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# A new family of shape invariantly deformed Darboux-Pöschl-Teller potentials with continuous $\ell$

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**Abstract**

We present a new family of shape invariant potentials which could be called a "continuous $\ell$ version" of the potentials corresponding to the exceptional ($X_\ell$) J1 Jacobi polynomials constructed recently by the present authors. In a certain limit, it reduces to a continuous $\ell$ family of shape invariant potentials related to the exceptional ($X_\ell$) L1 Laguerre polynomials. The latter was known as one example of the 'conditionally exactly solvable potentials' on a half line.

## 1 Introduction

We will present a new family of shape invariant [1], thus exactly solvable, potentials in one dimensional quantum mechanics. The inventory of exactly solvable quantum mechanics in one dimension [2, 3] has seen a rapid increase recently, thanks to the discovery of infinitely many shape invariant potentials connected with the *exceptional* Laguerre, Jacobi, continuous Hahn, Wilson and Askey-Wilson polynomials by the present authors [4, 5, 6, 7, 8, 9, 10]. The exceptional orthogonal polynomials are a new type of orthogonal polynomials satisfying second order differential (difference) equations. The $\ell$-th ($\ell = 1, 2, \ldots$) member of these families of orthogonal polynomials are sometimes called $X_\ell$ polynomials. They start with the degree $\ell$ instead of a degree zero constant term, which is the case for the ordinary orthogonal polynomials. Therefore they are not constrained by Bochner's theorem [11]. The concept of the exceptional orthogonal polynomials was introduced by Gómez-Ullate et al [12], and the explicit examples of the $X_1$ Laguerre and Jacobi polynomials were constructed within the framework of the Sturm-Liouville theory. Then Quesne [13] reformulated them in the language of quantum mechanics. These are the first members of the infinite families of the exceptional Laguerre and Jacobi polynomials [4]. Later another set of $X_2$ Laguerre polynomials was found [14], which was generalised to another family of $X_\ell$ Laguerre and Jacobi polynomials [6].

Roughly speaking, we are going to derive a "continuous $\ell$ version" of the potentials corresponding to the $X_\ell$ Jacobi polynomials. The prepotentials of the exceptional ($X_\ell$) Jacobi polynomials are obtained by deforming those for the Darboux-Pöschl-Teller (DPT) [15] potential in terms of a degree $\ell$ Jacobi polynomial of twisted parameters [4, 6]. As is well known, the Jacobi polynomials can be expressed in terms of a Gauss hypergeometric function, which is well defined for non-integer $\ell$, too. The new family of potentials are obtained by deforming the DPT potential in terms of the hypergeometric function, which would reduce to the Jacobi polynomial for integer $\ell$. There are, in fact, two types of exceptional Jacobi polynomials, called J1 and J2 [6, 7, 8]. It turns out that only the first type, the J1, deformations give rise to non-singular and shape invariant potentials. Naturally, the corresponding eigenfunctions are *no longer polynomials*. They are a "continuous $\ell$ version" of the J1 type $X_\ell$ polynomials, $\{P_{\ell,n}\}$. This is in good contrast to the most known cases of shape invariant potentials, in which the eigenfunctions are polynomials.

It is well known that the Laguerre polynomials (confluent hypergeometric functions) are obtained from the Jacobi polynomials (hypergeometric functions) in a certain limit [16]. Likewise the exceptional L1 and L2 Laguerre polynomials are derived from the exceptional J1 and J2 Jacobi polynomials, respectively, in the same limit [6]. This would mean that a "continuous $\ell$ version" of the potentials corresponding to the L1 $X_\ell$ Laguerre polynomials can be obtained from the above "continuous $\ell$ version" of the potentials corresponding to the J1 $X_\ell$ Jacobi polynomials. In fact, this "continuous $\ell$ version" of the potentials corresponding to the L1 $X_\ell$ Laguerre polynomials was derived by Junker and Roy [17] as one example of 'conditionally exactly solvable potentials' in the context of supersymmetric quantum mechanics. Somehow erroneously this type of potentials had been declared non-shape invariant [17, 18]. This is partly because the structure of the corresponding Hamiltonians (potentials) were not fully understood. The structure of the Hamiltonians (potentials) of the exceptional orthogonal polynomials are essentially the same for the J1 $X_\ell$ Jacobi and the L1 $X_\ell$ Laguerre polynomials, as shown in our previous papers [4, 6, 5, 7, 8]. Therefore we will present the "continuous $\ell$ versions" of the potentials corresponding to the J1 $X_\ell$ Jacobi and the L1 $X_\ell$ Laguerre polynomials in parallel. We will follow the notation of [7, 8].

This paper is organised as follows. In section two we will recapitulate the original systems, that is, the quantum mechanical systems of the DPT and the radial oscillator potentials, in order to set the stage and to introduce appropriate notation. In section three the new deforming functions, the "continuous $\ell$ versions" of the deforming polynomials $\xi_\ell(\eta)$, are introduced and their properties are demonstrated. Section four is the main part of this paper. In subsection 4.1, the deformed systems, that is, the "continuous $\ell$ versions" of the potentials corresponding to the J1 $X_\ell$ Jacobi and the L1 $X_\ell$ Laguerre polynomials are presented. Here we stress two points. Firstly, we show the concrete structure of the Hamiltonians. Secondly, the shape invariance is demonstrated explicitly. We will briefly mention why the "continuous $\ell$ versions" of the potentials corresponding to the J2 $X_\ell$ Jacobi and the L2 $X_\ell$ Laguerre polynomials do not exist. The Darboux-Crum transformations [19, 20] intertwining the Hamiltonians of the original systems with the new deformed systems are introduced in subsection 4.2. Various properties of the new deformed systems are derived from those of the original systems through the intertwining relations. In subsection 4.3 we briefly review the limiting procedure, from the Jacobi polynomials (hypergeometric function) to the Laguerre polynomials (confluent hypergeometric function), which connects the results of the "continuous $\ell$ versions" of the potentials corresponding to the J1 $X_\ell$ Jacobi and the L1 $X_\ell$ Laguerre polynomials. The final section is for a summary and comments. It is shown why the above recipe to construct a "continuous $\ell$ version" does not work for the Hamiltonians of the exceptional Askey type polynomials constructed by the present authors [9, 10].

## 2 Original systems

Here we summarise various properties of the original Hamiltonian systems, the two well known shape invariant systems, the Darboux-Pöschl-Teller [15] and the radial oscillator [2, 3] potentials. The eigenfunctions are described by the Jacobi and Laguerre polynomials, to be abbreviated as J and L. These results are to be compared with the specially modified systems to be presented in § 4. Let us start with the Hamiltonians, Schrödinger equations and eigenfunctions ($x_1 < x < x_2$):

$$\mathcal{H}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{A}(\boldsymbol{\lambda})^\dagger \mathcal{A}(\boldsymbol{\lambda}), \quad \mathcal{A}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{d}{dx} - \partial_x w_0(x;\boldsymbol{\lambda}), \quad \mathcal{A}(\boldsymbol{\lambda})^\dagger = -\frac{d}{dx} - \partial_x w_0(x;\boldsymbol{\lambda}), \tag{2.1}$$

$$\mathcal{H}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}) \quad (n = 0, 1, 2, \ldots), \tag{2.2}$$

$$\phi_n(x;\boldsymbol{\lambda}) = \phi_0(x;\boldsymbol{\lambda}) P_n(\eta(x);\boldsymbol{\lambda}), \quad \phi_0(x;\boldsymbol{\lambda}) = e^{w_0(x;\boldsymbol{\lambda})}. \tag{2.3}$$

Here $\eta(x)$ is the sinusoidal coordinate, $\boldsymbol{\lambda}$ is the set of parameters, $w_0(x;\boldsymbol{\lambda})$ is the prepotential and $\mathcal{E}_n(\boldsymbol{\lambda})$ is the $n$-th energy eigenvalue:

$$\eta(x) \stackrel{\text{def}}{=} \begin{cases} \cos 2x, & x_1 = 0, \quad x_2 = \frac{\pi}{2}, \quad : \mathrm{J} \\ x^2, & x_1 = 0, \quad x_2 = \infty, \quad : \mathrm{L} \end{cases}, \quad \boldsymbol{\lambda} \stackrel{\text{def}}{=} \begin{cases} (g,h), & g, h > 0 \quad : \mathrm{J} \\ g, & g > 0 \quad : \mathrm{L} \end{cases}, \tag{2.4}$$

$$w_0(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} g \log \sin x + h \log \cos x & : \mathrm{J} \\ -\frac{1}{2}x^2 + g \log x & : \mathrm{L} \end{cases}, \quad \mathcal{E}_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} 4n(n+g+h) & : \mathrm{J} \\ 4n & : \mathrm{L} \end{cases}. \tag{2.5}$$

The eigenfunction consists of an orthogonal polynomial $P_n(\eta;\boldsymbol{\lambda})$, a polynomial of degree $n$ in $\eta$, ($P_n(\eta;\boldsymbol{\lambda}) = 0$ for $n < 0$):

$$P_n(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} P_n^{(g-\frac{1}{2}, h-\frac{1}{2})}(\eta) & : \mathrm{J} \\ L_n^{(g-\frac{1}{2})}(\eta) & : \mathrm{L} \end{cases}. \tag{2.6}$$

Shape invariance [1] means in this setting [21, 22, 23]

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger = \mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger \mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}), \quad \boldsymbol{\delta} \stackrel{\text{def}}{=} \begin{cases} (1,1) & : \mathrm{J} \\ 1 & : \mathrm{L} \end{cases}, \tag{2.7}$$

or equivalently,

$$\big(\partial_x w_0(x;\boldsymbol{\lambda})\big)^2 - \partial_x^2 w_0(x;\boldsymbol{\lambda}) = \big(\partial_x w_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\big)^2 + \partial_x^2 w_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}). \tag{2.8}$$

It is straightforward to verify this for the given forms of the prepotential $w_0(x;\boldsymbol{\lambda})$ (2.5). The action of $\mathcal{A}(\boldsymbol{\lambda})$ and $\mathcal{A}(\boldsymbol{\lambda})^\dagger$ on the eigenfunction is

$$\mathcal{A}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})\phi_{n-1}\big(x;\boldsymbol{\lambda}+\boldsymbol{\delta}\big), \tag{2.9}$$

$$\mathcal{A}(\boldsymbol{\lambda})^\dagger \phi_{n-1}\big(x;\boldsymbol{\lambda}+\boldsymbol{\delta}\big) = b_{n-1}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}). \tag{2.10}$$

Here the coefficients $f_n(\boldsymbol{\lambda})$ and $b_{n-1}(\boldsymbol{\lambda})$ are the factors of $\mathcal{E}_n(\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})b_{n-1}(\boldsymbol{\lambda})$:

$$f_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} -2(n+g+h) & : \mathrm{J} \\ -2 & : \mathrm{L} \end{cases}, \quad b_{n-1}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -2n \ \ : \mathrm{J}\,\&\,\mathrm{L}. \tag{2.11}$$

The forward and backward shift operators, $\mathcal{F}(\boldsymbol{\lambda})$ and $\mathcal{B}(\boldsymbol{\lambda})$, are defined in the following way and they can be expressed in terms of $\eta$ only [7]:

$$\mathcal{F}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1} \circ \mathcal{A}(\boldsymbol{\lambda}) \circ \phi_0(x;\boldsymbol{\lambda}) = \frac{\phi_0(x;\boldsymbol{\lambda})}{\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}\frac{d}{dx} = c_{\mathcal{F}}\frac{d}{d\eta}, \tag{2.12}$$

$$
\begin{aligned}
\mathcal{B}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \phi_0(x;\boldsymbol{\lambda})^{-1}\circ\mathcal{A}(\boldsymbol{\lambda})^{\dagger}\circ\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\\
&= -\frac{\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}{\phi_0(x;\boldsymbol{\lambda})}\Big(\frac{d}{dx}+\partial_x\big(w_0(x;\boldsymbol{\lambda})+w_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\big)\Big)\\
&= -4c_{\mathcal{F}}^{-1}c_2(\eta)\Big(\frac{d}{d\eta}+\frac{c_1(\eta,\boldsymbol{\lambda})}{c_2(\eta)}\Big),
\end{aligned}
\tag{2.13}
$$

where $c_{\mathcal{F}}$, $c_1(\eta,\boldsymbol{\lambda})$ and $c_2(\eta)$ are

$$
c_{\mathcal{F}} \stackrel{\text{def}}{=} \begin{cases} -4 & : \mathrm{J}\\ 2 & : \mathrm{L}\end{cases},\quad c_1(\eta,\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} h-g-(g+h+1)\eta & : \mathrm{J}\\ g+\frac{1}{2}-\eta & : \mathrm{L}\end{cases},\quad c_2(\eta) \stackrel{\text{def}}{=} \begin{cases} 1-\eta^2 & : \mathrm{J}\\ \eta & : \mathrm{L}\end{cases}. \tag{2.14}
$$

Their action on the polynomial is

$$
\mathcal{F}(\boldsymbol{\lambda})P_n(\eta;\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})P_{n-1}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{2.15}
$$

$$
\mathcal{B}(\boldsymbol{\lambda})P_{n-1}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta}) = b_{n-1}(\boldsymbol{\lambda})P_n(\eta;\boldsymbol{\lambda}). \tag{2.16}
$$

These forward and backward shift relations are the factors of the second order differential equations for the polynomial $P_n$ :

$$
c_{\mathcal{F}}\partial_\eta P_n(\eta;\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})P_{n-1}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{2.17}
$$

$$
c_1(\eta,\boldsymbol{\lambda})P_{n-1}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta}) + c_2(\eta)\partial_\eta P_{n-1}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta}) = -\tfrac{1}{4}c_{\mathcal{F}}b_{n-1}(\boldsymbol{\lambda})P_n(\eta;\boldsymbol{\lambda}), \tag{2.18}
$$

$$
c_2(\eta)\partial_\eta^2 P_n(\eta;\boldsymbol{\lambda}) + c_1(\eta,\boldsymbol{\lambda})\partial_\eta P_n(\eta;\boldsymbol{\lambda}) = -\tfrac{1}{4}\mathcal{E}_n(\boldsymbol{\lambda})P_n(\eta;\boldsymbol{\lambda}), \tag{2.19}
$$

which correspond to the properties of the (confluent) hypergeometric function ${}_2F_1$ (3.16)–(3.18) or ${}_1F_1$ (3.22)–(3.24), respectively.

The orthogonality reads

$$
\int_{x_1}^{x_2}\phi_0(x;\boldsymbol{\lambda})^2\,P_n(\eta(x);\boldsymbol{\lambda})P_m(\eta(x);\boldsymbol{\lambda})dx = h_n(\boldsymbol{\lambda})\delta_{nm}, \tag{2.20}
$$

$$
h_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} \dfrac{\Gamma(n+g+\frac{1}{2})\Gamma(n+h+\frac{1}{2})}{2\,n!(2n+g+h)\Gamma(n+g+h)} & : \mathrm{J}\\ \frac{1}{2\,n!}\Gamma(n+g+\frac{1}{2}) & : \mathrm{L}\end{cases}. \tag{2.21}
$$

# 3 Deforming function with continuous $\ell$

In deriving the Hamiltonians of the exceptional Jacobi and Laguerre polynomials [4, 6], the original system is deformed in terms of a degree $\ell = 1, 2, \ldots$ polynomial $\xi_\ell(\eta;\boldsymbol{\lambda})$, which is the eigenpolynomial (Jacobi or Laguerre) with twisted parameters. We consider a real positive number $\ell$ instead of an integer.

Let us define the following deforming function $\xi_\ell(\eta;\boldsymbol{\lambda})$ with $\ell\in\mathbb{R}_{>0}$:

$$\text{J1}:\ \xi_\ell(\eta;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\frac{\Gamma(g+2\ell-\frac12)}{\Gamma(\ell+1)\Gamma(g+\ell-\frac12)}\,{}_2F_1\Bigl(\genfrac{}{}{0pt}{}{-\ell,\,g-h+\ell-1}{g+\ell-\frac12}\Bigm|\frac{1-\eta}{2}\Bigr) \tag{3.1}$$

$$=\frac{\Gamma(g+2\ell-\frac12)}{\Gamma(\ell+1)\Gamma(g+\ell-\frac12)}\Bigl(\frac{1+\eta}{2}\Bigr)^{h+\ell+\frac12}{}_2F_1\Bigl(\genfrac{}{}{0pt}{}{g+2\ell-\frac12,\,h+\frac12}{g+\ell-\frac12}\Bigm|\frac{1-\eta}{2}\Bigr), \tag{3.2}$$

$$\text{L1}:\ \xi_\ell(\eta;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\frac{\Gamma(g+2\ell-\frac12)}{\Gamma(\ell+1)\Gamma(g+\ell-\frac12)}\,{}_1F_1\Bigl(\genfrac{}{}{0pt}{}{-\ell}{g+\ell-\frac12}\Bigm|-\eta\Bigr) \tag{3.3}$$

$$=\frac{\Gamma(g+2\ell-\frac12)}{\Gamma(\ell+1)\Gamma(g+\ell-\frac12)}\,e^{-\eta}{}_1F_1\Bigl(\genfrac{}{}{0pt}{}{g+2\ell-\frac12}{g+\ell-\frac12}\Bigm|\eta\Bigr), \tag{3.4}$$

where the Kummer's transformation formula is used in the second equalities. In addition to the condition $g,h>0$ (2.4), we restrict the parameters as follows:

$$\left\{\begin{array}{ll} g>\frac32,\ \ h>\frac12 & :\text{J1}\\ g>\frac32, & :\text{L1}\end{array}\right., \tag{3.5}$$

then the deforming function $\xi_\ell(\eta(x);\boldsymbol{\lambda})$ has no zero in the domain $x_1<x<x_2$. This can be easily verified by using the power series definition of the (confluent) hypergeometric function (3.15),(3.21) and the alternative expressions (3.2) and (3.4).

Since the Jacobi and Laguerre polynomials are expressed as

$$P_n^{(\alpha,\beta)}(x)=\frac{(\alpha+1)_n}{n!}\,{}_2F_1\Bigl(\genfrac{}{}{0pt}{}{-n,\,n+\alpha+\beta+1}{\alpha+1}\Bigm|\frac{1-x}{2}\Bigr), \tag{3.6}$$

$$L_n^{(\alpha)}(x)=\frac{(\alpha+1)_n}{n!}\,{}_1F_1\Bigl(\genfrac{}{}{0pt}{}{-n}{\alpha+1}\Bigm|x\Bigr), \tag{3.7}$$

this deforming function reduces to the deforming polynomial in [4, 6] for integer $\ell$

$$\ell\in\mathbb{Z}_{>0}\quad\Rightarrow\quad \xi_\ell(\eta;\boldsymbol{\lambda})=\left\{\begin{array}{ll} P_\ell^{(g+\ell-\frac32,-h-\ell-\frac12)}(\eta) & :\text{J1}\\ L_\ell^{(g+\ell-\frac32)}(-\eta) & :\text{L1}\end{array}\right.. \tag{3.8}$$

We remark that we had restricted $g>h>0$ for J1 ($h>g>0$ for J2) in [4, 6, 7] for a positive integer $\ell$, but this restriction is unnecessary due to (3.2) and (3.4).

Here we present three formulas satisfied by the deforming function $\xi_\ell(\eta;\boldsymbol{\lambda})$ (3.9)–(3.11), which will play important roles in the derivation of various results in § 4:

$$c_2(\eta)\partial_\eta^2\xi_\ell(\eta;\boldsymbol{\lambda})+\tilde{c}_1(\eta,\boldsymbol{\lambda},\ell)\partial_\eta\xi_\ell(\eta;\boldsymbol{\lambda})=-\tfrac14\widetilde{\mathcal{E}}_\ell(\boldsymbol{\lambda})\xi_\ell(\eta;\boldsymbol{\lambda}), \tag{3.9}$$

$$d_1(\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\xi_\ell(\eta;\boldsymbol{\lambda})+d_2(\eta)\partial_\eta\xi_\ell(\eta;\boldsymbol{\lambda})=d_1(\boldsymbol{\lambda})\xi_\ell(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{3.10}$$

$$d_3(\boldsymbol{\lambda},\ell)\xi_\ell(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})+\frac{c_2(\eta)}{d_2(\eta)}\,\partial_\eta\xi_\ell(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})=d_3(\boldsymbol{\lambda}+\ell\boldsymbol{\delta},\ell)\xi_\ell(\eta;\boldsymbol{\lambda}),\tag{3.11}$$

where $\tilde{c}_1(\eta,\boldsymbol{\lambda},\ell)$, $d_1(\boldsymbol{\lambda})$, $d_2(\eta)$, $d_3(\boldsymbol{\lambda},\ell)$ and $\widetilde{\mathcal{E}}_\ell(\boldsymbol{\lambda})$ are given by [7]

$$\tilde{c}_1(\eta,\boldsymbol{\lambda},\ell)\stackrel{\text{def}}{=}\begin{cases}-\bigl(g+h+2\ell-1+(g-h)\eta\bigr) & :\mathrm{J1}\\ g+\ell-\frac{1}{2}+\eta & :\mathrm{L1}\end{cases},\tag{3.12}$$

$$d_1(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\begin{cases}h+\frac{1}{2} & :\mathrm{J1}\\ 1 & :\mathrm{L1}\end{cases},\quad d_2(\eta)\stackrel{\text{def}}{=}\begin{cases}-(1+\eta) & :\mathrm{J1}\\ 1 & :\mathrm{L1}\end{cases},\tag{3.13}$$

$$d_3(\boldsymbol{\lambda},\ell)\stackrel{\text{def}}{=}g+\ell-\tfrac{1}{2}\ :\mathrm{J1}\,\&\,\mathrm{L1},\quad \widetilde{\mathcal{E}}_\ell(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\begin{cases}4\ell(\ell+g-h-1) & :\mathrm{J1}\\ -4\ell & :\mathrm{L1}\end{cases}.\tag{3.14}$$

The first equation (3.9) is the differential equation for the deforming function. The eqs. (3.10)–(3.11) are identities relating $\xi_\ell(\eta;\boldsymbol{\lambda})$ and $\xi_\ell(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})$. The eqs. (3.9)–(3.11) are obtained from the properties of the hypergeometric function (3.16)–(3.20) and (3.22)–(3.26). It is interesting to note that (3.9) can be considered as a consequence of (3.10) and (3.11).

In the rest of this section we present some properties of the hypergeometric functions ${}_2F_1$ and the confluent one ${}_1F_1$. We assume that parameters $(a,b,c)$ are generic.

The hypergeometric function ${}_2F_1$ is defined by

$${}_2F_1\Bigl(\begin{matrix}a,b\\ c\end{matrix}\Bigm|x\Bigr)\stackrel{\text{def}}{=}\sum_{k=0}^\infty\frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!},\quad(|x|<1).\tag{3.15}$$

The following properties can be verified elementarily based on (3.15):

$$\frac{d}{dx}\,{}_2F_1\Bigl(\begin{matrix}a,b\\ c\end{matrix}\Bigm|x\Bigr)=\frac{ab}{c}\,{}_2F_1\Bigl(\begin{matrix}a+1,b+1\\ c+1\end{matrix}\Bigm|x\Bigr),\tag{3.16}$$

$$\Bigl(x(1-x)\frac{d}{dx}+c-(a+b+1)x\Bigr){}_2F_1\Bigl(\begin{matrix}a+1,b+1\\ c+1\end{matrix}\Bigm|x\Bigr)=c\,{}_2F_1\Bigl(\begin{matrix}a,b\\ c\end{matrix}\Bigm|x\Bigr),\tag{3.17}$$

$$\Bigl(x(1-x)\frac{d^2}{dx^2}+\bigl(c-(a+b+1)x\bigr)\frac{d}{dx}-ab\Bigr){}_2F_1\Bigl(\begin{matrix}a,b\\ c\end{matrix}\Bigm|x\Bigr)=0,\tag{3.18}$$

$$(a+b-c)\,{}_2F_1\Bigl(\begin{matrix}a,b\\ c\end{matrix}\Bigm|x\Bigr)+\frac{(c-a)(c-b)}{c}\,{}_2F_1\Bigl(\begin{matrix}a,b\\ c+1\end{matrix}\Bigm|x\Bigr)=(1-x)\frac{ab}{c}\,{}_2F_1\Bigl(\begin{matrix}a+1,b+1\\ c+1\end{matrix}\Bigm|x\Bigr),\tag{3.19}$$

$${}_2F_1\Bigl(\begin{matrix}a,b\\ c\end{matrix}\Bigm|x\Bigr)-{}_2F_1\Bigl(\begin{matrix}a,b\\ c+1\end{matrix}\Bigm|x\Bigr)=\frac{x}{c}\frac{ab}{c+1}\,{}_2F_1\Bigl(\begin{matrix}a+1,b+1\\ c+2\end{matrix}\Bigm|x\Bigr).\tag{3.20}$$

The confluent hypergeometric function ${}_1F_1$ is defined by

$${}_1F_1\Bigl(\begin{matrix}a\\ b\end{matrix}\Bigm|x\Bigr)\stackrel{\text{def}}{=}\sum_{k=0}^\infty\frac{(a)_k}{(b)_k}\frac{x^k}{k!}.\tag{3.21}$$

The following properties can be verified elementarily based on (3.21):

$$\frac{d}{dx}\,{}_1F_1\Big(\begin{matrix}a\\ b\end{matrix}\,\Big|\,x\Big)=\frac{a}{b}\,{}_1F_1\Big(\begin{matrix}a+1\\ b+1\end{matrix}\,\Big|\,x\Big), \tag{3.22}$$

$$\Big(x\frac{d}{dx}+b-x\Big){}_1F_1\Big(\begin{matrix}a+1\\ b+1\end{matrix}\,\Big|\,x\Big)=b\,{}_1F_1\Big(\begin{matrix}a\\ b\end{matrix}\,\Big|\,x\Big), \tag{3.23}$$

$$\Big(x\frac{d^2}{dx^2}+(b-x)\frac{d}{dx}-a\Big){}_1F_1\Big(\begin{matrix}a\\ b\end{matrix}\,\Big|\,x\Big)=0, \tag{3.24}$$

$${}_1F_1\Big(\begin{matrix}a\\ b\end{matrix}\,\Big|\,x\Big)+\frac{a-b}{b}\,{}_1F_1\Big(\begin{matrix}a\\ b+1\end{matrix}\,\Big|\,x\Big)=\frac{a}{b}\,{}_1F_1\Big(\begin{matrix}a+1\\ b+1\end{matrix}\,\Big|\,x\Big), \tag{3.25}$$

$${}_1F_1\Big(\begin{matrix}a+1\\ b\end{matrix}\,\Big|\,x\Big)-{}_1F_1\Big(\begin{matrix}a\\ b\end{matrix}\,\Big|\,x\Big)=\frac{x}{b}\,{}_1F_1\Big(\begin{matrix}a+1\\ b+1\end{matrix}\,\Big|\,x\Big). \tag{3.26}$$

To sum up, the deforming function $\xi_\ell(\eta(x);\boldsymbol{\lambda})$ possesses two important properties; (i) it has no zero in the domain $x_1<x<x_2$, (ii) it satisfies the three formulas (3.9)–(3.11), which are the essential properties of the deforming polynomials in the theory of the exceptional Jacobi and Laguerre polynomials [6, 7].

Before closing this section, let us briefly comment on the possible "continuous $\ell$ versions" corresponding to the J2 Jacobi and L2 Laguerre polynomials. The obvious candidates for the deforming function are:

$$\text{J2}:\ \xi_\ell(\eta;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\frac{\Gamma(-g+\frac12)}{\Gamma(\ell+1)\Gamma(-g-\ell+\frac12)}\,{}_2F_1\Big(\begin{matrix}-\ell,\,h-g+\ell-1\\ -g-\ell+\frac12\end{matrix}\,\Big|\,\frac{1-\eta}{2}\Big), \tag{3.27}$$

$$\text{L2}:\ \xi_\ell(\eta;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\frac{\Gamma(-g+\frac12)}{\Gamma(\ell+1)\Gamma(-g-\ell+\frac12)}\,{}_1F_1\Big(\begin{matrix}-\ell\\ -g-\ell+\frac12\end{matrix}\,\Big|\,\eta\Big). \tag{3.28}$$

For the above choices and a few other related candidates, we have not been able to find proper parameter ranges in which the above two properties (i) and (ii) are satisfied and at the same time invariant under the shifts, $g\to g+1$, $h\to h+1$, so that the shape invariance method is applicable. In other words, the J2 and L2 deformations are valid only for integer $\ell$.

# 4 Deformed systems and intertwining relations

We deform the original systems in terms of the deforming function $\xi_\ell(\eta(x);\boldsymbol{\lambda})$ in exactly the same manner as in the theory of the exceptional orthogonal Jacobi and Laguerre polynomials [4, 6, 7].

## 4.1 Deformed systems

For a real positive number $\ell$, we define a deformed Hamiltonian:

$$\mathcal{H}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger \mathcal{A}_\ell(\boldsymbol{\lambda}), \tag{4.1}$$

$$\mathcal{A}_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{d}{dx} - \partial_x w_\ell(x;\boldsymbol{\lambda}), \quad \mathcal{A}_\ell(x;\boldsymbol{\lambda})^\dagger = -\frac{d}{dx} - \partial_x w_\ell(x;\boldsymbol{\lambda}), \tag{4.2}$$

$$w_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} w_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}) + \log\frac{\xi_\ell(\eta(x);\boldsymbol{\lambda}+\boldsymbol{\delta})}{\xi_\ell(\eta(x);\boldsymbol{\lambda})}. \tag{4.3}$$

The overall normalisation of the deforming function $\xi_\ell$ is immaterial for the deformation and the original Hamiltonian corresponds to $\ell = 0$. This system is shape invariant,

$$\mathcal{A}_\ell(\boldsymbol{\lambda})\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger = \mathcal{A}_\ell(\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger\mathcal{A}_\ell(\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_{\ell,1}(\boldsymbol{\lambda}), \tag{4.4}$$

or equivalently,

$$\big(\partial_x w_\ell(x;\boldsymbol{\lambda})\big)^2 - \partial_x^2 w_\ell(x;\boldsymbol{\lambda}) = \big(\partial_x w_\ell(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\big)^2 + \partial_x^2 w_\ell(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_{\ell,1}(\boldsymbol{\lambda}). \tag{4.5}$$

As in the theory of the exceptional orthogonal Jacobi and Laguerre polynomials [5], this relation is reduced to an identity involving cubic products of $\xi_\ell$, which can be proved by using the three formulas (3.9)–(3.11). The shape invariance determines the whole spectrum and eigenfunctions in terms of the first excited state energy and the ground state wavefunction. Eqs. (4.10)–(4.12) and (4.15)–(4.16) are the consequences of the shape invariance and the normalization of the eigenfunctions.

We present various properties of these deformed systems, which will be derived without using shape invariance in the next subsection. The Schrödinger equation of this deformed system is

$$\mathcal{H}_\ell(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda}) = \mathcal{E}_{\ell,n}(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda}) \quad (n = 0,1,2,\ldots), \tag{4.6}$$

$$\phi_{\ell,n}(x;\boldsymbol{\lambda}) = \psi_\ell(x;\boldsymbol{\lambda})P_{\ell,n}(\eta(x);\boldsymbol{\lambda}), \quad \psi_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{\phi_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}{\xi_\ell(\eta(x);\boldsymbol{\lambda})}. \tag{4.7}$$

The spectrum and the main part of the eigenfunction are

$$\mathcal{E}_{\ell,n}(\boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}), \tag{4.8}$$

$$\begin{aligned} P_{\ell,n}(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{2}{\hat{f}_{\ell,n}(\boldsymbol{\lambda})}\Big(&d_2(\eta)\xi_\ell(\eta;\boldsymbol{\lambda})\partial_\eta P_n(\eta;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}}) \\ &- d_1(\boldsymbol{\lambda})\xi_\ell(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})P_n(\eta;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}})\Big), \end{aligned} \tag{4.9}$$

where $\hat{f}_{\ell,n}(\boldsymbol{\lambda})$ and $\tilde{\boldsymbol{\delta}}$ will be given in (4.21)–(4.22). The Sturm–Liouville's theorem ensures that the function $P_{\ell,n}(\eta(x);\boldsymbol{\lambda})$ has $n$ zeros in the domain $x_1<x<x_2$. The action of $\mathcal{A}_\ell(\boldsymbol{\lambda})$ and $\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger$ on the eigenfunction is

$$\begin{aligned}
&\mathcal{A}_\ell(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda})=f_{\ell,n}(\boldsymbol{\lambda})\phi_{\ell,n-1}\big(x;\boldsymbol{\lambda}+\boldsymbol{\delta}\big), &(4.10)\\
&\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\phi_{\ell,n-1}\big(x;\boldsymbol{\lambda}+\boldsymbol{\delta}\big)=b_{\ell,n-1}(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda}), &(4.11)\\
&f_{\ell,n}(\boldsymbol{\lambda})=f_n(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}),\quad b_{\ell,n-1}(\boldsymbol{\lambda})=b_{n-1}(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}). &(4.12)
\end{aligned}$$

The forward and backward shift operators are defined in a similar way as before

$$\begin{aligned}
\mathcal{F}_\ell(\boldsymbol{\lambda})&\stackrel{\text{def}}{=}\psi_\ell(x;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1}\circ\mathcal{A}_\ell(\boldsymbol{\lambda})\circ\psi_\ell(x;\boldsymbol{\lambda})\\
&=c_{\mathcal{F}}\frac{\xi_\ell(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})}{\xi_\ell(\eta;\boldsymbol{\lambda})}\Big(\frac{d}{d\eta}-\partial_\eta\log\xi_\ell(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})\Big), &(4.13)\\
\mathcal{B}_\ell(\boldsymbol{\lambda})&\stackrel{\text{def}}{=}\psi_\ell(x;\boldsymbol{\lambda})^{-1}\circ\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\circ\psi_\ell(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\\
&=-4c_{\mathcal{F}}^{-1}c_2(\eta)\frac{\xi_\ell(\eta;\boldsymbol{\lambda})}{\xi_\ell(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})}\Big(\frac{d}{d\eta}+\frac{c_1(\eta,\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}{c_2(\eta)}-\partial_\eta\log\xi_\ell(\eta;\boldsymbol{\lambda})\Big), &(4.14)
\end{aligned}$$

and their action on $P_{\ell,n}$ is

$$\begin{aligned}
&\mathcal{F}_\ell(\boldsymbol{\lambda})P_{\ell,n}(\eta;\boldsymbol{\lambda})=f_{\ell,n}(\boldsymbol{\lambda})P_{\ell,n-1}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta}), &(4.15)\\
&\mathcal{B}_\ell(\boldsymbol{\lambda})P_{\ell,n-1}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})=b_{\ell,n-1}(\boldsymbol{\lambda})P_{\ell,n}(\eta;\boldsymbol{\lambda}). &(4.16)
\end{aligned}$$

The second order differential operator $\widetilde{\mathcal{H}}_\ell(\boldsymbol{\lambda})$ acting on the functions $P_{\ell,n}(\eta;\boldsymbol{\lambda})$ is defined by

$$\begin{aligned}
\widetilde{\mathcal{H}}_\ell(\boldsymbol{\lambda})&\stackrel{\text{def}}{=}\mathcal{B}_\ell(\boldsymbol{\lambda})\mathcal{F}_\ell(\boldsymbol{\lambda})=\psi_\ell(x;\boldsymbol{\lambda})^{-1}\circ\mathcal{H}_\ell(\boldsymbol{\lambda})\circ\psi_\ell(x;\boldsymbol{\lambda})\\
&=-4\Big(c_2(\eta)\frac{d^2}{d\eta^2}+\big(c_1(\eta,\boldsymbol{\lambda}+\ell\boldsymbol{\delta})-2c_2(\eta)\partial_\eta\log\xi_\ell(\eta;\boldsymbol{\lambda})\big)\frac{d}{d\eta}\\
&\qquad-2d_2(\eta)d_3(\boldsymbol{\lambda},\ell)\partial_\eta\log\xi_\ell(\eta;\boldsymbol{\lambda})-\tfrac14\widetilde{\mathcal{E}}_\ell(\boldsymbol{\lambda})\Big), &(4.17)\\
\widetilde{\mathcal{H}}_\ell(\boldsymbol{\lambda})&P_{\ell,n}(\eta;\boldsymbol{\lambda})=\mathcal{E}_{\ell,n}(\boldsymbol{\lambda})P_{\ell,n}(\eta;\boldsymbol{\lambda}), &(4.18)
\end{aligned}$$

where we have used (3.9)–(3.11) in (4.17).

The orthogonality reads

$$\int_{x_1}^{x_2}\psi_\ell(x;\boldsymbol{\lambda})^2\,P_{\ell,n}(\eta(x);\boldsymbol{\lambda})P_{\ell,m}(\eta(x);\boldsymbol{\lambda})dx=h_{\ell,n}(\boldsymbol{\lambda})\delta_{nm}. \tag{4.19}$$

The normalisation constant $h_{\ell,n}(\boldsymbol{\lambda})$ is related to $h_n(\boldsymbol{\lambda})$ (2.21) as

$$h_{\ell,n}(\boldsymbol{\lambda})=\frac{\hat{b}_{\ell,n}(\boldsymbol{\lambda})}{\hat{f}_{\ell,n}(\boldsymbol{\lambda})}h_n(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}})=\frac{\hat{b}_{\ell,n}(\boldsymbol{\lambda})}{\hat{f}_{\ell,n}(\boldsymbol{\lambda})}\frac{\hat{f}_{0,n}(\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}{\hat{b}_{0,n}(\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}h_n(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}), \tag{4.20}$$

where $\hat{f}_{\ell,n}(\boldsymbol{\lambda})$, $\hat{b}_{\ell,n}(\boldsymbol{\lambda})$ and $\tilde{\boldsymbol{\delta}}$ are given by

$$\hat{f}_{\ell,n}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -2\times\left\{\begin{array}{ll} n+h+\frac{1}{2} & : \mathrm{J1} \\ 1 & : \mathrm{L1} \end{array}\right., \quad \hat{b}_{\ell,n}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -2(n+g+2\ell-\tfrac{1}{2}) \ \ : \mathrm{J1}\,\&\,\mathrm{L1}, \tag{4.21}$$

$$\tilde{\boldsymbol{\delta}} \stackrel{\text{def}}{=} \left\{\begin{array}{ll} (-1,1) & : \mathrm{J1} \\ -1 & : \mathrm{L1} \end{array}\right. . \tag{4.22}$$

In the second equality of (4.20) we have used the explicit expressions of $h_n(\boldsymbol{\lambda})$ (2.21).

## 4.2 Intertwining relations

### 4.2.1 General setting

For well-defined operators $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger$, let us define a pair of Hamiltonians $\hat{\mathcal{H}}_\ell^{(\pm)}(\boldsymbol{\lambda})$

$$\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}), \quad \hat{\mathcal{H}}_\ell^{(-)}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger, \tag{4.23}$$

and consider their Schrödinger equations, that is, the eigenvalue problems:

$$\hat{\mathcal{H}}_\ell^{(\pm)}(\boldsymbol{\lambda})\hat{\phi}_{\ell,n}^{(\pm)}(x;\boldsymbol{\lambda}) = \hat{\mathcal{E}}_{\ell,n}^{(\pm)}(\boldsymbol{\lambda})\hat{\phi}_{\ell,n}^{(\pm)}(x;\boldsymbol{\lambda}) \quad (n=0,1,2,\ldots). \tag{4.24}$$

By definition, all the eigenfunctions must be square integrable. Obviously the pair of Hamiltonians are intertwined:

$$\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda})\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger = \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger = \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\hat{\mathcal{H}}_\ell^{(-)}(\boldsymbol{\lambda}), \tag{4.25}$$

$$\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda}) = \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}) = \hat{\mathcal{H}}_\ell^{(-)}(\boldsymbol{\lambda})\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}). \tag{4.26}$$

If $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\hat{\phi}_{\ell,n}^{(+)}(x;\boldsymbol{\lambda}) \neq 0$ and $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\hat{\phi}_{\ell,n}^{(-)}(x;\boldsymbol{\lambda}) \neq 0$, then the two systems are exactly iso-spectral and there is one-to-one correspondence between the eigenfunctions:

$$\hat{\mathcal{E}}_{\ell,n}^{(+)}(\boldsymbol{\lambda}) = \hat{\mathcal{E}}_{\ell,n}^{(-)}(\boldsymbol{\lambda}), \tag{4.27}$$

$$\hat{\phi}_{\ell,n}^{(-)}(x;\boldsymbol{\lambda}) \propto \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\hat{\phi}_{\ell,n}^{(+)}(x;\boldsymbol{\lambda}), \quad \hat{\phi}_{\ell,n}^{(+)}(x;\boldsymbol{\lambda}) \propto \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\hat{\phi}_{\ell,n}^{(-)}(x;\boldsymbol{\lambda}). \tag{4.28}$$

This situation is called 'broken susy' case in the parlance of supersymmetric quantum mechanics [3, 17]. It should be stressed that in the ordinary setting of Crum's theorem, the zero mode of $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ is the groundstate of $\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda})$. In that case, $\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda})$ and $\hat{\mathcal{H}}_\ell^{(-)}(\boldsymbol{\lambda})$ are iso-spectral except for the groundstate of $\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda})$.

In the following we will present the explicit forms of the operators $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger$, which intertwine the original systems in § 2 and the deformed systems in § 4.1.

### 4.2.2 Intertwining the original and the deformed systems

Here we demonstrate that the Hamiltonian systems of the original polynomials reviewed in § 2 and the deformation summarised in § 4.1 are intertwined by the Darboux-Crum transformation.

The intertwining operators $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger$ are given by

$$\hat{\mathcal{A}}_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{d}{dx} - \partial_x \hat{w}_\ell(x;\boldsymbol{\lambda}), \quad \hat{\mathcal{A}}_\ell(x;\boldsymbol{\lambda})^\dagger = -\frac{d}{dx} - \partial_x \hat{w}_\ell(x;\boldsymbol{\lambda}), \tag{4.29}$$

$$\hat{w}_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \tilde{w}_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}) + \log \xi_\ell(\eta(x);\boldsymbol{\lambda}), \tag{4.30}$$

$$\tilde{w}_0(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} (g-1)\log\sin x - h\log\cos x & : \text{J1} \\ \frac{x^2}{2} + (g-1)\log x & : \text{L1} \end{cases}. \tag{4.31}$$

These have exactly the same form as those used for the exceptional J1 Jacobi and L1 Laguerre polynomials [8]. See also a similar work [24]. It is illuminating to compare these prepotential (4.30) with those of the original (2.5) and deformed (4.3) systems. Again it is obvious that the overall normalisation of the deforming polynomial $\xi_\ell$ is immaterial for $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger$.

For this choice of $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger$, one of the pair of Hamiltonians $\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda})$ (4.23) becomes proportional to the original Hamiltonian $\mathcal{H}(\boldsymbol{\lambda})$ (2.1) with $\boldsymbol{\lambda} \to \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}$ and the partner Hamiltonian $\hat{\mathcal{H}}_\ell^{(-)}(\boldsymbol{\lambda})$ is proportional to the deformed Hamiltonian $\mathcal{H}_\ell(\boldsymbol{\lambda})$ (4.1), up to a common additive constant:

$$\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda}) = \mathcal{H}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) + \hat{f}_{\ell,0}(\boldsymbol{\lambda})\hat{b}_{\ell,0}(\boldsymbol{\lambda}), \tag{4.32}$$

$$\hat{\mathcal{H}}_\ell^{(-)}(\boldsymbol{\lambda}) = \mathcal{H}_\ell(\boldsymbol{\lambda}) + \hat{f}_{\ell,0}(\boldsymbol{\lambda})\hat{b}_{\ell,0}(\boldsymbol{\lambda}). \tag{4.33}$$

These fundamental results can be obtained by explicit calculation, in which the three formulas (3.9)–(3.11) are used.

It is instructive to verify that the zero modes of $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger$ do not belong to the Hilbert space of the eigenfunctions. In fact, the zero mode $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ is

$$\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\chi = 0, \quad \chi = e^{\hat{w}_\ell(x;\boldsymbol{\lambda})} = e^{\tilde{w}_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}\xi_\ell(\eta(x);\boldsymbol{\lambda}), \tag{4.34}$$

which is non-square integrable for the chosen parameter range (3.5). The zero mode of $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger$ is

$$\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\rho = 0, \quad \rho = e^{-\hat{w}_\ell(x;\boldsymbol{\lambda})} = \frac{e^{-\tilde{w}_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}}{\xi_\ell(\eta(x);\boldsymbol{\lambda})} = \frac{1}{\chi}, \tag{4.35}$$

which is also non-square integrable for the chosen parameter range (3.5). Thus the 'broken susy' case is demonstrated [3, 17].

Based on the results (4.32)–(4.33), we have

$$\hat{\phi}^{(+)}_{\ell,n}(x;\boldsymbol{\lambda})=\phi_n(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}}),\quad \hat{\phi}^{(-)}_{\ell,n}(x;\boldsymbol{\lambda})=\phi_{\ell,n}(x;\boldsymbol{\lambda}),\tag{4.36}$$

$$\hat{\mathcal{E}}^{(\pm)}_{\ell,n}(\boldsymbol{\lambda})=\mathcal{E}_n(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}})+\hat{f}_{\ell,0}(\boldsymbol{\lambda})\hat{b}_{\ell,0}(\boldsymbol{\lambda})=\mathcal{E}_{\ell,n}(\boldsymbol{\lambda})+\hat{f}_{\ell,0}(\boldsymbol{\lambda})\hat{b}_{\ell,0}(\boldsymbol{\lambda}).\tag{4.37}$$

Then it is trivial to verify $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\hat{\phi}^{(+)}_{\ell,n}(x;\boldsymbol{\lambda})\neq 0$ and $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\hat{\phi}^{(-)}_{\ell,n}(x;\boldsymbol{\lambda})\neq 0$. For, if one of the eigenfunction is annihilated by $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ ($\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger$), the left hand side of (4.32)((4.33)) vanishes, whereas the right hand side is $\mathcal{E}_n(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}})+\hat{f}_{\ell,0}(\boldsymbol{\lambda})\hat{b}_{\ell,0}(\boldsymbol{\lambda})$ times the eigenfunction, which is obviously non-vanishing. Note that $\mathcal{E}_n(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}})=\mathcal{E}_n(\boldsymbol{\lambda}+\ell\boldsymbol{\delta})$.

The correspondence of the pair of eigenfunctions $\hat{\phi}^{(\pm)}_{\ell,n}(x)$ is expressed as

$$\hat{\phi}^{(-)}_{\ell,n}(x;\boldsymbol{\lambda})=\frac{\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\hat{\phi}^{(+)}_{\ell,n}(x;\boldsymbol{\lambda})}{\hat{f}_{\ell,n}(\boldsymbol{\lambda})},\quad \hat{\phi}^{(+)}_{\ell,n}(x;\boldsymbol{\lambda})=\frac{\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\hat{\phi}^{(-)}_{\ell,n}(x;\boldsymbol{\lambda})}{\hat{b}_{\ell,n}(\boldsymbol{\lambda})}.\tag{4.38}$$

Let us introduce operators $\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda})$ defined by

$$\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\psi_\ell(x;\boldsymbol{\lambda})^{-1}\circ\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\circ\phi_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}}),\tag{4.39}$$

$$\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\phi_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}})^{-1}\circ\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\circ\psi_\ell(x;\boldsymbol{\lambda}),\tag{4.40}$$

which can be expressed in terms of $\eta$:

$$\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})=2\Big(d_2(\eta)\xi_\ell(\eta;\boldsymbol{\lambda})\frac{d}{d\eta}-d_1(\boldsymbol{\lambda})\xi_\ell(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})\Big),\tag{4.41}$$

$$\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda})=\frac{-2}{\xi_\ell(\eta;\boldsymbol{\lambda})}\Big(\frac{c_2(\eta)}{d_2(\eta)}\frac{d}{d\eta}+d_3(\boldsymbol{\lambda},\ell)\Big).\tag{4.42}$$

The operators $\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda})$ act as the forward and backward shift operators connecting the original orthogonal polynomials $P_n(\eta)$ and the orthogonal functions $P_{\ell,n}(\eta)$:

$$\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})P_n(\eta;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}})=\hat{f}_{\ell,n}(\boldsymbol{\lambda})P_{\ell,n}(\eta;\boldsymbol{\lambda}),\tag{4.43}$$

$$\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda})P_{\ell,n}(\eta;\boldsymbol{\lambda})=\hat{b}_{\ell,n}(\boldsymbol{\lambda})P_n(\eta;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}}).\tag{4.44}$$

The former relation (4.43) with the explicit form of $\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})$ (4.41) provides the explicit expression (4.9) of the main part of the eigenfunction. Other simple consequences of these relations are

$$\hat{\mathcal{E}}^{(\pm)}_{\ell,n}(\boldsymbol{\lambda})=\hat{f}_{\ell,n}(\boldsymbol{\lambda})\hat{b}_{\ell,n}(\boldsymbol{\lambda}),\quad \mathcal{E}_n(\boldsymbol{\lambda}+\ell\boldsymbol{\delta})=\hat{f}_{\ell,n}(\boldsymbol{\lambda})\hat{b}_{\ell,n}(\boldsymbol{\lambda})-\hat{f}_{\ell,0}(\boldsymbol{\lambda})\hat{b}_{\ell,0}(\boldsymbol{\lambda}).\tag{4.45}$$

The operator $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ intertwines those of the original and deformed systems $\mathcal{A}(\boldsymbol{\lambda})$ and $\mathcal{A}_\ell(\boldsymbol{\lambda})$:

$$\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}+\boldsymbol{\delta})\mathcal{A}(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}}) = \mathcal{A}_\ell(\boldsymbol{\lambda})\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}), \tag{4.46}$$

$$\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}})^\dagger = \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}+\boldsymbol{\delta}). \tag{4.47}$$

These relations can be obtained by explicit calculation, in which the three formulas (3.9)–(3.11) are used. In terms of the definitions of the forward shift operators $\mathcal{F}(\boldsymbol{\lambda})$ (2.12), $\mathcal{F}_\ell(\boldsymbol{\lambda})$ (4.13), $\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})$ (4.39), and the backward shift operators $\mathcal{B}(\boldsymbol{\lambda})$ (2.13), $\mathcal{B}_\ell(\boldsymbol{\lambda})$ (4.14), the above relations are rewritten as:

$$\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda}+\boldsymbol{\delta})\mathcal{F}(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}}) = \mathcal{F}_\ell(\boldsymbol{\lambda})\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda}), \tag{4.48}$$

$$\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})\mathcal{B}(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}}) = \mathcal{B}_\ell(\boldsymbol{\lambda})\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda}+\boldsymbol{\delta}). \tag{4.49}$$

By applying $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}+\boldsymbol{\delta})$ and $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ to (2.9) and (2.10) with a replacement $\boldsymbol{\lambda}\to\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}}$ respectively, together with the use of (4.46), (4.47) and (4.38), we obtain

$$\begin{aligned}
\mathcal{A}_\ell(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda}) &= f_n(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}})\frac{\hat{f}_{\ell,n-1}(\boldsymbol{\lambda}+\boldsymbol{\delta})}{\hat{f}_{\ell,n}(\boldsymbol{\lambda})}\,\phi_{\ell,n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\\
&= f_n(\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\phi_{\ell,n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}), \qquad (4.50)\\
\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\phi_{\ell,n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) &= b_{n-1}(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}})\frac{\hat{f}_{\ell,n}(\boldsymbol{\lambda})}{\hat{f}_{\ell,n-1}(\boldsymbol{\lambda}+\boldsymbol{\delta})}\,\phi_{\ell,n}(x;\boldsymbol{\lambda})\\
&= b_{n-1}(\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\phi_{\ell,n}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}). \qquad (4.51)
\end{aligned}$$

In the calculation use is made of the explicit forms of $\hat{f}_{\ell,n}(\boldsymbol{\lambda})$, $f_n(\boldsymbol{\lambda})$ and $b_n(\boldsymbol{\lambda})$ in the second equalities. This provides a proof of (4.10)–(4.12) without recourse to the shape invariance. Likewise the above intertwining relations of the forward-backward shift operators (4.48)–(4.49) give a proof of (4.15)–(4.16), respectively, again without recourse to the shape invariance.

Eq. (4.20) is shown in the following way:

$$\begin{aligned}
&\hat{f}_{\ell,n}(\boldsymbol{\lambda})\hat{f}_{\ell,m}(\boldsymbol{\lambda})\int_{x_1}^{x_2}dx\,\phi_{\ell,n}(x;\boldsymbol{\lambda})\phi_{\ell,m}(x;\boldsymbol{\lambda})\\
&\overset{\text{(i)}}{=}\int_{x_1}^{x_2}dx\,\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}})\cdot\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\phi_m(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}})\\
&\overset{\text{(ii)}}{=}\int_{x_1}^{x_2}dx\,\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}})\cdot\phi_m(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}})
\end{aligned}$$

$$\stackrel{\text{(iii)}}{=} \hat{\mathcal{E}}^{(+)}_{\ell,n}(\boldsymbol{\lambda}) \int_{x_1}^{x_2} dx\, \phi_n(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}})\phi_m(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}})$$
$$\stackrel{\text{(iv)}}{=} \hat{f}_{\ell,n}(\boldsymbol{\lambda})\hat{b}_{\ell,n}(\boldsymbol{\lambda})h_n(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}+\tilde{\boldsymbol{\delta}})\delta_{nm}. \tag{4.52}$$

Here we have used (4.38) and (4.36) in (i), an integration by parts in (ii), (4.24) and (4.36) in (iii), (4.45) and (2.20) in (iv).

## 4.3 Limit: Jacobi $\to$ Laguerre

It is well known [16] that the Laguerre polynomial $L_n^{(\alpha)}(x)$ is obtained from the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ in a limit:

$$\lim_{\beta\to\infty} P_n^{(\alpha,\,\pm\beta)}\bigl(1-2x\beta^{-1}\bigr) = L_n^{(\alpha)}(\pm x). \tag{4.53}$$

It is also known that the radial oscillator potential can be obtained from the trigonometric DPT potential in the limit of infinite coupling $h\to\infty$ together with the rescaling of the coordinate:

$$x = \frac{x^{\text{L}}}{\sqrt{h}}, \quad 0 < x < \frac{\pi}{2} \Longleftrightarrow 0 < x^{\text{L}} < \frac{\pi}{2}\sqrt{h}\,. \tag{4.54}$$

The two prepotentials (2.5) are related [6]:

$$\eta^{\text{J}}(x) = 1 - 2\eta^{\text{L}}(x^{\text{L}})h^{-1} + O(h^{-2}), \tag{4.55}$$
$$\lim_{h\to\infty}\bigl(w_0^{\text{J}}(x;g,h) + \tfrac{1}{2}g\log h\bigr) = w_0^{\text{L}}(x^{\text{L}};g). \tag{4.56}$$

Here we will show that the above two "continuous $\ell$ versions", the J1 Jacobi and L1 Laguerre, are connected by the same limit. By using the series definitions of the (confluent) hypergeometric functions (3.15) and (3.21), one obtains

$$\lim_{h\to\infty} {}_2F_1\Bigl(\begin{matrix}-\ell,\, g-h+\ell-1\\ g+\ell-\frac{1}{2}\end{matrix}\Bigm|\frac{1-\eta^{\text{J}}(x)}{2}\Bigr) = {}_1F_1\Bigl(\begin{matrix}-\ell\\ g+\ell-\frac{1}{2}\end{matrix}\Bigm| -\eta^{\text{L}}(x^{\text{L}})\Bigr), \tag{4.57}$$
$$\lim_{h\to\infty} \xi^{\text{J}}_\ell(\eta^{\text{J}}(x);g,h) = \xi^{\text{L}}_\ell(\eta^{\text{L}}(x^{\text{L}});g). \tag{4.58}$$

At the same time we have

$$\lim_{h\to\infty}\bigl(\tilde{w}_0^{\text{J}}(x;g,h) + \tfrac{1}{2}(g-1)\log h\bigr) = \tilde{w}_0^{\text{L}}(x^{\text{L}};g). \tag{4.59}$$

Thus the limiting procedure connects the original systems, the deformed systems and the intertwining relations.

## 5 Summary and Comments

A new family of shape invariantly deformed Darboux-Pöschl-Teller potentials [15] is presented. It is a "continuous $\ell$ version" of the potentials corresponding to the exceptional ($X_\ell$) Jacobi polynomials [4, 6, 5, 7, 8, 9, 10]. The method of deformation, intertwining relations, etc are almost parallel with those in the theory of the exceptional orthogonal polynomials, that is, for integer $\ell$. In the well known limit leading from the Jacobi polynomials (the hypergeometric function) to the Laguerre polynomials (the confluent hypergeometric function), the family of shape invariantly deformed radial oscillator potentials with continuous $\ell$ is obtained. The latter is known as an example of 'conditionally exactly solvable potentials' [17]. It should be stressed that the "continuous $\ell$ version" of the exceptional orthogonal polynomials exists only for the first type, the J1 $X_\ell$ Jacobi and L1 $X_\ell$ Laguerre polynomials.

The Hamiltonian of the DPT (radial oscillator) potential is known to have infinitely many non-singular factorisations, up to an additive constant, related with the exceptional orthogonal ($X_\ell$) polynomials, $\ell = 1, 2, \ldots$, [8]. Now we have demonstrated that the same Hamiltonian, up to an additive constant, allow non-singular factorisations (4.32) parametrised by a continuous real number $\ell > 0$.

In discrete quantum mechanics [21, 22, 23], a similar deformation based on a deforming polynomial $\xi_\ell$ with integer $\ell = 1, 2, \ldots$, was studied [9, 10], in which the exceptional continuous Hahn, Wilson and Askey-Wilson polynomials were obtained. The deforming polynomial $\xi_\ell(\eta(x); \boldsymbol{\lambda})$ satisfies three formulas (2.67)–(2.69) in [10], which would correspond to (3.9)–(3.11) in the present paper. One naturally wonders if a similar "continuous $\ell$ version" could be constructed or not. The answer is negative. Let us explain by taking the simplest example of the continuous Hahn case. The theory has two parameters, $a_1$ and $a_2$, with $a_1 > 0$ and $\mathrm{Re}(a_2) > 0$. The continuous Hahn polynomial is defined by the *terminating* hypergeometric function

$$p_n(x; a_1, a_2, a_1, a_2^*) \stackrel{\text{def}}{=} i^n \frac{(2a_1)_n (a_1 + a_2^*)_n}{n!} \, {}_3F_2\Big( \begin{matrix} -n, \; n + 2a_1 + a_2 + a_2^* - 1, \; a_1 + ix \\ 2a_1, \; a_1 + a_2^* \end{matrix} \Big| \, 1 \Big). \quad (5.1)$$

If one replaces $n$ by a real positive number $\ell$, the hypergeometric series ${}_3F_2$ becomes *nonterminating* and divergent, for $\mathrm{Re}(a_2) > 1$. See Theorem 2.1.2 in [25]. Even for a very limited parameter range $0 < \mathrm{Re}(a_2) < \frac{1}{2}$, the hypergeometric series ${}_3F_2$ will be divergent after one step of shape-invariant transformation $a_2 \to a_2 + \frac{1}{2}$. Note that the "continuous $\ell$ version" implies $a_2 \to a_2 + \frac{1}{2}(\ell - 1)$, which does not improve the situation. Thus we conclude that the "continuous $\ell$ version" of the $X_\ell$ continuous Hahn polynomials does not exist. In contrast, the (basic) hypergeometric series ${}_4F_3$ (${}_4\phi_3$) appearing in the definition of the Wilson (Askey-Wilson) polynomial will not diverge for generic $\ell$. See Theorem 2.1.2 of [25] for the ${}_4F_3$ case. The convergence of ${}_4\phi_3$ in the Askey-Wilson case is due to the $q^k$ factor. However, no difference equation governing the nonterminating ${}_4F_3$ (${}_4\phi_3$) series corresponding to (2.67) in [10] is known. Thus the "continuous $\ell$ version" of the deformation is not possible.

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