Gauge-Higgs Unification In Spontaneously Created Fuzzy Extra Dimensions

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Abstract

We propose gauge-Higgs unification in fuzzy extra dimensions as a possible solution to the Higgs naturalness problem. In our approach, the fuzzy extra dimensions are created spontaneously as a vacuum solution of certain four-dimensional gauge theory. As an example, we construct a model which has a fuzzy torus as its vacuum. The Higgs field in our model is associated with the Wilson loop wrapped on the fuzzy torus. We show that the quadratic divergence in the mass of the Higgs field in the one-loop effective potential is absent. We then argue based on symmetries that the quantum corrections to the Higgs mass is suppressed including all loop contributions. We also consider a realization on the worldvolume theory of D3-branes probing $\mathbb{C}^3/(\mathbb{Z}_N \times \mathbb{Z}_N)$ orbifold with discrete torsion.

Contents

1 Introduction

The Standard Model has a naturalness problem regarding the mass of the Higgs field. The leading quantum corrections to the Higgs mass square δm_H^2 takes the form

$$\delta m_H^2 \sim \kappa \, \Lambda_{SM}^2, \tag{1.1}$$

where κ is a numerical coefficient and Λ_{SM} is the UV cut-off for the Standard Model. Λ_{SM} should be regarded as a physical energy scale above which the modification to the Standard Model becomes significant. The quadratic dependence on the UV cut-off Λ_{SM} in (1.1) is a generic feature of the quantum correction to the scalar mass in four dimensional space-time and this UV sensitivity is the origin of the Higgs naturalness problem. The Standard Model contributions to κ is of order $\sim 10^{-2}$. Since the Higgs mass is expected to be in the order of $\sim 10^2$ GeV, the formula (1.1) tells us that the mass of the Higgs field requires unnatural fine-tuning if the UV cut-off Λ_{SM} of the Standard Model goes too much beyond the TeV scale, not to mention the GUT scale or Planck scale. In order for the Standard Model to remain natural, new physics must enter at some TeV scale to modify the high energy behavior of the Standard Model. This is one of the main reasons why the Large Hadron Collider is likely to find not only the Higgs particle but also the new physics beyond the Standard Model.

The new physics relevant for solving the Higgs naturalness problem must replace (1.1) by

$$\delta m_H^2 \sim \kappa_{NP} \, \Lambda_{NP}^2, \tag{1.2}$$

with small enough[1] coefficient κ_{NP}, where Λ_{NP} is a UV cut-off of the model that describes the new physics. From the effective field theory point of view, small parameters in a theory, by which κ_{NP} is made small, must be associated with a (weakly broken) symmetry [1] in order for the model to be natural. Thus if the cut-off Λ_{NP} is hierarchically higher than some TeV scale, solving the Higgs naturalness problem in the framework of effective field theory boils down to identifying the relevant symmetry.[2] While (1.1) and (1.2) may look similar in the form, the limitation of the Standard Model was that it does not have any symmetry which can protect the Higgs mass from the quantum corrections.[3]

Local gauge symmetry is relevant up to the electroweak scale, and it is expected to be important even at much higher energy scale, as employed in candidates of the fundamental theory like string theory. It also forbids the mass term of the spin-one particles, and it is a vital candidate as a solution to the Higgs naturalness problem. In gauge-Higgs unification scenario, the Higgs field is the zero-mode of an extra-dimensional component of a gauge field in higher dimensions. It has been shown that the one-loop correction to the mass of the Higgs in this scenario is indeed insensitive to the UV cut-off [3, 4, 5, 6]. The Higgs field can be associated with the Wilson loop wrapped around a cycle in the extra dimensions.

Gauge theories in higher dimensions are non-renormalizable and inevitably effective field theories with a finite UV cut-off. This itself is not an immediate problem, though obviously another UV theory is required above the cut-off scale. Another issue is that the extra dimensions are given a priori in higher-dimensional gauge theories. The extra dimensional space is supposed to be determined by the dynamics of some gravitational theory in higher dimensions, which lies beyond the energy scale described by the gauge theory. It will be interesting if there is an alternative scenario based on four-dimensional quantum field theory where the extra dimensions effectively emerge: Four-dimensional quantum field theories have more chances to be renormalizable, and the emergent extra

[1]It is zero in some models, e.g. those based on supersymmetry.

[2]There are two other possible explanations to the smallness of the Higgs mass: 1. The fundamental scale is at some TeV scale. 2. The Higgs mass is fine-tuned, probably by the anthropic principle. The first gives a very strong restriction to the possible fundamental theory at the highest energy scale while the second is out of the framework of effective field theory. It is also hard to be very convincing. We will not pursue these possibilities in this paper.

[3]We refer to an amusing essay [2] for the historical background and relevant references regarding the Higgs naturalness problem.

dimensions are described within the framework of the four-dimensional quantum field theory. The idea of (de)construction [7, 8] (see also [9]) realizes such idea using quiver gauge theory. In this scenario, the quiver diagram (moose) prepares latticized extra dimensions, while the lattice spacing is dynamically determined by the four-dimensional quiver gauge theory.

On the other hand, fuzzy spaces are ubiquitous in multiple D-brane systems in string theory [10, 11, 12, 13]. In fact it has been known even before the (de)construction that the fuzzy extra dimensions can be described as a vacuum of lower dimensional quantum field theories. Moreover, it has also been shown that the fluctuations around the fuzzy vacuum contain excitations that can be identified with a gauge field on the fuzzy space. Thus fuzzy extra dimensions in string theory appear as a rather natural setting for the four-dimensional models of gauge-Higgs unification. Indeed, this possibility has been noticed for a while, see e.g. [14] and references therein. However, as far as we have noticed, there has been no detailed study of the quantum aspects of the gauge-Higgs unification in fuzzy extra dimensions which is relevant for the Higgs naturalness problem. The purpose of this work is to construct an explicit model that realizes the gauge-Higgs unification in emergent fuzzy extra dimensions, and study its quantum aspects in detail to make clear the issues in this scenario in the context of Higgs naturalness problem. We will be particularly interested in the fuzzy extra dimensions realized by finite size matrices. In this case, the KK mass spectrum in the fuzzy extra dimensions are truncated at finite level, and the difference from the ordinary extra dimensions becomes sharp.

2 A unitary matrix model of gauge-Higgs unification in fuzzy torus

In this section, we study a model which realizes the gauge-Higgs unification in fuzzy torus extra dimensions. At this stage, our model is a toy model and the "Higgs" field here means a scalar field in four dimensions and in some representation of a gauge group. The gauge group should contain the electroweak gauge group as a subgroup and the Higgs field should be in a certain representation in a realistic model, but we will not be concerned with these points too much in the following. This issue has been studied extensively in the ordinary gauge-Higgs unification models, and we leave this issue to the Discussion section.

In the study of the ordinary gauge-Higgs unification models, the torus has been a nice example of the extra dimensions in which one could make detailed studies as well as construct realistic models.[4] Therefore, it would be a good starting point to study a model

[4] Circle may be the simplest extra dimension, but fuzzy spaces should have coordinates which do not

which has a fuzzy version of the torus as its vacuum. A brief summary of fuzzy torus is provided in the appendix A.

2.1 The unitary matrix model

Let us consider the following four-dimensional action with $SU(kN)$ gauge group:

$$S = \int d^4x \, \mathrm{tr}_{SU(kN)} \Bigg[-\frac{1}{2} F_{\mu\nu}(x) F^{\mu\nu}(x) + \sum_{I=1,2} f_I^2 D_\mu U_I(x) D^\mu U_I^\dagger(x) + c_0 U_1 U_2 U_1^\dagger U_2^\dagger + c_0^* U_2 U_1 U_2^\dagger U_1^\dagger + \dots \Bigg] \quad (\mu, \nu = 0, \cdots, 3), \tag{2.1}$$

We regard the action (2.1) as an effective field theory with a UV cut-off Λ. Like in the chiral perturbation theory, a natural UV cut-off scale may be the energy scale where the perturbative loop expansion of the model breaks down [15, 16]. As explained in the appendix B, it is estimated to be

$$\Lambda \approx \frac{4\pi f}{\sqrt{kN}}. \tag{2.2}$$

Here, we consider the case where there is no big hierarchy between the scales f_1 and f_2: $f \approx f_1 \approx f_2$. The action (2.1) has two small expansion parameters and " ... " in (2.1) denotes the terms suppressed by powers of these small parameters: One is the inverse of the cut-off $1/\Lambda$, as is usual in effective field theory. Another is a small dimensionless $SU(kN)$ gauge coupling g, which in the perturbative expansion appears in the combination (see the appendix B)

$$\frac{gf}{\Lambda}. \tag{2.3}$$

The parameter (2.3) is associated with a breaking of a "chiral" symmetry in the action (2.1), as will be explained in more detail in section 2.4.

The fields U_I take values in special unitary matrix:

$$U_I^\dagger(x) = U_I^{-1}(x), \quad \det U_I(x) = 1 \quad (I = 1, 2). \tag{2.4}$$

The field strength of the gauge field is given as usual:

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig[A_\mu(x), A_\nu(x)]. \tag{2.5}$$

commute with each other, thus we need at least two dimensions.

The covariant derivatives are given by

$$D_\mu U_I(x) = \partial_\mu U_I(x) - ig[A_\mu(x), U_I(x)]. \tag{2.6}$$

The potential part of the action (2.1) has the same form with the finite rank version of the twisted Eguchi-Kawai model of lattice gauge theory [17, 18]. In the twisted Eguchi-Kawai model, the fields U_I are the link fields of the lattice gauge theory in the extra dimensions, where two extra dimensions are periodic lattice with just one lattice point. In the twisted Eguchi-Kawai model, larger size extra dimensions are effectively generated by the vacuum configuration, as we explain below.[5]

The action (2.1) is an extreme version of the one considered in (de)construction [8]. In (de)construction, latticized extra dimensions are constructed from the quiver diagram (moose) of a quiver gauge theory. In the language of the quiver gauge theory, our moose has only one node. The new ingredient of our model is that the large extra dimensions are generated not by the large moose but by the fuzzy torus vacuum. One may regard that the moose effectively gets large via (the inverse of) the twisted Eguchi-Kawai reduction.

From the point of view of effective field theory, there is a natural magnitude for the coefficient c_0 appearing in the action (2.1). It is estimated in the appendix B and can be parametrized as

$$c_0 = g^2 f_1^2 f_2^2 \, \tilde{c}_0, \tag{2.7}$$

where $\tilde{c}_0$ is a dimensionless complex number of order one.

As an effective field theory, it is important to specify the symmetries the action (2.1) has. We impose the four-dimensional Poincare symmetry and the $SU(kN)$ gauge symmetry as exact symmetries. We also require the action to be invariant under the following $\mathbb{Z}_{kN} \times \mathbb{Z}_{kN}$ global transformations:

$$U_I \to e^{\frac{2\pi i}{kN} n_I} U_I \quad (n_I \in \mathbb{Z}_{kN},\, I = 1, 2). \tag{2.8}$$

This is the so-called center symmetry which often appears in the study of gauge theories with $SU(kN)$ gauge group. It is particularly important in the Eguchi-Kawai reduction since the condition for the Eguchi-Kawai reduction to take place is that this symmetry (or the large part of it, see below) is not broken. The $\mathbb{Z}_N \times \mathbb{Z}_N$ subgroup of the $\mathbb{Z}_{kN} \times \mathbb{Z}_{kN}$ global symmetry (2.8) will be crucial for the suppression of the quantum corrections to the mass of our model Higgs field, as will be discussed in section 2.4.

In effective field theories, not only the exact symmetries but also approximate symmetries play important roles. Let us consider the CP transformation:

$$A_\mu \to A_\mu^T, \quad U_I \to U_I^T. \tag{2.9}$$

[5]The Eguchi-Kawai reduction [19] may be one of the earliest examples where the space(-time) effectively emerges from a lower dimensional quantum field theory.

CP symmetry is broken by the following term in the action (2.1):

$$i\,\mathrm{Im}\,c_0\,\mathrm{tr}_{SU(kN)}\left[U_1U_2U_1^\dagger U_2^\dagger - U_2U_1U_2^\dagger U_1^\dagger\right]. \tag{2.10}$$

This means that it is natural for the coefficient $\mathrm{Im}\,\tilde{c}_0$ to be small in the sense of 't Hooft [1].

The following transformations which can be regarded as the reflections of coordinates in the extra dimensional lattice directions are also weakly broken by the term (2.10):

$$P_1 \;:\; U_1 \to U_1^{-1} = U_1^\dagger, \tag{2.11}$$
$$P_2 \;:\; U_2 \to U_2^{-1} = U_2^\dagger. \tag{2.12}$$

In addition to the symmetries mentioned above, the leading terms presented in the action (2.1) has a weakly broken global $(SU_L(kN)\times SU_R(kN))^2$ "chiral" symmetry[6] which recovers when the gauge coupling g is turned off:[7]

$$U_I \to L_I U_I R_I^\dagger \qquad (I=1,2), \tag{2.13}$$

where L_I and R_I are independent $SU(kN)$ matrices. A subgroup of this chiral symmetry is the origin of the small expansion parameter (2.3) and will be crucial for the suppression of the quantum corrections to the mass of the Higgs field, as we discuss in section 2.4.

The potential term in the action (2.1) which is leading in the expansions in $1/\Lambda$ and fg/Λ is

$$V_0(U_I) \equiv -\mathrm{tr}_{SU(kN)}\left[c_0U_1U_2U_1^\dagger U_2^\dagger + c_0^*U_2U_1U_2^\dagger U_1^\dagger\right]. \tag{2.14}$$

This can be rewritten in the form of the perfect square:

$$V_0(U_I) = g^2f_1^2f_2^2|\tilde{c}_0|\,\mathrm{tr}_{SU(kN)}\left[\left|U_1U_2 - e^{-i\theta}U_2U_1\right|^2 - 2\right], \tag{2.15}$$

where we have used the parametrization (2.7) and θ is the phase of the complex number $\tilde{c}_0$:

$$\tilde{c}_0 = |\tilde{c}_0|e^{i\theta}. \tag{2.16}$$

Then, the absolute minimum of the potential (2.14) is given by the configuration satisfying

$$U_1U_2 - e^{-i\theta}U_2U_1 = 0. \tag{2.17}$$

[6]The reason we call it chiral symmetry is that chiral gauge theories are candidates of the UV completion of this effective field theory, and the origin of this symmetry in this case is the approximate chiral symmetry [8].

[7]We have taken into account (2.7).

However, (2.17) is satisfied only for specific values of θ, as we describe shortly.

We would like to consider the vacuum configuration of the form

$$\begin{aligned} U_1 = V_1 &\equiv W_1 \otimes e^{\frac{N-1}{N}\pi i}\mathbb{1}_k, \\ U_2 = V_2 &\equiv W_2 \otimes e^{\frac{1}{N}\pi i}\mathbb{1}_k, \end{aligned} \tag{2.18}$$

where W_1 and W_2 are $N \times N$ constant unitary matrices satisfying the relation

$$W_1 W_2 = e^{-i\theta} W_2 W_1. \tag{2.19}$$

The phases in (2.18) are put to make the fields U_I to be special unitary matrices. Notice that in order to satisfy (2.19) by finite size matrices, the parameter θ has to take a special value

$$\theta = \frac{2\pi}{N}\,\ell, \tag{2.20}$$

where ℓ is an integer. This can be understood by taking the determinant of both sides of (2.19). We will discuss the case when θ is away from the value (2.20) in section 2.3. Our purpose here is to explain the mechanism that suppresses the quantum corrections to the Higgs mass. Therefore we may choose the simplest case $\ell = 1$ as an example.[8] The case for other ℓ is similar as long as ℓ is small compared with N.

The vacuum (2.18) breaks the gauge symmetry to $SU(k)$. In a realistic model, the electroweak gauge group should be in a subgroup of this $SU(k)$ gauge group. The vacuum (2.18) also breaks the global $\mathbb{Z}_{kN} \times \mathbb{Z}_{kN}$ symmetry to $\mathbb{Z}_N \times \mathbb{Z}_N$. The global $\mathbb{Z}_N \times \mathbb{Z}_N$ symmetry

$$U_I \to e^{\frac{2\pi i}{N} n_I}\, U_I \quad (n_I \in \mathbb{Z}_N, I = 1, 2), \tag{2.21}$$

is not broken since the $\mathbb{Z}_N \times \mathbb{Z}_N$ transformation (2.21) to the vacuum (2.18) is equivalent to a gauge transformation due to (2.18):

$$\begin{aligned} e^{\frac{2\pi i}{N}} V_1 &= V_2 V_1 V_2^\dagger, \\ e^{\frac{2\pi i}{N}} V_2 &= V_1^\dagger V_2 V_1. \end{aligned} \tag{2.22}$$

This unbroken $\mathbb{Z}_N \times \mathbb{Z}_N$ symmetry will be crucial for the suppression of the quantum corrections to the mass of the Higgs field. This will be explained in section 2.4.

The configuration (2.18) can be interpreted as a fuzzy torus [20, 21]. The reason that it can be regarded as a fuzzy version of the *torus* is that the mass spectrum on this vacuum

[8] If ℓ is a divisor of N, we can redefine $N_{new} = N/\ell$, $k_{new} = \ell k$ to have $\ell_{new} = 1$. If we assume that the CP violation due to this phase is small, ℓ/N is naturally small. Put it differently, the model with $SU(N')$ gauge group ($N' = kN$) can have fuzzy torus solution with $\theta = 2\pi\ell/N'$, and we chose the $\ell = k$ case with k being a divisor of N'.

approximates the low-lying KK modes on the ordinary torus, as we will see below. On the other hand, the higher mass spectrum deviates from that of the KK modes on the ordinary torus and is truncated at a finite level.[9] Thus one cannot probe arbitrarily small distance on the fuzzy torus, and the notion of a point becomes obscure. This is the reason we call it *fuzzy*.

2.2 One-loop effective potential around the fuzzy torus vacuum

For concreteness, below we study the case $k = 2$. Generalization to arbitrary k is straightforward. In the ordinary gauge-Higgs unification scenario, the Higgs field is identified with a zero-mode of an extra dimensional component of a gauge field. We identify corresponding zero-modes in our fuzzy torus model below. Then we calculate the 1PI effective potential for the zero-modes.

The fluctuations of the field U_I around the fuzzy torus vacuum (2.18) are analogous to the link variables in the lattice gauge theory (see the appendix A). Thus these fluctuations can be identified with the exponential of the components of the gauge field in the extra dimensions. On the other hand, the commutators with the vacuum configuration V_I (2.18) are related to the discrete counterparts of the derivatives on the fuzzy torus (see the appendix A). Thus the zero-modes in the extra dimensions are those which commute with the matrices V_I. Without loss of generality, we can parametrize the zero-modes by $u_0^I(x)$ as

$$U_I = U_I^{(0)} \equiv e^{i\frac{u_0^I(x)}{\sqrt{4N}f_I}\Sigma} V_I \qquad (I = 1, 2), \tag{2.23}$$

where

$$\Sigma = \mathbb{1}_N \otimes \sigma_3, \tag{2.24}$$

and σ_i $(i = 1, 2, 3)$ are the Pauli matrices. In (2.23) we have canonically normalized the zero-modes $u_0^I(x)$. Notice that since the zero-modes $u_0^I(x)$ commute with the matrices V_I as well as with each other by definition, $U_I^{(0)}$ also satisfy the relation

$$U_1^{(0)} U_2^{(0)} = e^{-i\theta} U_2^{(0)} U_1^{(0)}. \tag{2.25}$$

Thus the configuration $U_I^{(0)}$ is also a classical minimum of the leading potential (2.15). In other words, the zero-modes u_0^I parametrize the classical flat directions of (2.15).

Now we calculate the 1-PI effective potential for $u_0^I(x)$ from the one-loop diagrams made from the leading terms explicitly shown in the action (2.1).[10] For simplicity, we

[9] When the size N of the matrices which describe the fuzzy torus is finite. In this paper, we will mostly consider this case. See section 2.4 for further discussions.

[10] In the current choice of the UV cut-off Λ (2.2), the natural magnitude of a term at the tree level is the same to that of the loop contributions.

present the calculation for $\tilde{c}_0 = 1$. When $\tilde{c}_0 \neq 1$, the gauge field A_μ and the field u^I introduced below feel different fuzzy torus radii. Because of this the calculation for $\tilde{c}_0 \neq 1$ case is slightly more complicated compared with the $\tilde{c}_0 = 1$ case. However, $\tilde{c}_0 = 1$ case is enough for understanding the mechanism that suppresses the quantum corrections to the Higgs mass. We will also give more general analysis for the suppression of the quantum corrections based on symmetries in section 2.4.

We fix the gauge as

$$\partial_\mu A^\mu + \Delta_I^0 u^I = 0, \tag{2.26}$$

where we have parametrized U_I as

$$U_I = e^{i\frac{u^I(x)}{\sqrt{4N}f_I}} U_I^{(0)}. \tag{2.27}$$

We have also defined

$$\Delta_I^0 \varphi \equiv \frac{1}{a_I}\left(U_I^{(0)}\varphi U_I^{(0)\dagger} - \varphi\right), \tag{2.28}$$

for a field φ in the adjoint representation of $SU(kN)$, where

$$a_I \equiv \frac{1}{gf_I} \quad (I = 1, 2). \tag{2.29}$$

Since we have assumed $f_1 \approx f_2 \approx f$, $a_1 \approx a_2 \approx a$. (2.28) are discrete counterparts of the covariant derivatives with the background gauge field. Including the contributions from the ghost fields, the one-loop effective potential is given as

$$\begin{aligned} V_{1-loop}(u_0^I) &= i \ln \det((D^0)^2)^{-6/2} + i \ln \det((D^0)^2)^{+1} \\ &= -2i \operatorname{Tr} \ln((D^0)^2), \end{aligned} \tag{2.30}$$

where we have defined

$$(D^0)^2 \equiv \partial_\mu \partial^\mu + \Delta_I^0 \Delta_I^0. \tag{2.31}$$

After the Wick rotation (2.30) becomes

$$V_{1-loop}(u_0^I) = 2 \sum_{m_1,m_2} \sum_{i,j=1}^{2} \int \frac{d^4k}{(2\pi)^4} \ln\left(k^2 + m^2_{(m_1,m_2)(i,j)}(u_0^I)\right), \tag{2.32}$$

where

$$\begin{aligned} m^2_{(m_1,m_2)(i,j)}(u_0^I) &\equiv \sum_{I=1,2} \left(\frac{2}{a_I}\right)^2 \sin^2 \frac{1}{2}\left(m_I\theta + (u_i^I - u_j^I)\right) \\ &= \sum_{I=1,2} \frac{2}{a_I^2}\left(1 - \cos\left(m_I\theta + (u_i^I - u_j^I)\right)\right), \end{aligned} \tag{2.33}$$

with

$$u_1^I = -u_2^I = \frac{1}{\sqrt{4Nf_I}} u_0^I. \tag{2.34}$$

In (2.30) the sum over m_I ($I = 1, 2$) run over integers in $-\frac{N}{2} \le m_I < \frac{N}{2}$. Notice that when $m_I \ll N$, the mass spectrum of the fluctuations around the fuzzy torus vacuum (2.18) ($u_0^I(x) = 0$ in (2.33)) approximates the low-lying KK modes of the ordinary torus

$$\frac{2}{a_I^2} (1 - \cos(m_I\theta)) \approx \left(\frac{m_I}{R_I}\right)^2, \tag{2.35}$$

where the radii R_I of the torus are given by

$$2\pi R_I \equiv N a_I \qquad (I = 1, 2). \tag{2.36}$$

This is the reason why the vacuum (2.18) is regarded as the fuzzy version of the torus. Below we will call $R_I = a_I N/2\pi$ as the radii of the fuzzy torus.

With the momentum UV cut-off at Λ, (2.32) becomes

$$\begin{aligned} V_{1-loop}(u_0^I) \; = \; & \sum_{m_1,m_2} \sum_{i,j=1}^{2} \left[\frac{\Lambda^2}{8\pi^2} m^2_{(m_1,m_2)(i,j)}(u_0^I) \right. \\ & \left. + \frac{1}{16\pi^2} (m^2_{(m_1,m_2)(i,j)}(u_0^I))^2 \ln \frac{m^2_{(m_1,m_2)(i,j)}(u_0^I)}{\Lambda^2} + \mathcal{O}(\Lambda^{-2}) \right]. \end{aligned} \tag{2.37}$$

Now, notice that the sum of $m^2_{(m_1,m_2)(i,j)}(u_0^I)$ over m_1 and m_2 does not depend on $u_0^I(x)$ for $N \ge 2$. Similarly, the sum of $(m^2_{(m_1,m_2)(i,j)}(u_0^I))^2$ over m_1 and m_2 does not depend on $u_0^I(x)$ for $N \ge 3$, due to the cancellations between phases. Recall that we are considering the case $\ell = 1$ in (2.20)), i.e. $\theta = 2\pi/N$. Thus the divergences associated with $\Lambda \to \infty$[11] contribute only to the constant term in the effective potential for the zero modes $u_0^I(x)$ for $N \ge 3$, while there remains only logarithmic divergences for $N = 2$.[12]

Similarly, we observe that for given N the first non-zero correction in the inverse power expansion of the UV cut-off Λ is proportional to

$$\sum_{m_1,m_2} \sum_{i,j=1}^{2} \Lambda^4 \left(\frac{m^2_{(m_1,m_2)(i,j)}(u_0^I)}{\Lambda^2} \right)^N. \tag{2.38}$$

We will discuss this structure from the point of view of effective field theory below.

[11] Below we will call divergences associated with taking the cut-off Λ to infinity as "divergence" for briefness, although the natural UV cut-off scale is as in (2.2) in our model.

[12] This result is similar to that in the (de)construction models [8].

The mass m_0 of the zero-modes u_0^I in the one-loop effective potential (2.37) is calculated in the appendix C and of the order

$$m_0^2 \approx \frac{g_{SU(k)}^2}{16\pi^2}\frac{1}{R^2}, \tag{2.39}$$

where the four-dimensional $SU(k)$ gauge coupling $g_{SU(k)}$ is given in (2.66) and $R_1 \approx R_2 \approx R$ follows from our earlier assumption $f_1 \approx f_2 \approx f$. This is as expected since $1/R$ is the scale where the effect of the new physics appears, and $g_{SU(k)}^2/16\pi^2$ is the one-loop factor.

To generalize the above $k = 2$ result to general k, just notice that the zero-modes are parametrized by the Cartan of the $SU(k)$ group. The result has the same form with (2.37), with the sum over the indices i and j run from 1 to k.

2.3 Fuzzy torus local minima for general θ

So far we have been considering a special case where the parameter θ takes the particular discrete value (2.20). Below we will argue that our previous analysis can be applied to the case when θ is away from this value, with just minor modifications.

We would like to analyze the minima of the effective potential[13]

$$V_0(U_I;\theta) = g^2 f_1^2 f_2^2 |\tilde{c}_0| \tilde{V}_0(U_I;\theta), \tag{2.40}$$

where

$$\tilde{V}_0(U_I;\theta) \equiv -\mathrm{tr}_{SU(kN)}\left[e^{i\theta}U_1U_2U_1^\dagger U_2^\dagger + e^{-i\theta}U_2U_1U_2^\dagger U_1^\dagger\right]. \tag{2.41}$$

(2.40) is the leading term of the effective potential in the expansions in $1/\Lambda$ and gf/Λ.

Let us define fuzzy torus background $V_I(\ell)$ for θ_ℓ as follows:

$$\theta_\ell \equiv \frac{2\pi}{N}\ell \quad (\ell:\text{integer}), \tag{2.42}$$

$$V_1(\ell)V_2(\ell) = e^{-i\theta_\ell}V_2(\ell)V_1(\ell). \tag{2.43}$$

Let us put $U_I = V_I(\ell)$ into the potential (2.40) and study the fluctuations around the background. We can rewrite (2.41) as

$$\begin{aligned}\tilde{V}_0(U_I;\theta) \;\; = \;\; & -\cos\delta\theta\,\mathrm{tr}_{SU(kN)}\left[e^{i\theta_\ell}U_1U_2U_1^\dagger U_2^\dagger + e^{-i\theta_\ell}U_2U_1U_2^\dagger U_1^\dagger\right] && (2.44)\\ & -i\sin\delta\theta\,\mathrm{tr}_{SU(kN)}\left[e^{i\theta_\ell}U_1U_2U_1^\dagger U_2^\dagger - e^{-i\theta_\ell}U_2U_1U_2^\dagger U_1^\dagger\right], && (2.45)\end{aligned}$$

[13]Rather than the value of θ in the action, the corresponding parameter in the 1-PI effective potential is directly relevant for the determination of the vacuum. Below we use the same symbols θ, $\tilde{c}_0$ etc. to express the parameters in the 1-PI effective potential.

where

$$\delta\theta \equiv \theta - \theta_\ell. \tag{2.46}$$

The fluctuation spectrum around the $U_I = V_I(\ell)$ in the first line (2.44) can be analyzed similarly as in the case $\ell = 1$. When $\cos\delta\theta > 0$, the fuzzy torus vacuum is a local minimum of the term (2.44) and there are massless and massive fluctuations but no tachyonic modes. On the other hand, if we put $U_I = V_I(\ell)$ into (2.45) and expand U_I around this background, one can explicitly check that the terms linear or quadratic in the fluctuations are absent. This means that the fuzzy torus configuration stays in the local minimum of the potential (2.41) as long as $\cos\delta\theta > 0$.

We can understand the absence of the linear and the quadratic fluctuations around the fuzzy torus in the term (2.45) from the symmetry. Consider the CP transformation (2.9):

$$\begin{aligned} &A_\mu \to A_\mu^T, \quad U_I \to U_I^T, \\ &\theta \to -\theta. \end{aligned} \tag{2.47}$$

Here, we have associated a transformation property to the parameter θ under the CP transformation so that the whole action becomes symmetric under the CP transformations. Under the CP transformation, both the term

$$\mathrm{tr}_{SU(kN)}\left[e^{i\theta_\ell}U_1U_2U_1^\dagger U_2^\dagger - e^{-i\theta_\ell}U_2U_1U_2^\dagger U_1^\dagger\right], \tag{2.48}$$

and the coefficient $i\sin\delta\theta$ are odd, so that the whole potential is even under the CP transformation. Here, the CP transformation property of θ_ℓ and $\delta\theta$ are induced from that of θ: $\theta_\ell \to -\theta_\ell$, $\delta\theta \to -\delta\theta$. On the other hand, From the CP symmetry and the gauge symmetry, the only possible terms linear and quadratic in the fluctuations corresponds to the following term in the continuum limit $N \to \infty$:

$$i\delta\theta \int_{T^2} \mathrm{tr}_{SU(k)}\, F_{12}, \tag{2.49}$$

(F_{12} is the field strength of the $SU(k)$ gauge theory on the torus) which is identically zero (it would be a total derivative if we were considering the $U(k)$ gauge group instead of $SU(k)$, which is again zero after the integration.) Even without taking the continuum limit, the terms linear and quadratic in fluctuations have essentially the same structure to that of the commutative limit (2.49). One can also use the reflection symmetries (2.11), (2.12) to obtain the same conclusion.

The energy of the local minimum $U_I = V_I(\ell)$ is calculated to be

$$V_0(V_I;\theta) = -2g^2f^4|c_0|kN\cos\delta\theta. \tag{2.50}$$

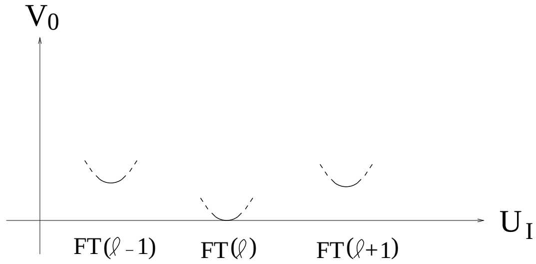

Figure 1: Schematic figure of the potential $V_0(U_I;\theta)$ for $\theta = 2\pi\ell/N$. FT(ℓ) denotes the fuzzy torus configuration (2.43).

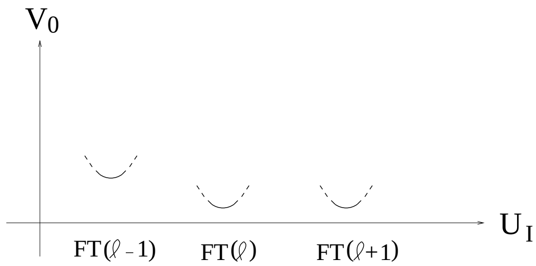

Figure 2: The potential $V_0(U_I;\theta)$ for $2\pi\ell/N < \theta < 2\pi(\ell+1)/N$. Among the fuzzy torus minima, the one whose θ_ℓ is closest to θ has the smallest energy.

This means that among the fuzzy torus minima labeled by ℓ, the one whose θ_ℓ is closest to θ has the smallest energy. See Figs. 1-3. Although we have not completely sought out the all minima of the potential (2.40), from (2.50) it seems reasonable to assume that the fuzzy torus configuration with θ_ℓ closest to θ is the absolute minimum. If this is the case, no fine tuning for θ is required. Notice that when N is large, there is always θ_ℓ which is close ($\delta\theta \lesssim 2\pi/N$) to θ.

The fuzzy torus looks closer to the ordinary torus when ℓ/N is small. By assuming that the CP violation (or the violation of the reflection symmetry) is small, $\theta \ll 1$ is preferred, thus small ℓ/N is preferred.

Comment on the possibility of dynamical tuning of θ

There is another scenario with an attractive feature, which however has a problem in the naturalness. We will explain this possibility below.

Let us consider the case where θ is not a constant but a dynamical field depending on the four-dimensional coordinates x:

$$V_0(U_I(x),\theta(x)) = g^2 f_1^2 f_2^2 |\tilde{c}_0| \tilde{V}_0(U_I(x),\theta(x)), \tag{2.51}$$

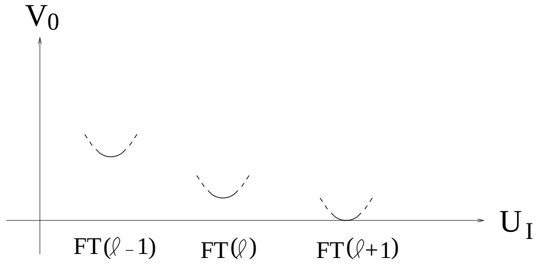

Figure 3: The potential $V_0(U_I;\theta)$ for $\theta = 2\pi(\ell+1)/N$.

where

$$\tilde{V}_0(U_I(x), \theta(x)) \equiv -\mathrm{tr}_{SU(kN)} \left[e^{i\theta} U_1 U_2 U_1^\dagger U_2^\dagger + e^{-i\theta} U_2 U_1 U_2^\dagger U_1^\dagger \right]. \tag{2.52}$$

Then, the combination

$$\theta = \frac{2\pi}{N} \ell, \quad U_I = V_I(\ell) \quad (\ell : \text{integer}), \tag{2.53}$$

is the minimum of the potential (2.51). In string theory, the non-commutative parameter usually arise from form field background, and thus it is natural to expect that it becomes dynamical at some energy scale. An attractive feature of this scenario is that this provides a model for the spontaneous CP symmetry breaking.

However, regarding (2.51) as the leading potential is problematic from the point of view of effective field theory. The following term can be induced by the quantum correction

$$V_q(U_I) = c_q \mathrm{tr} \left[U_1 U_2 U_1^\dagger U_2^\dagger \right] + h.c. \tag{2.54}$$

where the natural magnitude of the complex coefficient c_q is estimated to be

$$c_q \sim g^2 f^4. \tag{2.55}$$

The combined potential $V_0 + V_q$ no longer has the combination (2.53) as its minimum.

The problem is that no symmetry guarantees the form of the potential (2.51) to be the leading term in our effective field theory. The best one can do may be to promote the imaginary part of $\tilde{c}_0$ in (2.41) to be dynamical. The real part of $\tilde{c}_0$ is a coefficient of the CP even term and the imaginary part is that of the CP odd term. Thus the separation of the real part and imaginary part of $\tilde{c}_0$ is protected from the quantum corrections by the CP symmetry, and it is possible to promote only the imaginary part dynamical. The quantum correction (2.54) contributes to $\mathrm{Im}\,\tilde{c}_0$ as a constant shift. This can be absorbed into the field redefinition of $\mathrm{Im}\,\tilde{c}_0$ if one requires this shift to be a symmetry of the other part of the action. This is reminiscent of the Peccei-Quinn symmetry for the axion [22]. However, here we are interested in the CP breaking vacua. When $\theta \ll 1$, $\theta \sim \mathrm{Im}\,\tilde{c}_0/\mathrm{Re}\,\tilde{c}_0$ and in this case the combination (2.53) may become an approximate solution, with dynamical $\mathrm{Im}\,\tilde{c}_0$ helping for lowering the potential energy.[14]

2.4 Arbitrary loop diagrams and the symmetry constraints

Analysis at small N

We first consider the case when N is small and can be neglected for a rough order estimate. The N dependence will be incorporated after this analysis.

What is important for the suppression of the quantum corrections to the zero-modes is the $SU_L(kN) \times SU_R(kN)$ subgroup of the weakly broken chiral symmetry (2.13). The breaking of the approximate chiral symmetry can be parametrized by introducing non-propagating "spurion" s which takes value in $kN \times kN$ complex matrix. We assign transformation laws for s under the chiral transformation so that the covariant derivatives transform homogeneously under the chiral transformation. Thus we replace the covariant derivative (2.6) as follows:

$$D_\mu U_I = \frac{1}{g}\partial_\mu U_I s - iA_\mu U_I s + iU_I s A_\mu. \tag{2.56}$$

Then we can make the $SU_L(kN) \times SU_R(kN)$ part of the $(SU_L(kN) \times SU_R(kN))^2$ chiral transformation to a global symmetry of the action:

$$\begin{aligned} U_I &\to LU_I R^\dagger, && (2.57)\\ A_\mu &\to LA_\mu L^\dagger, && (2.58)\\ s &\to RsL^\dagger. && (2.59)\end{aligned}$$

After using the spurion s to define the global chiral symmetry, we set $s = g\mathbb{1}_{kN}$. This breaks the chiral $SU_L(kN) \times SU_R(kN)$ to its diagonal subgroup in which $L = R$.

Now, notice that in the fuzzy torus vacuum (2.18), the vacuum expectation value of $\mathrm{tr}_{SU(kN)}\, {U_I}^\ell$ vanishes except for the special value of ℓ due to the remaining $\mathbb{Z}_N \times \mathbb{Z}_N$ symmetry (2.21) on the vacuum:

$$\mathrm{tr}_{SU(kN)}\, {U_1}^\ell\Big|_{U_I=V_I} = 0 \quad (\ell \neq (\text{multiple of } N)). \tag{2.60}$$

Thus in the leading order in the power series expansion in $1/\Lambda$, the operator which is consistent with the global symmetry $\mathbb{Z}_{kN} \times \mathbb{Z}_{kN}$ (2.8) and the spurious symmetry (2.57) which contributes to the mass term of the zero mode $u_0^1(x)$ is

$$\left|\mathrm{tr}_{SU(kN)}\, (U_1 s)^N\right|^2. \tag{2.61}$$

[14] While for the leading potential (2.40) with $\mathrm{Im}\,\tilde{c}_0$ dynamical the configuration $\mathrm{Im}\,\tilde{c}_0 = 0$ will be the minimum of the potential, it might be modified by the higher order corrections.

The analysis for the mass of the zero-mode $u_0^2(x)$ is similar. The operators $\left|\mathrm{tr}_{SU(kN)}\ (U_1 s)^{\ell}\right|^2$ with $\ell \neq$ (multiple of N) can be generated in the effective action, but they do not contribute to the mass term of the zero modes due to (2.60). From the identification of the fields U_I with the link variables in the lattice gauge theory, the operator (2.61) can be regarded as the square of the Wilson loop wrapped on the 1-cycle of the torus with the radius R_1.

After setting $s = g\mathbb{1}_{kN}$, (2.61) is proportional to g^{2N}. Comparing with (B.10) in the appendix B, the power factor $2N$ on g should be equal to the power $2I_A - 2G$ on the suppression factor f/Λ, where G is the number of purely gauge interaction vertices and I_A is the number of the gauge field propagators in a Feynman diagram under consideration. Thus the contribution to the coefficient of the operator (2.61) from the L-loop Feynman diagrams is estimated as

$$f^2\Lambda^2\left(\frac{\Lambda}{4\pi f}\right)^{2L}\left(\frac{gf}{\Lambda}\right)^{2N}, \tag{2.62}$$

with a dimensionless numerical coefficient of order one.

From (2.62), we observe that at the one loop $L = 1$, the quadratic dependence on the cut-off Λ only appears when $N = 1$, and the logarithmic dependence appears when $N = 2$. However, for $N = 1$ we do not have the fuzzy torus solution. Therefore there is no quadratic divergence in the radiative corrections to the mass of the zero-modes. On the other hand, our one-loop results (2.37) and (2.38) are consistent with the above argument: When $N = 2$ there is a log divergence while for $N \geq 3$ the radiative corrections to the mass of the zero-modes is finite.

Before setting $\Lambda = 4\pi f$ (since we are considering small N case here we neglect the factor $1/\sqrt{N}$) the expression (2.62) is seemingly more divergent for higher loop contributions, since it is proportional to $\Lambda^{2+2L-2I_A}$. However, our effective field theory is valid up to the cut-off scale $\Lambda \approx 4\pi f$. At this cut-off scale all loops contributes in the same order in the magnitude. After setting $\Lambda = 4\pi f$, the coefficient (2.62) is estimated as

$$f^4 16\pi^2\left(\frac{g}{4\pi}\right)^{2N}. \tag{2.63}$$

This is the natural magnitude of the coefficient for the operator (2.61).

To summarize, two global symmetries played the major roles in suppressing the quantum corrections to the mass of the zero modes: the weakly broken chiral symmetry introduces the suppression factor gf/Λ, while how much powers are on the suppression factor is determined by the unbroken $\mathbb{Z}_N \times \mathbb{Z}_N$ symmetry (2.21).

The natural magnitude of the coefficient c_0 in the action (2.1) can be estimated in a similar way to give the explained order.

Incorporating N dependence

When comparing the fuzzy torus with different N, we should fix the radii of the fuzzy torus:

$$2\pi R_I = a_I N : fixed \quad (I = 1, 2). \tag{2.64}$$

This means when we take N to be large, we should scale a_I as

$$a_I \sim \frac{1}{N}. \tag{2.65}$$

$1/a_I = N/R_I$ is the energy scale where the discrepancy between the fuzzy torus and the ordinary torus becomes large. Thus "divergences" associated with the limit $N \to \infty$ is related to the UV divergences in the extra dimensional directions.[15] We should also compare the theory with the same four-dimensional $SU(k)$ gauge coupling. Thus we obtain the scaling[16]

$$\frac{g}{\sqrt{N}} \equiv g_{SU(k)} : fixed. \tag{2.66}$$

Together with (2.65) and (2.29), this means

$$\frac{f_I}{\sqrt{N}} : fixed. \tag{2.67}$$

Taking into account the N dependence as (B.12) in the appendix B, the coefficient (2.62) of the operator (2.61) which leads to the mass of the zero-modes is modified as

$$\frac{f^2\Lambda^2}{kN}\left(\frac{\Lambda\sqrt{kN}}{4\pi f}\right)^{2L}\left(\frac{gf}{\Lambda}\right)^{2N}. \tag{2.68}$$

The first $1/kN$ factor comes from the fact that (2.61) is a double trace operator.

We restrict ourselves to the case where the factor gf/Λ is small and provides a suppression factor in (2.68), which in turn suppresses the mass of the zero-modes. From the definition (2.29), it amounts to the case when the highest end of the mass spectrum around the fuzzy torus vacuum is below the UV cut-off scale Λ determined from the validity of the loop expansion:

$$\Lambda = \frac{4\pi f}{\sqrt{kN}} \gg gf = \frac{1}{a} = \frac{N}{2\pi R}. \tag{2.69}$$

[15] Here we use the term "divergence" in the same sense to that in the footnote 11. While we will not take N to infinity, the dependence on the large N is the dependence on the UV scale in the extra dimensional directions.

[16] It is interesting to observe that this scaling is the same to that of the (de)construction with one-dimensional periodic lattice rather than that of the two-dimensional periodic lattice.

This provides the upper limit N_c of N around which the expansion in terms of gf/Λ breaks down:

$$N \ll N_c \approx \frac{4\pi}{g_{SU(k)}\sqrt{k}}. \tag{2.70}$$

In the application to the electroweak symmetry breaking, the subgroup of $SU(k)$ should be identified with the standard model gauge group $SU(2) \times U(1)$. This would constrain the value of $g_{SU(k)}$ to be around ~ 0.5 (a typical value for an order estimate). Thus we obtain a rough estimate for the upper limit N_c:

$$N_c \approx \frac{25}{\sqrt{k}}. \tag{2.71}$$

3 A D-brane inspired model

While UV completions are not necessary from the effective field theory point of view, they do provide good motivations for the assumed symmetries in the effective field theory. As in the case of (de)construction [7], quiver gauge theories are possible UV completions of the unitary matrix model discussed in the previous section. Here we present another UV completion inspired by the worldvolume theory on D-branes probing $\mathbb{C}^3/(\mathbb{Z}_N \times \mathbb{Z}_N)$ orbifold with discrete torsion. A difference between the two UV completions is that chiral symmetry breaking of the quiver gauge theory which is expected to lead to the unitary matrix model via the non-linear realization occurs at the strong coupling regime, whereas the D-brane inspired model is perturbative. Since we are motivated by the fact that fuzzy spaces are ubiquitously realized by D-branes, the D-brane inspired model is a natural direction to investigate. Additionally, D-branes on $\mathbb{C}^3/(\mathbb{Z}_N \times \mathbb{Z}_N)$ with discrete torsion [23, 24, 25] has an explanation for the special value of θ (2.20) which is required for the existence of the fuzzy torus vacuum.

Let us consider the following action:

$$\begin{aligned} S \;=\; & \int d^4x \, \mathrm{tr}_{SU(N\times k)} \Bigg[-\frac{1}{2} F_{\mu\nu}(x) F^{\mu\nu}(x) \\ & + \sum_{I=1}^{3} \left\{ D_\mu Z_I(x) D^\mu Z_I^\dagger(x) - \frac{g^2}{2} [Z_I, Z_I^\dagger]^2 \right\} \\ & - g^2 \sum_{I=1}^{3\,(mod\,3)} \left[e^{i\theta} Z_I Z_{I+1} - Z_{I+1} Z_I \right] \left[e^{-i\theta} Z_{I+1}^\dagger Z_I^\dagger - Z_I^\dagger Z_{I+1}^\dagger \right] \\ & - \sum_{I=1,2} \frac{M_I^2}{4 f_I^2} \left(Z_I Z_I^\dagger - f_I^2 \right)^2 - M_3^2 Z_3 Z_3^\dagger \Bigg]. \end{aligned} \tag{3.1}$$

Here,

$$Z_1 = X_1 + iX_2, \quad Z_2 = X_3 + iX_4, \quad Z_3 = X_5 + iX_6, \tag{3.2}$$

where X_I are $kN \times kN$ Hermite matrices which are adjoint representations of $SU(kN)$. The covariant derivative for Z_I is given by

$$D_\mu Z_I = \partial_\mu Z_I - ig[A_\mu, Z_I]. \tag{3.3}$$

Except for the last line, the action (3.1) is the bosonic part of the low-energy effective action realized on D-branes on an orbifold $\mathbb{C}^3/(\mathbb{Z}_N \times \mathbb{Z}_N)$ with discrete torsion [23, 24, 25].[17] At the tree level, θ takes the following discrete value:

$$\theta = \frac{2\pi}{N}\,\ell. \tag{3.4}$$

This discreteness of θ at the tree level is understood as discrete torsion of the $\mathbb{C}^3/(\mathbb{Z}_N \times \mathbb{Z}_N)$ orbifold. As before, we will consider the case $\ell = 1$ as an example.

The last line in the action (3.1) was introduced to stabilize the radius of the fuzzy torus vacuum at the tree level, so that scaler fields lighter than the model Higgs do not appear.

The action (3.1) has the following global $U(1)^3$ symmetry:

$$Z_I \to e^{i\alpha_I} Z_I, \quad (\alpha_I \in \mathbb{R} \mod 2\pi). \tag{3.5}$$

This symmetry plays the similar role to that of the $\mathbb{Z}_{kN} \times \mathbb{Z}_{kN}$ symmetry in the unitary matrix model.

The minimum of the potential is given by

$$Z_1 = f_1 V_1, \quad Z_2 = f_2 V_2, \quad Z_3 = 0. \tag{3.6}$$

where the matrices V_I are the same as the ones given in (2.18). The vacuum breaks the global $U(1)^3$ symmetry (3.5) to $\mathbb{Z}_N \times \mathbb{Z}_N \times U(1)$, as described around (2.22).

Now, any complex matrix can be decomposed by a unitary matrix and an Hermite matrix. Thus we may decompose the fields Z_I by unitary matrix U_I and an Hermite matrix H_I as follows:

$$Z_I = U_I H_I \quad (\text{for } I = 1, 2). \tag{3.7}$$

This decomposition is convenient and thus appropriate for perturbative analysis around the vacuum (3.6). The unitary matrices U_I plays the similar role to the unitary matrix

[17] Similar action has been studied as an extension of the (de)construction to fuzzy spaces [26, 27].

fields U_I in the previous section and thus we have used the same symbols. The covariant derivative satisfies the Leibniz rule:

$$D_\mu Z_I = (D_\mu U_I)H_I + U_I(D_\mu H_I) \quad (\text{for } I = 1, 2), \tag{3.8}$$

where

$$\begin{aligned} D_\mu U_I &= \partial_\mu U_I - ig[A_\mu, U_I], & (3.9)\\ D_\mu H_I &= \partial_\mu H_I - ig[A_\mu, H_I]. & (3.10)\end{aligned}$$

We can extend the weakly broken global $SU_L(kN) \times SU_R(kN)$ chiral symmetry on the unitary matrix model discussed in section 2.4 to the current model:

$$\begin{aligned} &U_I \to LU_I R^\dagger, \quad H_I \to RH_I R^\dagger \quad (\text{for } I = 1, 2),\\ &Z_3 \to LZ_3 R^\dagger,\\ &s \to RsL^\dagger. \end{aligned} \tag{3.11}$$

When using the spurion s to describe the chiral symmetry, the covariant derivatives for H_I and Z_3 are modified to

$$\begin{aligned} D_\mu H_I &= \partial_\mu H_I + \frac{1}{g}\left(is^\dagger A_\mu s H_I - iH_I s^\dagger A_\mu s\right), & (3.12)\\ D_\mu Z_3 &= \frac{1}{g}\partial_\mu Z_3 s - iA_\mu Z_3 s + iZ_3 s A_\mu. & (3.13)\end{aligned}$$

Since this model has the same symmetries to those in the unitary matrix model, the mass of the zero-modes is suppressed by essentially the same mechanism.

Notice that when $M_I \ll \Lambda$, this model is not just a UV completion but also introduces other fields to the unitary matrix model. In order for the additional fields not to be lighter or have similar mass to the zero-modes, we may require

$$M_I \gtrsim \frac{1}{R}. \tag{3.14}$$

On the other hand, we would also like to require that the coupling constant in front of the quartic coupling $\mathrm{tr}(Z_I Z_I^\dagger)^2$ in the last line of (3.1) remains in the perturbative regime. Thus we require

$$M_I \lesssim f_I. \tag{3.15}$$

One may also like to require that the modification from this model appears before the perturbative expansion of the unitary matrix model breaks down. This gives the bound

$$M_I \lesssim \Lambda. \tag{3.16}$$

However, (3.15) tends to give stronger constraint, as we have seen in section 2.4.

This model without the last line in (3.1) is an asymptotically free gauge theory and theoretically we can take its UV cut-off to infinity. As a solution to the Higgs naturalness problem, this is an advantageous feature. However, the model has its own naturalness problem due to the last line of (3.1), since the masses M_I of the scalar fields Z_I are not protected by any symmetry. This point may be refined by considering a different stabilization mechanism for the fields H_I in (3.7). Since this is a model dependent detail, we leave this issue to the future investigations. We expect the idea of the gauge-Higgs unification in spontaneously created fuzzy extra dimensions to be general and have rich varieties of realizations.

4 Discussions

In this paper we focused on the naturalness issue regarding the mass of the Higgs field. We hope our results provide a basis for the construction of more realistic models of the electroweak symmetry breaking. In order to construct more realistic models, one should introduce the standard model fermions. Notice that as in the ordinary gauge-Higgs unification, the coupling between the fermions and the Higgs field are tightly constrained by the gauge symmetry. Fuzzy spaces are known to give additional constraints to the possible gauge group and the representations of the matter fields, see e.g. [28] for more about the issue and some direction in the case of the fuzzy torus. It will be interesting to examine how much of the mechanisms employed in the gauge-Higgs unification in ordinary extra dimensions can be extended to the case of the fuzzy extra dimensions. We would like to point out that the fuzzy extra dimensions may give rise to interesting Yukawa texture, as has been discussed in [29, 30, 31, 32]. Moreover, the gauge field in the extra dimensions is one of the candidates for the Higgs field in these models. So far the weak-scale supersymmetry has been employed to solve the Higgs naturalness problem in these models. Our work may provide an economical alternative solution to the Higgs naturalness problem in such scenarios, because the fuzzy extra dimensions are already built in in these scenarios.

In this work we studied the models with the fuzzy torus extra dimensions. The fuzzy torus might be special in that it circumvent the following issue. In the D-brane setting, the fuzzy space appears as a vacuum solution for the matrix version of the embedding coordinate fields of D-branes. In the situation where the fuzzy space is embedded in higher dimensional space, one would need to separate the fluctuations of the matrix coordinate fields around the fuzzy space vacuum into the gauge field components and the scalar field components. This is because in the gauge-Higgs unification, the zero-modes of the

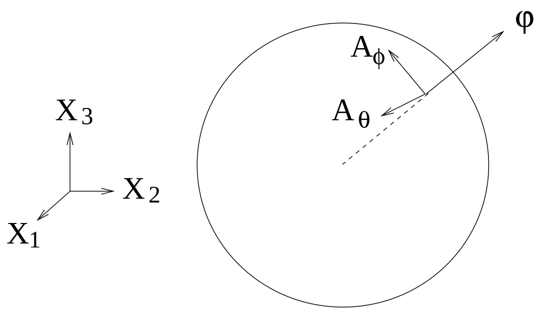

Figure 4: Decomposition of the vector in $\mathbb{R}^3$ (parametrized by the Cartesian coordinates X^1, X^2, X^3) into components tangent to the sphere (A_θ, A_ϕ) and perpendicular to the sphere (φ). The sphere is embedded in the Euclidean space $\mathbb{R}^3$. It is not clear how to make such decomposition for the fuzzy sphere with finite size matrix coordinates.

gauge field in the extra dimensions are to be identified with the Higgs field. The above separation amounts to the separation of the matrix coordinate fields into the direction tangent to the fuzzy space and perpendicular to the fuzzy space. However, it is not clear how to make such separation when the fuzzy space is made from finite size matrices. On the fuzzy space described by finite size matrices, the KK modes on the fuzzy space are truncated at some finite level. This means that we may not have enough functions to make a coordinate transformation which is needed for the separation of the matrix coordinate fields in higher dimensions into the components tangent- and perpendicular- to the embedded fuzzy space. Moreover, we have to make the coordinate transformation with the non-commutative matrix product.[18] Let us explain with the fuzzy sphere [35] as an example. Fig. 4 is a figure of a sphere embedded in $\mathbb{R}^3$. The matrix coordinate fields X^I $(I = 1, 2, 3)$ of D-branes are associated with the Cartesian coordinates of $\mathbb{R}^3$ in the figure. The fuzzy sphere is described by the matrix coordinate fields satisfying

$$[X^I, X^J] = i\alpha\epsilon^{IJK}X^K, \tag{4.1}$$

where α is a real number. To discuss gauge-Higgs unification in the fuzzy sphere extra dimensions, one should extract the zero-modes of the gauge fields from the fluctuations around the (4.1). However, as explained above, it is not clear how to extract the gauge field on the fuzzy sphere when the size of the matrices are finite [36, 37]. The fuzzy torus was little special in that it can be described by unitary matrices, and any complex matrix can be decomposed into a product of a unitary matrix and an Hermite matrix, see (3.7). Thus the generalizations of the gauge-Higgs unification to other fuzzy extra dimensions remain as interesting future directions. On the other hand, torus has been one of the

[18] See [33, 34] and references for a closely related issue.

most useful backgrounds in the ordinary gauge-Higgs unification, and the fuzzy torus may also remain as the most basic background in the study of the gauge-Higgs unification in fuzzy extra dimensions. One possibility may be that the gauge-Higgs unification picture may not be necessary and one may do without the separation mentioned above. When N is small the extra dimensions look far from the ordinary space. And in the general discussions in section 2.4, we were mostly working with the Wilson lines rather than the gauge field on the fuzzy space, and we may generalize pushing along this direction.

Acknowledgments

The authors thank Pei-Ming Ho, Hirotaka Irie, Yoshiharu Kawamura, C. S. Lim, Chia-Min Lin, Eibun Senaha, Tomohisa Takimi, Dan Tomino and Satoshi Watamura for useful discussions. K.F. thanks IPMU for the support and the hospitality during his visit. He also thanks Particle Physics Group of Chuo University for the hospitality. T.I. thanks NCTS for the support and the hospitality during his visits at the final stage of this work. T.I. is partially supported by the grants of the Ministry of Education, Kiban A, 21540278 and Kiban C, 21244063. K.O. would like to thank Hajime Aoki, Naoki Sasakura and Asato Tsuchiya for valuable discussions during the YITP workshop "Field Theory and String Theory". K.O. is supported in part by JSPS Grant-in-Aid for Young Scientists (B) 23740178.

A Fuzzy torus and emergent gauge field

A.1 Fuzzy torus

Fuzzy torus is described by unitary matrices W_1, W_2 subject to the relation

$$W_1 W_2 = e^{-i\theta} W_2 W_1, \tag{A.1}$$

where

$$\theta = \frac{2\pi}{N}. \tag{A.2}$$

Here, N is the size of the matrices W_1 and W_2, which is a parameter of the fuzzy torus. An explicit realization of W_1 and W_2 satisfying (A.2) is given by the so-called 't Hooft-Weyl matrices:

$$W_1 = \begin{pmatrix} 1 & & & & \\ & e^{-i\theta} & & & \\ & & e^{-i2\theta} & & \\ & & & \ddots & \\ & & & & e^{-i(N-1)\theta} \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}, \tag{A.3}$$

(the empty entries should be read as zero). Any $N \times N$ matrix φ can be expanded in terms of W_1 and W_2, which can be interpreted as the Fourier expansion on the fuzzy torus:

$$\varphi = \sum_m \sum_n \varphi_{(m,n)} e^{imn\theta} W_1^m W_2^n \times \frac{1}{\sqrt{2N}}, \tag{A.4}$$

Here, $T^{(m,n)} \equiv e^{imn\theta} W_1^m W_2^n \times \frac{1}{\sqrt{2N}}$ are normalized so that $\mathrm{tr}\, T^{(m,n)} T^{(m',n')} = \frac{1}{2}\delta_{m+m',0}\delta_{n+n',0}$. m and n in the summation in (A.4) run over integers in $-\frac{N}{2} \leq m, n < \frac{N}{2}$. The phase factor in (A.4) is chosen so that for an Hermite matrix $\varphi = \varphi^\dagger$, $\varphi_{(m,n)} = \varphi^*_{(-m,-n)}$. The inverse transformation of (A.4) is given by

$$\varphi_{(m,n)} = \sqrt{\frac{2}{N}} \mathrm{tr}\, \varphi\, W_1^{-m} W_2^{-n}. \tag{A.5}$$

As can be foreseen from calling it as Fourier expansion on the fuzzy torus, W_1 and W_2 are analogous to $e^{-i\frac{\phi_1}{R_1}}$ and $e^{i\frac{\phi_2}{R_2}}$ respectively, where ϕ_1 and ϕ_2 are the periodic coordinates on ordinary torus: $\phi_1 \sim \phi_1 + 2\pi R_1$, $\phi_2 \sim \phi_2 + 2\pi R_2$.

From (A.2), it follows that

$$\begin{aligned} W_1(W_1^m W_2^n)W_1^\dagger &= e^{-in\theta}(W_1^m W_2^n), \\ W_2(W_1^m W_2^n)W_2^\dagger &= e^{im\theta}(W_1^m W_2^n). \end{aligned} \tag{A.6}$$

This means that the unitary transformation by W_1 generates an analogue of the translation of the ϕ_2 coordinate by $-\theta R_2$, and the unitary transformation by W_2 generates an analogue of the translation by θR_1 for the ϕ_1 coordinate.

Thus if we define the difference operators

$$\begin{aligned} \delta_1\varphi &\equiv W_1 \varphi W_1^\dagger - \varphi, \\ \delta_2\varphi &\equiv W_2 \varphi W_2^\dagger - \varphi, \end{aligned} \tag{A.7}$$

it follows from (A.6) that

$$\begin{aligned} \delta_1(W_1^m W_2^n) &= (e^{-in\theta} - 1)(W_1^m W_2^n) = -2ie^{-\frac{in\theta}{2}} \sin\left(\frac{n\theta}{2}\right)(W_1^m W_2^n), \\ \delta_2(W_1^m W_2^n) &= (e^{im\theta} - 1)(W_1^m W_2^n) = 2ie^{\frac{im\theta}{2}} \sin\left(\frac{m\theta}{2}\right)(W_1^m W_2^n). \end{aligned} \tag{A.8}$$

In the commutative limit $N \to \infty$, W_1 and W_2 can be identified with the commutative periodic coordinates ϕ_1, ϕ_2 on the torus ($\phi_{1,2} \sim \phi_{1,2} + 2\pi R_{1,2}$) as

$$W_1 \to e^{-i\frac{\phi_1}{R_1}}, \quad W_2 \to e^{i\frac{\phi_2}{R_2}}. \tag{A.9}$$

This is based on the following algebraic relation in the $N \to \infty$ limit with fixed fuzzy torus radii:

$$2\pi R_I = N a_I : fixed, \tag{A.10}$$

$$\begin{aligned} \frac{1}{a_1}\delta_1 W_2^n &\to -i\frac{n}{R_2}W_2^n \leftrightarrow \partial_{\phi_2} e^{-i\frac{n}{R_2}\phi_2}, \\ \frac{1}{a_2}\delta_2 W_1^m &\to i\frac{m}{R_1}W_1^m \leftrightarrow \partial_{\phi_1} e^{i\frac{m}{R_1}\phi_1}. \end{aligned} \tag{A.11}$$

where $\leftrightarrow$ indicates that the same algebraic relations are satisfied with the identification (A.9). Thus in the $N \to \infty$ limit, those algebraic relations reduce to those of differentiations on the periodic functions on a commutative torus. On the other hand, the trace becomes the integration on the torus:

$$\frac{1}{N}\mathrm{tr} \to \int \frac{d\phi_1}{2\pi R_1}\frac{d\phi_2}{2\pi R_2}. \tag{A.12}$$

A.2 Emergent gauge field on the fuzzy torus

In the ordinary gauge-Higgs unification, the inhomogeneous part of the local gauge transformation forbids the mass term of the gauge field. Therefore, it would be useful to observe that the fluctuations around the fuzzy torus vacuum contain an excitation which can be identified with the components of the gauge field in the fuzzy torus directions.

Emergence of the inhomogeneous gauge transformation

Let us expand the fields U_I around the fuzzy torus background (2.18):

$$U_I = e^{ia_I \mathcal{A}_I} V_I. \tag{A.13}$$

Originally, the fields U_I transform homogeneously under the $SU(kN)$ local gauge transformation:

$$U_I \to e^{i\lambda} U_I e^{-i\lambda}, \tag{A.14}$$

where $kN \times kN$ Hermite matrix $\lambda(x)$ is a gauge transformation parameter. The inhomogeneous gauge transformation of the gauge field components in the fuzzy torus directions appear by requiring that the form (A.13), i.e. the separation of the fuzzy torus background part and the fluctuation part is fixed under the $SU(kN)$ gauge transformation.

$$U_I = e^{ia_I \mathcal{A}_I} V_I \to e^{i\lambda}\left(e^{ia_I \mathcal{A}_I} V_I\right) e^{-i\lambda} = e^{i\lambda} e^{ia_I \mathcal{A}_I} e^{-i\lambda'} V_I, \tag{A.15}$$

where

$$e^{-i\lambda'} \equiv V_I e^{-i\lambda} V_I^{-1} = e^{-iV_I \lambda V_I^{-1}}, \tag{A.16}$$

or

$$\lambda' = V_I \lambda V_I^{-1}. \tag{A.17}$$

By the requirement that the form of our parametrization (A.13) is fixed, the gauge transformation law for the fluctuation $\mathcal{A} \to \mathcal{A}^\lambda$ should be defined by

$$e^{ia_I \mathcal{A}_I^\lambda} = e^{i\lambda} e^{ia_I \mathcal{A}_I} e^{-i\lambda'}. \tag{A.18}$$

For an infinitesimal λ,

$$\begin{aligned} e^{ia_I \mathcal{A}^\lambda} &= e^{i\lambda} e^{ia_I \mathcal{A}_I} e^{-i\lambda'} \\ &= e^{i\lambda} e^{ia_I \mathcal{A}_I} e^{-i\lambda} e^{i\lambda} e^{-i\lambda'} \\ &= \exp\left[ia_I \mathcal{A}_I - iV_I \lambda V_I^{-1} + i\lambda - a_I[\lambda, \mathcal{A}_I] + \mathcal{O}(\lambda^2)\right] \\ &= \exp\left[ia_I \mathcal{A}_I - i\delta_I \lambda - a_I[\lambda, A_I] + \mathcal{O}(\lambda^2)\right], \end{aligned} \tag{A.19}$$

where δ_I is defined in (A.7). Thus in terms of the field $\mathcal{A}_I$, the gauge transformation is given by

$$\mathcal{A}_I \to \mathcal{A}_I^\lambda = \mathcal{A}_I - \frac{1}{a_I} \delta_I \lambda + i[\lambda, A_I] + \mathcal{O}(\lambda^2). \tag{A.20}$$

Recalling that $\frac{1}{a_I}\delta_I$ can be regarded as a derivative on the fuzzy torus (A.11), we can regard (A.20) as the inhomogeneous $SU(k)$ gauge transformation for the gauge field $\mathcal{A}_I$ on the fuzzy torus.

Wilson loop operator

Next we explain why the operator

$$\frac{1}{kN} \operatorname{tr} {U_I}^N, \tag{A.21}$$

can be regarded as the Wilson loop operator on the fuzzy torus [21]. Using (A.13), (A.21) can be rewritten as

$$\begin{aligned} & kN \operatorname{tr}_{SU(kN)} {U_I}^N \\ = \ & \operatorname{tr}_{SU(kN)} \left[\left(V_I e^{ia_I \mathcal{A}_I}\right)^N\right] \\ = \ & \operatorname{tr}_{SU(kN)} \left[\left(V_I e^{ia_I \mathcal{A}_I} V_I^{-1}\right)\left(V_I^2 e^{ia_I \mathcal{A}_I} V_I^{-2}\right) \cdots \left(V_I^N e^{ia_I \mathcal{A}_I} V_I^{-N}\right)\right] \\ = \ & \operatorname{tr}_{SU(kN)} \left[e^{ia_I \mathcal{A}_I}\left(V_I e^{ia_I \mathcal{A}_I} V_I^{-1}\right)\left(V_I^2 e^{ia_I \mathcal{A}_I} V_I^{-2}\right) \cdots \left(V_I^{N-1} e^{ia_I \mathcal{A}_I} V_I^{-(N-1)}\right)\right], \end{aligned} \tag{A.22}$$

where we have used

$$V_I^N = 1, \tag{A.23}$$

and the cyclic property of the trace. From the correspondence between the fuzzy torus and the ordinary torus through the $N \to \infty$ limit with $a_I N = 2\pi R_I$ fixed discussed previously, (A.22) can be regarded as a discretized version of

$$\lim_{N\to\infty} \mathrm{tr}_{SU(kN)} \left[e^{ia_I \mathcal{A}_I(\phi_I)} e^{ia_I \mathcal{A}_I(\phi_I + a_I)} e^{ia_I \mathcal{A}_I(\phi_I + 2a_I)} \cdots e^{ia_I \mathcal{A}_I(\phi_I + (N-1)a_I)} \right]. \tag{A.24}$$

Rewriting using (A.12), this formally has the form of the Wilson loop operator wrapping the I-th direction once and integrated over the fuzzy torus:

$$\int \frac{d\phi_1}{2\pi R_1} \frac{d\phi_2}{2\pi R_2} \frac{1}{k} \mathrm{tr}_{SU(k)} P \exp i \oint \mathcal{A}_I \phi_I, \tag{A.25}$$

Here, $\mathrm{tr}_{SU(k)} P$ denotes the path ordered trace. However, $N \to \infty$ limit is a little bit formal since for any finite N, one needs the integration over the fuzzy torus corresponding to the trace of $U(N)$ gauge subgroup in order for the operator to be gauge invariant observable.

B The estimate of the UV cut-off scale Λ

In this appendix, we estimate the natural UV cut-off scale Λ for the effective field theory (2.1). We will follow the argument of [16].

Let us parametrize the unitary matrix fields as

$$U_I = e^{i\frac{\pi_I}{f}} V_I, \tag{B.1}$$

where V_I is the vacuum expectation value of U_I.[19] Consider general possible vertex involving the gauge field A_μ and π_I.

Analysis at small N

We first study the case when N is small and one can neglect the N dependence in the rough order estimate. The N dependence will be included after this analysis.

A coefficient consistent with naive dimensional analysis is

$$(2\pi)^4 \delta^4 \left(\sum p\right) \left(\frac{gA_\mu}{\Lambda}\right)^A \left(\frac{\pi}{f}\right)^B \left(\frac{p}{\Lambda}\right)^C f^2 \Lambda^2. \tag{B.2}$$

[19]The difference from the parametrization in (A.13) is just a matter of taste. In the appendix A, the analogy with the lattice gauge theory was useful and therefore we used the notation closer to those often used in the lattice gauge theory. In this appendix, the analogy with the chiral perturbation theory is useful, thus we use the notation similar to those used in the description of pion.

This correctly estimates the coefficient of the kinetic term for U_I but underestimates the gauge field kinetic term. This can be corrected by multiplying the factor

$$\left(\frac{\Lambda}{gf}\right)^2, \tag{B.3}$$

for the terms purely made from the gauge fields and their derivatives.

Now, consider arbitrary Feynman diagram involving a total of V vertices of the form (B.2) with (A, B, C) values equals to (A_i, B_i, C_i), $i = 1, \cdots, V$.

The diagram simply gives

$$\begin{aligned}
& (2\pi)^4\delta^4\left(\sum p\right)\left(\frac{gA_\mu}{\Lambda}\right)^A\left(\frac{\pi}{f}\right)^B\left(\frac{p}{\Lambda}\right)^C f^2\Lambda^2 \\
\times\ & g^{-A+\sum_i A_i} f^{B-\sum_i B_i+2V-2}\Lambda^{A+C+2V-2-\sum_i(A_i+C_i)} \\
\times\ & k^{-C+\sum_i C_i}(2\pi)^{4(V-1)}\left[\delta^{(4)}\left(\sum p_i\right)\right]^{V-1}\left[\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\right]^I\left(\frac{\Lambda^2}{g^2f^2}\right)^G,
\end{aligned} \tag{B.4}$$

where G is the number of the purely gauge interaction vertices and I is the number of the internal propagators in the Feynman diagram. Thus we use

$$\sum A_i = A + 2I_A, \tag{B.5}$$

$$\sum B_i = B + 2I_\pi, \tag{B.6}$$

(conservation of ends of propagators) where I_A and I_π are the number of internal propagators of A_μ and π_I in the Feynman diagram, respectively, and $I_A + I_\pi = I$. We also have the equality

$$L = I - V + 1, \tag{B.7}$$

where L is the number of the loops in the Feynman diagram.

Since all the momentum integrals are cut off at Λ, we can estimate them by replacing all internal momenta by Λ:

$$k \rightarrow \Lambda, \tag{B.8}$$

$$\left[\int\frac{d^4k}{(2\pi)^4}\right]^L \rightarrow \frac{\Lambda^{4L}}{(4\pi)^{2L}}, \tag{B.9}$$

Thus we obtain

$$\begin{aligned}
& (2\pi)^4\delta^4\left(\sum p\right)\left(\frac{gA_\mu}{\Lambda}\right)^A\left(\frac{\pi}{f}\right)^B\left(\frac{p}{\Lambda}\right)^C f^2\Lambda^2 \\
\times\ & \left[(4\pi)^{-2L}(f^{-1}\Lambda)^{2L}\right]\left(\frac{gf}{\Lambda}\right)^{2I_A-2G}.
\end{aligned} \tag{B.10}$$

In order for the naive dimensional counting to be correct when $I_A = 0$, the factor inside the square bracket should be of order one. Thus we set

$$\Lambda \lesssim 4\pi f. \tag{B.11}$$

When $I_A \neq 0$, there is a suppression factor $(gf/\Lambda)^{2I_A - 2G}$. ($I_A \geq G$ since there cannot be a term with negative powers of g.) The terms which do not preserve the weakly broken chiral symmetry (2.13) are suppressed by this factor.

Taking into account the N dependence

In the above, we have neglected the N dependence, related to the size of the gauge group. It can be taken into account by the following consideration [38]. For vacuum diagrams which have no external legs the number of the index loop is at most $L + 1$, which is the case when the diagram is planar. Vacuum diagrams just contribute to the cosmological constant which is only relevant when we consider the coupling to gravity and we will not discuss it further in our effective field theory. If there are external lines in the diagram, they break at least one index loop compared with vacuum diagrams. Thus the maximum number of the index loop is L.[20] This modifies the L dependent factor in (B.10) as

$$\left[(4\pi)^{-2L}(f^{-1}\Lambda)^{2L}(kN)^{L}\right]. \tag{B.12}$$

Thus the appropriate UV cut-off is given by

$$\Lambda \approx \frac{4\pi f}{\sqrt{kN}}. \tag{B.13}$$

C The mass of the zero-modes in the one-loop effective potential

Below we will estimate the mass of the zero-modes in the effective potential at the one-loop level (2.32):

$$V_{1-loop}(u_0^I) \;=\; 2\sum_{m_1,m_2}\sum_{i,j=1}^{2}\int \frac{d^4k}{(2\pi)^4}\ln\left(k^2 + m^2_{(m_1,m_2)(i,j)}(u_0^I)\right). \tag{C.1}$$

[20] For the estimation of the UV cut-off one should consider the largest contributions to general operators in the model. For a specific operator, the number of the index loops in the Feynman diagrams which contribute to that operator might be constrained to be smaller. In that case, this term may be suppressed by the inverse powers of N. The double trace operator (2.61) is such an example.

Let us define

$$\zeta_{D^2}(s) \equiv \int \frac{d^4k}{(2\pi)^4} \sum_{m_1,m_2} \sum_{i,j=1}^{2} \left(k^2 + m^2_{(m_1,m_2)(i,j)}(u_0^I)\right)^{-s} . \tag{C.2}$$

Then, (C.1) can be written as

$$V_{1-loop}(u_0^I) = -2 \frac{d\zeta_{D^2}(s)}{ds}\bigg|_{s=0} . \tag{C.3}$$

On the other hand, ζ_{D^2} can be rewritten as

$$\zeta_{D^2}(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau\, \tau^{s-1} \sum_{m_1,m_2} \sum_{i,j=1}^{2} \exp\left[-\tau\left(k^2 + m^2(u_0^I)\right)\right] , \tag{C.4}$$

where we have introduced a shorthand notation $m^2(u_0^I)$ for $m_{(m_1,m_2)(i,j)}(u_0^I)$. After performing the Gaussian integral, we obtain

$$\zeta_{D^2} = \frac{1}{(4\pi)^2} \frac{1}{\Gamma(s)} \sum_{m_1,m_2} \sum_{i,j=1}^{2} \int_0^\infty d\tau\, \tau^{s-3} \exp\left[-\tau m^2(u_0^i)\right] . \tag{C.5}$$

Thus

$$\frac{d\zeta_{D^2}(s)}{ds} = \frac{1}{(4\pi)^2} \sum_{m_1,m_2} \sum_{i,j=1}^{2} \int_0^\infty d\tau\, \tau^{s-3} \left(\frac{\ln\tau\, \Gamma(s) - \Gamma'(s)}{(\Gamma(s))^2}\right) \exp\left[-\tau m^2(u_0^I)\right] . \tag{C.6}$$

Using

$$\lim_{s\to 0} \Gamma(s) = \frac{1}{s} + finite, \tag{C.7}$$

we obtain

$$\frac{d\zeta_{D^2}(s)}{ds}\bigg|_{s\to 0} = -\frac{1}{2(4\pi)^2} \sum_{m_1,m_2} \sum_{i,j=1}^{2} \int_0^\infty d\tau\, \tau^{s-3} \exp\left[-\tau m^2(u_0^I)\right]\bigg|_{s\to 0} . \tag{C.8}$$

Now, recall (2.33):

$$\begin{aligned} m^2_{(m_1,m_2)(i,j)}(u_0^I) &\equiv \sum_{I=1,2} \left(\frac{2}{a_I}\right)^2 \sin^2 \frac{1}{2}\left(m_I\theta + (u_i^I - u_j^I)\right) , \\ &= \sum_{I=1,2} \frac{2}{a_I^2} \left(1 - \cos\left(m_I\theta + (u_i^I - u_j^I)\right)\right) , \end{aligned} \tag{C.9}$$

with

$$u_1^I = -u_2^I = \frac{1}{\sqrt{4N} f_I} u_0^I. \tag{C.10}$$

From (C.8) and (C.9), we obtain

$$
\begin{aligned}
V_{1-loop}(u_0^I) &= -\frac{1}{(4\pi)^2}\sum_{m_1,m_2}\sum_{i,j=1}^{2}\int_0^\infty d\tau\,\tau^{s-3}\exp\left[-\tau\sum_{I=1,2}\frac{2}{a_I^2}\right] \\
&\qquad \times\exp\left[\tau\sum_{I=1,2}\frac{2}{a_I^2}\cos(m_I\theta+u_{ij}^I)\right]\Bigg|_{s\to 0}, \qquad (C.11)
\end{aligned}
$$

where

$$
u_{ij}^I \equiv u_i^I - u_j^I. \qquad (C.12)
$$

We can safely set $s=0$ when $N\geq 3$, while for $N=2$ we have logarithmic divergence as a function of s. Below we will consider the case $N\geq 3$ and set $s=0$. In (C.11) the sum over m_I $(I=1,2)$ run over integers in $-\frac{N}{2}\leq m_I<\frac{N}{2}$. Here we are considering the case $\theta=2\pi/N$ and due to the cancellation of the phases only the terms proportional to $\cos(\ell_I N(m_I\theta+u_{ij}^I))$ with integer ℓ_I survive in the sum over m_I. Thus the net effect of the sum over m_I with $\theta=2\pi/N$ is equivalent to the following Fourier transform:

$$
\frac{1}{N}\sum_{m_I} f(\cos(m_I\theta+u)) = \sum_{\ell_I=-\infty}^{\infty}\left(\frac{1}{\pi}\int_0^{2\pi}d\theta'\,f(\cos\theta')\cos(N\ell_I\theta')\right)\cos(N\ell_I u). \qquad (C.13)
$$

Using the identity for the modified Bessel function $I_\nu(z)$ with integer ν:

$$
e^{z\cos\theta} = I_0(z) + 2\sum_{\nu=1}^{\infty} I_\nu(z)\cos\nu\theta, \qquad (C.14)
$$

(C.11) can be rewritten as

$$
\begin{aligned}
V_{1-loop}(u_0^I) &= \frac{4N^2}{(4\pi)^2}\int_0^\infty\frac{d\tau}{\tau^3}\exp\left[-\tau\sum_I\frac{2}{a_I^2}\right]\sum_{\ell_1,\ell_2=0}^{\infty} I_{N\ell_1}\left(\frac{2\tau}{a_1^2}\right)I_{N\ell_2}\left(\frac{2\tau}{a_2^2}\right) \\
&\qquad \times\cos(N\ell_1 u_{ij}^1)\cos(N\ell_2 u_{ij}^2). \qquad (C.15)
\end{aligned}
$$

When comparing the theories with different N, we should fix the radii of the fuzzy torus (A.10) and the $SU(k)$ gauge coupling as in (2.66):

$$
\begin{aligned}
a_I N &= 2\pi R_I : fixed, \qquad (C.16)\\
g_{SU(k)} &\equiv \frac{g}{\sqrt{N}} : fixed. \qquad (C.17)
\end{aligned}
$$

Notice that this also fixes the scaling of f through (2.29):

$$
\frac{f}{\sqrt{N}} : fixed. \qquad (C.18)
$$

We will express the calculations in terms of these parameters fixed for different N below.

Using the integral representation of the modified Bessel function which follows from (C.14):

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi d\theta' \, e^{z\cos\theta'} \cos\nu\theta' \quad (\nu : integer), \tag{C.19}$$

and defining

$$V_{1-loop}(u_0^I) = 4\sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty} V^{1-loop}_{(N\ell_1,N\ell_2)} \cos(N\ell_1 u_{12}^1)\cos(N\ell_2 u_{12}^2) + const., \tag{C.20}$$

we obtain

$$\begin{aligned}
& V^{1-loop}_{(N\ell_1,N\ell_2)} \\
= \;& \frac{4N^2}{(4\pi)^2}\int_0^\infty \frac{d\tau}{\tau^3}\prod_{I=1,2}\frac{1}{\pi}\int_0^\pi d\theta_I \exp\left[-\tau\frac{2N^2}{(2\pi R_I)^2}(1-\cos\theta_I)\right]\frac{e^{iN\ell_I\theta_I}+e^{-iN\ell_I\theta_I}}{2} \\
= \;& \frac{N^2}{(4\pi)^2}\int_0^\infty \frac{d\tau}{\tau^3}\prod_{I=1,2}\frac{1}{\pi}\int_0^\pi d\theta_I\left(\exp\left[-\tau\frac{1}{(2\pi R_I)^2}\left(N^2\theta_I^2 + i\frac{(2\pi R_I)^2}{\tau}\ell_I N\theta_I\right)\right]\right. \\
& \qquad\qquad + \exp\left[-\tau\frac{1}{(2\pi R_I)^2}\left(N\theta_I^2 - i\frac{(2\pi R_I)^2}{\tau}\ell_I N\theta_I\right)\right] + \mathcal{O}(N^{-2}) \\
= \;& \frac{1}{(4\pi)^2}\int_0^\infty \frac{d\tau}{\tau^3}\prod_{I=1,2}\frac{1}{\pi}\int_0^{N\pi} d\tilde\theta_I\left(\exp\left[-\frac{\tau}{(2\pi R_I)^2}\left(\tilde\theta_I^2 + i\frac{(2\pi R_I)^2}{\tau}\ell_I\tilde\theta_I\right)\right]\right. \\
& \qquad\qquad \left.+ \exp\left[-\frac{\tau}{(2\pi R_I)^2}\left(\tilde\theta_I^2 - i\frac{(2\pi R_I)^2}{\tau}\ell_I\tilde\theta_I\right)\right]\right) + \mathcal{O}(N^{-2}) \\
& \quad (\tilde\theta = N\theta) \\
= \;& \frac{1}{(4\pi)^2}\int_0^\infty \frac{d\tau}{\tau^3}\frac{4}{(2\pi)^2}\left(\prod_{I=1,2}\sqrt{\frac{\pi(2\pi R_I)^2}{\tau}}e^{-\frac{(2\pi R_I)^2\ell_I^2}{4\tau}}\right) + \mathcal{O}(N^{-2}) + \mathcal{O}\left(\frac{1}{N}e^{-(\pi N)^2}\right) \\
= \;& \frac{1}{(4\pi)^2}\int_0^\infty d\tilde\tau\tilde\tau^2\frac{1}{\pi}\left(\prod_{I=1,2}2\pi R_I e^{-\frac{(2\pi R_I)^2\ell_I^2}{4}\tilde\tau}\right) + \mathcal{O}(N^{-2}) + \mathcal{O}\left(\frac{1}{N}e^{-(\pi N)^2}\right) \\
& \quad \left(\tilde\tau = \frac{1}{\tau}\right) \\
= \;& \frac{1}{(4\pi)^2}\Gamma(3)\frac{1}{\pi}(2\pi R_1)(2\pi R_2)\left(\frac{4}{(2\pi R_1)^2\ell_1^2 + (2\pi R_2)^2\ell_2^2}\right)^3 + \mathcal{O}(N^{-2}) + \mathcal{O}\left(\frac{1}{N}e^{-(\pi N)^2}\right).
\end{aligned} \tag{C.21}$$

The leading term in the large N expansion coincides with the one in the gauge-Higgs unification in the ordinary torus extra dimensions [3, 4, 5, 6]. Noticing that in terms of the canonically normalized field u_0^I,

$$Nu_{12}^I = N\frac{1}{\sqrt{4N}f_I}2u_0^I = g_{SU(k)}(2\pi R_I)u_0^I, \tag{C.22}$$

the mass m_0 of the zero-modes can be read off from (C.21) and is of order

$$m_0^2 \approx \frac{g_{SU(k)}^2}{16\pi^2}\frac{1}{R^2}, \tag{C.23}$$

where the four-dimensional effective $SU(k)$ gauge coupling $g_{SU(k)}$ is given in (C.17) and as before we have assumed $R_1 \approx R_2 \approx R$. This is as expected since $1/R$ is the scale where the effect of the new physics appears, and $g_{SU(k)}^2/16\pi^2$ is the one-loop factor.

$\mathcal{O}(N^{-2})$ etc. in (C.21) refers to the relative magnitude compared with the leading term. Thus even when N is small, it gives a correction at most of the same order. Thus (C.23) is still a valid order estimate even when N is small.

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