SELF-AERATED FLOWS ON CHUTES AND SPILLWAYS

by H. CHANSON

Abstract: In open channel flows an important design parameter is the amount of entrained air. The presence of air within the flow increases not only the bulk of the flow but also the transfer of atmospheric gases (e.g. oxygenation). Further aeration of high-velocity flows may prevent or reduce cavitation damage. This paper reviews the characteristics of self-aerated flows on spillways and chutes: uniform flows and gradually varied flows. First the uniform flow conditions are presented, with new prototype results. Similarities with suspended sediments flows and extremely rough flows are developed, and the interaction between air bubbles and turbulence is discussed. Then the basic equations for gradually varied flows are developed using the same method as WOOD (1985). The results are applied to chutes and tunnel spillways, and compared with experimental data.

INTRODUCTION

The presence of air in open channel flows increases the bulk of the flow which must be taken into account when designing spillway and chute sidewalls (FALVEY 1980). Also the presence of air within the boundary layer reduces the shear stress, and the resulting increase of momentum must be considered when designing a ski jump downstream of a spillway (ACKERS and PRIESTLEY 1985). Further the presence of air within high-velocity flows may prevent or reduce the damage caused by cavitation (MAY 1987; FALVEY 1990). Recently air entrainment on chutes has been recognized for its contribution to the air-water transfer of atmospheric gases such as oxygen and nitrogen (WILHELMS and GULLIVER 1989).

In the first part of this paper uniform self-aerated flows are studied using the same method as that used by WOOD (1983). Comparisons are made with experimental data. The results are discussed with analogy to flows over rockfill dams and suspended sediments flows.

In the second part of the paper the gradually varied flow region is described using the same method as that used by WOOD (1985). The results are discussed and compared with experimental data.

Mechanisms of air entrainment on chutes

In high speed flow down a steep chute, air is entrained at the free surface, and this process is called self-aeration. Several explanations were proposed to describe the mechanisms of self-aeration. KEULEGAN and PATTERSON (1940) analysed wave instability in open channel flows, and their work suggest that air may be

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entrained by breaking waves at the free surface, if the flow conditions satisfy : \( Fr > 1.5 \). VOLKART (1980) indicated that air is entrained by water drops falling back into the water flow. HINO (1961) and ERVINE and FALVEY (1987) suggested that air is entrapped by turbulent velocity fluctuations on the free surface.

It is believed that air entrainment occurs when the turbulence level is large enough to overcome both surface tension and gravity effects. The turbulent velocity normal to the free surface \( v' \) must be large enough to overcome the surface tension pressure of the entrained bubble (RAO and RAJARATNAM 1961, ERVINE and FALVEY 1987) and greater than the bubble rise velocity component for the bubble to be carried away. These conditions become:

\[
\begin{align*}
  v' &> \sqrt{\frac{8 \cdot \sigma}{\rho_w \cdot d_b}} \quad (1) \\
  v' &> u_r \cdot \cos \alpha \quad (2)
\end{align*}
\]

where \( \sigma \) is the surface tension, \( \rho_w \) the water density, \( d_b \) the bubble diameter, \( u_r \) the bubble rise velocity and \( \alpha \) the spillway slope. Air entrainment occurs when the turbulent velocity \( v' \) satisfies both the equations (1) and (2). It must be noted that the equation (1) derives from ERVINE and FALVEY’s (1987) work. Figure 1 shows both conditions for bubble sizes in the range 1 to 100 mm and slopes from 0 to 75 degrees. The rise velocity of individual bubbles in still water was computed as by the method of COMOLET (1979). Figure 1 suggests that self-aeration will occur for turbulent velocities normal to the free surface greater than 0.1 to 0.3 m/s, and bubbles in the range 8 to 40 mm. For steep slopes the action of the buoyancy force is reduced and larger bubbles are expected to be carried away.

**Self-aeration on spillway chute**

For a spillway flow the entraining region follows a region where the flow over the spillway is smooth and glassy. Next to the boundary however turbulence is generated and the boundary layer grows until the outer edge of the boundary layer reaches the surface. This point is called the point of inception (figure 2). Downstream of the point of inception, a layer containing a mixture of both air and water extends gradually through the fluid. The rate of growth of the layer is small and the air concentration distribution varies gradually with distance. Far downstream the flow will become uniform. This region is defined as the uniform equilibrium flow region.

**Definitions**

The local air concentration is defined as the volume of air per unit volume. The characteristic flow depth \( d \) is defined as:

\[
d = \frac{Y_{90}}{\int_0^{Y_{90}} (1 - C) \cdot dy} \quad (3)
\]

where \( y \) is measured perpendicular to the spillway surface and \( Y_{90} \) is the depth where the local air concentration is 90%. The model and prototype data presented in this paper were obtained using
conductivity probes: a conductivity probe records the average time of passage of air bubbles, and provides the air concentration only if the air velocity equals the water velocity (i.e. $V_{air}/V_w = 1$). Above 90% of air concentration the slip ratio $V_{air}/V_w$ no longer equals 1 (CAIN 1978, CHANSON 1988), and the characteristic depth $d$ must be defined from 0 to 90% only. The depth averaged mean air concentration $C_{mean}$ is defined as:

$$(1 - C_{mean}) \times Y_{90} = d$$

The average water velocity $U_w$ is defined as:

$$U_w = \frac{q_w}{d}$$

where $q_w$ is the water discharge per unit width. The characteristic velocity $V_{90}$ is defined as that at $Y_{90}$.

**UNIFORM FLOW REGION**

**Equilibrium air concentration distribution**

WOOD (1983) re-analyzed STRAUB and ANDERSON's (1958) set of self-aerated flow measurements. The analysis showed that the average air concentration for uniform flow conditions $C_e$ is independent of the upstream geometry (i.e. discharge, Froude number, relative roughness) and is a function of the slope only (table 1, column 2). Figure 3 shows the average air concentration $C_e$ as a function of the slope $\alpha$ for STRAUB and ANDERSON's (1958) data obtained on a model and field data presented by AIVAZYAN (1986). The agreement between the model and prototype data is good. For slopes flatter than 50 degrees, the average air concentration may be estimated as:

$$C_e = 0.9 \times \sin \alpha$$

It must be noted that AIVAZYAN's (1986) and JEVDJEVICH and LEVIN's (1953) data were initially presented with reference of the depth $Y_{98}$ corresponding to 98% air concentration and were recalculated with reference to $Y_{90}$.

HARTUNG and SCHEUERLEIN (1970) studied open channel flows with large natural roughness ($k_s$ in the range 0.1 to 0.35 m) and steep slopes ($\alpha$ in the range 6 to 34 degrees). The extremely rough bottom induced a highly turbulent flow with air entrainment. KNAUSS (1979) indicated that the quantity of air entrained was estimated as:

$$C_e = 1.44 \times \sin \alpha - 0.08$$

This result is of similar form as equation (6). Both equations (6) and (7) are plotted on figure 3.

For a given mean air concentration the diffusion of air bubbles within the air-water mixture can be represented by a simple model developed by WOOD (1984):

$$C = \frac{B'}{B' + e^{(G' \cos \alpha \times y^2)}}$$
where $B'$ and $G'$ are functions of $C_e$ only (table 1, columns 3 and 4) and $y' = y/Y_90$. However next to the spillway surface, CAIN's (1978) and CHANSON's (1988) data depart from the equation (8), and show consistently that the air concentration tends to zero at the bottom. A re-analysis of the data indicates the presence of an air concentration boundary layer, in which the air concentration distribution may be estimated as:

$$C = k \times \frac{3}{\sqrt{\frac{y}{\delta_{ac}}}}$$

(9)

where $k$ is a constant that satisfies the continuity between the equations (8) and (9), and $\delta_{ac}$ is the air concentration boundary layer thickness: $\delta_{ab} = 10$ to $15$ mm (CHANSON 1989).

Velocity distribution

Measurements of velocity within self-aerated flows were performed on Aviemore dam by CAIN (1978), and CAIN and WOOD (1981) showed that the velocity distribution can be approximated by:

$$\frac{V}{V_90} = \left( \frac{y}{Y_90} \right)^{1/n}$$

(10)

where the exponent $n$, for the roughness of the Aviemore dam ($k_s = 1$ mm, CAIN 1978), is: $n = 6.00$ (CHANSON 1989). For uniform non-aerated flows CHEN (1990) derived a theoretical relation between the exponent $n$ and the friction factor $f$ as:

$$n = K \times \sqrt{\frac{8}{f}}$$

(11)

where $K$ is the Von Karman universal constant ($K = 0.4$) and $f$ is dependent upon the Reynolds number and the roughness. On Aviemore dam equation (11) would imply $f = 0.0356$ which is higher than the values computed from the Colebrook-White formula for non-aerated flow (i.e. $f = 0.023$ and $0.022$).

CAIN's (1978) measurements were made in the gradually varied flow region with the mean air concentration in the range 0 to 50%. To a first approximation, the dimensionless velocity distribution $V/V_90$ is independent of the air concentration. It is reasonable to believe that this also applies in the uniform flow region (WOOD 1985). The characteristic velocity $V_90$ may be deduced by combining equation (10) with the continuity equation for the water phase, and this yields:

$$\frac{q_w}{Y_90 \times V_90} = \int_0^1 (1 - C) \times y^{1/n} \times dy'$$

(12)

where $C$ is computed from the equation (8). For the air-water mixture, it is also possible to define a momentum correction parameter $M$ and a kinetic energy parameter $E$ as:

\[
M = \frac{\int_{0}^{Y_{90}} (1 - C) \cdot V^2 \cdot dy}{\frac{1}{d} \cdot \left( \int_{0}^{Y_{90}} (1 - C) \cdot V \cdot dy \right)^2}
\]  

(13)

\[
E = \frac{\int_{0}^{Y_{90}} (1 - C) \cdot V^3 \cdot dy}{\frac{1}{d^2} \cdot \left( \int_{0}^{Y_{90}} (1 - C) \cdot V \cdot dy \right)^3}
\]  

(14)

where \(d\) is defined in equation (3). Using equation (10) the dimensionless formulation of these parameters becomes:

\[
M = (1 - C_e) \cdot \frac{\int_{0}^{1} (1 - C) \cdot y^{2/n} \cdot dy'}{\left( \int_{0}^{1} (1 - C) \cdot y^{1/n} \cdot dy' \right)^2}
\]  

(15)

\[
E = (1 - C_e)^2 \cdot \frac{\int_{0}^{1} (1 - C) \cdot y^{3/n} \cdot dy'}{\left( \int_{0}^{1} (1 - C) \cdot y^{1/n} \cdot dy' \right)^3}
\]  

(16)

The equations (12), (15) and (16) provide the analytical solutions for \(V_{90}\), \(M\) and \(E\). For \(n = 6.0\), they are plotted on figures 4 and 5 as a function of the average air concentration, and compared with experimental data obtained on prototype (JEVDJEVICH and LEVIN 1953, CAIN 1978) and model (CHANSON 1988). The scatter of the model data differs from the prototype results because of the limitation of the instrumentation on the spillway model.

**Friction factor of self-aerated flows**

For uniform aerated flow the energy equation yields:

\[
f_e = \frac{8 \cdot g \cdot \sin \alpha \cdot d^2}{q_w^2} \cdot \left( \frac{D_H}{4} \right)
\]  

(17)

where \(f_e\) is the friction factor for the uniform air-water mixture and \(D_H\) the hydraulic diameter. WOOD (1983) analysed STRAUB and ANDERSON's (1958) data and showed that the friction factor for aerated flow \(f_e\) decreases when the average air concentration increases. Prototype data (JEVDJEVICH and LEVIN 1953, AIVAZYAN 1986) confirmed the reduction in the friction factor observed on model. The data were re-analysed using equation (17) formulated in terms of hydraulic diameters to take into account the shape of the channel cross-section. The results are presented in figure 6 where the ratio \(f_e/f\) is plotted as a function of
the average air concentration, \( f \) being the non-aerated friction factor calculated using the Colebrook-White formula. Details of the range of roughness and Reynolds numbers are reported in table 2.

Dimensional analysis suggests that the ratio \( f_{e}/f \) is a function of the average air concentration, Reynolds number and roughness:

\[
f_{e}/f = \Phi(C_{e}; \text{Re}; k_{s}/D_{H})
\]  

(18)

The author investigated the effect of the Reynolds number and the roughness on the ratio \( f_{e}/f \). For the data of JEVDJEVICH and LEVIN (1953), STRAUB and ANDERSON (1958) and AIVAZYAN (1986), the equation (18) is estimated by (appendix I):

\[
f_{e}/f = 0.5 \left( 1 + \tanh\left( \lambda \cdot \frac{C_{0.5} - C_{e}}{C_{e} \cdot (1 - C_{e})} \right) \right)
\]  

(19)

where \( C_{0.5} \) represents the mean air concentration for \( f_{e} = 0.5 \cdot f \):

\[
C_{0.5} = 0.0593 + 0.07494 \cdot \log_{10}(\text{Re})
\]

\[
\lambda = 0.4726 \left( 1 + (3.6644 - 0.4729 \cdot \log_{10}(\text{Re})) \cdot \left( 2.5915 + \log_{10}\left( \frac{k_{s}}{D_{H}} \right) \right) \right)
\]

A comparison between the equation (19) and the experimental data is presented on figure 7. The results are within the accuracy of the data. The general trend is that, for a given average air concentration, the aerated friction factor increases with the Reynolds number toward the non-aerated friction factor. When the Reynolds number increases, the average shear stress increases and the average bubble size decreases. For small bubble sizes the turbulence is less affected by the presence of the bubbles which may explain the increase of the aerated friction factor toward the non-aerated values when the Reynolds number increases.

The ratio \( f_{e}/f \) is less affected by the roughness. A close scrutiny of the data suggests that, for low air concentration (i.e. \( C_{e} < 0.40 \)), the ratio \( f_{e}/f \) increases with the relative roughness for a given Reynolds number. But the lack of data for large air concentrations and large roughness prevents a generalisation of that trend. For a given air concentration and Reynolds number, the shear stress increases with the roughness and the turbulence would be less affected by smaller bubble sizes.

HARTUNG and SCHEUERLEIN (1970) performed experiments on extremely rough bottom channels (i.e. \( k_{s}/D_{H} \) in the range 0.02 to 0.2) and their results are presented as:

\[
f_{e}/f = \frac{1}{(1 - 3.2 \cdot \sqrt{f} \cdot \log_{10}(1 - C_{e}))^{2}}
\]  

(20)

where \( C_{e} \) is estimated from the equation (7). This result shows also a reduction of the ratio \( f_{e}/f \) with an increase of air concentration. Further in fully rough turbulent flows the equation (20) suggests that the ratio \( f_{e}/f \) is independent of the Reynolds number and decreases with increasing roughness.
Similar reductions of the friction factor were observed with suspended sediment in water flows: VANONI (1946) suggested that the effect of the sediment is to reduce the turbulence, and for neutrally buoyant particles ELATA and IPPEN (1961) indicated that suspended particles change the structure of the turbulent motion. Observations obtained in sediment flows (VANONI 1946) and aerated flows (KILLEN 1968) suggest that the Von Karman universal constant $\kappa$ decreases as the concentration of particles increases. On Aviemore dam the shape of the velocity distribution implies a value of $\kappa = 0.318$ (eq. (10)). But RAO and KOBUS (1971) showed that the presence of air concentration increases the value of $\kappa$, while COLEMAN (1981) indicated that the Karman coefficient does not change with increasing suspended sediment. The complete process is not yet clear but the author believes that the interactions between the turbulent shear stress, the velocity distribution and the air concentration boundary layer next to the channel bottom, play a major role in the drag reduction process. The presence of air bubbles is expected to affect the turbulence, and any change in the Von Karman constant means that the turbulent mixing mechanism has been altered.

The bubble size is an important parameter in the alterations of the turbulence. In aerated flows, the size of the bubbles varies across the flow from large sizes near the free surface ($d_b > 10$ mm) down to small diameters next to the channel bottom ($d_b < 1$ mm) (CAIN 1978). In a turbulent shear flow the mean bubble size is determined by the balance between the capillary force and the inertial force caused by the velocity changes over distances of the order of the bubble diameter. HINZE (1955) showed that the splitting of air bubbles occurs for:

$$\frac{\rho_w \cdot v^2 \cdot d_b}{2 \cdot \sigma} > (\text{We})_c$$  \hspace{1cm} (21)

where $v^2$ the spatial average value of the square of the velocity differences over a distance equal to $d_b$, and $(\text{We})_c$ is a critical Weber number. Experiments showed that $(\text{We})_c$ is a constant near unity (0.59, HINZE 1955; 1.26, SEVIK and PARK 1973; 1.02, KILLEN 1982). A maximum bubble size $(d_b)_c$ can be defined from the equation (21). Assuming that the term $v^2$ is order of magnitude of:

$$v^2 \sim \left( \frac{d}{dy} + d_b \right)^2$$  \hspace{1cm} (22)

and for a power law velocity distribution (eq. (10)), the maximum bubble size is in order of magnitude of:

$$\frac{(d_b)_c}{Y_{90}} \sim \sqrt{\frac{3}{2 \cdot n^2 \cdot \frac{(\text{We})_c}{\rho_w \cdot V_{90}^2 \cdot Y_{90}^2} \cdot \left( \frac{Y}{Y_{90}} \right)^{2(n-1)/n}}}$$  \hspace{1cm} (23)

where $n$ is the exponent of the power law. This simple formulation satisfies the common sense that the maximum bubble size increases with the depth as the shear stress decreases. For CAIN's (1978) and CHANSON's (1988) flow conditions, with $(\text{We})_c = 1$ and $n = 6.0$, equation (23) is presented on figure 8.
Although equation (21) was developed for individual bubbles in shear flows, a comparison between the results of equation (23) and observations of average bubble sizes (table 3) shows a good agreement.

**GRADUALLY VARIED FLOW REGION**

**Continuity equation for air**

Downstream of the point of inception CAIN’s (1978) data indicate a slow increase of the quantity of air entrained along the spillway. WOOD (1985) showed that, for a given mean air concentration, the air concentration distribution has a shape that is close to the equilibrium air concentration distribution. In the gradually varied flow region, assuming a slow variation of the rate of air entrainment, slow variations of the velocity with distance and a hydrostatic pressure distribution, the continuity equation for the air phase and the energy equation can be solved.

The continuity equation for the air phase yields (WOOD 1985):

\[
\frac{d}{ds} q_{\text{air}} = V_e - C_{\text{mean}} u_r \cos \alpha
\]

where \(q_{\text{air}}\) is the quantity of air entrained within the flow, \(u_r\) the local bubble rise velocity and \(V_e\) is the local entrainment velocity. The entrainment velocity characterizes the quantity of air entrained by turbulent eddies close to the free surface. In uniform flow the limit of the equation (24) is:

\[
0 = [V_e]_e - C_e [u_r]_e \cos \alpha
\]

where \([V_e]_e\) and \([u_r]_e\) are the entrainment velocity and rise velocity at equilibrium, in the uniform flow region. Denoting:

\[
K_e = \frac{V_e}{[V_e]_e}, \quad K_r = \frac{u_r}{[u_r]_e}
\]

the continuity equation yields:

\[
\frac{d}{ds} q_{\text{air}} = (K_e C_e - K_r C_{\text{mean}}) [u_r]_e \cos \alpha
\]

After transformation the continuity equation for the air phase becomes:

\[
\frac{d}{ds'} C_{\text{mean}} = (1 - C_{\text{mean}}) *

\left( \frac{[u_r]_e d^* \cos \alpha}{q_w} * (K_e C_e - K_r C_{\text{mean}}) * (1 - C_{\text{mean}}) + \frac{C_{\text{mean}} d W'}{d s'} \right)
\]

where \(W\) is the channel width, \(d^*\) is the flow depth at the origin (\(s = 0\)), \(s' = s/d^*\) and \(W' = W/d^*\).

The parameters \(K_e\) and \(K_r\) are expected to be functions of the turbulence level and bubble size. If the turbulence level and the bubble size distribution vary from the point of inception down to the uniform flow region, the parameters \(K_e\) and \(K_r\) will be different from unity but will tend to one in uniform equilibrium flow. For turbulent air-water pipe flows, WANG et al. (1990) showed that the presence of bubbles increases

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2The quantity of air entrained within the flow and the mean air concentration are related by:

\[
q_{\text{air}}/q_w = C_{\text{mean}}/(1 - C_{\text{mean}}) \quad \text{(see discussion by CHANSON 1989)}
\]
the level of turbulence. For self-aerated flows the presence of bubbles is thus expected also to increase the level of turbulence, and the entrainment velocity may increase from the point of inception to the uniform region (i.e. $K_e < 1$). Downstream of an aeration device, the high level of turbulence (CHANSON 1989) suggests that $K_e > 1$. Also, as the size of the bubbles increases with the mean air concentration and hence the rise velocity, $K_r$ is expected to be less than 1 for $C_{\text{mean}} < C_e$. Examples of the author's expectations for the trend for $K_e$ and $K_r$ are presented in table 4.

Assuming that the entrainment and rise velocities are the same in gradually varied flow as in uniform flow (i.e. $K_e = K_r = 1$), equation (27) can be written as:

$$\frac{d}{ds'} C_{\text{mean}} = (1 - C_{\text{mean}}) \left( \frac{u_r \cdot d* \cdot \cos \alpha}{q_w} \right) (C_e - C_{\text{mean}}) \cdot (1 - C_{\text{mean}}) + \frac{C_{\text{mean}} \cdot W' \cdot d W'}{ds'}$$

(28)

It must be noted that the equation (28) allows the calculations of the average air concentration $C_{\text{mean}}$ as a function of the distance along the chute independently of the velocity, roughness and flow depth. For a channel of constant width the continuity equation for air becomes the equation obtained by WOOD (1985):

$$\frac{d}{ds'} C_{\text{mean}} = \frac{u_r \cdot d* \cdot \cos \alpha}{q_w} (C_e - C_{\text{mean}}) \cdot (1 - C_{\text{mean}})$$

(29)

and for a constant channel slope, the analytical solution is:

$$\frac{1}{(1 - C_e)^2} \cdot \ln \left( \frac{1 - C_{\text{mean}}}{C_e - C_{\text{mean}}} \right) - \frac{1}{(1 - C_e) \cdot (1 - C_{\text{mean}})} = k \cdot s' + K_0$$

(30)

where $K_0$ and $k$ are:

$$K_0 = \frac{1}{1 - C_e} \cdot \ln \left( \frac{1 - C_\ast}{C_e - C_\ast} \right) - \frac{1}{1 - C_\ast}$$

$$k = \frac{u_r \cdot d* \cdot \cos \alpha}{q_w}$$

and $C_\ast$ and $d_\ast$ are the mean air concentration and flow depth at the origin ($s = 0$).

Experimental data on model and prototype self-aerated flows (STRAUB and LAMB 1956, ISACHENKO 1965, RAO and KOBUS 1971, CAIN 1978, XI 1988) and downstream of an aeration device (CHANSON 1988) were used to verify the equation (30). The data provides straight lines with a mean normalized coefficient of correlation of 0.90 (table 5, column 7). The slope of these lines implies values of the rise velocity $u_r$ in the range 0.2 to 41 cm/s (table 5, column 5). These values of $u_r$ may be interpreted as the average value for each experiment and are plotted as a function of the flow velocity at the start of air entrainment (i.e. $U_w = q_w/d_\ast$) on figure 9.

For self-aerated flows the flow velocity $U_w = q_w/d_\ast$ is that at the point of inception where the flow is nearly uniform and the turbulence quasi-homogeneous. Hence the velocity $[q_w/d_\ast]$ characterizes the turbulence of the flow, and figure 9 would suggest that the rise velocity may increase with the turbulence.
Even so this result must be balanced by the effects of the bubble size. The observations of bubble sizes, reported in table 3, indicate that the largest values of $u_r$ presented on figure 9, were obtained with large bubble sizes (i.e. CAIN 1978).

**Energy equation**

In the gradually varied flow region, assuming a quasi-hydrostatic pressure distribution and slow variations of the velocity, the energy equation yields (WOOD 1985, CHANSON 1989):

\[
\frac{d}{ds'} \left( d' \right) = \frac{\sin \alpha \left( 1 + d' \frac{d \alpha}{ds'} \right) - S_f + E \frac{d'}{W'} \frac{Fr^2 \frac{d W'}{ds'}}{d'^3} - \frac{d W'}{ds'} \cos \alpha - E \frac{Fr^2 \frac{d W'}{ds'}}{d'^3}}{\cos \alpha - E \frac{Fr^2 \frac{d W'}{ds'}}{d'^3}}
\]

where $E$ is the kinetic energy correction parameter (eq. (14) & (16)), $d' = d/d*$, $Fr = q_w/\sqrt{g * d^3}$, and $S_f$ is the friction slope for aerated flow defined as:

\[
S_f = \frac{q_w^2 * f_c}{8 * g * d^2} * \left( \frac{4}{D_H} \right)
\]

where $f_c$ is the local value of the aerated friction factor. The slow increase of flow aeration, observed on both prototype and model, suggests that the friction factor $f_c$ and the energy parameter $E$ can be computed as in uniform flow using the local value of $C_{mean}$.

**Discussion**

The equations (27) and (31) provide two simultaneous differential equations in terms of the mean air concentration and the flow depth, and these equations can be solved with a simple explicit numerical scheme. The knowledge of $C_{mean}$ and $d$ at any point along the chute enables the calculation of $Y_{90}$ (eq. (4)), $V_{90}$ (eq. (12)), the integration constants $B'$ and $G' \cos \alpha$, the air concentration distribution (eq. (8)) and the velocity distribution (eq. (10)).

It must be emphasized that the calculations depend on the assumed rise velocity $u_r$. Further equation (27) depends also on the assumed coefficients $K_c$ and $K_r$. At the present time little information is available on these parameters. As a first approximation it is convenient to use $K_c = K_r = 1$ but these parameters may also be determined empirically from existing experiments as described in the next paragraph.

**APPLICATION**

The equations (29) and (31) were used to reproduce air entrainment on a prototype spillway (Aviemore) and on a large model (Meishan Hydraulics Lab.). Figure 10 presents a comparison between calculations and data for CAIN’s (1978) and XI’s (1988) experiments, where $L$ is the distance from the point of inception. The
location and flow depth at the point of inception were measured by CAIN (1978) at Aviemore dam, and were computed for XI's (1988) experiment using WOOD's (1985) formula. The rise velocity was obtained from the table 5, and the roughness heights were taken as: \( k_s = 1 \text{ mm} \) (CAIN 1978) and \( k_s = 0.03 \text{ mm} \) (XI 1988). In each case the agreement between the data and the analytical results is good.

**Application to tunnel spillway flow - Grande-Dixence**

VOLKART and RUTSCHMANN (1984) performed air concentration measurements in a tunnel spillway of rectangular cross-section \((W = 0.8 \text{ m})\). Using equation (29) their data suggest computed rise velocities \( u_r = 0.023 \text{ and } 0.005 \text{ m/s} \) for \( q_w = 2.75 \text{ and } 5.5 \text{ m}^2/\text{s} \). These results are not consistent with those obtained in table 5. Indeed in a tunnel spillway the amount of air available above the flow is limited. Further as the air flow above the water is accelerated, the air pressure will decrease. The entrainment velocity (and hence \( K_e \)) is expected to decrease along the tunnel spillway as the air flow above the water surface is accelerated, and as air is entrained within the air-water flow. The buoyancy force is not affected because the pressure gradient across the flow remains quasi-hydrostatic, and the bubble rise velocity may be assumed constant: \( u_r = [u_r]_e \)

Figure 11 shows a comparison between the experimental data obtained by VOLKART and RUTSCHMANN (1984) and the equations (27) and (31) as a function of the distance along the spillway from the intake. The calculations were done assuming \( k_s = 0.1 \text{ mm} \) (steel lining), \( K_r = 1 \), the coefficient \( K_e \) was selected by trial-and-error, decreasing from 0.6 down to 0.5, and the rise velocity \( [u_r]_e \) was taken as that computed for XI's (1988) experiments (i.e. \( u_r = 0.17 \text{ m/s} \)) on a flume of similar size and flow velocity.

**CONCLUSION**

In uniform aerated flows a complete flow description can be obtained as a function of the channel slope, the water discharge and the non-aerated friction factor. From the spillway geometry and the discharge, the main flow parameters (i.e. \( C_w \), \( d \), \( f_w \), \( V_{90} \)) and the air concentration and velocity distributions can be computed. Comparisons were made with experimental data obtained on model and prototype for slopes in the range 7.5 to 75 degrees. It is believed that the interactions between the shear stress, the air concentration boundary layer and the velocity distribution next to the channel surface might explain the drag reduction process. But further experimental work is required to obtain a better understanding of the drag reduction mechanisms in self-aerated flows.

In the gradually varied flow region the continuity equation for air and the energy equation provides two simultaneous differential equations in terms of the average air concentration and the flow depth. Predictions of self-aeration will however depend upon the estimation of the rise velocity, the entrainment velocity and the non-aerated friction factor. At the present time insufficient data are available and additional work on the turbulence parameters and bubble sizes is required.
A first analysis of air entrainment in tunnel spillway was developed. The uniform flow conditions are expected to be different to those obtained in chutes. They are function of the boundary conditions: the geometry of the tunnel, the water flow rate and the initial air intake geometry.

ACKNOWLEDGEMENTS

The author wishes to thank the Department of Civil Engineering, University of Queensland (Australia) for its support, MM. D. BAXTER and D. SARTOR who made the calculations of the Grande-Dixence tunnel spillway, and Professor I.R. WOOD for many helpful discussions. The author wish also to acknowledge the helpful comments of one of the reviewers.

APPENDIX I. Correlations between the ratio $f_e/f$, the air concentration, the Reynolds number and the roughness

The author investigated the effect of the Reynolds number and roughness on the ratio $f_e/f$, using the data of JEVDJEVICH and LEVIN (1953), STRAUB and ANDERSON (1958) and AIVAZYAN (1986).

Firstly the author analysed the data neglecting the effects of the Reynolds number and relative roughness, and found that the ratio $f_e/f$ may be estimated as:

$$
\frac{f_e}{f} = 0.5 \left( 1 + \tanh \left( 0.7 \cdot \frac{0.490 - C_e}{C_e \cdot (1 - C_e)} \right) \right)
$$

with a correlation of $R = 0.898$ (for 104 data points).

For engineering applications a simple correlation between the ratio $f_e/f$, the mean air concentration $C_{\text{mean}}$ and the Reynolds number $Re$ is:

$$
\frac{f_e}{f} = 0.307 + 0.1446 \cdot \log_{10}(Re) - 1.40 \cdot C_{\text{mean}}
$$

for $C_{\text{mean}} > 0.25$ and $2 \times 10^5 < Re < 4 \times 10^7$, with a correlation of $R = 0.913$. A more general correlation that works well at the limits is:

$$
\frac{f_e}{f} = 0.5 \left( 1 + \tanh \left( 0.7 \cdot \frac{C_{0.5} - C_{\text{mean}}}{C_{\text{mean}} \cdot (1 - C_{\text{mean}})} \right) \right)
$$

where $\tanh(x) = (\exp(x) - \exp(-x))/(\exp(x) + \exp(-x))$ and $C_{0.5}$ represents the mean air concentration for $f_e = 0.5f$:

$$
C_{0.5} = 0.1032857 \cdot \log_{10}(Re) - 0.1378571
$$

The equation (I.3) was established for Reynolds numbers in the range $2 \times 10^5$ to $4 \times 10^7$ and the normalized coefficient of correlation is $R = 0.914$. A more sophisticated correlation for the relationship $f_e/f = \Phi(C_{\text{mean}}, Re, k_s/D_H)$ that takes into account the influence of the roughness is:
$f_c \frac{T}{1} = 0.5 \left(1 + \tanh\left(\lambda \frac{C_{0.5} - C_{\text{mean}}}{C_{\text{mean}} (1 - C_{\text{mean}})}\right)\right)$  \hspace{1cm} (1.4)

where:

$C_{0.5} = 0.0593 + 0.07494 \log_{10}(Re)$

$\lambda = 0.4726 \left(1 + (3.6644 - 0.4729 \log_{10}(Re)) \cdot \left(2.5915 + \log_{10}\left(\frac{ks}{DH}\right)\right)\right)$

with a correlation of $R = 0.920$.

APPENDIX II. REFERENCES


APPENDIX III. NOTATION.

The following symbols are used in this paper:

- $B'$ = integration constant of the equilibrium air concentration distribution;
- $C$ = air concentration defined as the volume of air per unit volume;
- $C_e$ = depth averaged equilibrium air concentration (mean air concentration of uniform flow);
- $C_{\text{mean}}$ = depth averaged mean air concentration defined as: $(1 - C_{\text{mean}}) \cdot Y_{90} = d$;
- $C*$ = mean air concentration at the start of the gradually varied flow region;
- $C_{0.5}$ = mean air concentration for $f_e = 0.5 \cdot f$;
- $D_H$ = hydraulic diameter (m) defined as: $D_H = 4 \frac{\text{Area}}{\text{Wetted Perimeter}} = \frac{4 \cdot W \cdot d}{W + 2 \cdot d}$;
- $d$ = characteristic flow depth (m);
- $d_b$ = air bubble diameter (m);
- $(d_p)_c$ = maximum bubble diameter in shear flows;
- $d*$ = characteristic flow depth (m) at the start of the gradually varied flow region;
- $d'$ = dimensionless characteristic flow depth: $d' = d/d*$;
- $E$ = kinetic energy correction parameter;
- $Fr*$ = Froude number at the start of the gradually varied flow region: $Fr* = \sqrt{\frac{q_w}{g \cdot d*^3}}$;
- $f$ = friction factor of non-aerated flow;
- $f_e$ = friction factor of aerated flow;
- $G'$ = integration constant of the equilibrium air concentration distribution;

\( g \) = gravity constant (m/s\(^2\));
\( K \) = Von Karman universal constant;
\( K_e \) = ratio of the entrainment velocity over the equilibrium entrainment velocity:
\[ K_e = \frac{V_e}{[V_e]_e}; \]
\( K_r \) = ratio of the local rise velocity over the equilibrium rise velocity:
\[ K_r = \frac{u_r}{[u_r]_e}; \]
\( k_s \) = equivalent uniform sand roughness (m);
\( L \) = distance along the spillway (m);
\( M \) = momentum correction parameter;
\( n \) = exponent of the velocity power law;
\( q \) = discharge per unit width (m\(^2\)/s);
\( R \) = normalized coefficient of correlation;
\( Re \) = Reynolds number defined as:
\[ Re = \frac{\rho \cdot U \cdot D}{\mu}; \]
\( S_f \) = friction slope;
\( s \) = curvilinear coordinate (m);
\( s' \) = dimensionless curvilinear coordinate:
\[ s' = \frac{s}{d_s}; \]
\( U_w \) = average water velocity (m/s) defined as:
\[ U_w = \frac{q_w}{d}; \]
\( u_r \) = bubble rise velocity (m/s);
\([u_r]_e\) = bubble rise velocity in the uniform equilibrium flow region (m/s);
\( V \) = velocity (m/s);
\( V_e \) = entrainment velocity (m/s);
\([V_e]_e\) = entrainment velocity in the equilibrium flow region (m/s);
\( V_{90} \) = characteristic velocity at \( Y_{90} \) (m/s);
\( v' \) = root mean square of lateral component of turbulent velocity (m/s);
\( v^2 \) = spatial average value of the square of the velocity differences over a distance equal to \( d_b \) (m\(^2\)/s\(^2\));
\( W \) = channel width (m);
\((We)_c\) = critical Weber number characterizing the bubble splitting;
\( W' \) = dimensionless channel width:
\[ W' = \frac{W}{d_s}; \]
\( Y_{90} \) = characteristic depth (m) where the air concentration is 90%;
\( Y_{98} \) = characteristic depth (m) where the air concentration is 98%;
\( y \) = distance from the bottom measured perpendicular to the spillway surface (m);
\( y' \) = dimensionless depth:
\[ y' = \frac{y}{Y_{90}}; \]
\( \alpha \) = spillway slope;

\[ \mu = \text{dynamic viscosity (N.s/m}^2) \];
\[ \rho = \text{density (kg/m}^3) \];
\[ \sigma = \text{surface tension between air and water (N/m)} \].

**Subscript**
- air = air flow;
- e = equilibrium uniform aerated flow;
- w = water flow.
Table 1. Dimensionless air concentration and velocity distribution in uniform self-aerated flow

<table>
<thead>
<tr>
<th>Slope (degrees)</th>
<th>$C_{\text{mean}}$</th>
<th>$G^* \cos \alpha$</th>
<th>$B'$</th>
<th>$\frac{q_w}{V_{90} \cdot Y_{90}}$</th>
<th>$M$</th>
<th>$E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.5</td>
<td>0.161</td>
<td>8.000</td>
<td>0.003021</td>
<td>0.688</td>
<td>1.029</td>
<td>1.075</td>
</tr>
<tr>
<td>15.0</td>
<td>0.241</td>
<td>5.745</td>
<td>0.02880</td>
<td>0.609</td>
<td>1.039</td>
<td>1.097</td>
</tr>
<tr>
<td>22.5</td>
<td>0.310</td>
<td>4.834</td>
<td>0.07157</td>
<td>0.554</td>
<td>1.033</td>
<td>1.085</td>
</tr>
<tr>
<td>30.0</td>
<td>0.410</td>
<td>3.825</td>
<td>0.19635</td>
<td>0.467</td>
<td>1.042</td>
<td>1.105</td>
</tr>
<tr>
<td>37.5</td>
<td>0.569</td>
<td>2.675</td>
<td>0.6203</td>
<td>0.335</td>
<td>1.061</td>
<td>1.148</td>
</tr>
<tr>
<td>45.0</td>
<td>0.622</td>
<td>2.401</td>
<td>0.8157</td>
<td>0.301</td>
<td>1.038</td>
<td>1.097</td>
</tr>
<tr>
<td>60.0</td>
<td>0.680</td>
<td>1.894</td>
<td>1.354</td>
<td>0.241</td>
<td>1.107</td>
<td>1.249</td>
</tr>
<tr>
<td>75.0</td>
<td>0.721</td>
<td>1.574</td>
<td>1.864</td>
<td>0.206</td>
<td>1.138</td>
<td>1.318</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>+ infinite</td>
<td>0.00000</td>
<td>$\frac{n}{n+1}$</td>
<td>$\frac{(n+1)^2}{n \cdot (n+2)}$</td>
<td>$\frac{(n+1)^3}{n^2 \cdot (n+3)}$</td>
</tr>
</tbody>
</table>

Note:

(I) Data from STRAUB and ANDERSON (1958)

(II) Computed from STRAUB and ANDERSON's data

(III) Analytical formula for a power law velocity distribution (eq. (10))

(IV) Computed value for $n = 6.0$
Table 2. Flow conditions for JEVDJEVICH and LEVIN's (1953), STRAUB and ANDERSON's (1958) and AIVAZYAN's (1986) data

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Slope (degrees)</th>
<th>k_s (mm)</th>
<th>k_s/D_H</th>
<th>Re</th>
<th>C_mean</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mostarsko Blato (I)</td>
<td>---</td>
<td>10 to 20</td>
<td>0.015 to</td>
<td>8.3E+4 to 3E+7</td>
<td>0.58 to</td>
<td>Prototype. Wide channel (W = 5.75 m). Stone lining.</td>
</tr>
<tr>
<td>STRAUB and ANDERSON (II)</td>
<td>7.5 to 75</td>
<td>0.71</td>
<td>3E-3 to 4.7E+5</td>
<td>0.15 to</td>
<td>5.75 m.</td>
<td></td>
</tr>
<tr>
<td>AIVAZYAN (III)</td>
<td>14 to 31</td>
<td>0.1 to 10</td>
<td>5E-4 to 1.7E+5</td>
<td>0.21 to</td>
<td>0.3 to 4 Flow downstream of an aerator.</td>
<td></td>
</tr>
</tbody>
</table>

Note: (I) JEVDJEVICH and LEVIN (1953)  
      (II) STRAUB and ANDERSON (1958)  
      (III) AIVAZYAN (1986)

Table 3. Average bubble size in self-aerated flows: observed values

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Slope (degrees)</th>
<th>q_w (m^2/s)</th>
<th>V_90 (m/s)</th>
<th>Y_90 (m)</th>
<th>d_b (mm)</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aviemore (I)</td>
<td>45.00</td>
<td>2.23 ; 3.16</td>
<td>20.5 ; 21.7</td>
<td>0.26 ; 0.31</td>
<td>3 to 20</td>
<td>Prototype. Wide channel.</td>
</tr>
<tr>
<td>Clyde dam model (II)</td>
<td>52.33</td>
<td>0.20 to 0.40</td>
<td>12.3 to 17.8</td>
<td>0.036 to 0.054</td>
<td>0.3 to 4</td>
<td>Flow downstream of an aerator.</td>
</tr>
<tr>
<td>St Anthony Falls (III)</td>
<td>---</td>
<td>0.136 to 0.793</td>
<td>---</td>
<td>---</td>
<td>0.7 to 2.7</td>
<td>STRAUB and ANDERSON's experiment.</td>
</tr>
</tbody>
</table>

Note: (I) CAIN (1978)  
      (II) CHANSON (1988)  
      (III) GULLIVER et al. (1990)
Table 4. Entrainment and rise velocity parameters $K_e$ and $K_r$

<table>
<thead>
<tr>
<th>$K_e$</th>
<th>$K_r$</th>
<th>Application</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt; 1</td>
<td>&lt; 1</td>
<td>Flow downstream of the point of inception.</td>
<td>Low turbulence and small bubbles.</td>
</tr>
<tr>
<td>&gt; 1</td>
<td>&gt; 1</td>
<td>Flow on a ski jump.</td>
<td>Low turbulence and $u_r$ increases with the pressure gradient.</td>
</tr>
<tr>
<td>= 1</td>
<td>= 1</td>
<td>Uniform equilibrium flow.</td>
<td></td>
</tr>
<tr>
<td>&lt; 1</td>
<td>&lt; 1</td>
<td>Flow downstream of an aeration device with steep slope.</td>
<td>High turbulence and $C_{mean} &lt; C_e$.</td>
</tr>
<tr>
<td>= 1</td>
<td>= 1</td>
<td>Open channel flow in a tunnel spillway.</td>
<td>$V_e$ is reduced by the limited amount of air.</td>
</tr>
<tr>
<td>&gt; 1</td>
<td>&gt; 1</td>
<td>Flow downstream of an aeration device with flat slope.</td>
<td>High turbulence and $C_{mean} &gt; C_e$.</td>
</tr>
</tbody>
</table>

Table 5. Bubble rise velocity : computed values

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Slope degrees</th>
<th>$q_w$ m$^2$/s</th>
<th>$U_w$ m/s</th>
<th>$u_r$ m/s</th>
<th>$k_s$ mm</th>
<th>$R$ correl.</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aviemore dam</td>
<td>45.0</td>
<td>2.23</td>
<td>14.7</td>
<td>0.40</td>
<td>1.0</td>
<td>0.998</td>
<td>SELF-AERATED FLOW</td>
</tr>
<tr>
<td></td>
<td>3.16</td>
<td>16.3</td>
<td>0.39</td>
<td>1.0</td>
<td>0.981</td>
<td></td>
<td>CAIN (1978)</td>
</tr>
<tr>
<td>Meishan</td>
<td>52.5</td>
<td>0.320</td>
<td>8.3</td>
<td>0.17</td>
<td>0.07</td>
<td>0.995</td>
<td>XI (1988)</td>
</tr>
<tr>
<td>St Anthony Falls</td>
<td>30.0</td>
<td>0.396</td>
<td>7.4</td>
<td>0.11</td>
<td>0.05</td>
<td>0.988</td>
<td>STRAUB and LAMB (1956)</td>
</tr>
<tr>
<td>Isachenko</td>
<td>21.25</td>
<td>0.15</td>
<td>4.56</td>
<td>0.039</td>
<td>0.1</td>
<td>0.966</td>
<td>ISACHENKO (1965)</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>5.47</td>
<td>0.033</td>
<td>0.1</td>
<td>0.991</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.40</td>
<td>6.47</td>
<td>0.046</td>
<td>0.1</td>
<td>0.983</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.60</td>
<td>7.48</td>
<td>0.047</td>
<td>0.1</td>
<td>0.994</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.90</td>
<td>8.65</td>
<td>0.043</td>
<td>0.1</td>
<td>0.844</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>4.85</td>
<td>0.016</td>
<td>3.0</td>
<td>0.682</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.40</td>
<td>5.74</td>
<td>0.053</td>
<td>3.0</td>
<td>0.989</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.60</td>
<td>6.63</td>
<td>0.069</td>
<td>3.0</td>
<td>0.991</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.90</td>
<td>7.66</td>
<td>0.076</td>
<td>3.0</td>
<td>0.991</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>4.71</td>
<td>0.056</td>
<td>7.0</td>
<td>0.903</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.40</td>
<td>5.57</td>
<td>0.061</td>
<td>7.0</td>
<td>0.945</td>
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<tr>
<td></td>
<td>0.60</td>
<td>6.43</td>
<td>0.069</td>
<td>7.0</td>
<td>0.903</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.90</td>
<td>7.44</td>
<td>0.082</td>
<td>7.0</td>
<td>0.928</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rao and Kobus</td>
<td>31.98</td>
<td>0.095</td>
<td>4.1</td>
<td>0.003</td>
<td>0.91</td>
<td>0.339</td>
<td>RAO and KOBUS (1971)</td>
</tr>
<tr>
<td>Clyde dam model</td>
<td>52.33</td>
<td>0.212</td>
<td>8.2</td>
<td>0.11</td>
<td>0.1</td>
<td>0.985</td>
<td>FLOW DOWNSTREAM OF</td>
</tr>
<tr>
<td></td>
<td>0.396</td>
<td>12.0</td>
<td>0.16</td>
<td>0.1</td>
<td>0.567</td>
<td></td>
<td>AN AERATION DEVICE</td>
</tr>
<tr>
<td></td>
<td>0.345</td>
<td>10.5</td>
<td>0.10</td>
<td>0.1</td>
<td>0.714</td>
<td></td>
<td>CHANSON (1988)</td>
</tr>
<tr>
<td></td>
<td>0.340</td>
<td>9.2</td>
<td>0.048</td>
<td>0.1</td>
<td>0.639</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.273</td>
<td>8.3</td>
<td>0.049</td>
<td>0.1</td>
<td>0.867</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.210</td>
<td>6.4</td>
<td>0.044</td>
<td>0.1</td>
<td>0.977</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.429</td>
<td>6.8</td>
<td>0.011</td>
<td>0.1</td>
<td>0.685</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: $q_w$ discharge per unit width - $k_s$ roughness height - $R$ coefficient of correlation

$u_r$ rise velocity (eq. (30)) - $U_w$ flow velocity at the start of self-aeration : $U_w = q_w / d^*$
Figure 1 - Critical turbulent velocity $v'$ for air entrainment

Fig. 2 - Self-aeration on chute spillway
Fig. 3 - Equilibrium air concentration as a function of the channel slope - STRAUB and ANDERSON (1958) and AIVAZYAN (1986)
Fig. 4 - Characteristic velocity $V_{90}$ and momentum correction coefficient $M$ as a function of the equilibrium air concentration $C_e$
Fig. 5 - Kinetic energy correction coefficient $E$ as a function of the equilibrium air concentration $C_e$
Fig. 6 - Relative friction factor $f_e/f$ as a function of the equilibrium air concentration $C_e$
Fig. 7 - Comparison between the equation (19) and the data of JEVDJEVICH and LEVIN (1953), STRAUB and ANDERSON (1958) and AIVAZYAN (1986)
Fig. 8 - Maximum bubble size distribution in a turbulent shear flow - Equation (23)
Figure 9 - Bubble rise velocity $u_r$ estimated from the equation (30) as a function of the flow velocity at the start of self-aeration $U_w = q_w/d$.
Figure 10 - Self-aerated flow calculations (A) $\alpha = 45$ deg., $q_w = 2.16$ m$^2$/s - $d^* = 0.152$ m - $u_r = 0.40$ m/s - $k_s = 1$ mm - CAIN (1978); (B) $\alpha = 52.5$ deg., $q_w = 0.32$ m$^2$/s - $d^* = 0.039$ m - $u_r = 0.17$ m/s - $k_s = 0.03$ mm - XI (1988)
Figure 11 - Air entrainment on the Grande-Dixence tunnel spillway
L : distance from the tunnel spillway intake - VOLKART and RUTSCHMANN (1984)
α = 31 and 34.5 degrees - \( q_w = 2.75 \text{ m}^2/\text{s} \) - \( u_r = 0.17 \text{ m/s} \) - \( k_s = 0.1 \text{ mm} \)