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# AN INTEGRAL EXPRESSION OF THE FIRST NONTRIVIAL ONE-COCYCLE OF THE SPACE OF LONG KNOTS IN $\mathbb{R}^3$

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**Our main object of study is a certain degree-one cohomology class of the space $\mathcal{K}_3$ of long knots in $\mathbb{R}^3$. We describe this class in terms of graphs and configuration space integrals, showing the vanishing of some anomalous obstructions. To show that this class is not zero, we integrate it over a cycle studied by Gramain. As a corollary, we establish a relation between this class and ($\mathbb{R}$-valued) Casson's knot invariant. These are $\mathbb{R}$-versions of the results which were previously proved by Teiblyum, Turchin and Vassiliev over $\mathbb{Z}/2$ in a different way from ours.**

## 1. Introduction

A *long knot* in $\mathbb{R}^n$ is an embedding $f : \mathbb{R}^1 \hookrightarrow \mathbb{R}^n$ that agrees with the standard inclusion $\iota(t) = (t, 0, \dots, 0)$ outside $[-1, 1]$. We denote by $\mathcal{K}_n$ the space of long knots in $\mathbb{R}^n$ equipped with $C^\infty$-topology.

In [Cattaneo et al. 2002] a cochain map $I : \mathcal{D}^* \to \Omega^*_{DR}(\mathcal{K}_n)$ from a certain *graph complex* $\mathcal{D}^*$ was constructed for $n > 3$. The cocycles of $\mathcal{K}_n$ corresponding to *trivalent graph cocycles* via $I$ generalize an integral expression of finite type invariants for (long) knots in $\mathbb{R}^3$ [Altschuler and Freidel 1997; Bott and Taubes 1994; Kohno 1994; Volić 2007]. In [Sakai 2008] the author found a *nontrivalent* graph cocycle $\Gamma \in \mathcal{D}^*$ and proved that, when $n > 3$ is odd, it gives a nonzero cohomology class $[I(\Gamma)] \in H^{3n-8}_{DR}(\mathcal{K}_n)$. On the other hand, when $n = 3$, some obstructions to $I$ being a cochain map (called *anomalous obstructions*; see for example [Volić 2007, Section 4.6]) may survive, so even the closedness of $I(\Gamma)$ was not clear. However, the obstructions for trivalent graph cocycles $X$ (of "even orders") in fact vanish [Altschuler and Freidel 1997], hence the map $I$ still yields closed zero-forms $I(X)$ of $\mathcal{K}_3$ (they are finite type invariants). This raises our hope

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that all obstructions for any graphs may vanish and hence the map $I$ could be a cochain map even when $n = 3$.

In this paper we will show (in Theorem 2.4) that the obstructions for the non-trivalent graph cocycle $\Gamma$ mentioned above also vanish, hence the map $I$ yields the first example of a closed one-form $I(\Gamma)$ of $\mathcal{K}_3$. To show that $[I(\Gamma)] \in H^1_{DR}(\mathcal{K}_3)$ is not zero, we will study in part how $I(\Gamma)$ fits into a description of the homotopy type of $\mathcal{K}_3$ given in [Budney 2010; 2007; Budney and Cohen 2009]. It is known that on each component $\mathcal{K}_3(f)$ that contains $f \in \mathcal{K}_3$, there exists a one-cycle $G_f$ called the *Gramain cycle* [Gramain 1977; Budney 2010; Turchin 2006; Vassiliev 2001]. The Kronecker pairing gives an isotopy invariant $V : f \mapsto \langle I(\Gamma), G_f \rangle$. We show in Theorem 3.1 that $V$ coincides with *Casson's knot invariant* $v_2$, which is characterized as the coefficient of $z^2$ in the Alexander–Conway polynomial. This result will be generalized in Theorem 3.6 for one-cycles obtained by using an action of *little two-cubes operad* on the space $\tilde{\mathcal{K}}_3$ of *framed* long knots [Budney 2007].

Closely related results have appeared in [Turchin 2006; Vassiliev 2001], where the $\mathbb{Z}/2$-reduction of a cocycle $v^1_3$ of $\mathcal{K}_n$ ($n \geq 3$), appearing in the $E_1$-term of Vassiliev's spectral sequence [Vassiliev 1992], was studied. A natural quasi-isomorphism $\mathcal{D}^* \to E_0 \otimes \mathbb{R}$ maps our cocycle $\Gamma$ to $v^1_3$. In this sense, our results can be seen as "lifts" of those in [Turchin 2006; Vassiliev 2001] to $\mathbb{R}$.

The invariant $v_2$ can also be interpreted as the linking number of colinearity manifolds [Budney et al. 2005]. Notice that in each formulation (including the one in this paper) the value of $v_2$ is computed by counting some colinearity pairs on the knot.

## 2. Construction of a close differential form

***Configuration space integral.*** We review briefly how we can construct (closed) forms of $\mathcal{K}_n$ from graphs. For full details see [Cattaneo et al. 2002; Volić 2007].

Let $X$ be a *graph* in the sense of those references (see Figure 1 for examples). Let $v_{\mathrm{i}}$ and $v_{\mathrm{f}}$ be the numbers of the *interval vertices* (or *i-vertices* for short; those on the specified oriented line) and the *free* vertices (or *f-vertices*; those which are not interval vertices) of $X$, respectively. With $X$ we associate a configuration space

$$C_X := \left\{ \begin{array}{l} (f; x_1, \dots, x_{v_{\mathrm{i}}}; x_{v_{\mathrm{i}}+1}, \dots, x_{v_{\mathrm{i}}+v_{\mathrm{f}}}) \\ \in \mathcal{K}_n \times \mathrm{Conf}\,(\mathbb{R}^1, v_{\mathrm{i}}) \times \mathrm{Conf}\,(\mathbb{R}^n, v_{\mathrm{f}}) \end{array} \;\middle|\; \begin{array}{l} f(x_i) \neq x_j \text{ for any} \\ 1 \leq i \leq v_{\mathrm{i}} < j \leq v_{\mathrm{i}} + v_{\mathrm{f}} \end{array} \right\},$$

where $\mathrm{Conf}\,(M, k) := M^{\times k} \setminus \bigcup_{1 \leq i < j \leq k} \{x_i = x_j\}$ for a space $M$.

Let $e$ be the number of the edges of $X$. Define $\omega_X \in \Omega^{(n-1)e}_{DR}(C_X)$ as the wedge of closed $(n-1)$-forms $\varphi^*_\alpha \mathrm{vol}_{S^{n-1}}$, where $\varphi_\alpha : C_X \to S^{n-1}$ is the *Gauss map*, which assigns a unit vector determined by two points in $\mathbb{R}^n$ corresponding to the vertices adjacent to an edge $\alpha$ of $X$ (for an i-vertex corresponding to $x_i \in \mathbb{R}^1$, we consider the point $f(x_i) \in \mathbb{R}^n$). Here we assume that $\mathrm{vol}_{S^{n-1}}$ is "(anti)symmetric", namely $i^*\mathrm{vol}_{S^{n-1}} = (-1)^n \mathrm{vol}_{S^{n-1}}$ for the antipodal map $i : S^{n-1} \to S^{n-1}$. Then $I(X) \in \Omega_{DR}^{(n-1)e-v_\mathrm{i}-nv_\mathrm{f}}(\mathcal{K}_n)$ is defined by

$$I(X) := (\pi_X)_* \omega_X,$$

the integration along the fiber of the natural fibration $\pi_X : C_X \to \mathcal{K}_n$. This fiber is a subspace of $\mathrm{Conf}\,(\mathbb{R}^1, v_\mathrm{i}) \times \mathrm{Conf}\,(\mathbb{R}^n, v_\mathrm{f})$. Such integrals converge, since the fiber can be compactified in such a way that the forms $\varphi_\alpha^* \mathrm{vol}_{S^{n-1}}$ are still well-defined on the compactification [Bott and Taubes 1994, Proposition 1.1]. We extend $I$ linearly onto $\mathcal{D}^*$, a cochain complex spanned by graphs. The differential $\delta$ of $\mathcal{D}^*$ is defined as a signed sum of graphs obtained by "contracting" the edges one at a time.

One of the results of [Cattaneo et al. 2002] states that $I : \mathcal{D}^* \to \Omega_{DR}^*(\mathcal{K}_n)$ is a cochain map if $n > 3$. The proof is outlined as follows. By the generalized Stokes theorem, $dI(X) = \pm(\pi_X^\partial)_* \omega_X$, where $\pi_X^\partial$ is the restriction of $\pi_X$ to the codimension one strata of the boundary of the (compactified) fiber of $\pi_X$. Each codimension one stratum corresponds to a collision of subconfigurations in $C_X$, or equivalently to $A \subset V(X) \cup \{\infty\}$ (here $V(X)$ is the set of vertices of $X$) with a consecutiveness property: if two i-vertices $p, q$ are in $A$, then all the other i-vertices between $p$ and $q$ are in $A$. Here "$\infty \in A$" means that the points $x_l$ ($l \in A$) escape to infinity. When $\infty \notin A$, the interior $\mathrm{Int}\,\Sigma_A$ of the corresponding stratum $\Sigma_A$ to $A$ is described by the pullback square

(2-1)
$$\begin{array}{ccccc}
 & & \mathrm{Int}\,\Sigma_A & \longrightarrow & \hat{B}_A \\
 & \swarrow^{\pi_X^{\partial_A}} & \downarrow & & \downarrow \rho_A \\
\mathcal{K}_n & \xleftarrow[\pi_{X/X_A}]{} & C_{X/X_A} & \xrightarrow[D_A]{} & B_A
\end{array}$$

Here

- $X_A$ is the maximal subgraph of $X$ with $V(X_A) = A$, and $X/X_A$ is a graph obtained by collapsing the subgraph $X_A$ to a single vertex $v_A$;
- $B_A = S^{n-1}$ if $A$ contains at least one i-vertex, and $B_A = \{*\}$ otherwise;
- if $A$ consists of i-vertices $i_1, \dots, i_s$ ($s > 0$) and f-vertices $i_{s+1}, \dots, i_{s+t}$, then

$$\hat{B}_A := \left\{ \begin{array}{l} (v; (x_{i_1}, \dots, x_{i_s}; x_{i_{s+1}}, \dots, x_{i_{s+t}})) \\ \in S^{n-1} \times \mathrm{Conf}\,(\mathbb{R}^1, s) \times \mathrm{Conf}\,(\mathbb{R}^n, t) \end{array} \middle| \begin{array}{l} x_{i_p} v \neq x_{i_q} \text{ for any} \\ 1 \le p \le s < q \le s+t \end{array} \right\} \Big/ \sim,$$

where $\sim$ is defined by

$$\big(v; (x_{i_1}, \dots, x_{i_s}; x_{i_{s+1}}, \dots, x_{i_{s+t}})\big) \sim \big(v; (a(x_{i_1} + r), \dots, a(x_{i_s} + r); a(x_{i_{s+1}} + rv), \dots, a(x_{i_{s+t}} + rv))\big),$$

for any $a \in \mathbb{R}_{>0}$ and $r \in \mathbb{R}$ (if $A$ consists only of $t$ f-vertices, then

$$\hat{B}_A := \mathrm{Conf}\,(\mathbb{R}^n, t)/(\mathbb{R}^1_{>0} \rtimes \mathbb{R}^n),$$

where $\mathbb{R}^1_{>0} \rtimes \mathbb{R}^n$ acts on $\mathrm{Conf}\,(\mathbb{R}^n, t)$ by scaling and translation);

- $\rho_A$ is the natural projection;
- when $A$ contains at least one i-vertex, $D_A : C_{X/X_A} \to S^{n-1}$ maps $(f; (x_i))$ to $f'(x_{v_A})/|f'(x_{v_A})|$.

We omit the case $\infty \in A$; see [Cattaneo et al. 2002, Appendix].

By properties of fiber integrations and pullbacks, the integration of $\omega_X$ along $\mathrm{Int}\,\Sigma_A$ can be written as $(\pi_{X/X_A})_*(\omega_{X/X_A} \wedge D_A^*(\rho_A)_*\hat{\omega}_{X_A})$, where $\hat{\omega}_{X_A} \in \Omega^*_{DR}(\hat{B}_A)$ is defined similarly to $\omega_X \in \Omega^*_{DR}(C_X)$.

The stratum $\Sigma_A$ is called *principal* if $|A| = 2$, *hidden* if $|A| \geq 3$, and *infinity* if $\infty \in A$. Since two-point collisions correspond to contractions of edges, we have $dI(X) = I(\delta X)$ modulo the integrations along hidden and infinity faces. When $n > 3$, the hidden/infinity contributions turn out to be zero; in fact $(\rho_A)_*\hat{\omega}_{X_A} = 0$ if $n > 3$ and if $A$ is not principal; see [Cattaneo et al. 2002, Appendix] or the next example. This proves that the map $I$ is a cochain map if $n > 3$.

**Example 2.1.** Here we show one example of vanishing of an integration along a hidden face $\Sigma_A$. Let $X$ be the seventh graph in Figure 1 and $A := \{1, 4, 5\}$. Then in (2-1), $B_A = S^{n-1}$ since $A$ contains an i-vertex 1, and

$$\hat{B}_A = \{(v; x_1; x_4, x_5) \in S^{n-1} \times \mathbb{R}^1 \times \mathrm{Conf}\,(\mathbb{R}^n, 2) \mid x_1 v \neq x_4, x_5\}/\sim,$$

where $(v; x_1; x_4, x_5) \sim (v; a(x_1 + r); a(x_4 + rv), a(x_5 + rv))$ for any $a > 0$ and $r \in \mathbb{R}^1$. The subgraph $X_A$ consists of three vertices $1, 4, 5$ and three edges 14, 15 and 45. The open face $\mathrm{Int}\,\Sigma_A$, where three points $f(x_1)$, $x_4$ and $x_5$ collide with each other, is a hidden face and is described by the square (2-1). Then the integration of $\omega_X$ along $\mathrm{Int}\,\Sigma_A$ is $(\pi_{X/X_A})_*(\omega_{X/X_A} \wedge D_A^*(\rho_A)_*\hat{\omega}_{X_A})$, where

$$\hat{\omega}_{X_A} = \varphi_{14}^*\mathrm{vol}_{S^{n-1}} \wedge \varphi_{15}^*\mathrm{vol}_{S^{n-1}} \wedge \varphi_{45}^*\mathrm{vol}_{S^{n-1}} \in \Omega^{3(n-1)}_{DR}(\hat{B}_A),$$

$$\varphi_{1j} := \frac{x_j - x_1 v}{|x_j - x_1 v|} \quad (j = 4, 5), \qquad \varphi_{45} := \frac{x_5 - x_4}{|x_5 - x_4|}.$$

In this case we can prove that $(\rho_A)_*\hat{\omega}_{X_A} = 0$, hence the integration of $\omega_X$ along $\mathrm{Int}\,\Sigma_A$ vanishes. Indeed a fiberwise involution $\chi : \hat{B}_A \to \hat{B}_A$ defined by

$$\chi(v; x_1; x_4, x_5) := (v; x_1; 2x_1 v - x_4, 2x_1 v - x_5)$$

preserves the orientation of the fiber but $\chi^*\hat{\omega}_{X_A} = -\hat{\omega}_{X_A}$ (here we use that $\mathrm{vol}_{S^{n-1}}$ is antisymmetric), hence we have $(\rho_A)_*\hat{\omega}_{X_A} = -(\rho_A)_*\hat{\omega}_{X_A}$.

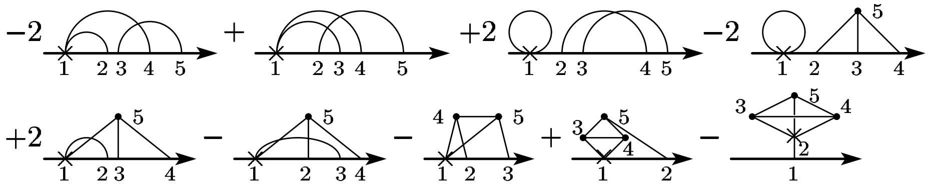


**Figure 1.** A graph cocycle $\Gamma$.

***Nontrivalent cocycle.*** It is shown in [Cattaneo et al. 2002] that, when $n > 3$, the induced map $I$ on cohomology restricted to the space of trivalent graph cocycles is injective. In [Sakai 2008], the author gave the first example of a nontrivalent graph cocycle $\Gamma$ (Figure 1) which also gives a nonzero class $[I(\Gamma)] \in H^{3n-8}_{DR}(\mathcal{K}_n)$ when $n > 3$ is odd.

In Figure 1, nontrivalent vertices and trivalent f-vertices are marked by $\times$ and $\bullet$, respectively, and other crossings are not vertices. Here we say an i-vertex $v$ is trivalent if there is exactly one edge emanating from $v$ other than the specified oriented line. Each edge $ij$ $(i < j)$ is oriented so that $i$ is the initial vertex.

**Remark 2.2.** An analogous nontrivalent graph cocycle for the space of embeddings $S^1 \hookrightarrow \mathbb{R}^n$ for even $n \geq 4$ can be found in [Longoni 2004].

If $n = 3$, integrations along some hidden faces (called *anomalous contributions*) might survive, so the map $I$ might fail to be a cochain map. However, nonzero anomalous contributions arise from limited hidden faces.

**Theorem 2.3.** *Let $X$ be a graph and $A \subset V(X) \cup \{\infty\}$ be such that $\Sigma_A$ is not principal. When $n = 3$, the integration of $\omega_X$ along $\Sigma_A$ can be nonzero only if the subgraph $X_A$ is trivalent.*

Our main theorem is proved by using Theorem 2.3.

**Theorem 2.4.** *$I(\Gamma) \in \Omega^1_{DR}(\mathcal{K}_3)$ is a closed form.*

*Proof.* We call the nine graphs in Figure 1 $\Gamma_1, \dots, \Gamma_9$, respectively. The graphs $\Gamma_i$, $i \neq 3, 4, 9$, do not contain trivalent subgraphs $X_A$ satisfying the *consecutive property*; see the paragraph just before (2-1). So $dI(\Gamma_i) = I(d\Gamma_i)$ for $i \neq 3, 4, 9$ by Theorem 2.3.

Possibly the integration of $\omega_{\Gamma_i}$ $(i = 3, 4, 9)$ along $\Sigma_A$ $(A := \{2, \dots, 5\})$ might survive, since the corresponding subgraph $X_A$ is trivalent. However, we can prove $(\rho_A)_* \hat{\omega}_{X_A} = 0$ (and hence $dI(\Gamma_i) = I(d\Gamma_i)$) as follows: $(\rho_A)_* \hat{\omega}_{X_A} = 0$ for $\Gamma_3$, because there is a fiberwise free action of $\mathbb{R}_{>0}$ on $\hat{B}_A$ given by translations of $x_2$ and $x_4$ [Volić 2007, Proposition 4.1] which preserves $\hat{\omega}_{X_A}$. Thus $(\rho_A)_* \hat{\omega}_{X_A} = 0$ by dimensional reason. The proof for $\Gamma_4$ has appeared in [Bott and Taubes 1994, page 5271]; $\hat{\omega}_{X_A}=0$ on $\hat{B}_A$ since the image of the Gauss map $\varphi : B_A \to (S^2)^3$ corresponding to three edges of $X_A$ is of positive codimension. As for $\Gamma_9$, $(\rho_A)_*\hat{\omega}_{X_A}=0$ follows from $\deg(\rho_A)_*\hat{\omega}_{X_A} = 4$ which exceeds $\dim B_A$ (in fact $B_A = \{*\}$ in this case). □

*Proof of Theorem 2.3.* Let $A$ be a subset of $V(X)$ with $|A| \geq 3$ or $\infty \in A$, and $X_A$ is nontrivalent. We must show the vanishing of the integrations along the nonprincipal face $\Sigma_A$ of the fiber of $C_X \to \mathcal{K}_3$. To do this it is enough to show $(\rho_A)_*\hat{\omega}_{X_A}=0$. By dimensional arguments [Cattaneo et al. 2002, (A.2)] the contributions of infinite faces vanish. So below we consider the hidden faces $\Sigma_A$ with $|A| \geq 3$.

If $X_A$ has a vertex of valence $\leq 2$, then $(\rho_A)_*\hat{\omega}_{X_A}=0$ is proved by dimensional arguments or existence of a fiberwise symmetry of $B_A$ which reverses the orientation of the fiber of $\rho_A : \hat{B}_A \to B_A$ but preserves the integrand $\hat{\omega}_{X_A}$ (like $\chi$ from Example 2.1, see also [Cattaneo et al. 2002, Lemmas A.7–A.9]).

Next, consider the case that there is a vertex of $X_A$ of valence $\geq 4$. Let $e$, $s$ and $t$ be the numbers of the edges, the i-vertices and the f-vertices of $X_A$, respectively. Then $\deg \hat{\omega}_{X_A} = 2e$ and the dimension of the fiber of $\rho_A$ is $s + 3t - k$, where $k = 2$ or 4 according to whether $s > 0$ or $s = 0$ [Cattaneo et al. 2002, (A.1)]. Thus $(\rho_A)_*\hat{\omega}_{X_A} \in \Omega^*_{DR}(B_A)$ is of degree $2e - s - 3t + k$. It is not difficult to see $2e - s - 3t > 0$ because at least one vertex of $X_A$ is of valence $\geq 4$. Hence $\deg(\rho_A)_*\hat{\omega}_{X_A}$ exceeds $\dim B_A$ (= 0 or 2) and hence $(\rho_A)_*\hat{\omega}_{X_A}=0$.

Thus only the integrations along $\Sigma_A$ with $X_A$ trivalent can survive. □

**Remark 2.5.** Every finite type invariant $v$ for long knots in $\mathbb{R}^3$ can be written as a sum of $I(\Gamma_v)$ ($\Gamma_v$ is a trivalent graph cocycle) and some “correction terms” which kill the contributions of hidden faces corresponding to trivalent subgraphs [Altschuler and Freidel 1997; Bott and Taubes 1994; Kohno 1994; Volić 2007]. So by Theorem 2.3 the problem whether $I : \mathcal{D}^* \to \Omega^*_{DR}(\mathcal{K}_3)$ is a cochain map or not is equivalent to the problem whether one can eliminate all the correction terms from integral expressions of finite type invariants.

## 3. Evaluation on some cycles

Here we will show that $[I(\Gamma)] \in H^1_{DR}(\mathcal{K}_3)$ restricted to some components of $\mathcal{K}_3$ is not zero.

We introduce two assumptions to simplify computations.

**Assumption 1.** The support of (antisymmetric) $\mathrm{vol}_{S^2}$ is contained in a sufficiently small neighborhood of the poles $(0, 0, \pm 1)$ as in [Sakai 2008]. So only the configurations with the images of the Gauss maps lying in a neighborhood of $(0, 0, \pm 1)$ can nontrivially contribute to various integrals below. Presumably $[I(\Gamma)] \in H^1_{DR}(\mathcal{K}_3)$ may be independent of choices of $\mathrm{vol}_{S^2}$ [Cattaneo et al. 2002, Proposition 4.5].

**Assumption 2.** Every long knot in $\mathbb{R}^3$ is contained in $xy$-plane except for over-arc of each crossing, and each over-arc is in $\{0 \le z \le h\}$ for a sufficiently small $h > 0$ so that the projection onto $xy$-plane is a regular diagram of the long knot.

***The Gramain cycle.*** For any $f \in \mathcal{K}_3$, we denote by $\mathcal{K}_3(f)$ the component of $\mathcal{K}_3$ which contains $f$. Regarding $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and fixing $f$, we define the map $G_f : S^1 \to \mathcal{K}_3(f)$, called the *Gramain cycle*, by $G_f(s)(t) := R(s)f(t)$, where $R(s) \in \mathrm{SO}(3)$ is the rotation by the angle $s$ fixing the "long axis" (the $x$-axis). $G_f$ generates an infinite cyclic subgroup of $\pi_1(\mathcal{K}_3(f))$ if $f$ is nontrivial [Gramain 1977]. The homology class $[G_f] \in H_1(\mathcal{K}_3(f))$ is independent of the choice of $f$ in the connected component; if $f_t \in \mathcal{K}_3$ $(0 \le t \le 1)$ is an isotopy connecting $f_0$ and $f_1$, then $G_{f_t} : [0, 1] \times S^1 \to \mathcal{K}_3$ gives a homotopy between $G_{f_0}$ and $G_{f_1}$. Therefore the Kronecker pairing gives an isotopy invariant $V(f) := \langle I(\Gamma), G_f \rangle$ for long knots.

**Theorem 3.1.** *The invariant $V$ is equal to Casson's knot invariant $v_2$.*

**Corollary 3.2.** $[I(\Gamma)|_{\mathcal{K}_3(f)}] \in H^1_{DR}(\mathcal{K}_3(f))$ *is not zero if* $v_2(f) \neq 0$. □

We will prove two statements that characterize Casson's knot invariant: $V$ is of finite type of order two and $V(3_1) = 1$, where $3_1$ is the long trefoil knot. To do this, we will represent $G_f$ using a *Browder operation*, as in [Sakai 2008].

*Little cubes action.* Let $\tilde{\mathcal{K}}_n$ be the space of *framed* long knots in $\mathbb{R}^n$ (embeddings $\tilde{f} : \mathbb{R}^1 \times D^{n-1} \hookrightarrow \mathbb{R}^n$ that are standard outside $[-1, 1] \times D^{n-1}$). There is a homotopy equivalence $\Phi : \tilde{\mathcal{K}}_3 \simeq \mathcal{K}_3 \times \mathbb{Z}$ [Budney 2007] that maps $\tilde{f}$ to the pair $(\tilde{f}|_{\mathbb{R}^1 \times \{(0,0)\}}, \mathrm{fr}\, \tilde{f})$, where the *framing number* $\mathrm{fr}\, \tilde{f}$ is defined as the linking number of $\tilde{f}|_{\mathbb{R}^1 \times \{(0,0)\}}$ with $\tilde{f}|_{\mathbb{R}^1 \times \{(1,0)\}}$. Since $\mathrm{fr}\, \tilde{f}$ is additive under the connected sum, $\Phi$ is a homotopy equivalence of $H$-spaces. In general, $\tilde{\mathcal{K}}_n \simeq \mathcal{K}_n \times \Omega\, \mathrm{SO}(n-1)$ as $H$-spaces, where $\Omega$ stands for the based loop space functor.

In [Budney 2007] an action of the *little two-cubes operad* on the space $\tilde{\mathcal{K}}_n$ was defined. Its second stage gives a map $S^1 \times (\tilde{\mathcal{K}}_n)^2 \to \tilde{\mathcal{K}}_n$ up to homotopy, which is given as "shrinking one knot $f$ and sliding it along another knot $g$ by using the framing, and repeating the same procedure with $f$ and $g$ exchanged" [Budney 2007, Figure 2]. Fixing a generator of $H_1(S^1)$, we obtain the *Browder operation* $\lambda : H_p(\tilde{\mathcal{K}}_n) \otimes H_q(\tilde{\mathcal{K}}_n) \to H_{p+q+1}(\tilde{\mathcal{K}}_n)$, which is a graded Lie bracket satisfying the Leibniz rule with respect to the product induced by the connected sum. The author proved in [Sakai 2008] that $\langle I(\Gamma), r_*\lambda(e, v)\rangle = 1$ when $n > 3$ is odd, where $r : \tilde{\mathcal{K}}_n \to \mathcal{K}_n$ is the forgetting map, $e \in H_{n-3}(\tilde{\mathcal{K}}_n)$ comes from the space of framings, and $v \in H_{2(n-3)}(\tilde{\mathcal{K}}_n)$ is the first nonzero class of $\mathcal{K}_n$ represented by a map $(S^{n-3})^{\times 2} \to \mathcal{K}_n$ (see below).

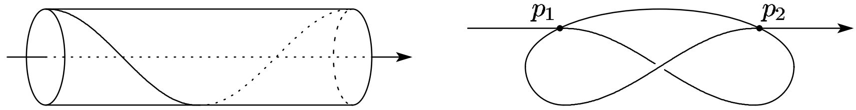



**Figure 2.** The cycles $e$ and $v = v(T)$.

*The case* $n = 3$. In [Sakai 2008] the assumption $n > 3$ was used only to deduce the closedness of $I(\Gamma)$ from the results of Cattaneo et al. [2002]. The cycles $e$ and $v$ are defined even when $n = 3$:

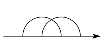
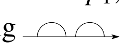
- Under the homotopy equivalence $\tilde{\mathcal{K}}_3 \simeq \mathcal{K}_3 \times \mathbb{Z}$, the zero-cycle $e$ is given by $(\iota, 1)$ where $\iota$ is the trivial long knot ($\iota(t) = (t, 0, 0)$ for any $t \in \mathbb{R}^1$).
- The zero-cycle $v = v(T)$ is given by $\sum_{\varepsilon_i=\pm 1} \varepsilon_1\varepsilon_2 T_{\varepsilon_1,\varepsilon_2}$, where $T = 3_1$ and $T_{\varepsilon_1,\varepsilon_2}$ is $T$ with its crossing $p_i$, for $i = 1, 2$ changed to be positive if $\varepsilon_i = +1$ and negative if $\varepsilon_i = -1$ (see Figure 2).

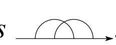
Notice that, for any $f \in \mathcal{K}_3$ and any pair $(p_1, p_2)$ of its crossings, an analogous zero-cycle $v = v(f; p_1, p_2)$ can be defined.

Regard $f \in \mathcal{K}_3$ as a zero-cycle of $\tilde{\mathcal{K}}_3$ (with $\operatorname{fr} f = 0$) and consider $r_*\lambda(e, f)$. During a knot $f$ "going through" $e$, $f$ rotates once around the $x$-axis. Thus the one-cycle $r_*\lambda(e, f)$ is homologous to the Gramain cycle $G_f$. This leads us to the fact that, for $v = v(f; p_1, p_2)$, the one-cycle $r_*\lambda(e, v)$ is homologous to the sum $\sum_{\varepsilon_i=\pm 1} \varepsilon_1\varepsilon_2 G_{f_{\varepsilon_1,\varepsilon_2}}$. This is why we can apply the method in [Sakai 2008] to compute

$$D^2V(f) := \sum_{\varepsilon_j=\pm 1} \varepsilon_1\varepsilon_2 V(f_{\varepsilon_1,\varepsilon_2}) = \sum_{\varepsilon_j=\pm 1} \varepsilon_1\varepsilon_2 \langle I(\Gamma), G_{f_{\varepsilon_1,\varepsilon_2}}\rangle = \langle I(\Gamma), r_*\lambda(e, v(f))\rangle.$$

Recall that our graph cocycle $\Gamma$ is a sum of nine graphs $\Gamma_1, \dots, \Gamma_9$ (see Figure 1). By Assumption 1, the integration $\langle I(\Gamma_i), G_f\rangle$ can be computed by "counting" the configurations with all the images of the Gauss maps corresponding to edges of $\Gamma_i$ being around the poles of $S^2$. Lemma 3.4 below was proved in such a way in [Sakai 2008] when $n > 3$. Since $[v(f)] \in H_0(\mathcal{K}_3(f))$ is independent of small $h > 0$ (see Assumption 2), we may compute $D^2V(f)$ in the limit $h \to 0$.

**Definition 3.3.** We say that a pair $(p_1, p_2)$ of crossings of $f$ *respects the diagram*  if there exist $t_1 < t_2 < t_3 < t_4$ where $f(t_1)$ and $f(t_3)$ correspond to $p_1$, while $f(t_2)$ and $f(t_4)$ correspond to $p_2$. The notion of $(p_1, p_2)$ respecting  or  is defined analogously.

**Lemma 3.4** [Sakai 2008]. *Suppose that* $(p_1, p_2)$ *respects* . *Then, in the limit* $h \to 0$, $P_i(f) := \sum_{\varepsilon_j=\pm 1} \varepsilon_1\varepsilon_2 \langle I(\Gamma_i), G_{f_{\varepsilon_1,\varepsilon_2}}\rangle$ *converges to zero for* $i \neq 2$, *and* $P_2(f)$ *converges to* 1. *Thus* $D^2V(f) = 1$.

*Outline of proof.* Let $\hat{C}_{\Gamma_i} \to S^1$ be the pullback of $C_{\Gamma_i} \to \mathcal{K}_3$ via $G_f$, and let $\hat{G}_f : \hat{C}_{\Gamma_i} \to C_{\Gamma_i}$ be the lift of $G_f$. By the properties of pullbacks and fiber integrations,

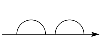
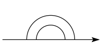
$$P_i(f) = \sum_{\varepsilon_i=\pm 1} \varepsilon_1 \varepsilon_2 \int_{\hat{C}_{\Gamma_i}} \hat{G}^*_{f_{\varepsilon_1,\varepsilon_2}} \omega_{\Gamma_i}. \tag{3-1}$$

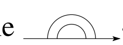
Let $t_1 < \cdots < t_4$ be such that $f(t_1)$ and $f(t_3)$ correspond to $p_1$, while $f(t_2)$ and $f(t_4)$ correspond to $p_2$. Define the subspace $C'_{\Gamma_i} \subset \hat{C}_{\Gamma_i}$ as consisting of $(G_f(s); (x_j))$ $(s \in S^1)$ such that, for each $j = 1, 2$, there is a pair $(l, m)$ of i-vertices of $\Gamma_i$ such that $x_l$ is on the over-arc of $p_j$, $x_m$ is on the under-arc of $p_j$, and there is a sequence of edges in $\Gamma_i$ from $l$ to $m$.

*First observation*: The integration over $\hat{C}_{\Gamma_i} \setminus C'_{\Gamma_i}$ does not essentially contribute to $P_i(f)$ in the limit $h \to 0$. This is because, over $\hat{C}_{\Gamma_i} \setminus C'_{\Gamma_i}$, the integrals in (3-1) are well defined and continuous even when $h = 0$ ($p_j$ becomes a double point), so two terms in $P_i(f)$ corresponding to $\varepsilon_j = \pm 1$ cancel each other. This implies $\lim_{h\to 0} P_i(f) = 0$ for $i = 7, 8, 9$, since $C'_{\Gamma_i} = \varnothing$ if $\sharp\{\text{i-vertices}\} \leq 3$.

*Second observation*: Consider the configurations $(x_i) \in C'_{\Gamma_i}$ such that, for any pair $(l, m)$ of i-vertices of $\Gamma_i$ with $x_l$ on the over-arc of $p_j$ and $x_m$ on the under-arc of $p_j$, all the points $x_k$ ($k$ is in a sequence in $\Gamma_i$ from $l$ to $m$) are not near $p_j$. Such configurations also do not essentially contribute to $P_i(f)$ in the limit $h \to 0$, by the same reason as above. This implies $\lim_{h\to 0} P_i(f) = 0$ for $i = 4, 5, 6$; the configurations $(x_l) \in C'_{\Gamma_i}$ $(4 \leq i \leq 6)$ must be such that the point $x_l \in \mathbb{R}^1$ $(1 \leq l \leq 4)$ is near $t_l$. By the second observation, the "free point" $x_5$ must be near $p_1$ or $p_2$. But then $\omega_{\Gamma_i} = 0$, since at least one Gauss map $\varphi_{l5}$ has its image outside the support of $\mathrm{vol}_{S^2}$ (see Assumption 1). Thus $\lim_{h\to 0} P_i(f) = 0$.

Finally consider the $P_i(f)$, for $i = 1, 2, 3$. For $i = 1$ we have $\omega_{\Gamma_1} = 0$ over $C'_{\Gamma_1}$, since the Gauss map corresponding to the edge 12 has its image outside of the support of $\mathrm{vol}_{S^2}$. The same reasoning, using the loop edge 11, shows that $\omega_{\Gamma_3} = 0$ over $C'_{\Gamma_3}$. Only $P_2(f)$ survives, since the configurations with $x_1$ near $t_1$, $x_2$ near $t_2$, $x_3$ and $x_4$ near $t_3$, and $x_5$ near $t_4$, contribute nontrivially to the integral [Sakai 2008, Lemma 4.6]. □

**Lemma 3.5.** *If $(p_1, p_2)$ respects*  *or* , *then $D^2V(f) = 0$.*

*Proof.* For $i = 4, \ldots, 9$, we see in the same way as in Lemma 3.4 that $P_i(f)$ approaches 0 as $h \to 0$. That $\lim_{h\to 0} P_i(f)$ for $i = 2, 3$ and the -case for $i = 1$ is proved by the first observation in the proof of Lemma 3.4.

In the -case for $P_1(f)$ over $C'_{\Gamma_1}$ only the configurations with $x_j$ near $t_j$, with $j = 1, 2, 3$, and $x_5$ near $t_4$ may essentially contribute to $P_1(f)$; in this case the edges 12 and 35 join the over/under arcs of $p_1$ and $p_2$ respectively. However, the Gauss map $\varphi_{14}$ cannot have its image in the support of $\mathrm{vol}_{S^2}$, so $\omega_{\Gamma_1}$ vanishes. □

*Proof of Theorem 3.1.* For three crossings $(p_1, p_2, p_3)$ of $f \in \mathcal{K}_3$, consider the third difference

$$D^3V(f) := \sum_{\varepsilon_j=\pm 1} \varepsilon_1\varepsilon_2\varepsilon_3 V(f_{\varepsilon_1,\varepsilon_2,\varepsilon_3}) = D^2V(g_{+1}) - D^2V(g_{-1}),$$

where $g_{\pm 1} := f_{+1,+1,\pm 1}$ and $D^2V(g_{\pm 1})$ are taken with respect to $(p_1, p_2)$. Since the pair $(p_1, p_2)$ of $g_{+1}$ respects the same diagram as $(p_1, p_2)$ of $g_{-1}$, we have $D^2V(g_{+1}) = D^2V(g_{-1})$ by the above Lemmas 3.4, 3.5. Thus $D^3V = 0$ and hence $V$ is finite type of order two. Moreover $V(\iota) = 0$ for the trivial long knot $\iota$ since $\mathcal{K}_3(\iota)$ is contractible [Hatcher 1983]; therefore $G_\iota \sim 0$, and $V(3_1) = 1$ by Lemma 3.4 and $V(\iota) = 0$. These properties uniquely characterize Casson's knot invariant $v_2$. □

***The Browder operations.*** We denote a framed long knot corresponding to $(f, k)$ under the equivalence $\tilde{\mathcal{K}}_3 \simeq \mathcal{K}_3 \times \mathbb{Z}$ by $f^k \in \tilde{\mathcal{K}}_3$ (unique up to homotopy). As mentioned above, the Gramain cycle can be written as $[G_f] = [r_*\lambda(f^k, \iota^1)]$ ($k$ may be arbitrary). Below we will evaluate $I(\Gamma)$ on more general cycles $r_*\lambda(f^k, g^l)$ of $\mathcal{K}_3$ for any nontrivial $f, g \in \mathcal{K}_3$ and $k, l \in \mathbb{Z}$. This generalizes Theorem 3.1.

**Theorem 3.6.** *We have* $\langle I(\Gamma),\ r_*\lambda(f^k, g^l)\rangle = l v_2(f) + k v_2(g)$ *for any* $f, g \in \mathcal{K}_3$ *and* $k, l \in \mathbb{Z}$.

**Corollary 3.7.** *If at least one of* $v_2(f)$ *and* $v_2(g)$ *is not zero, then*

$$[I(\Gamma)|_{\mathcal{K}_3(f\sharp g)}] \in H^1_{DR}(\mathcal{K}_3(f\sharp g)) \neq 0,$$

*where* $\sharp$ *stands for the connected sum.*

*Proof.* This is because $r_*\lambda(f^k, g^l)$ is a one-cycle of $\mathcal{K}_3(f\sharp g)$ for any $k, l \in \mathbb{Z}$. Since $v_2(f)$ or $v_2(g)$ is not zero, there exist some $k, l$ such that $l v_2(f) + k v_2(g) \neq 0$, so $\langle I(\Gamma),\ r_*\lambda(f^k, g^l)\rangle \neq 0$ by Theorem 3.6. □

**Remark 3.8.** If $v_2(f) = -v_2(g)$, then $v_2(f\sharp g) = 0$ since it is known that $v_2$ is additive under $\sharp$. Hence we cannot deduce $[I(\Gamma)|_{\mathcal{K}_3(f\sharp g)}] \neq 0$ from Corollary 3.2. Moreover if $v_2(f) = -v_2(g) \neq 0$, then Corollary 3.7 implies $[I(\Gamma)|_{\mathcal{K}_3(f\sharp g)}] \neq 0$.

To prove Theorem 3.6, first we remark that $f^m \sim f^0\sharp\iota^m$. Since $\lambda$ satisfies the Leibniz rule, $\lambda(f^k, g^l)$ is homologous to

$$\lambda(f^0, g^0)\sharp\iota^{k+l} + \lambda(f^0, \iota^l)\sharp g^k + \lambda(\iota^k, g^0)\sharp f^l + \lambda(\iota^k, \iota^l)\sharp f^0\sharp g^0.$$

Since by definition $r_*\lambda(f^k, \iota^m) \sim mG_f$ $(k, m \in \mathbb{Z})$ and $G_\iota \sim 0$,

$$r_*\lambda(f^k, g^l) \sim r_*\lambda(f^0, g^0) + lG_f\sharp g + kf\sharp G_g. \tag{3-2}$$

Notice that $\sharp$ makes $\mathcal{K}_3$ an $H$-space and induces a coproduct $\Delta$ on $H^*_{DR}(\mathcal{K}_3)$.

**Lemma 3.9.** $\Delta([I(\Gamma)]) = 1 \otimes [I(\Gamma)] + [I(\Gamma)] \otimes 1 \in H^*_{DR}(\mathcal{K}_3)^{\otimes 2}$.

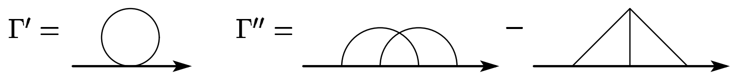



**Figure 3.** Graph cocycles $\Gamma'$ and $\Gamma''$.

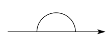
*Proof.* $\mathfrak{D}$ also admits $\Delta$ defined as a "separation" of the graphs by removing a point from the specified oriented line [Cattaneo et al. 2005, Section 3.2]. Theorem 6.3 of [Cattaneo et al. 2005] shows, without using $n > 3$, that $(I \otimes I)\Delta(X) = \Delta I(X)$ if $X$ satisfies $dI(X) = I(\delta X)$.

As for our graphs in Figure 1, $\Delta\Gamma_i = 1 \otimes \Gamma_i + \Gamma_i \otimes 1$ $(i \neq 3, 4)$ and

$$\Delta(\Gamma_3 - \Gamma_4) = 1 \otimes (\Gamma_3 - \Gamma_4) + (\Gamma_3 - \Gamma_4) \otimes 1 + \Gamma' \otimes \Gamma'' + \Gamma'' \otimes \Gamma',$$

where $\Gamma'$ and $\Gamma''$ are as shown in Figure 3. Thus

$$\Delta I(\Gamma) = 1 \otimes I(\Gamma) + I(\Gamma) \otimes 1 + I(\Gamma') \otimes I(\Gamma'') + I(\Gamma'') \otimes I(\Gamma').$$

But in fact $\Gamma' = \delta\Gamma_0$ where $\Gamma_0 =$ , and $I(\Gamma') = dI(\Gamma_0)$ since there is no hidden face in the boundary of the fiber of $\pi_{\Gamma_0}$. □

By (3-2), Lemma 3.9 and Theorem 3.1,

$$\langle I(\Gamma),\, r_*\lambda(f^k, g^l)\rangle = \langle I(\Gamma),\, r_*\lambda(f^0, g^0)\rangle + l v_2(f) + k v_2(g).$$

Thus it suffices to prove Theorem 3.6 in the case $k = l = 0$.

*Proof of Theorem 3.6.* Fix $g$ and regard $\langle I(\Gamma),\, r_*\lambda(f^0, g^0)\rangle$ as an invariant $V_g(f)$ of $f$. We choose two crossings $p_1$ and $p_2$ from the diagram of $f$ in $xy$-plane, and compute $D^2 V_g(f) := \sum_{\varepsilon_1,\varepsilon_2} \varepsilon_1\varepsilon_2 \langle I(\Gamma), r_*\lambda(f^0_{\varepsilon_1,\varepsilon_2}, g^0)\rangle$ in the limit $h \to 0$ as on page 414. If this is zero for any $(p_1, p_2)$, then the arguments similar to that in the proof of Theorem 3.1 show that $V_g$ is of order two and takes the value zero for the trefoil knot, thus identically $V_g = 0$ for any $g$. This will complete the proof.

We will compute each $P'_i := \sum_{\varepsilon=\pm 1} \langle I(\Gamma_i), r_*\lambda(f^0_{\varepsilon_1,\varepsilon_2}, g^0)\rangle$ $(1 \leq i \leq 9)$ in the limit $h \to 0$. The two observations appearing in the proof of Lemma 3.4 allow us to conclude $P'_i \to 0$ for $4 \leq i \leq 9$ in the same way as before, so we compute $P'_i$ for $i = 1, 2, 3$ below. We may concentrate on the integration over $C'_{\Gamma_i}$ by the first observation. Recall $C'_{\Gamma_i} \subset S^1 \times \mathrm{Conf}\,(\mathbb{R}^1, s) \times \mathrm{Conf}\,(\mathbb{R}^3, t)$ by definition. We take the $S^1$-parameter $\alpha \in S^1 = \mathbb{R}^1/2\pi\mathbb{Z}$ so that $g$ goes through $f$ during $0 \leq \alpha \leq \pi$, and $f$ goes through $g$ during $\pi \leq \alpha \leq 2\pi$.

First consider the integration over $0 \leq \alpha \leq \pi$. We may shrink $g$ sufficiently small. Then the sliding of $g$ through $f$ does not affect the integration, so almost all the integrations converge to zero for the same reasons as in Lemmas 3.4 and 3.5. Only the configurations $(x_i) \in C'_{\Gamma_1}$ with $x_1$ and $x_2$ near $p_1$ may essentially contribute to $P'_1$ when $g$ comes around $p_1$; the form $\varphi^*_{12}\mathrm{vol}_{S^2}$ may detect the knotting of $g$. However, the two terms for $\varepsilon_1 = \pm 1$ cancel each other.

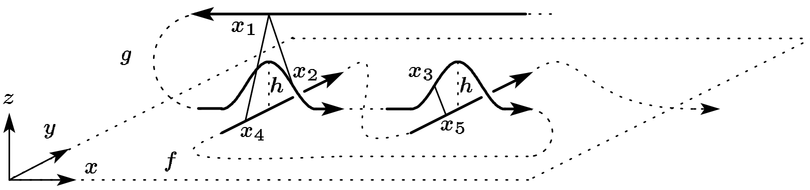
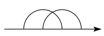



**Figure 4.** When $f$ comes near an under-arc of $g$.

Next consider the integration over $\pi \leq \alpha \leq 2\pi$. There may be two types of contributions to $P_i'$. One type comes from the configurations in which all the points on the knot concentrate in a neighborhood of $f$. Such a contribution depends only on the framing number $\mathrm{fr}\, g$ of $g$, not on the global knotting of $g$. Since $\mathrm{fr}\, g^0 = 0$ here, such configurations do not essentially contribute to $P_i'$.

The other possible contributions arise when $f$ comes near the crossings of $g$. For example, consider the case that $(p_1, p_2)$ respects . When $f$ comes near a crossing of $g$, a configuration $(x_1, \dots, x_5) \in C_{\Gamma_1}$ as in Figure 4 is certainly in $C'_{\Gamma_1}$, so it may contribute to $P_1'$.

However, such contributions converge to zero in the limit $h \to 0$, because $x_1$ cannot be near $p_1$ (see the second observation in the proof of Lemma 3.4). For $\Gamma_3$, we should take the configuration $(x_1, \dots, x_5)$ with $x_j$ $(2 \leq j \leq 5)$ near $t_{j-1}$ into account; but in this case the Gauss map $\varphi_{11}$ cannot have the image in the support of $\mathrm{vol}_{S^2}$. In such ways we can check that all such contributions of $\Gamma_i$ $(i = 1, 2, 3)$ can be arbitrarily small. □

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