq|$; we can average over dW to see the behavior of the density operator for the ensemble of measurement results.

Some more routine Ito calculus gives the evolution of this:

$$d|pq| = -|pq| \left[\frac{1}{8} (\langle x \rangle_\beta - \langle x \rangle_\alpha)^2 dt + \frac{1}{2} (|p|^2 - |q|^2)(\langle x \rangle_\beta - \langle x \rangle_\alpha) dW \right]. \tag{4.8}$$

Since the average of dW over different measurement results is zero, on average

$$d|pq| = -\frac{1}{8} (\langle x \rangle_\beta - \langle x \rangle_\alpha)^2 |pq| dt. \tag{4.9}$$

If the difference between the charges associated with states $|0\rangle$ and $|1\rangle$ is e, then in the state $p|\alpha 0\rangle + q|\beta 1\rangle$, the uncertainty in the charge is given by

$$[\langle (ne)^2 \rangle - \langle ne \rangle^2]^{1/2} = e|pq|. \tag{4.10}$$

From Eq. (4.9), this decreases exponentially as the measurement progresses, at a rate

$$\frac{1}{8} (\langle x \rangle_\beta - \langle x \rangle_\alpha)^2 = \frac{8\Delta^2 E^2}{(1+4\Delta^2)^2}. \tag{4.11}$$

This differs from the square-root decay of classical uncertainty as measurements are averaged over time, but exponential decay is what we would expect for decay of coherence [13]. For the device described in [8], this is almost equal to the decoherence rate. In real devices, thermal noise will cause the trajectory states to be mixed, however, the evolution of such mixed states is much harder to calculate.

V. DISCUSSION

To estimate the time required for our measurement, we have calculated how long it takes for an initially pure superposition of charge states to be reduced to a mixture, and how long (in some sense) it takes us to find out which charge eigenstate we have been left with. While these questions are interesting in their own right (they composed the deepest mystery of physics for the best part of a century), it could be argued that they do not reflect the way measurements would be used in a real computer.

The most that we could do with measurements on pure states is state preparation. In a coherent quantum computer, this would be rather pointless though, since if we know the initial state, we could just rotate it into the eigenstate we want. We carry out measurements to find out something we don't know: in other words, we apply them to mixed states, with a view to finding out which of the possibilities is real.

Information theory provides tools to quantify this, such as conditional entropy and mutual information. Unfortunately, calculating any of these requires knowledge of the ensemble of trajectories generated by each component of the mixture, and the overlaps between them. In general, it is hard to find the probability distribution of trajectories; we usually just calculate averages. It might be worth doing this numerically, however.

There is a more straightforward limitation to our analysis: in present-day devices, the thermal effects that we have neglected in the trajectory treatment utterly dominate the vacuum noise we have considered. Hence, the measurement time will be limited by the need to average classical fluctuations. It is possible that future devices operating at higher frequencies will reduce the level of thermal noise so that quantum effects will be important. This presents the remarkable prospect of a solid cantilever with position and momentum known to the limit allowed by the uncertainty principle.

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