Buyer-Seller Quantity-Discount Strategies Under Profit Maximizing Objectives: A Game-Theoretical Analysis

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Abstract:
This paper is intended to provide a detailed game-theoretical analysis of the buyer-vendor coordination problem embedded in the price-discount inventory model. Pure and mixed, cooperative and non-cooperative strategies are developed. Highlights of the paper include the full characterization of the Pareto optimal set, the determination of profit-sharing mechanisms for the cooperative case and the derivation of a set of parameter-specific non-cooperative mixed strategies. A numerical example is presented to illustrate the main features of the model.
Keywords: Game Theory, Inventory, Price-Discount, Buyer-Vendor

Article:

1. Introduction

This paper presents a game-theoretical approach to the analysis of the buyer-vendor coordination problem embedded in the price discount model (PDM). The basic PDM, whereby existing purchasing practices may be modified in exchange for a discount in the price of the merchandise, is well known in the literature [41] and so is the rationale for giving price discounts (e.g. [8], [10]). Spurred by calls for an improved buyer-vendor relationship (e.g. [6], [18]), the last few years have witnessed a revival of interest in this problem, in an effort to explore tile feasibility of finding coordinated solutions to the PDM. As a result, there exists a large number of PDM versions available, to the point where the term PDM is becoming simply a convenient notational simplification. Goyal and Gupta [16], Kohli and Park [25] and Joglekar and Tharthare [22] review this literature. At issue here is whether market-induced PDM solutions, where rational players acting independently of others attempt to optimize their own objectives, are superior to strategies involving some degree of coordination among the parties involved.

The process of developing the PDM needed to evaluate these strategies has focused on the objectives of the parties and on the assumptions of the PDM itself. The pioneering work of Goyal [12], Monahan [26] and Banerjee [2] extended the basic PDM to consider the vendor's, in addition to the buyer's, problem of setting pricing and ordering policies, within a cost minimization framework. Subsequent PDM formulations have been developed to satisfy a variety of objectives. Included here are the optimization of joint costs (e.g. [2], [12], [19], [22]), of joint profits (e.g. [9], [26]) or the interaction between two return on investment (ROI) maximizers (e,g. [1]) or two profit maximizers (e.g. [35]) or a profit maximizing vendor and a cost minimizing buyer (e.g, [15], [25], [30]). In addition, efforts have been directed towards rendering the various assumptions of the PDM more in tune with modern purchasing practices. This has resulted in research on the number of buyers (e,g, [7], [19], [20], [26], [35]), on the specification of the vendor's cost (e.g. [41, [27], [19]) and profit (e.g. [3]) functions, on the appropriateness of the original lot-for-lot policy assumption (e.g. [1], [15], f21], [22], [27]), on the type of discount offered (e.g. [23], [38]) and on the nature of the vendor-buyer relationship (e.g. [7], [25], [35], [38]).

In spite of the numerous studies on the PDM, the fundamental nature of the PDM buyer-seller interaction has not been addressed fully. For example, Rubin and Carter [1990] deal only with cost reductions without an indication as to possible sharing arrangements. Chakravarti and Martin [7] consider a cost-saving sharing policy agreed upon in advance by a single seller and multiple buyers, while retaining many of the assumptions of earlier PDM versions. Parlar and Wang [35] consider a fixed margin, linear demand model and thus the buyer's demand is not optimally determined as a function of price. Finally, Kohli and Park [25] adapt the profit-sharing solutions of Nash [311 Kalai and Smordinsky [24] and Eliashberg [11] to an earlier PDM, with lot-for-lot policies and Monahan's [30] assumption of holding costs being unaffected by the discount.

However, without exploring further the nature of the resulting solutions, it is certainly not enough (i) to show the existence or lack thereof of a unique solution to a particular version of the PDM; nor (ii) to identify a potential negotiating range within which a feasible agreement may evolve; nor even (iii) to define a-priori profit-sharing rules, such as passing all the savings to one party (e.g. [27]) or a profit-sharing parameter (e.g. [13], [14], [421) where cooperation is possible. What is needed is to capture the complexity of the conflicts and possible cooperative forces which may affect the sharing arrangement between the parties (e.g. [32]). For example, modern manufacturing practices (e.g. [29]) lean toward longer-term, single-sourcing (or at least very few sources), more information sharing, buyer-vendor arrangements. The resulting negotiating experience gained from these closer links tend to increase the incentives for cooperative behaviour (e.g. [36]), to the point of sometimes producing Nash-equilibrium solutions, even within a freely competitive framework (e.g. [5], [32]).

The above discussion suggests game theory as the appropriate vehicle for the analysis of these interaction issues. To that effect, this paper derives a whole gamut of game-theoretical solutions to the coordination problem, embedded in a generalized PDM, with profits as optimizing objectives. Section 2 sets the stage for the analysis by introducing the mathematical formulation of the PDM, by studying the properties of the resulting profit functions and by defining the ensuing game. The model considers a one buyer one seller environment, but it may also be applicable to a homogeneous set of buyers (e.g. [9], [10], [25], [26], [35]). Sections 3, 4 and 5 evaluate respectively non-cooperative (pure and mixed strategies) and cooperative pure strategies forms of the PDM game. A numerical example is introduced in section 6 to highlight the main features of the solutions. Fingly, a Conclusions section completes the paper. Proofs of all Theorems and Lemmas are provided in Appendix B.

2. Model Formulation

This section first describes formally the generalized PDM formulation and then defines the game which results from this characterization.

2.1 The Generalized PDM Formulation

Before proceeding to the development of the model, some notation is needed. The buyer and the seller are identified by b and s, respectively. Currently, they are assumed to be operative at a procurement level of Q_0* units per buyer order. However, the seller is interested in inducing a higher purchasing level by offering a discount. If the discount is rejected, the current Q_0* order policy remains in force. Let d be the unit discount, expressed as a fraction of the item's dollar value. Then, the parameter vector for the buyer consists of the unit selling price, p_b, the unit purchasing cost, $c_b(1 - d)$, the unit profit margin, m_b, the holding cost per dollar per unit of time, h_b, and the ordering cost per order, a_b. The corresponding vector for the vendor is denoted by $p_s(1 - d) = c_b(1 - d)$, m_s, c_s, h_s, and a_s, respectively. The profit margins are defined as

$$m_b = p_b - c_b(1 - d)$$

$$\tag{1}$$

$$m_s = c_b(1 - \mathrm{d}) - c_s$$

Confronted with the seller's offer of a fixed discount level, d, the buyer purchases in lots of Q units to satisfy a constant demand of D units per year (e.g. [25], [26], for the implications of the constant demand assumption). As a result, the profit function, π_b, is

$$\pi_b\,(Q) = m_bD — a_bD/Q — c_b(1 — \mathrm{d})\, h_bQ/2 \qquad (2)$$

Confronted with orders of Q units, the selling firm must decide on the number of orders, $n \geq 1$, of size Q to purchase/manufacture, as well as on the discount level, d, it is willing to offer. Note that the $n \geq 1$ condition is tantamount to dropping the lot-for-lot policy assumption of a few of the earlier PDM versions, in favour of the more common formulation adopted by some of the more recent studies cited earlier. The seller's profit function, π_s, is given by (e.g. [27], [38])

$$\pi_s(d, nQ) = m_sD — a_sD/nQ — h_sc_s(n — 1)\, Q\,/2 \qquad (3)$$

Observe that in (2) and (3), Q is unconstrained from above. "This follows from the usual PDM assumption that the seller's production rate is essentially infinite. Further, the properties of the profit functions in (2) and (3) with respect to the order quantity, Q, may be summarized as follows

Property 1. Properties of the profit functions with respect to the order quantity

1a. $\pi_b{}^*(Q)$ has a unique maximum at

$$Q^* = \sqrt{2a_bD\,/c_b(1-d)h_b} \qquad (4)$$

1b. For n real and a given Q, π_s (d, nQ) has a unique maximum at

$$n^* = \mathrm{Max}\{\,\sqrt{2a_sD\,/h_sc_s/}\; Q, 1\} \qquad (5)$$

*1c. π_s (d, n*Q) is an increasing and linear function of Q.*

Note that the seller's decision problem in 1b is somewhat different from the buyer's. The dropping of the lot-for-lot assumption requires that n be also a decision variable. To substantially simplify the analysis, as in [35], n is assumed to be real. Appendix A demonstrates that the n-real assumption represents a good approximation to the n integer case. To be feasible, n^* in (5) must be at least 1. Whether n^* equals one or not has important implications in deriving the optimal cooperative-pure-strategy order quantity [see (40) below].

The results for π_s are based on Lee and Rosenblatt [27], For the remainder of the paper, the seller's profit function is evaluated at its optimum value of n, as defined by (5). Further, the seller is assumed to have fixed the value of d. The two most common reasons for this situation are competitive pressures and lack of authority on the part of the seller to modify, for a particular customer, a discount policy developed elsewhere in the firm (e.g. [10]). As a result, only the order quantity is negotiable. This assumption is not as restrictive as it sounds. Within the range of feasibility, for any order quantity, there exists an optimum discount level. Also, for a specific discount level, there corresponds an optimal order quantity. Thus, a simple reparameterization of Q on d renders possible the setting of the bargaining problem in terms of d rather than Q. Hence, to simplify notation, both n and d are dropped from the characterization of this function. Then, the function $\pi_s(d, nQ)$ may now be rewritten simply as $\pi_s(Q)$. Combining (3) and (5), it is evident that

$$\begin{aligned} \pi_s(Q) &= m_sD — \sqrt{2a_sh_sc_sD} + h_sc_sQ/2 \quad \text{if } n^* > 1 \\ &= m_sD – a_sD/Q, \qquad\qquad \text{if } n^* = 1 \end{aligned} \qquad (6)$$

It should also be observed that the expressions for π_i, $i = b, s,$ in (3) and (6), implicitly assume a positive discount rate. Only when the optimal no-discount order quantity, $Q_0{}^*$, is needed, $d = 0$ is implied. Finally, the yearly demand, D , is considered to be independent of fluctuations in either d or Q. This assumption follows the usual practice in the PDM literature referred to earlier. It also concentrates the applicability of the ensuing model to relatively high price-inelastic products, for which inventory control, rather than market fluctuations, is paramount.

Property 1 justifies the graphical representation of the profit functions of Figures 1 and 2. Without discount, the buyer purchases $Q_0{}^*$ units, as defined in (4), for a profit of $\pi_b{}^*(Q_0{}^*)$. Hence, $\pi_b{}^*(Q_0{}^*)$ represents the lower profitability limit for the buyer in any negotiation with the seller. At the same time, the corresponding seller's no discount profit, $\pi_s\,(Q_0{}^*)$, becomes the vendor's lower bound. In addition, these bounds represent the sectuity levels for each party and in turn define the upper (for the buyer) and the lower (for the vendor) quantity limits from which bargaining ranges may be constructed. These results are summarized in Property 2 below.

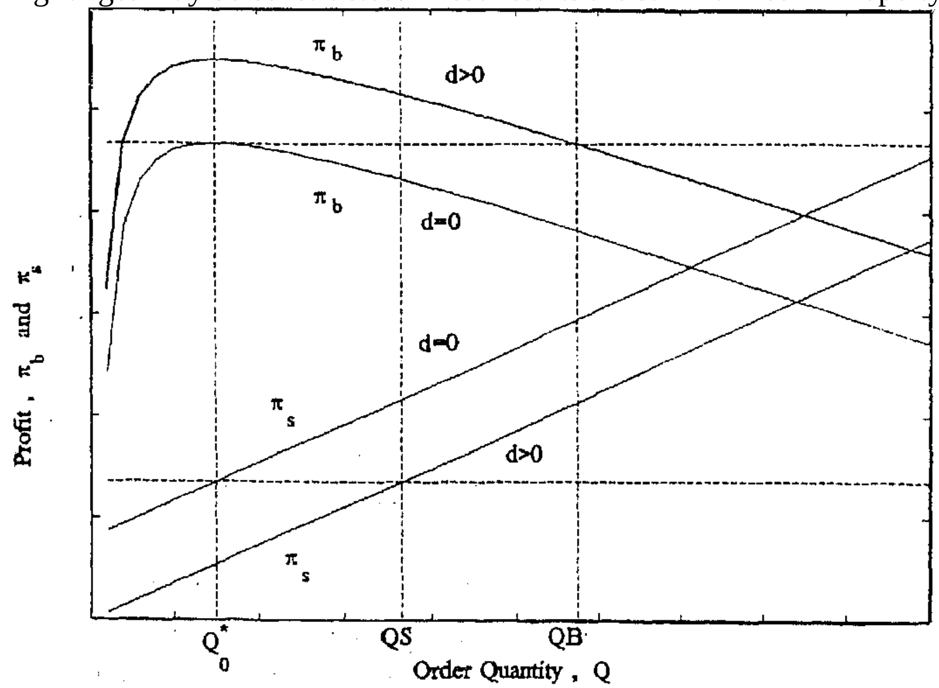

Figure 1 *Buyer-Seller Profit Functions - QB > QS*

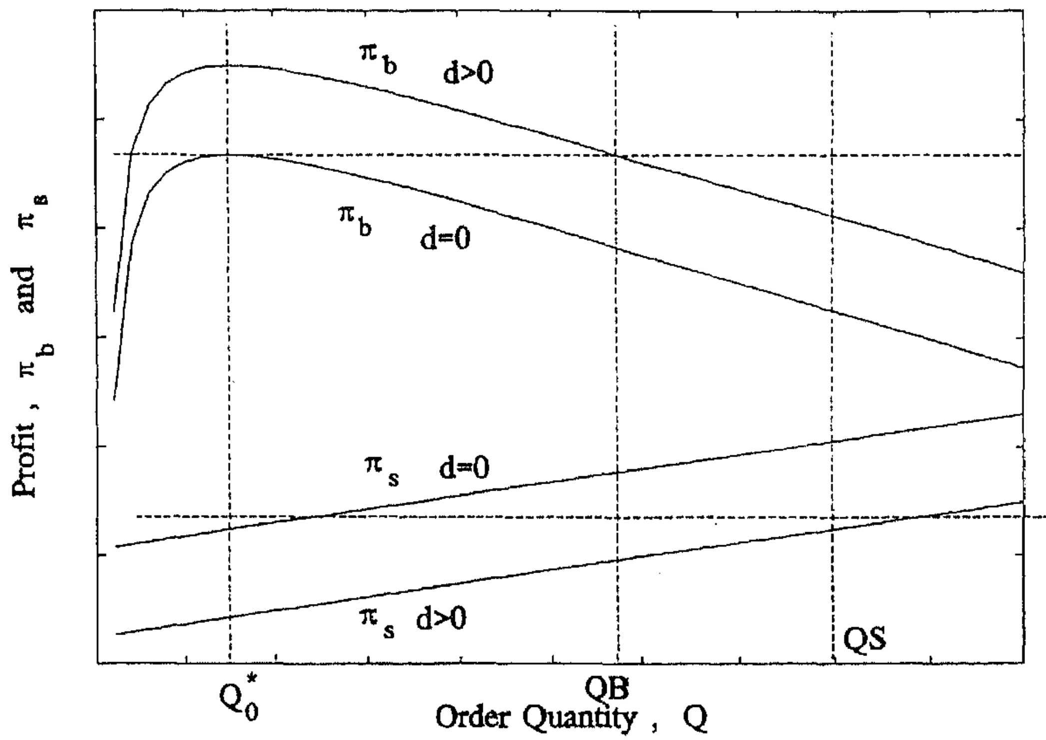

Figure 2 *Buyer-Seller Profit Functions - QS > QB*

Property 2. Bargaining range limits

2a. The buyer's largest order quantity, QB, is such that

$$\pi_b\,(QB) = \pi_b\,(Q_0^*) \qquad (7)$$

and corresponds to

$$QB = [\; Q_0^* + dD/h_b + \sqrt{(Q_0^* + \frac{dD}{h_b)^2} - \; Q_0^{*2}(1-d)]} \;/\; (1-d) \qquad (8)$$

2b. The vendor's lowest order quantity, QS, is such that

$$\pi_s\,(QS) = \pi_s(Q_0^*) \qquad (9)$$

and corresponds to

$$QS = Q_0^* + 2c_b dD/h_s c_s \qquad (10)$$

2c. Both QB and QS exceed the value of Q_0.*
2d. The bargaining range is the interval of order quantities between QS and QB if QS ≤ QB or the empty set, otherwise.

Two observations as to the nature of Property 2 are in order. First, since the expression for *QB* in (8) is based upon the definition of π_b in (2), it does not include the widely used approximation of Monahan [30], whereby the buyer's holding cost savings, $c_b\, dQ/2$, are assumed to be negligible. Even though such a simplifying assumption is computationally more tractable and provides a very tight lower bound on *QB*, it does affect the existence of a pure cooperative optimal strategy, as shown in the next section. Second, both *QB* and *QS*, as defined by (8) and (10) respectively, are functions of the discount rate, which in effect links the discount to the price break point. This is as it should be. The buyer will not accept any quantity above *QB* nor will the seller accept any quantity below *QS*. Hence, it is rational to expect that the magnitude of these limits be dependent upon, and in fact increasing along with, the size of the discount.

Once the quantity limits, *QB* and *QS*, have been computed, the bargaining ranges are readily established, by finding the discount values which render (*QB* - *QS*) either positive or negative. Table 1 summarizes these conditions. The results of the Table form the basis for subsequent derivations of the various buyer-seller strategies. But, first, the PDM game must be properly characterized.

2.2 The PDM Game

The PDM game is a two-person general-sum simultaneous game on a continuous strategy space. As mentioned earlier, the two players, buyer and seller, are denoted by b and s. Since the discount Q is fixed, the only strategy available to a player is to choose a favourable order quantity. Let q_b (q_s) be the variable that denotes the strategy chosen (i.e. the order quantity proposed) by b(s) respectively. Then, the strategy space, S, is given by the two-dimensional set of all positive order quantities, i.e.

$$S = \{(q_b, q_s) \mid q_b > 0,\ q_s > 0\} \qquad (11)$$

However, it will be shown that in different versions of the PDM, the set *S* that needs to be considered is smaller than that defined by (11). Hence, there exists a subset of *S*, heretofore referred to as the Effective Strategy Space and denoted by S_E, such that any pair of strategies in the set $\{S|S_E\}$ is dominated by at least one pair of strategies in S_E.

Note that (11) is defined as a function of the order quantity only and thus excludes the discount level as a strategic issue for either player. This results from the nature of the PDM considered here, namely a simultaneous game, where the seller offers a discount under the condition that the buyer buys more than before and where the price discount is not for negotiation. Hence, the only strategy available to both parties is choosing the order quantity. As is commonly done in such games, this paper focuses on the two most important solution concepts for the PDM game, namely pure and mixed strategy Nash Equilibrium.

Let $P_i(q_b, q_s)$, $i = b, s$, the payoff to player i, when a set of strategies q_b, q_s is chosen by b and s, respectively. 'Then, it is evident that the payoff function for the PDM game is:

$$(P_b(q_b, q_s),\ P_s(q_b, q_s)) = \begin{cases} (\pi_b(Q),\ \pi_s(Q)), & \text{when } q_b = q_s = Q \geq Q_0^* \\ (\pi_b(Q_0^*),\ \pi_s(Q_0^*)), & \text{when } q_b \neq q_s \end{cases} \qquad (12)$$

Table 1
Conditions for QB - QS to be Positive or Negative

IF	$QB \geq QS$	$QB \leq QS$
$2c_bh_b > c_bh_b > c_sh_s$	When $d \leq H_b$ **	When $d \geq H_b$ **
$2c_bh_b > c_sh_s > c_bh_b$	Always	Never
$c_sh_s > 2c_bh_b > c_bh_b$	Never	Always

** $H_b = Q^* [c_s^2 h_s^2 (2D / h_b - Q^*)]/[4(c_b h_b - c_s h_s) c_b h_b]$

Note that (12) indicates that the discount offer will only be accepted when the two players choose the exact same quantity. Such characterization reflects the fact that both parties have the option of disagreeing at any iteration of the game and of proposing a new quantity at the next iteration. As a result, if $q_b \neq q_s$, settling at either quantity will leave one party with a perceived loss of potential gain which could have been realized if settled at the other quantity. It is this behavioral assumption that justifies the payoff structure of (12). Further, the second condition of (12) also implies that if the two players are unable to choose the same quantity, the no-discount solution prevails. Hence, rejection of the discounted offer does not imply zero purchases. Rather, as assumed in section 2.1, it results in orders of size Q_0*. Observe also that no enforcement costs are explicitly included in the payoff to either player. This is done mainly to simplify notation, since the payoffs can be simply defined as net of enforcement costs.

Both cooperative and non-cooperative versions of the PDM game are possible, The difference between them lies on whether or not side payments are allowed between the parties. Thus, the solution to a cooperative game is defined as the order quantity and the side payment agreed upon by each party, whereas only agreement on the order quantity is required for a solution to a non-cooperative game. Two additional variations, pure and mixed strategies, are also evaluated. At issue here is whether the strategies are chosen deterministically (pure strategy) or according to a probability distribution (mixed strategies), defined on the game's strategy space. Combining these two sets of PDM variations yields three possible types of PDM games, namely (i) non-cooperative, pure strategy; (ii) non-cooperative, mixed strategy; and (iii) cooperative, pure strategy. The resulting models are developed sequentially in the next three sections. The concept of the solution to a particular game is first defined in each case and then it is demonstrated how the solution may be obtained. Note that the cooperative-mixed-strategy model is not being considered, since it can be readily verified that both players will choose the cooperative-pure-strategy solution with probability 1.

3. Non Cooperative, Pure Strategies

The solution to this version of the PDM game (i) is deterministic; (ii) includes no side payments; and (iii) corresponds to the order quantity which yields the Nash equilibria. The latter is given by a set of strategies, q_b*, q_s*, such that

$$P_b(q_b^*, q_s^*) \geq P_b(q_b, q_s^*), \quad \forall\ q_b \in S \text{ and} \qquad (13)$$
$$P_s(q_b^*, q_s^*) \geq P_s(q_b^*, q_s), \quad \forall\ q_s \in S$$

Theorem 1. ***Non-Cooperative Pure Strategy Nash Equilibria Order Quantity***
The Nash equilibria, given by a set of strategies q_b, q_s*, depends upon the relationship between QB and QS in this model.*

Case A: If QB < QS, the only Nash Equilibria that exists is $q_b^* = q_s^* = Q_0^*$.

Case B: If QB ≥ QS, the Nash Equilibria is given by either $q_b^* = q_s^* = Q_0^*$ or any set of strategies within the following effective strategy space

$$S_E = \{q_b, q_s \mid QS \leq q_b, q_s \leq QB\} \qquad (14)$$

An intuitive explanation of Theorem 1, case B is based upon Properties la and lc, which show that, within the [*QS*, *QB*] bargaining range, (i) π_b is a decreasing function of *Q* and hence the buyer's best solution is to buy the minimum amount possible, *QS*; and (ii) π_s is an increasing function of *Q* and thus its optimal strategy tends to the largest possible order, *QB*. Clearly, if *QB* < *QS*, the set of feasible quantities within the bargaining range is empty, as Figure 2 depicts. Hence, if Case A occurs, the players will choose strategies corresponding to the no discount position, i.e. $q_b^* = q_s^* = Q_0^*$. On the other hand, when *QB* > *QS*, it will be optimal for them to choose strategies that are in the interval [*QS*, *QB*], defined in (14), since they yield no worse payoffs to both players than those associated with the no discount alternative. Thus, in Case B, unless some kind of cooperation is possible, no unique common order quantity exists, even though any solution within the bargaining set benefits both parties. This result is also useful in finding the Nash Equilibrium of the PDM game if mixed strategies are allowed, as discussed in the next section.

4. Non Cooperative, Mixed Strategies

This section starts by examining the main characteristics of the mixed-strategy non-cooperative PDM game to be followed by the derivation of the Nash Equilibrium solution. Its salient feature is the derivation of unique non-cooperative equilibrium mixed strategies for a particular family of probability distributions, rather than merely proving the existence of such a solution. Given the nature of S_E for this version of the game, the Nash Equilibrium is trivial to find if Case A occurs, since in that situation S_E is a singleton. Hence, the emphasis here is on the Effective Strategy Space defined in (14), valid only for Case B, when $QB \geq QS$.

4.1 Characteristics of the Non-Cooperative Mixed-Strategy PDM Game

In this case, it is assumed that the two players choose strategies from the strategy space in accordance with probability distributions, to be defined shortly. The mixed strategy PDM game of this section may be called a game of agreement, in that it exhibits the following characteristics.

Property 3. Characteristics of the Non-Cooperative Mixed Strategy PDM Game/

3a. The buyer and the seller choose from identical sets of allowable quantities, defined by the effective strategy space, S_E, defined by (14). Hence, the kernel is square.

3b. The payoffs to both players, π_b and π_s respectively, are continuous functions of the order quantity and identify a two-person, general-sum continuous game.

3c. The buyer and the seller select a quantity from S_E, in accordance with probability distributions (F_b and F_s) and their respective density functions (f_b and f_s), all of which are assumed to be continuous over the bargaining range.

3d. The incremental profits of moving to a new equilibrium quantity are non-zero if both players choose the same quantity.

Property 3a follows from the characterization of S_E in Theorem 1. The continuity of the payoff functions in 3b is a direct result of Property 1. The definition of a mixed strategy is represented by 3c and 3d is based on the second condition of (12). As shall be seen shortly, its usefulness lies on the fact that it characterizes a special two-person general-sum game, from which Nash solutions can be easily obtained, without resorting to the more complex generalized approach of Owen [34].

4.2 Nash-Equilibrium Non-Cooperative Mixed Strategies

The expected payoffs to the buyer and seller are given in the following lemma.

Lemma 1: *Expected Payoffs to the Buyer and the Seller*

For the effective strategy space, S_E, defined in (14), the expected payoffs to the buyer and the seller may' be expressed as

$$E_i(f_b,\ f_s) = \int_{QS}^{QB} \pi_i(q) f_b(q) f_s(q)\, dq + \pi_i(Q_0^*)\ , \quad \text{for } i = b,\ s \tag{15}$$

Hence, using Lemma 1, if a pair of probability distributions, characterized by the density functions $f_b^*(q)$ and $f_s^*(q)$, defined on the interval $[QS, QB]$ for the buyer and the seller respectively, define a Nash equilibrium, then it can be claimed that

$$\begin{aligned} & E_b(f_b^*, f_s^*) \geq E_b(f_b, f_s^*)\ , \quad \forall\ f_b(\ .) \\ \Rightarrow & \int_{QS}^{QB} \pi_b(q) f_b^*(q) f_s^*(q)\, dq \geq \int_{QS}^{QB} \pi_b(q) f_b(q) f_s^*(q)\, dq, \quad \forall\ f_b(\ .) \end{aligned} \tag{16}$$

and

$$(17)\quad E_s(f_b^*, f_s^*) \geq E_s(f_b^*, f_s), \quad \forall f_s(.)$$
$$\Rightarrow \int_{QS}^{QB} \pi_s(q) f_b^*(q) f_s^*(q)\, dq \geq \int_{QS}^{QB} \pi_s(q) f_b^*(q) f_s(q)\, dq, \quad \forall f_s(.)$$

For two-person general-sum games, existence proofs for Nash Equilibrium are well known [33]. However, by using Property 3 and Lemma 1, the following theorem provides an explicit mixed strategy Nash Equilibrium for the PDM game and a characterization of the conditions under which it is unique.

Theorem 2. ***A Particular Non-Cooperative Mixed Strategy Equilibrium Solution***

2a. A set of non-cooperative equilibrium mixed strategies is defined by the following density functions

$$f_i^*(Q) = k_i/\pi_j(Q), \quad (i, j) = [(b, s), (s, b)], \quad QS \leq Q \leq QB \qquad (18)$$

where k_i is positive, finite and satisfies the condition

$$\int_{QS}^{QB} f_i^*(Q)\, dQ = k_i \int_{QS}^{QB} dQ/\pi_j(Q) = 1 \qquad (i, j) = [(b, s), (s, b)] \qquad (19)$$

2b. The equilibrium expected payoffs are given by

$$E_i^* = k_j, \quad (i, j) = [(b, s), (s, b)] \qquad (20)$$

2c. The value of k_i yields the following expression

$$k_i = \begin{cases} \delta_1/\ln[(\delta_0 + \delta_1 QB)/(\delta_0 + \delta_1 QS)], & i = s \\ 2r_2/[\ln H_1 - 2r_1 H_3/\sqrt{r_3}], & i = b, r_3 > 0 \\ 2r_2/\{[\ln H_1 - 2r_1[(1/r_4) - (1/r_5)]\}, & i = b, r_3 = 0 \\ 2r_2/[\ln H_1 - (r_1 \ln H_2)/\sqrt{-r_3}], & i = b, r_3 < 0 \end{cases} \qquad (21)$$

where

$$\begin{aligned} H_1 &= (r_2 QB^2 + r_1 QB + r_0)/(r_2 QS^2 + r_1 QS + r_0) \\ H_2 &= [(r_4 - \sqrt{-r_3})(r_5 + \sqrt{-r_3})] / [(r_4 + \sqrt{-r_3})(r_5 - \sqrt{-r_3})] \\ H_3 &= \arctan(r_4/\sqrt{r_3}) - \arctan(r_5/\sqrt{r_3}) \end{aligned} \qquad (22)$$

and

$$\begin{bmatrix} r_0 & r_1 \\ r_2 & r_3 \\ r_4 & r_5 \\ \delta_0 & \delta_1 \end{bmatrix} = \begin{bmatrix} -a_b D & m_b D \\ -c_b(1-d)h_b/2 & 4r_0 r_2 - r_1^2 \\ r_1 + 2r_2 QB & r_1 + 2r_2 QS \\ m_s D - \sqrt{2a_s h_s c_s D} & h_s c_s/2 \end{bmatrix} \qquad (23)$$

2d. If it is required for any pair of equilibrium strategies to have the property that $f_i^*(Q) > 0$, $i = b, s$, for all Q, *s.t.* $QS \leq Q \leq QB$, then the strategies given by Theorem 2a are unique.

The competitive strategies described in Theorem 2 are characterized by two players (i) defining their probability distributions in terms of each other's profit function; (ii) normalizing it; and then (iii) yielding a unique pair of equilibrium strategies. Such result is only possible by taking advantage of the characteristics of games of agreement listed in Property 3. Further, observe that although the assumption of continuous payoff functions is still being made only for ease of computations of (21)-(23), the results of this section straightforwardly extend to the more realistic discrete case. Moreover, if desired for the purposes of comparability, a discrete version of the PDM game may be constructed, by approximating the continuous set of quantities, with a suitable number of discrete values. This reduces the problem to the standard discrete two-person general-sum game. The resulting bimatrix game is characterized by a payoff matrix, P, whose typical element, $[p_{ij}]$, is defined as follows:

$$p_{ij} = \begin{cases} \pi_b(q_i), \pi_s(q_j), & i = j \\ 0, & \text{otherwise} \end{cases} \qquad (24)$$

It is well known [31] that at least one pair of equilibrium strategies exists for this game. One such pair may be computed using Lemke and Howson's [28] complementary pivot algorithm.

5. Cooperative Pure Strategies

It is obvious from Theorem 1 that no unique pure strategy solution exists without side payments. Hence, some form of cooperation is needed. This implies that the cooperative pure strategy solution requires not only the determination of an appropriate order quantity, Q_C*, but also a side payment, $\alpha(Q_C^*)$, that is mutually agreeable to the two players. Thus, the main issues in this section are the determination of Q_C* (section 5.4) and, subsequently, the profit-sharing mechanism (section 5,5), For these purposes, it is first required that the relative strength of the parties be assessed (section 5.1), the reward structure determined (section 5.2), and the Pareto optimal set (POS) fully characterized (section 5.3). In addition, for this section's version of the PDM game, it is assumed that (i) the two players negotiate to choose a mutually agreeable strategy; (ii) utility is freely transferable, to capture the strength of the players [39]; (iii) agreements about side payments are enforceable; and (iv) in being transferred, the side payment preserves its utility [37].

5.1 Relative Strength of the Parties

The following definition of "strength" is needed.

Definition 1. A player is said to be the strongest if there is no incentive to move from the no discount position without receiving a side payment from the other side.

From the definition of the strongest player given above, it is possible to identify the order quantity set which determines the relative strength of each party. Such characterization is useful to identify the POS and, by implication, the party having to make the first move if an agreement is to be reached. The next result helps elucidate which party is the strongest.

Property 4. ***Relative Strength between Parties***

4a. The buyer is the strongest player if

$$QB \le QS \le Q \quad \text{or} \quad QS \le QB \le Q \tag{25}$$

4b. The seller is the strongest player if

$$Q \le QB \le QS \quad \text{or} \quad Q \le QS \le QB \tag{26}$$

4c. Both parties are better off if

$$QS \le Q \le QB \tag{27}$$

4d. Both parties are worse off if

$$QB \le Q \le QS \tag{28}$$

4e. The cases identified in 4a-4d are mutually exclusive and collectively exhaustive.

The import of Property 4 can be depicted graphically, Consider Figure 1, where $QS < QB$. As shown in Theorem 1, any $Q \in [QS, QB]$ represents a Nash equilibrium order quantity without side payment. For any Q greater than the upper limit (or lower than the lower limit), the buyer (or the seller) is the strongest player since $\pi_i\, Q) < \pi_i\,(Q_0^*)$, $i = b$ (or s). Similarly, in the $QB < QS$ case, neither party has any incentive to move away from the no discount option if $Q \in [QB, QS]$, whereas, without side payment, the buyer (or the seller) will not move if the order quantity exceeds (falls below) QS (or QB).

5.2 The Reward Structure With Side Payments

Let $L_b(Q)$ [or $G_b(Q)$] be the loss (or gain) to the buying firm, when it is the strongest (or weakest) player and $G_s(Q)$ [or $L_s(Q)$], the corresponding gain (or loss) to the seller. Then, since the standard for comparison is the no-discount purchase of Q_0* units or its equivalent, QB or QS units at a particular discount rate, it follows that

$$\begin{aligned} L_b(Q) &= -G_b(Q) = \pi_b(Q_0^*) - \pi_b(Q) \\ G_s(Q) &= -L_s(Q) = \pi_s(Q) - \pi_s(Q_0^*) \end{aligned} \tag{29}$$

Let (i) $\alpha(Q)$ be the proportion of the weaker party's incremental profits (G_b- L_s or G_s - L_b) shared by the stronger player; and (ii) $R_b(Q)$ and $R_s(Q)$ be the rewards for the buyer and the vendor, respectively, resulting from the reallocation of resources. Further, assume that $\alpha(Q)$, $R_b(Q)$ and $R_s(Q)$ are, along with the profit functions defined in (2) and (3), all continuous and twice differentiable. Then, $R_b(Q)$ and $R_s(Q)$ are defined as follows.

Property 5. Reward Structures with Side Payments

5a. If the buyer is the strongest player, then

$$\begin{aligned} R_b(Q) &= \pi_b(Q_0^*) + \alpha(Q)\,[G_s(Q) - L_b(Q)] \\ R_s(Q) &= \pi_s(Q_0^*) + [1-\alpha(Q)]\,[G_s(Q) - L_b(Q)] \end{aligned} \tag{30}$$

5b. If the seller is the strongest player, then

$$\begin{aligned} R_b(Q) &= \pi_b(Q_0^*) + [1-\alpha(Q)]\,[G_b(Q) - L_s(Q)] \\ R_s(Q) &= \pi_s(Q_0^*) + \alpha(Q)\,[G_b(Q) - L_s(Q)] \end{aligned} \tag{31}$$

5c. A necessary condition for a negotiated solution to exist is that

$$\begin{aligned} R_b(Q) &\geq \pi_b(QB) = \pi_b(Q_0^*) \\ R_s(Q) &\geq \pi_s(QS) = \pi_s(Q_0^*) \end{aligned} \tag{32}$$

The justification for Property 5 is straightforward. The necessary condition in 5c follows from the assumption that no rational player is expected to move from the no discount position, unless the discount rewards, [$R_i(Q)$, $i = b, s$], exceed in value their no discount counterparts [$\pi_i(Q_0*)$, $i = b , s$]. Furthermore, when the buyer (or the seller) is the strongest player, as in 5a (or 5b), then, by definition of "strongest", $\pi_i(Q) < \pi_i(Q_0*)$, $i = b$ or s. Thus, agreement is possible only if vendor (or buyer), as the weaker party, is willing to share a proportion, $\alpha(Q)$, of the incremental benefits [$G_i(Q)$ - $L_j(Q)$, (i, j) {(s, b) or (b, s)}] of moving to the discount position.

5.3 The Parcto Optimal Set (POS)

The next step in the search for a cooperative solution is to identify the POS from among the six possible regions in Property 4. But first, the POS must be defined.

Definition 2. A quantity, Q, is said to be an element of the Pareto optimal set, if the combined total profit to both parties exceeds that of its no-discount counterpart. Formally,

$$\text{POS} = \{Q/\pi_b(Q) + \pi_s(Q) \geq \pi_b(Q_0^*) + \pi_s(Q_0^*)\} \tag{33}$$

With Definition 2, the identification from Property 4 of the order quantities in the POS is straightforward. It is clear that the case defined by Property 4d, where both parties are worse off, can be dropped and that the entire [QS, QB] range of 4c, where both sides benefit, belongs to the POS. In the four regions of 4a and 4b, the existence of a solution hinges upon whether the gain to the weaker party of moving to a given quantity, Q, outside the bargaining range limits defined in Property 2 (i.e. where $Q \notin [QS, QB]$ or $Q \notin [QB, QS]$, as the case may be) is large enough to offset the corresponding loss suffered by the strongest player. To help elucidate these issues, consider the following property, based upon the definitions in (29) of gains and losses to the players and of the profit functions in (2) and (6).

Property 6. ***Indifference Order Quantities***

$$\begin{aligned} L_b(Q) &= (Q - QB)[h_b c_b(1-d)/2 - (a_b/Q)(D/QB)] \\ G_s(Q) &= (Q - QS)c_s h_s/2 \end{aligned} \tag{34}$$

6b. $L_b(Q)$ is an increasing and concave function of Q $\forall$ $Q \geq QB$.
$G_b(Q)$ is a decreasing and concave function of Q $\forall$ $Q_0* \leq Q \leq QB$.
$L_s(Q)$ is a decreasing and linear function of Q $\forall$ $Q \leq QS$.
$G_s(Q)$ is and increasing and linear function of Q $\forall$ $Q \geq QS$.

6c. The indifference order quantity, Q_p, such that

$$\begin{aligned} L_b(Q_p) &= G_s(Q_p) \\ G_b(Q_p) &= L_s(Q_p) \end{aligned} \tag{35}$$

and corresponds to one of the roots of the following quadratic equation

$$h_1 Q^2 + h_2 Q + a_b D = 0 \tag{36}$$

where

$$\begin{aligned} h_1 &= [h_b c_b (1-d) - h_s c_s]/2 \\ h_2 &= [h_s c_s QS - h_b c_b (1-d)\, QB]/2 - a_b D/QB \end{aligned} \tag{37}$$

Property 6b places the necessary limits on the order quantity to characterize the POS in the regions where one of the parties is stronger. Property 6c follows from (29) and provides the indifference order quantities, where neither party is weaker or stronger. Any other point within the appropriate regions defined in 6b may be a part of the POS.

The next step is to establish the conditions under which a given order quantity belongs to the POS. Table 2 summarizes the various stages needed to characterize the POS. First, it is helpful to evaluate each of the regions of Property 4, in order to determine for which order quantity, if any, each party is the stronger (see Table 2, column 1). Second, for each region, the feasibility or lack thereof of Q_p is determined. For this purpose, Q_p is said to be feasible in a given region, if its value is finite and falls within that region. Consequently, as shown in the second column of Table 2, in the cases of Property 4a (or 4b) where the buyer (or the seller) is the strongest, Q_p is feasible if its value exceeds Max(QS, QB) (or is between Q_0* and Min(QS, QB)). Finally, on the basis of these two stages, the values of $Q \in$ POS are identified (see Table 2, column 3). Formally, the following lemma characterizes the POS.

Table 2

Characterization of the Pareto Optimal Set

Stronger Party		**Is Q_p feasible ?**	**When is $Q \in$ POS ?**
Either Party	$QS \le Q \le QB$	N/A	any $Q \in [QS, QB]$
	$QB \le Q \le QS$	N/A	POS is empty
Buyer	$QB \le QS \le Q$	Yes, $Q_p > QB$	any $Q \in [QB, Q_p]$
		No, $Q_p \le QB$	any $Q \ge$ QB
	$QB \le QS \le Q$	Yes, $Q_p > QS$	any $Q \ge Q_p$
		No, $Q_p \le QS$	POS is empty
Seller	$Q \le QS \le QB$	Yes, $Q_0 \le Q_p \le QS$	any $Q \in [Q_p, QS]$
		No, $Q_p \notin [Q_0, QS]$	any $Q \in [Q_0^*, QS]$
	$Q \le QB \le QS$	Yes, $Q_0 \le Q_p \le QB$	any $Q \in [Q_0^*, Q_p]$
		No, $Q_p \notin [Q_0, QB]$	POS is empty

Lemma 2. ***The Pareto optimal set is characterized by the following conditions***

2a. If $QS \le Q \le QB$, then any $Q \in [QS, QB]$ belongs to the POS.

2b. If the buyer is stronger and $QS \le QB$, then

Case 1. If Q_p is feasible, any $Q \in [B, Q_p]$ is an element of the POS.

Case 2. If Q_p is not feasible, any $Q \ge QB$ belongs to the POS.

2c. If the buyer is stronger and $QB \le QS$, then

Case I. If Q_p is feasible, any $Q > Q_p$ belongs to the POS.

Case 2. If Q_p is not feasible, the POS is empty.

2d. If the seller is stronger and $QS \le QB$, then

Case 1. If Q_p is feasible, any $Q \in [Q_p, QS]$ belongs to the POS.
Case 2. If Q_p is not feasible, any $Q \in [Q_0{*}, QS]$ is an element of the POS.

2e. If the seller is stronger and $QB \leq QS$, then
Case 1. If Q_p is feasible, any $Q \in [Q_0{*}, Q_p]$ is an element of the POS,
Case 2. If Q_p is not feasible, the POS is empty.

The POS may now be defined explicitly, in Theorem 3 below, by adroit selection of the regions in Lemma 2, on the basis of the relative magnitude of QB and QS and of the feasibility of Q_p.

Theorem 3. ***Cooperative Pure Strategy Pareto Optimal Set***
3a. If $QB \geq QS$, then

$$POS = \{Q \mid LL \leq Q \leq UL\} \tag{38}$$

where (i) the POS lower limit, LL, equals Q_p, if feasible; $Q_0{*}$, otherwise.
(ii) the POS upper limit, UL, equals Q_p, if feasible; unbounded, otherwise.

3b. If $QB < QS$, then

$$POS = \begin{cases} Q \mid Q \in (Q_0^*, Q_p), & \text{if } Q_0^* \leq Q_p \leq QB \\ Q \mid Q > Q_p, & \text{if } Q_p \geq QS \\ \emptyset, & \text{otherwise} \end{cases} \tag{39}$$

Note that, in 3a(ii), even if the POS is unbounded, there exists a practical upper limit, defined as the maximum stock level available for the discount.

5.4 Optimal Order Quantity

Once the POS is identified, the next question is to determine the optimal order quantity, $Q_C{*}$, of the cooperative pure strategy. $Q_C{*}$ is optimal in the sense that, along with the profit sharing mechanism of the next section, it represents the solution to the cooperative version of the PDM game. The theorem below characterizes Qc*.

Theorem 4. ***Cooperative-Pure-Strategy Optimal Order Quantity, $Q_C{*}$.***
4a. $Q_C{*}$ is the order quantity which maximizes $G_b - L_s$ (or $G_s - L_b$), $\pi_b + \pi_s$ and $R_b + R_s$.
4b. The value of $Q_C{*}$ is given by

$$\begin{aligned} Q_C^* &= \sqrt{2a_b D / h_1}, && \text{if } h_1 > 0 \text{ and } n^* > 1 \\ &= \sqrt{2(a_b + a_s) D / [c_b h_b (1-d)]}, && \text{otherwise} \end{aligned} \tag{40}$$

with h_1 defined by (37) and n^*, by (5).
4c. $Q_C{*}$ is the cooperative pure strategy optimal order quantity.

Several implications of interest may be derived from this theorem. First, as expected, (37) suggests that the feasibility of $Q_C{*}$ is parameter specific, as was the existence of the POS in Theorem 3. Second, Monahan's [30] approximation, which consists of ignoring the effect of the discount on the buyer's holding costs, reduces the $h_1 > 0$ feasibility space and hence, the size of the POS, Third, a smaller POS has also interesting implications for other studies. For example, it tightens the feasibility of the pricing scheme proposed by Dada and Srikanth [9] and a negative h_1, where the seller's holding costs are higher than the buyer's, renders the scheme infeasible. Fourth, observe that the value of $Q_c{*}$ in (40) depends upon the values of n^* from (5) and h_1 in (37) and hence upon the interplay between the relative magnitudes of the ordering and the holding costs of both parties, When $n^* > 1$ and $h_1 > 0$, and the seller's holding costs are lower than those of the buyer, the standard ordering/holding trade-off dictates a finite magnitude for $Q_C{*}$. When $n^* = 1$, the seller has no holding costs [see (3)]. Hence, the appropriate value for $Q_C{*}$ results from the interplay of the buyer's holding costs vis à vis the sum of the ordering costs from both parties. Finally, when $n^* > 1$ and $h_1 < 0$, the seller's holding costs dominate over those of the buyer. As a result, it is more beneficial for both parties to have the buyer hold as much stock as possible and then share the resulting overall benefits. However as the quantity bought by the buyer increases, the seller's n^* -

1. Thus, the $n^* = 1$ order quantity given by the second expression of (40) is also valid in this case.

5.5 Profit-Sharing Mechanism

Having discussed that Q_C^* is the optimal order quantity of the cooperative version of the PDM, the next issue concerns the determination of the value of α (Q_C^*). It is assumed without loss in generality that

$$\Delta = [\pi_b(Q_c^*) + \pi_s(Q_C^*)] - [\pi_b(Q_0^*) + \pi_s(Q_0^*)] > 0 \qquad (41)$$

i.e. at the negotiated solution Q_C^*, there is a positive amount of profit that can be shared between the two players by means of side payments from the weaker to the stronger party. Hence, the problem of profit sharing at the negotiated solution, Q_C^*, can be formulated as follows: there is a net utility of A that is to be shared between the two players and the issue of determining $\alpha(Q_C^*)$ is therefore equivalent to the problem of deciding the optimal amount of utility to be transferred from the weaker to the stronger party, given the utility function of the two players. As a result, any profit-sharing mechanism requires the identification of three factors. These are (i) the shape of the utility functions of the players; (ii) their attitude towards risk; and (iii) an allocation scheme which reflects each side's relative strength. Three of the best known mechanisms are presented here. They include the solutions to the Nash's bargaining problem provided by Nash [31] and Kalai and Smordinsky [24] and the allocation scheme of Eliashberg [11].

Nash's Bargaining Problem

In terms of the PDM game, consider two players of different strengths, with utility functions U and V, for the weaker and the stronger respectively. Their aim is to distribute the incremental return, defined as the difference between the joint return, $R_T(Q_C^*)$, which comprises the sum of the individual returns as given in Property 5 and the joint no-discount optimum profits, $\pi_b(Q_0^*) + \pi_s(Q_0^*)$. The allocation scheme is based upon a pair of strength factors β_U, β_V, whereby each utile the weaker party gives to the stronger brings additional β_U / β_V utiles to the stronger side, In addition, the players attitude toward risk is summarized by the set (γ_U, γ_S), $0 < \gamma_1 < 1$, $i = U, V$ of parameters denoting either neutrality or aversion towards risk. Another element of Nash's model needed is the assumption that the maximum of the minimum utility each party can get by acting on its own is zero. Then, Nash's solution to the PDM game exhibits the following characteristics.

Theorem 5. ***Nash's Solution to the PDM Game***

5a. The utility functions of the parties are given by

$$\text{Weakest Party} \qquad U = R_T^{\gamma_U}(Q_C^*) \qquad 0 \le \gamma_U \le 1$$
$$\text{Strongest Party} \qquad V = R_T^{\gamma_V}(Q_C^*) \qquad 0 \le \gamma_V \le 1 \qquad (42)$$

5b. The Pareto optimal frontier is described by

$$V = \beta_V R_T^{\gamma_V} - (\beta_V / \beta_U)\, R_T^{(\gamma_V - \gamma_U)} U \qquad (43)$$

5c. Nash's solution results in

$$V^* = \beta_U \mathrm{R}_T^{\gamma_V}/2 \qquad (44)$$
$$U^* = \beta_V \mathrm{R}_T^{\gamma_U}/2$$

.

5d. Kalai and Smordinsky's solution results in

$$V^* = U^* = \beta_V \beta_U R_T^{\gamma V} / [\beta_U + \beta_V R_T^{(\gamma V - \gamma U)}] \qquad (45)$$

Eliashberg's Allocation Scheme

Instead of maximizing the product of the two utilities, expressed on the basis of the entire amount to be shared, Eliashberg [111 maximizes the sum of the individual utilities, constrained by the Pareto optimal frontier. The form of the individual utility functions determines the type of possible agreements between the players. Hence, it is possible to derive a multitude of profit-sharing arrangements given the wide range of multiattribute utility functions available in the literature. In fact, Eliashberg's example 1, where the individual utilities are exponential in form and independent of the other party's payoff yields the Kalai-Smordinsky solution of

Theorem 5d above. Similar profit-sharing mechanisms may readily be derived for any other set of individual utilities.

6. Numerical Example

To illustrate the main features of the model, the following example is being used. The basic data consists of

$$\begin{bmatrix} p_b, & c_b, & h_b, & a_b, & d \\ p_s, & c_s, & h_s, & a_s, & D \end{bmatrix} = \begin{bmatrix} \$10, & \$5, & .1, & \$400, & .1 \\ \$5, & \$4, & .1, & \$650, & 2000 \end{bmatrix} \quad (46)$$

Then, the no-discount optimal policy and the security levels (Property 2) for each party, *QB* and *QS*, are given by

$$[Q_0^*, \pi_b(0, Q_0^*), \pi_s(0, Q_0^*), QB, QS] = [1789, \$9106, \$338, 7974, 6789] \quad (47)$$

for a total undiscounted profit of $9,444. Note that the value of the error bound, Δ^*, from (A2), at the Q_0^* = 1,789 of (47) is below a minuscule 5.53, with an optimal real n^* = 1.425. Further, since $QS < QB$, the graphical representation is akin to Figure 1. In addition, any order quantity in between *QS* and *QB* represents a feasible non-cooperative pure strategy solution (Theorem 1, Case B).

To derive the POS for the cooperative pure strategy, it suffices to note that the indifference order quantity (Property 6) is Q_p = 15,953,661 units and hence the POS is defined by the following set

$$POS = \{Q \mid Q_0^* = 1,789 \le Q \le Q_p = 15,953,661\} \quad (48)$$

The end result is an optimum cooperative pure strategy, defined by (Theorem 4)

$$[Q_C^*; \pi_b(.1, Q_C^*); \pi_s(.1, Q_C^*)] = [8000; \$9,100; \$1,581] \quad (49)$$

for a total combined profit of $10,681. 'This amount exceeds its non-discount counterpart by a total of $1,237 to be shared by the buyer and the seller. Observe by comparing the individual profits, that the buyer is the strongest player (Definition 1), given the lack of financial incentive to move from the more profitable no-discount position, unless a side payment is present, As for the sharing arrangement, the Nash (Theorem 5c) and Kalai-Smordinsky (Theorem 5d) solutions for two risk 'neutral ($\gamma_u = \gamma_v$ = 1. Theorem 5a) players of similar strengths ($\beta_V = \beta_U$ = 1, Theorem 5b) consists of distributing the said amount on an equal basis, for optimal returns of

$$[R_b(.1, Q_C^*); R_s(.1, Q_C^*)] = [\$9,724; \$956] \quad (50)$$

For the non-cooperative mixed strategy, note that the value of r_3 in (21) is negative. Hence, the third expression for k_b in (20) is needed here. Then, after some tedious computations, the equilibrium expected payoffs (Theorem 2) are given by

$$[E_s^* = k_b;\ E_b^* = k_s] = [7.354; 1.226] \quad (51)$$

7. Conclusions

This paper has provided a comprehensive set of game theoretical solutions to the price-discount problem, under profit maximizing objectives. Cooperative and non-cooperative, pure and mixed strategy solutions have been provided. The differences among strategies yield clearly defined managerial choices. Theorem 1 hints at the potential difficulties in reaching a decision. If the players' objectives lead them in opposite directions (Case A, $QB < QS$), no feasible non-cooperative solution is possible and the no discounted optimal solution appears to be the best strategy. Otherwise, when $QS < QB$, the profit functions overlap (Case B) and a range of quantities may be acceptable to the parties. One way of improving the chances of reaching an agreement is for each player to take into consideration the other's profit objectives. As Theorem 2 indicates, this may be done in the non-cooperative case, by explicitly defining density functions dependent upon the other player' profit function, Under these conditions, unique solutions are possible for the $QS < QB$ case. Nevertheless, regardless of its nature, a non-cooperative solution never yields a better return to both players than its cooperative counterpart. Side payments which take into consideration the risk attitude and the relative strength of both parties will ensure that both players benefit from the agreement.

Finally, two main research contributions of this paper are worth highlighting again. The first appears in section

4 and deals with the derivation of unique non-cooperative equilibrium mixed strategies for a particular family of probability distributions. Topics for further analysis in this area include the search for other families of probability distributions that may also (i) produce unique non-cooperative equilibrium mixed strategies; and (ii) help in identifying a unifying structure, if any, in terms of the distributions themselves and of the various types of strategies. The second main research contribution of this paper, from section 5, defines a new way of sharing profits, by incorporating into the relative strength of the parties their utility functions as indicators of their attitude toward risk. Additional work on this area, specially on the use of bargaining with utility functions is also warranted. The study of these and other issues justifies additional research.

Appendix A

Error Bound by Approximating n ≥ 1 Integer With n ≥ 1 Real

In this Appendix, justification is provided for making the critical assumption of *n* real for computational purposes only. At issue here is that the game theory methodology for *n* integer is not tractable mathematically. Thus, some sort of an approximation is needed, when developing the cooperative PDM version. Two possible approximations are: (i) approximate the integer-*n* objective function of (3) with its *n*-continuous counterpart and then develop the price discount cooperative model; or (ii) take the theoretically correct *n*-integer formulation of (3), approximate the vendor's profit with the *n*-real approximation of (6) and then develop the price discount cooperative model. Whereas in principle the first approach appears to be a valid approximation, further analysis suggests that it is not. It can be easily shown that the π_s*(*n*-continuous) approximation is independent of *Q* and hence invariant to the buyer's ordering policies. Under these circumstances, there is no incentive for the buyer to entertain any discount offer. Thus the question of cooperation does not even arise within this context.

The second and preferred approximation allows for cooperative solutions more in tune with the realities of today's marketplace. The end result is a practical approximation, which yields near-(*n*-integer)-optimal solutions and is proven below to be quite acceptable, in terms of the error it introduces. Hence, the approach of this paper is to produce an *n*-real optimal solution that overestimates the seller's *n*-integer optimal profit and then show that the overestimation cannot exceed a certain bound, that is a decreasing function of *Q*. The latter is important, since the paper deals with a price-discount model and discounts lead to larger order sizes.

To compute the bound, start by assuming that *n* is real in (3). Then the slope of the profit function for the seller, as a function of *n*, *S*(*n*), may written as

$$S(n) = \partial\pi_s(n, Q)/\partial n = a_s D/(Qn^2) - h_s c_s Q/2 \qquad \text{(A1)}$$

The function in (A1) is clearly a decreasing function of *n*. Hence, its maximum feasible value, *S**, occurs at *n* = 1 and may be expressed as

$$S^* = S(n = 1) = a_s D/Q - h_s c_s Q/2 \qquad \text{(A2)}$$

Let [*n**] be the largest integer not exceeding real *n**. Since the following inequalities hold

$$\begin{gathered} [n^*] \le \text{real } n^* \le [n^*] + 1 \\ \text{and for any given value of } Q, \\ \pi_S[n^*, Q] \le \pi_S(\text{real } n^*, Q) \\ \text{and } \pi_S([n^*] + 1, Q) \le \pi_S(\text{real } n^*, Q), \end{gathered} \qquad \text{(A3)}$$

then,

$$\begin{gathered} S^* \ge S(n^*) \ge \pi_S(\text{real } n^*, Q) - \pi_S[n^*, Q] \\ \text{and } S^* \ge \pi_S(\text{real } n^*, Q) - \pi_S([n^*] + 1, Q) \end{gathered} \qquad \text{(A4)}$$

Further, if [*n**] is the optimal integer, (Al) represents the tightest upper bound on the error. Should [*n**] + 1 be optimal, (Al) becomes a looser upper bound. In any case, *S*(*n**) in (A4) represents the upper bound on the error, which, as stated earlier is maximum at *n* = 1 and decreases in inverse proportion to the order quantity.

Appendix B
Proofs of Theorems 1-5 and of Lemma 1

First, it will be shown that $q_b^* = q_s^* = Q_0^*$, i.e. the no-discoun position is always a Nash Equilibrium. That is obvious, because if $q_b^* = q_s^* = Q_0^*$, then

$$
\begin{aligned}
P_b(q_b^*, q_s^*) &= \pi_b(Q_0^*) \geq P_b(q_b, q_s^*), \quad \forall\, q_b \in S \text{ and} \\
P_s(q_b^*, q_s^*) &= \pi_s(Q_0^*) \geq P_s(q_b^*, q_s), \quad \forall\, q_s \in S
\end{aligned}
\tag{B1}
$$

Now consider Case A, where $Q_0^* < QB \leq QS$. Suppose that the statement is not true. Then, $\exists$ a $q' \neq Q_0^*$ such that if $q_b^* = q_s^* = q'$, the following holds

$$
\begin{aligned}
P_b(q_b^*, q_s^*) &= \pi_b(q') \geq P_b(q_b, q_s^*), \quad \forall\, q_b \in S \text{ and} \\
P_s(q_b^*, q_s^*) &= \pi_s(q') \geq P_s(q_b^*, q_s), \quad \forall\, q_s \in S
\end{aligned}
\tag{B2}
$$

If $q' \leq QB$ (and therefore $q' < QS$), then $P_s(q_b^*, Q_0^*) = \pi_s(Q_0^*) > P_s(q_b^*, q_s^*) = \pi_s(q')$ (since $q' < QS$) and hence $q_b^* = q_s^* = q'$ cannot be Nash Equilibrium. Similarly, if $q' > QB$, then $P_b(Q_0^*, q_s^*) = \pi_b(Q_0^*) > P_b(q_b^*, q_s^*) = \pi_b(q')$ and so $q_b^* = q_s^* = q'$ can again not be Nash Equilibrium.

It now remains to be shown that if Case B occurs, then any set of strategies, q_b^*, q_s^*, satisfying $QS \leq q_b^*, q_s^* \leq QB$ yields a Nash Equilibrium. Note that in this case $Q_0^* < QS$. Now, choose any q' such that $QS \leq q_b^* = q_s^* = q' \leq QB$. Then, because $QS \leq q' \leq QB$, $P_b(q_b^*, q_s^*) = \pi_b(Q_0^*)$ and $P_s(q_b^*, q_s^*) = \pi_s(q') \geq \pi_s(Q_0^*)$, with strictly inequality holding when $QS < q' < QB$. This immediately proves that $q_b^* = q_s^* = q'$ is a Nash Equilibrium.

Proof of Lemma 1. From (12)-(14) and Property 3, it follows that the expected payoff to the buyer (seller) is given by

$$
E_i(f_b, f_s) = \iint_{S_E} P_i(q_b, q_s) f_b(q_b) f_s(q_s)\, dq_b\, dq_s, \quad \text{for } i = b, s \tag{B3}
$$

Since, by (12), the payoff to the buyer (seller) is $\pi_b(Q_0^*)$ ($\pi_s(Q_0^*)$) unless they both choose the same quantity, the expression in (B3) for the expected values reduces to

$$
E_i(f_b, f_s) = \int_{QS}^{QB} \pi_i(q) f_b(q) f_s(q)\, dq + \pi_i(Q_0^*) \left[\iint_{S_E \setminus \{q_b, q_s \mid q_b = q_s\}} f_b(q_b) f_s(q_s)\, dq_b\, dq_s \right] \tag{B4}
$$

for $i = b, s$. Since it can be readily shown that the value of the term in brackets in (B4) equals 1, then (B4) yields (15).

Proof of Theorem 2. The condition on k_i, $i = b, s$, in (19) is needed for $f_i^*(Q)$, $i = b, s$, to be a density function. positivity and finiteness of k_i, $i = b, s$, are a direct consequence of the fact that

$$
0 < \pi_i(Q) < \text{infinity}, \quad \text{for } QS \leq Q \leq QB \tag{B5}
$$

Then, combining (15), (18) and (19) results in (20). Furthermore, solving for k_i in (19), using the definitions of the profit functions given by (2) and (6) and, for the buyer, integration formulas 2.172 and 2.175 of Gradshteyn and Ryzhik [17] leads to 2c.

The proof of 2d is by contradiction. Consider an alternate pair of density functions, $v_i(Q)$, $i = b, s$, which are not characterized by the functional form of (18)-(23) and denote by $V_i(Q)$, $i = b, s$, the corresponding cumulative probability distributions. It is clear that $\pi_b(Q)v_s(Q)$, even if continuous, is not uniform. Now, choose $\bar{Q}$, such that

$$
\bar{Q} = \{Q \mid \pi_b(\bar{Q})\, v_s(\bar{Q}) \geq \pi_b(Q)\, v_s(Q), \quad \forall\, QS \leq Q \leq QB\} \tag{B6}
$$

and let

$$
\Delta = \int_{Q \in \bar{Q}} v_b(Q)\, dQ \tag{B7}
$$

Clearly, $\Delta < 1$, by the assumption that $v_b(Q) > 0\ \forall\ QS \leq Q \leq QB$. Moreover, it can be readily seen that one can always choose another density function, $w_b(Q)$ and its associated probability distribution, $W_b(Q)$, with the property that

$$\int_{Q \in \bar{Q}} w_b(Q)\, dq = \Delta + \varepsilon = 1 - \int_{Q \notin \bar{Q}} w_b(Q)\, dQ, \quad \text{for some } \varepsilon > 0 \tag{B8}$$

This implies that

$$E_b(W_b, V_s) \;>\; E_b(V_b, V_s) \tag{B9}$$

which is impossible, if the pair of strategies, v_b and v_s, is to yield a non-cooperative Nash equilibrium; hence, the contradiction.

Proof of Lemma 2. Lemma 2a is obvious, since, as shown in Figure 1, both parties are better off throughout the [QS, QB] range. Lemma 2b (see Figure 1) follows from Property 4a. At $Q = QB$, it is clear that $L_b = 0 < G_s$. As Q increases, both L_b and G_s increase in value, albeit the latter at a constant rate and the former at an increasing rate. Whether or not $L_b > G_s$ is parameter specific and so is the feasibility of Q_p. But only when feasibility can be established is the POS finite. For Lemma 2c, Figure 2 shows that at $Q = QB$, there is no possible solution. As Q increases, the POS continues to be empty, unless Q_p is feasible and hence the smallest element of the open POS. In Lemma 2d, as $Q \rightarrow QS$, then (see Figure 1) $L_s < G_b$ and the existence of a POS is established. Whether the POS covers the entire [Q_0*, QS] range is once again parameter specific. What is known is that as $Q \rightarrow Q_0$* , G_b decreases and L_s increases. If at Q_0*, $G_b(Q_0^*) > L_s(Q_0^*)$, then $G_b > L_s$ throughout the [Q_0*, QS] range and hence all $Q \in [Q_0^*, QS]$ are elements of the POS. Otherwise, Q_p is feasible and renders the point at which the gain-loss relationship is reversed, as well as the lower limit of the POS. Finally, for Lemma 2e, it is clear from Figure 2 that any $Q \rightarrow QB$ does not belong to the POS. As Q decreases G_b (or L_s) increases (decreases) in value. Again, feasibility of Q_p implies that $G_b(Q_0^*) > L_s(Q_0^*)$, at which point any $Q \in [Q_0^*, Q_p]$ is an element of the POS. Otherwise, the non-feasibility of Q_p yields an empty POS.

Proof of Theorem 4. For 4a, it follows, from Property 6 and Lemma 2, that the optimal order quantity, Q_C*, maximizes the difference between the weaker party's gain and the corresponding stronger player's loss. Using (29), it can be readily shown that Q_C* also maximizes the sum of the profits, $\pi_b + \pi_s$ and, from (30) and (31), the sum of the rewards, $R_b + R_s$. To prove 4b, it is necessary to consider the two possible values for n* in (5). When n* = 1, the seller has no holding costs. Hence, combining (2) and (3), it is readily seen that Q_C* results from the tradeoff between the buyer's costs and the joint ordering costs, to yield the second expression in (40). If n* > 1, it follows from the first order conditions that the feasibility and magnitude of Q_C* hinges upon the differential between the buyer's and the seller's holding costs, i.e. upon the sign of h_1, as defined in (37). A positive h_1 gives the first expression in (40). Its negative counterpart suggests that the quantity be unbounded. However, as the quantity increases in value, n* moves in the opposite direction. Hence, the quantity upper bound is that associated with n* = 1. Finally, for 4c, it suffices to show that any sharing ratio, α_1, and its order quantity, Q_1, yield lower rewards for both the buyer, $R_b(\alpha_1, Q_C^*)$, and the seller, $R_s(\alpha_1, Q_1)$, than the corresponding rewards, $R_b(\alpha_1, Q_C^*)$ and $R_s(\alpha_1, Q_C^*)$, associated with the same sharing ratio, α_1, and Q_C*. Hence, Qc* is the optimal quantity for any sharing ratio. The proof is by contradiction. Assume that the claim is incorrect and thus that there exists some quantity, Q_1, such that

$$\begin{aligned} R_b(\alpha_1, Q_1) &> R_b(\alpha_1, Q_C^*) \\ R_s(\alpha_1, Q_1) &> R_s(\alpha_1, Q_C^*) \end{aligned} \tag{B10}$$

Then, it can be readily seen that, regardless of which party is stronger, the use of (30), (31) and (B10) yields

$$\begin{aligned} G_s(\alpha_1, Q_1) - L_b(\alpha_1, Q_1) &> G_s(\alpha_1, Q_C^*) - L_b(\alpha_1, Q_C^*) \\ G_b(\alpha_1, Q_1) - L_s(\alpha_1, Q_1) &> G_b(\alpha_1, Q_C^*) - L_s(\alpha_1, Q_C^*) \end{aligned} \tag{B11}$$

Neither inequality in (B11) is possible according to Theorem 4a, hence the contradiction.

Proof of Theorem 5. 5a is simply a formal representation of the discussion preceding the theorem. It contains the total return to be shared by the two parties, $R_T(Q_C^*)$, conditioned by each player's attitude towards risk. 5b is derived in the usual way. V and U are inversely proportional to each other. The coefficients of this linear relationship are computed from the observation that if a party's utility is zero, the other gets the full share. 5c follows from Nash's objective of maximizing UV, the product of the two utilities. Finally, 5d is the solution to the simultaneous system formed by $U = V$ and to the Pareto frontier equation of 5b. Note that for the simplifying case of a one-to-one utility transfer (i.e. $\beta_U = \beta_V = 1$) and risk neutrality on both sides (i.e. $\gamma_u = \gamma_v = 1$), the solutions from 5c and 5d to the PDM game are identical. Q.E.D.

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