SPECTRAL ANALYSIS OF NON-COMMUTATIVE HARMONIC OSCILLATORS: THE LOWEST EIGENVALUE AND NO CROSSING

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ABSTRACT. The lowest eigenvalue of non-commutative harmonic oscillators $Q(\alpha,\beta)$ $(\alpha>0,\beta>0,\alpha\beta>1)$ is studied. It is shown that $Q(\alpha,\beta)$ can be decomposed into four self-adjoint operators,

$$Q(\alpha,\beta)=\bigoplus_{\sigma=\pm,\mathrm{p}=1,2}Q_{\sigma\mathrm{p}},$$

and all the eigenvalues of each operator $Q_{\sigma\mathrm{p}}$ are simple. We show that the lowest eigenvalue of $Q(\alpha,\beta)$ is simple whenever $\alpha\neq\beta$. Furthermore a Jacobi matrix representation of $Q_{\sigma\mathrm{p}}$ is given and spectrum of $Q_{\sigma\mathrm{p}}$ is considered numerically.

1. Introduction

The non-commutative harmonic oscillator is introduced by A. Parmeggiani and M. Wakayama [PW01, PW02, PW03] as a non-commutative extension of harmonic oscillators. We also refer to [Par10] which is a first account about non-commutative harmonic oscillators and of their spectral properties. It is defined by

$$Q=Q(\alpha,\beta)=A\otimes\left(-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2\right)+J\otimes\left(x\frac{d}{dx}+\frac{1}{2}\right), \tag{1.1}$$

as an operator in $\mathcal{H}=\mathbb{C}^2\otimes L^2(\mathbb{R})$. Here $A,J\in\mathrm{Mat}_2(\mathbb{R})$, A is positive definite symmetric, and J skew-symmetric. Furthermore $A+iJ$ is positive definite. It is shown in [PW02, PW03] that A and J can be assumed to be $A=\begin{pmatrix}\alpha&0\\0&\beta\end{pmatrix}$, $J=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, and α and β satisfy

$$\alpha>0,\quad \beta>0,\quad \alpha\beta>1. \tag{1.2}$$

We fix A and J as above, and throughout this paper we assume (1.2). Under (1.2), Q is self-adjoint on the domain $D(Q)=\mathbb{C}^2\otimes(D(d^2/dx^2)\cap D(x^2))$ and has purely discrete spectrum $E_0\leq E_1\leq E_2\leq\cdots\nearrow\infty$. When $\alpha=\beta$, $Q(\alpha,\beta)$ is equivalent to the direct sum of a harmonic oscillator. Then $E_j=E_{j+1}=\frac{1}{2}(1+j)\sqrt{\alpha^2-1}$ for $j=0,2,4,\cdots$. In the case of $\alpha\neq\beta$, however, the spectrum of $Q(\alpha,\beta)$ is nontrivial, and exploring properties of the spectrum is the main purpose of the present paper.

An eigenvector associated with the lowest eigenvalue $E=E_0$ is called a ground state in this paper. A long-standing problem concerning eigenvalues of $Q(\alpha,\beta)$ is to

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determine their multiplicity explicitly. Let $\alpha \neq \beta$. Let $E_n = E_n(\alpha, \beta)$ denote the n-th eigenvalue of $Q(\alpha, \beta)$. The map $c_n : (\alpha, \beta) \mapsto E_n(\alpha, \beta) \in \mathbb{R}$ is called an eigenvalue-curve. To consider the multiplicity of eigenvalues is reduced to considering crossing or no crossing of eigenvalue-curves.

We state a short history concerning studies of the multiplicity of eigenvalues of Q. In [PW03] it is shown that the multiplicity of any eigenvalues of Q is at most three and an alternative proof is given in [Och01]. At a numerical level it is found in [NNW02] that eigenvalue-curves cross at some points but the lowest eigenvalue is simple. The multiplicity of eigenvalues of Q is also considered in [IW07], where it is derived that

$$\left(n-\frac{1}{2}\right)\min\{\alpha,\beta\}\sqrt{\frac{\alpha\beta-1}{\alpha\beta}} \leq E_{2n-1} \leq E_{2n} \leq \left(n-\frac{1}{2}\right)\max\{\alpha,\beta\}\sqrt{\frac{\alpha\beta-1}{\alpha\beta}}$$

for $n = 1, 2, 3, \cdots$. From this we can see that the multiplicity of E is at most two if $\beta < 3\alpha$ or $\alpha < 3\beta$. In [Par04] it is shown that E is simple but for sufficiently large $\alpha\beta$. Furthermore in [HS12] it is proven that the lowest eigenvalue is at most two and all the ground states are even for $(\alpha, \beta) \in D_{\sqrt{2}}$, where $D_{\sqrt{2}} = \{(\alpha, \beta) | \alpha, \beta > \sqrt{2}\}$, and it is also shown that E is simple for $(\alpha, \beta) \in D$ for some subset $D \subset D_{\sqrt{2}}$. Recently Wakayama [Wak12] breaks through in studying the multiplicity of E, in that he proves that if all the ground states are even, then E is simple whenever $\alpha \neq \beta$. Combining [Wak12] with [HS12], it is immediate to see that E is simple for $(\alpha, \beta) \in D_{\sqrt{2}}$.

In this paper we settle down the question concerning the multiplicity of the lowest eigenvalue of Q, i.e., we prove that E is simple for all values of α and β ($\alpha \neq \beta$), see Theorem 3.1. Moreover no crossing between eigenvalue-curves associated with an odd eigenvector and an even eigenvector can occur, as proved in Corollary 5.2.

This paper is organized as follows. In Section 2, we decompose $Q(\alpha, \beta)$ into four self-adjoint operators: $Q(\alpha, \beta) = \bigoplus_{\sigma=\pm, \mathrm{p}=1,2} Q_{\sigma\mathrm{p}}$. It is shown that each $Q_{\sigma\mathrm{p}}$ is equivalent to some Jacobi matrix $\widehat{Q}_{\sigma\mathrm{p}}$, and all the eigenvalues of $Q_{\sigma\mathrm{p}}$ are simple. In Section 3, we show that the lowest eigenvalue of $Q(\alpha, \beta)$ is simple. In Section 4, we construct a unitary transformation $U_{\sigma\mathrm{p}}$ such that $e^{-tU_{\sigma\mathrm{p}}^{-1} Q_{\sigma\mathrm{p}} U_{\sigma\mathrm{p}}}$ is positivity improving, and it is shown that the ground state is in a positive cone. In Section 5, we show that $\widehat{Q}_{-\mathrm{p}} - \widehat{Q}_{+\mathrm{p}} \geq \Delta(\alpha, \beta)$, $\mathrm{p} = 1, 2$, for some $\Delta(\alpha, \beta)$. In particular, if $\Delta(\alpha, \beta) > 0$, then there is no crossing between the n-th eigenvalue-curve of $Q_{-\mathrm{p}}$ and that of $Q_{+\mathrm{p}}$. In Section 6, we show some numerical results.

2. Decomposition of $Q(\alpha, \beta)$ and Jacobi matrix

2.1. **Decomposition of $Q(\alpha, \beta)$.** Let $a = \frac{1}{\sqrt{2}}(x + \frac{d}{dx})$ and $a^* = \frac{1}{\sqrt{2}}(x - \frac{d}{dx})$ be the annihilation operator and the creation operators, respectively. In terms of a and a^*, Q can be expressed as

$$Q = A(a^*a + \frac{1}{2}) + \frac{J}{2}(aa - a^*a^*). \tag{2.1}$$

Let $\mathcal{H}_+$ (resp. $\mathcal{H}_-$) be the set of even (resp. odd) functions in $\mathcal{H}$, and P_+ (resp. P_-) be the orthogonal projection onto $\mathcal{H}_+$ (resp. $\mathcal{H}_-$). Let $|n\rangle$ be the n-th normalized eigenvector of a^*a, i.e., $|n\rangle = \frac{1}{\sqrt{n!}}(a^*)^n |0\rangle$ with $|0\rangle = \pi^{-1/4}e^{-x^2/2}$. Let $\mathbb{C}|n\rangle$ be the one-dimensional subspace spanned by $|n\rangle$ over $\mathbb{C}$. Hence the Wiener-Itô decomposition $L^2(\mathbb{R}) = \bigoplus_{n=0}^{\infty} \mathbb{C}|n\rangle$ follows. The total Hilbert space is

$$\mathcal{H} \cong \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \,\middle|\, X, Y \in \bigoplus_{n=0}^{\infty} \mathbb{C}|n\rangle \right\} \cong \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \quad \mathcal{H}_n = \begin{pmatrix} \mathbb{C}|n\rangle \\ \mathbb{C}|n\rangle \end{pmatrix}.$$

We use this equivalence without further notice. Since $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^*|n\rangle = \sqrt{n+1}|n+1\rangle$, we see that $aa : \mathcal{H}_n \to \mathcal{H}_{n-2}$ and $a^*a^* : \mathcal{H}_n \to \mathcal{H}_{n+2}$. Furthermore a^*a leaves $\mathcal{H}_n$ invariant. Then we have $Q : \mathcal{H}_n \to \mathcal{H}_{n-2} \oplus \mathcal{H}_n \oplus \mathcal{H}_{n+2}$. From these observations, we can find the invariant domains of Q. We denote the orthogonal projection onto $\mathbb{C}|n\rangle$ by $|n\rangle\langle n|$, and define orthogonal projections on $\mathcal{H}$ by

$$P_\uparrow(n) = \begin{pmatrix} |n\rangle\langle n| & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_\downarrow(n) = \begin{pmatrix} 0 & 0 \\ 0 & |n\rangle\langle n| \end{pmatrix}. \tag{2.2}$$

Note that $1 = \sum_{n=0}^{\infty}(P_\uparrow(n) + P_\downarrow(n))$. In order to decompose Q, we define the following orthogonal projections:

$$T_{+1} = \sum_{n=0}^{\infty}(P_\uparrow(4n) + P_\downarrow(4n+2)), \qquad T_{+2} = \sum_{n=0}^{\infty}(P_\downarrow(4n) + P_\uparrow(4n+2)),$$
$$T_{-1} = \sum_{n=0}^{\infty}(P_\uparrow(4n+1) + P_\downarrow(4n+3)), \qquad T_{-2} = \sum_{n=0}^{\infty}(P_\downarrow(4n+1) + P_\uparrow(4n+3)).$$

Since $|2n\rangle$ is even and $|2n+1\rangle$ is odd, one has $T_{+1} + T_{+2} = P_+$ and $T_{-1} + T_{-2} = P_-$. We set $\mathcal{H}_{\sigma\mathrm{p}} = \mathrm{Ran}(T_{\sigma\mathrm{p}})$. Then $\mathcal{H}$ is decomposed as

$$\mathcal{H} = \bigoplus_{\sigma=\pm,\mathrm{p}=1,2} \mathcal{H}_{\sigma\mathrm{p}}. \tag{2.3}$$

Theorem 2.1. *The operator Q is reduced by $\mathcal{H}_{\sigma\mathrm{p}}$, $\sigma = \pm$, $\mathrm{p} = 1, 2$.*

Proof. Recall that $A \subset B$ means that $D(A) \subset D(B)$ and $Av = Bv$ for all $v \in D(A)$. We see that $a^2P_j(n) \supset P_j(n-2)a^2$, $a^*a^*P_j(n) \supset P_j(n+2)a^*a^*$ and $a^*aP_j(n) \supset P_j(n)a^*a$ for all $n = 0, 1, 2, \cdots$, and $j = \uparrow, \downarrow$. Clearly it holds that $AP_j(n) = P_j(n)A$, $JP_\uparrow(n) = P_\downarrow(n)J$ and $JP_\downarrow(n) = P_\uparrow(n)J$. Then $QT_{\sigma\mathrm{p}} \supset T_{\sigma\mathrm{p}}Q$ and the theorem follows. □

Let us set $Q_{\sigma\mathrm{p}} = Q\lceil_{\mathcal{H}_{\sigma\mathrm{p}}}$. Then it holds that

$$Q = \bigoplus_{\sigma=\pm,\mathrm{p}=1,2} Q_{\sigma\mathrm{p}}. \tag{2.4}$$

2.2. **Jacobi matrix representation of $Q_{\sigma\mathrm{p}}$.** We construct a unitary operator implementing the equivalence between $Q_{\sigma\mathrm{p}}$ and a Jacobi matrix. Set

$$U_{+1} = \sum_{n=0}^{\infty}(P_\uparrow(8n) + P_\downarrow(8n+2)) - \sum_{n=0}^{\infty}(P_\uparrow(8n+4) + P_\downarrow(8n+6)). \tag{2.5}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$\uparrow$	■	□	□	□	■	□	□	□	■	□	□	□	■	$\cdots$
$\downarrow$	□	□	■	□	□	□	■	□	□	□	■	□	□	$\cdots$

FIGURE 1. $\operatorname{Ran} T_{+1}$ is supported on "■"

n	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$\uparrow$	□	□	■	□	□	□	■	□	□	□	■	□	□	$\cdots$
$\downarrow$	■	□	□	□	■	□	□	□	■	□	□	□	■	$\cdots$

FIGURE 2. $\operatorname{Ran} T_{+2}$ is supported on "■"

n	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$\uparrow$	□	■	□	□	□	■	□	□	□	■	□	□	□	$\cdots$
$\downarrow$	□	□	□	■	□	□	□	■	□	□	□	■	□	$\cdots$

FIGURE 3. $\operatorname{Ran} T_{-1}$ is supported on "■"

n	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$\uparrow$	□	□	□	■	□	□	□	■	□	□	□	■	□	$\cdots$
$\downarrow$	□	■	□	□	□	■	□	□	□	■	□	□	□	$\cdots$

FIGURE 4. $\operatorname{Ran} T_{-2}$ is supported on "■"

This operator is unitary on $\mathcal{H}_{+1}$ and we have

$$\bar{Q}_{+1} = U_{+1}^{-1} Q_{+1} U_{+1} = T_{+1} \left(A(a^* a + \frac{1}{2}) - \frac{\mathrm{S}}{2}(aa + a^* a^*) \right) T_{+1}, \tag{2.6}$$

where $\mathrm{S} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In a way similar to that of U_{+1} one can define the unitary operators U_{+2}, U_{-1} and U_{-2} on $\mathcal{H}_{+2}$, $\mathcal{H}_{-1}$ and $\mathcal{H}_{-2}$, respectively, such that

$$\bar{Q}_{+2} = U_{+2}^{-1} Q_{+1} U_{+2} = T_{+2} \left(A(a^* a + \frac{1}{2}) - \frac{\mathrm{S}}{2}(aa + a^* a^*) \right) T_{+2},$$
$$\bar{Q}_{-1} = U_{-1}^{-1} Q_{-1} U_{-1} = T_{-1} \left(A(a^* a + \frac{1}{2}) - \frac{\mathrm{S}}{2}(aa + a^* a^*) \right) T_{-1},$$
$$\bar{Q}_{-2} = U_{-2}^{-1} Q_{-2} U_{-2} = T_{-2} \left(A(a^* a + \frac{1}{2}) - \frac{\mathrm{S}}{2}(aa + a^* a^*) \right) T_{-2}.$$

For sequences $a = (a_0, a_1, a_2, \cdots)$ and $b = (b_0, b_1, , b_2, \cdots)$, we define the Jacobi matrix

$$J(a, b) = \begin{pmatrix} b_0 & a_0 & & & 0 \\ a_0 & b_1 & a_1 & & \\ & a_1 & b_2 & \ddots & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & \ddots \end{pmatrix}, \tag{2.7}$$

which acts in the set of square summable sequences, $\ell^2 := \ell^2(\mathbb{N}_0)$, where $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Set $a_\sigma = (a_\sigma(0), a_\sigma(1), \cdots)$ and $b_{\sigma\mathrm{p}} = (b_{\sigma\mathrm{p}}(0), b_{\sigma\mathrm{p}}(1), \cdots)$, where

$$
\begin{aligned}
&a_+(n) = -\sqrt{(2n+1)(2n+2)}, && a_-(n) = -\sqrt{(2n+2)(2n+3)},\\
&b_{+1}(n) = \begin{cases} \alpha(1+4n) & \text{for even } n \\ \beta(1+4n) & \text{for odd } n, \end{cases} && b_{+2}(n) = b_{+1}(n)\Big|_{(\alpha,\beta)\to(\beta,\alpha)},\\
&b_{-1}(n) = \begin{cases} \alpha(3+4n) & \text{for even } n \\ \beta(3+4n) & \text{for odd } n, \end{cases} && b_{-2}(n) = b_{-1}(n)\Big|_{(\alpha,\beta)\to(\beta,\alpha)}.
\end{aligned}
$$

For $\sigma = \pm$ and $\mathrm{p} = 1, 2$, we define the Jacobi matrix $\widehat{Q}_{\sigma\mathrm{p}}$ by

$$
\widehat{Q}_{\sigma\mathrm{p}} = \frac{1}{2} J(a_\sigma, b_{\sigma\mathrm{p}}). \tag{2.8}
$$

Let $e_n = (\delta_{n,j})_{j=0}^\infty \in \ell^2$ be the standard basis of ℓ^2. Note that the space $\mathcal{H}_{+1}$ is spanned by the vectors $\left\{ \begin{pmatrix} |4n\rangle \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ |4n+2\rangle \end{pmatrix}, n = 0, 1, 2, ... \right\}$. We define the unitary operator $Y_{+1} : \mathcal{H}_{+1} \to \ell^2$ by $Y_{+1} \begin{pmatrix} |4n\rangle \\ 0 \end{pmatrix} = e_{2n}$ and $Y_{+1} \begin{pmatrix} 0 \\ |4n+2\rangle \end{pmatrix} = e_{2n+1}$. Then one can compute the matrix element of $\bar{Q}_{+1}$ as $\widehat{Q}_{+1} = Y_{+1} \bar{Q}_{+1} Y_{+1}^{-1}$. Similarly one can define the unitary transformations such that the following theorem holds.

Theorem 2.2 (Jacobi matrix representations)**.** *For $\sigma = \pm, \mathrm{p} = 1, 2$, the operators $Q_{\sigma\mathrm{p}}$ are unitarily equivalent to the Jacobi matrix $\widehat{Q}_{\sigma\mathrm{p}}$.*

Remark. In the case of $\alpha = \beta$, $\widehat{Q}_{\sigma 1} = \widehat{Q}_{\sigma 2}$ for $\sigma = \pm$. Explicitly, each $\widehat{Q}_{\sigma \mathrm{p}}$ is expressed as

$$\widehat{Q}_{+1} = \frac{1}{2}\begin{pmatrix} \alpha & -\sqrt{1\cdot 2} & & & & & 0 \\ -\sqrt{1\cdot 2} & 5\beta & -\sqrt{3\cdot 4} & & & & \\ & -\sqrt{3\cdot 4} & 9\alpha & -\sqrt{5\cdot 6} & & & \\ & & -\sqrt{5\cdot 6} & 13\beta & -\sqrt{7\cdot 8} & & \\ & & & -\sqrt{7\cdot 8} & 17\alpha & -\sqrt{9\cdot 10} & \\ & & & & -\sqrt{9\cdot 10} & 21\beta & \ddots \\ 0 & & & & & \ddots & \ddots \end{pmatrix},$$

$$\widehat{Q}_{+2} = \frac{1}{2}\begin{pmatrix} \beta & -\sqrt{1\cdot 2} & & & & & 0 \\ -\sqrt{1\cdot 2} & 5\alpha & -\sqrt{3\cdot 4} & & & & \\ & -\sqrt{3\cdot 4} & 9\beta & -\sqrt{5\cdot 6} & & & \\ & & -\sqrt{5\cdot 6} & 13\alpha & -\sqrt{7\cdot 8} & & \\ & & & -\sqrt{7\cdot 8} & 17\beta & -\sqrt{9\cdot 10} & \\ & & & & -\sqrt{9\cdot 10} & 21\alpha & \ddots \\ 0 & & & & & \ddots & \ddots \end{pmatrix},$$

$$\widehat{Q}_{-1} = \frac{1}{2}\begin{pmatrix} 3\alpha & -\sqrt{2\cdot 3} & & & & & 0 \\ -\sqrt{2\cdot 3} & 7\beta & -\sqrt{4\cdot 5} & & & & \\ & -\sqrt{4\cdot 5} & 11\alpha & -\sqrt{6\cdot 7} & & & \\ & & -\sqrt{6\cdot 7} & 15\beta & -\sqrt{8\cdot 9} & & \\ & & & -\sqrt{8\cdot 9} & 19\alpha & -\sqrt{10\cdot 11} & \\ & & & & -\sqrt{10\cdot 11} & 23\beta & \ddots \\ 0 & & & & & \ddots & \ddots \end{pmatrix},$$

$$\widehat{Q}_{-2} = \frac{1}{2}\begin{pmatrix} 3\beta & -\sqrt{2\cdot 3} & & & & & 0 \\ -\sqrt{2\cdot 3} & 7\alpha & -\sqrt{4\cdot 5} & & & & \\ & -\sqrt{4\cdot 5} & 11\beta & -\sqrt{6\cdot 7} & & & \\ & & -\sqrt{6\cdot 7} & 15\alpha & -\sqrt{8\cdot 9} & & \\ & & & -\sqrt{8\cdot 9} & 19\beta & -\sqrt{10\cdot 11} & \\ & & & & -\sqrt{10\cdot 11} & 23\alpha & \ddots \\ 0 & & & & & \ddots & \ddots \end{pmatrix}.$$

Theorem 2.3. *Each eigenvalue of* $Q_{\sigma \mathrm{p}}$, $\sigma = \pm$, $\mathrm{p} = 1, 2$, *is simple.*

Proof. Let λ be any eigenvalue of $\widehat{Q}_{+1}$. Then any vector $u = (u_n)_{n=0}^{\infty} \in \ker(\widehat{Q}_{+1} - \lambda)$ satisfies the recurrence relations:

$$u_{n+1} = a_+(n)^{-1}\left\{(\lambda - b_{+1}(n))u_n - a_+(n-1)u_{n-1}\right\}, \quad n \in \mathbb{N}_0, \tag{2.9}$$

$$u_{-1} = 0. \tag{2.10}$$

Note that $a_+(n) \neq 0$. Solutions of system (2.9)-(2.10) are uniquely determined by the term $u_0 \in \mathbb{C}$, i.e.,

$$u_1 = a_+(0)^{-1}(\lambda - b_{+1}(0))u_0 \tag{2.11}$$

$$u_2 = a_+(1)^{-1}\{(\lambda - b_{+1}(1))a_+(0)^{-1}(\lambda - b_{+1}(0)) - a_+(0)\}u_0 \tag{2.12}$$

$$\vdots \tag{2.13}$$

Hence the set of solutions of (2.9)-(2.10) forms a one dimensional subspace. Therefore the multiplicity of any eigenvalue of $\widehat{Q}_{+1}$ is one. Proofs for other cases are similar. $\square$

Let $\lambda_{\sigma \mathrm{p}}(n) = \lambda_{\sigma \mathrm{p}}(n, \alpha, \beta)$ be the n-th eigenvector of $Q_{\sigma \mathrm{p}}$. Then $\{\lambda_{\sigma \mathrm{p}}(n)\}_{n=0}^{\infty} = Spec(Q_{\sigma \mathrm{p}})$ and $\lambda_{\sigma \mathrm{p}}(n) \le \lambda_{\sigma \mathrm{p}}(n+1)$ for $n = 0, 1, 2, \cdots$. The following result follows immediately from the above theorem.

Corollary 2.4. *For each $\sigma = \pm$ and $\mathrm{p} = 1, 2$, eigenvalue-curves*

$$\{(\alpha, \beta) \mapsto \lambda_{\sigma \mathrm{p}}(n) = \lambda_{\sigma \mathrm{p}}(n, \alpha, \beta), n = 0, 1, 2, 3, \cdots\}$$

have no crossing, i.e., for arbitrary (α, β) and $n \neq m$, $\lambda_{\sigma \mathrm{p}}(n, \alpha, \beta) \neq \lambda_{\sigma \mathrm{p}}(m, \alpha, \beta)$.

3. Simplicity of the lowest eigenvalue of $Q(\alpha, \beta)$

In this section, we state the main theorem in this paper.

Theorem 3.1. *Assume that $\alpha\beta > 1$ and $\alpha \neq \beta$. Then the lowest eigenvalue of $Q(\alpha, \beta)$ is simple and the ground state is even.*

In order to show Theorem 3.1 we introduce a remarkable result given by Wakayama [Wak12].

Theorem 3.2. *Assume that (1) $\alpha \neq \beta$; (2) all the ground states of $Q(\alpha, \beta)$ are even, i.e.,* $\ker(Q(\alpha, \beta) - E) \subset \mathcal{H}_+$. *Then the lowest eigenvalue of $Q(\alpha, \beta)$ is simple.*

Let $Q_\sigma = Q_{\sigma 1} \oplus Q_{\sigma 2}$, $\sigma = +, -$. Then Q is decomposed into the direct sum of even part and odd part, $Q = Q_+ \oplus Q_-$. Let $E_\sigma = \inf Spec(Q_\sigma)$.

Lemma 3.3. *It follows that $E_+ \le E_-$.*

Proof. Let $\Phi_- = \begin{pmatrix} \Phi_{-1} \\ \Phi_{-2} \end{pmatrix}$ be a normalized ground state of Q_-. Note that Φ_{-j}, $j = 1, 2$, are odd functions. We set

$$\theta(x) := \begin{cases} +1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{cases} \tag{3.1}$$

We define an even function $\widetilde{\Phi}_- \in \mathcal{H}_+$ by

$$\widetilde{\Phi}_- := \begin{pmatrix} \widetilde{\Phi}_{-1} \\ \widetilde{\Phi}_{-2} \end{pmatrix}, \qquad \widetilde{\Phi}_{-j}(x) := \theta(x)\Phi_{-j}(x)$$

Since Φ_- is also an eigenfunction of Q, it is a Schwartz function (see [Par10, Theorem 3.3.13]). So $\theta\Phi_{-j}$ is a distribution over the real line. Since the distributional derivative of θ is $2\delta_0$, where δ_0 is the Dirac mass concentrated at the origin, then $(\theta\Phi_{-j})' = \theta\Phi'_{-j} + 2\delta_0\Phi_{-j}$. Since $\Phi_{-j}(0) = 0$ and δ_0 being a measure, we get $\delta_0\Phi_{-j} = 0$. Hence $(\theta\Phi_{-j})' = \theta\Phi'_{-j} \in L^2(\mathbb{R})$, which shows that $\theta\Phi_{-j} \in D(-d/dx)$ and

$$\|(d/dx)\widetilde{\Phi}_{-j}\|^2 = \|(d/dx)\Phi_{-j}\|^2, \quad \left(\widetilde{\Phi}_{-j'}, x\frac{d}{dx}\widetilde{\Phi}_{-j}\right) = \left(\Phi_{-j'}, x\frac{d}{dx}\Phi_{-j}\right), \quad j', j = 1, 2.$$

Thus one has

$$E_+ \le \left(\widetilde{\Phi}_-, Q\widetilde{\Phi}_-\right) = (\Phi_-, Q\Phi_-) = E_-. \tag{3.2}$$

Therefore $E_+ \le E_-$ follows. □

Lemma 3.4. *It follows that $E_+ < E_-$.*

Proof. Assume that $E_+ = E_-$. Then by (3.2) we have $E_+ = \left(\widetilde{\Phi}_-, Q\widetilde{\Phi}_-\right)$, which implies that $\widetilde{\Phi}_-$ is a ground state of Q_+. In other words, $\widetilde{\Phi}_-$ is an eigenvector of Q with eigenvalue E_+. Thus $\widetilde{\Phi}_{-1}$ and $\widetilde{\Phi}_{-2}$ are in the Schwartz class. We normalize $\widetilde{\Phi}$ as $\|\widetilde{\Phi}\| = 1$. From the fact that Φ_{-j} is odd (resp. $\widetilde{\Phi}_{-j}$ is even), it follows that $\Phi_-(0) = \begin{pmatrix}0\\0\end{pmatrix} = \widetilde{\Phi}_-(0)$ (resp. $\frac{d}{dx}\widetilde{\Phi}_{-j}(0) = \begin{pmatrix}0\\0\end{pmatrix}$). Therefore $\widetilde{\Phi}_{-j}$ satisfies the ordinary differential equations:

$$\frac{d}{dx}\begin{pmatrix}\widetilde{\Phi}_{-1}\\ \widetilde{\Phi}_{-2}\\ \widetilde{\Phi}'_{-1}\\ \widetilde{\Phi}'_{-2}\end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ x^2 + \frac{2E_+}{\alpha} & -\frac{1}{\alpha} & 0 & -\frac{2x}{\alpha}\\ \frac{1}{\beta} & x^2 - \frac{2E_+}{\beta} & \frac{2x}{\beta} & 0\end{pmatrix}\begin{pmatrix}\widetilde{\Phi}_{-1}\\ \widetilde{\Phi}_{-2}\\ \widetilde{\Phi}'_{-1}\\ \widetilde{\Phi}'_{-2}\end{pmatrix} \tag{3.3}$$

$$\begin{pmatrix}\widetilde{\Phi}_{-1}(0)\\ \widetilde{\Phi}_{-2}(0)\\ \widetilde{\Phi}'_{-1}(0)\\ \widetilde{\Phi}'_{-2}(0)\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}. \tag{3.4}$$

Since the right hand side of (3.3) is smooth in $(\widetilde{\Phi}_{-1}, \widetilde{\Phi}_{-2}, \widetilde{\Phi}'_{-1}, \widetilde{\Phi}'_{-2}, x)$, the differential equation (3.3) with initial condition (3.4) has the unique solution $\begin{pmatrix}\widetilde{\Phi}_{-1}(x)\\ \widetilde{\Phi}_{-2}(x)\\ \widetilde{\Phi}'_{-1}(x)\\ \widetilde{\Phi}'_{-2}(x)\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$, which contradicts $\|\widetilde{\Phi}_-\| = 1$. Therefore, $E_+ < E_-$. □

Proof of Theorem 3.1. Assume that $\alpha \ne \beta$. By Theorem 3.2, it is enough to show that $\ker(Q - E) \subset \mathcal{H}_+$. By Lemma 3.3, we have $E_+ < E_-$. Hence all the ground states are even. Therefore the theorem follows. □

4. Positivity of ground state

Let

$$\mathscr{C}^+ = \left\{ \sum_{n=0}^\infty a_n \begin{pmatrix}|4n\rangle\\ 0\end{pmatrix} + \sum_{n=0}^\infty b_n \begin{pmatrix}0\\ |4n+2\rangle\end{pmatrix} \middle| a_n > 0, b_n > 0, n \ge 0 \right\},$$

$$\mathscr{C}_0^+ = \left\{ \sum_{n=0}^\infty a_n \begin{pmatrix}|4n\rangle\\ 0\end{pmatrix} + \sum_{n=0}^\infty b_n \begin{pmatrix}0\\ |4n+2\rangle\end{pmatrix} \middle| a_n \ge 0, b_n \ge 0, n \ge 0 \right\}.$$

Then $\mathscr{C}^+$ is a positive cone of $\mathcal{H}_{+1}$ and $\mathscr{C}_0^+$ a nonnegative cone of $\mathcal{H}_{+1}$. We say that Ψ is nonnegative iff $\Psi \in \mathscr{C}_0^+$, which we denote by $\Psi \geq 0$, and Ψ is strictly positive iff $\Psi \in \mathscr{C}^+$, which we denote by $\Psi > 0$. A *bounded* operator T on $\mathcal{H}_{+1}$ is *positivity preserving* if and only if $T\mathscr{C}_0^+ \subset \mathscr{C}_0^+$, and *positivity improving* if and only if $T\mathscr{C}_0^+ \subset \mathscr{C}^+$.

Proposition 4.1. *Suppose that a bounded self-adjoint operator T is positivity improving on $\mathcal{H}_{\sigma \mathrm{p}}$ and $\|T\|$ is an eigenvalue. Then the multiplicity of $\|T\|$ is simple and the corresponding eigenvector is strictly positive.*

Proof. See [Far72]. □

Theorem 4.2. *For all $t > 0$, $\sigma = \pm$ and $\mathrm{p} = 1, 2$, $e^{-t\bar{Q}_{\sigma \mathrm{p}}}$ is positivity improving on $\mathcal{H}_{\sigma \mathrm{p}}$. In particular, the lowest eigenvalue of $Q_{\sigma \mathrm{p}}$ is simple and corresponding eigenvector is strictly positive.*

Proof. We prove the theorem only for the case of $\sigma = +$ and $\mathrm{p} = 1$. For other cases the proof is similar and is left to the reader. We shall show below that $e^{-t\bar{Q}_{+1}}$ is positivity improving. We define

$$H_0 = A(a^*a + \frac{1}{2})T_{+1}, \qquad V = \frac{\mathrm{S}}{2}(aa + a^*a^*)T_{+1}. \tag{4.1}$$

Note that $\bar{Q}_{+1} = H_0 - V$. Since $a\,|n\rangle = \sqrt{n}\,|n-1\rangle$ and $a^*\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$, and H_0 is the multiplication by $\alpha(n + \frac{1}{2})$, we see that e^{-tH_0} is positivity preserving. Since $\begin{pmatrix} |4n\rangle \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ |4n+2\rangle \end{pmatrix}$ are analytic vectors of V, we see that

$$e^{tV}\begin{pmatrix} |4n\rangle \\ 0 \end{pmatrix} = \sum_{j=0}^{\infty} \frac{t^j}{j!}(aa + a^*a^*)^j \left(\frac{\mathrm{S}}{2}\right)^j \begin{pmatrix} |4n\rangle \\ 0 \end{pmatrix} \in \mathscr{C}^+, \tag{4.2}$$

$$e^{tV}\begin{pmatrix} 0 \\ |4n+2\rangle \end{pmatrix} = \sum_{j=0}^{\infty} \frac{t^j}{j!}(aa + a^*a^*)^j \left(\frac{\mathrm{S}}{2}\right)^j \begin{pmatrix} 0 \\ |4n+2\rangle \end{pmatrix} \in \mathscr{C}^+. \tag{4.3}$$

From this $e^{tV}\mathscr{C}_0^+ \subset \mathscr{C}^+$ follows. Let $\Psi, \Phi \in \mathscr{C}_0^+$. By the Trotter-Kato product formula, we have

$$\left(\Psi, e^{-t\bar{Q}_{+1}}\Phi\right) = \lim_{j\to\infty}\left(\Psi, (e^{-tH_0/j}e^{tV/j})^j\Phi\right) \geq 0. \tag{4.4}$$

Therefore $e^{t\bar{Q}_{+1}}$ is positivity preserving. Next we show that $e^{-t\bar{Q}_{+1}}$ is positivity improving. We can assume that $\alpha \leq \beta$ without loss of generality. Let $P_{\leq k}$ be the projection defined by

$$P_{\leq k} = \begin{pmatrix} \sum_{4n\leq k} |2n\rangle\langle 4n| & 0 \\ 0 & \sum_{4n+2\leq k} |4n+2\rangle\langle 4n+2| \end{pmatrix}$$

It is immediately seen that $\Psi \ge P_{\le k}\Psi$ for any $\Psi \in \mathscr{C}_0^+$ and $e^{tV/j}\Psi \ge (1+tV/j)\Psi$. For $k' \ge k$, we set $v = \begin{pmatrix} |4k\rangle \\ 0 \end{pmatrix}$ and $v' = \begin{pmatrix} 0 \\ |4k'\rangle \end{pmatrix}$. Then we have

$$\begin{aligned}\left(v', e^{-t\bar{Q}_{+1}}v\right) &= \lim_{j\to\infty}\left(v', (e^{-tH_0/j}e^{tV/j})^j v\right) \ge \overline{\lim_{j\to\infty}}\left(v', (e^{-tH_0/j}P_{\le k'}e^{tV/j})^j v\right) \\ &\ge \overline{\lim_{j\to\infty}}\left(v', (e^{-t(k'+(1/2))\beta/j}P_{\le k'}e^{tV/j})^j v\right) \\ &\ge e^{-t(k'+(1/2))\beta}\,\overline{\lim_{j\to\infty}}\left(v', (P_{\le k'}(1+tV/j))^j v\right).\end{aligned}$$

Note that $e^{-tH_0}(1+tV/j)$ is still positivity preserving. For all $\ell = 2k'-2k$, we have

$$\begin{aligned}&\overline{\lim_{j\to\infty}}\left(v', (P_{\le k'}(1+tV/j))^j v\right) \ge \overline{\lim_{j\to\infty}}\left(v', {}_jC_\ell (tV/j)^\ell v\right) \ge \overline{\lim_{j\to\infty}}\left(v', {}_jC_\ell (t(a^*)^2/j)^\ell v\right) \\ &= t^\ell\,\overline{\lim_{j\to\infty}}\,{}_jC_\ell j^{-\ell}\left(v', (a^*)^{4k'-4k}v\right) = \frac{t^\ell}{\ell!}\overline{\lim_{j\to\infty}}\frac{j(j-1)\cdots(j-\ell-1)}{j^\ell}\left(v', (a^*)^{4k'-4k}v\right) \\ &= \frac{t^\ell}{\ell!}\left(v', (a^*)^{4k'-4k}v\right) > 0,\end{aligned}$$

where ${}_jC_k$ denotes the binomial coefficient. Thus we have $\left(v', e^{-t\bar{Q}_{+1}}v\right) > 0$. Similarly $\left(\begin{pmatrix} |4n\rangle \\ 0\end{pmatrix}, e^{-t\bar{Q}_{+1}}\begin{pmatrix} 0 \\ |4n+2\rangle\end{pmatrix}\right) > 0$ is derived for all n. Thus $e^{-t\bar{Q}_{+1}}$ is positivity improving. □

5. No crossings

Recall that $E_n(\alpha,\beta)$ be the n-th eigenvalue of $Q(\alpha,\beta)$, and the map $(\alpha,\beta) \mapsto E_n(\alpha,\beta) \in \mathbb{R}$ is an eigenvalue-curve. It will be shown here that the spectrum of Q is $Spec(Q) = \bigcup_{\sigma=\pm,\mathrm{p}=1,2} Spec(Q_{\mathrm{p}\sigma})$, and all the eigenvalues in $Spec(Q_{\mathrm{p}\sigma})$ are simple. Now we are interested in operators, $\widehat{Q}_{-1} - \widehat{Q}_{+1}$ and $\widehat{Q}_{-2} - \widehat{Q}_{+2}$.

Theorem 5.1. *Assume that*

$$\sqrt{\alpha\beta} > 1 + \frac{1}{1600000000} \tag{5.1}$$

Then $\widehat{Q}_{-1} - \widehat{Q}_{+1} \ge \Delta(\alpha,\beta)$ *and* $\widehat{Q}_{-2} - \widehat{Q}_{+2} \ge \Delta(\alpha,\beta)$, *where*

$$\Delta(\alpha,\beta) = 2\min\{\sqrt{\alpha/\beta}, \sqrt{\beta/\alpha}\}(\sqrt{\alpha\beta} - 1 - 1/1600000000) > 0.$$

In particular $\lambda_{-1}(n) \ge \lambda_{+1}(n) + \Delta(\alpha,\beta)$ *and* $\lambda_{-2}(n) \ge \lambda_{+2}(n) + \Delta(\alpha,\beta)$.

Proof. We have

$$\widehat{Q}_{-1} - \widehat{Q}_{+1} = \frac{1}{2}\begin{pmatrix} 2\alpha & -\gamma_0 & & & & & & 0 \\ -\gamma_0 & 2\beta & -\gamma_1 & & & & & \\ & -\gamma_1 & 2\alpha & -\gamma_2 & & & & \\ & & -\gamma_2 & 2\beta & -\gamma_3 & & & \\ & & & -\gamma_3 & 2\alpha & -\gamma_4 & & \\ & & & & -\gamma_4 & 2\beta & -\gamma_5 & \\ & & & & & -\gamma_5 & 2\alpha & \ddots \\ 0 & & & & & & \ddots & \ddots \end{pmatrix}, \tag{5.2}$$

where $\gamma_n = \sqrt{(2n+2)(2n+3)} - \sqrt{(2n+1)(2n+2)}$. We set

$$S_1 = \mathrm{diag}[(\beta/\alpha)^{1/4}, (\alpha/\beta)^{1/4}, (\beta/\alpha)^{1/4}, (\alpha/\beta)^{1/4}, \cdots], \tag{5.3}$$

$$S_2 = \mathrm{diag}[(\alpha/\beta)^{1/4}, (\beta/\alpha)^{1/4}, (\alpha/\beta)^{1/4}, (\beta/\alpha)^{1/4}, \cdots]. \tag{5.4}$$

Then we have

$$S_1(\widehat{Q}_{-1} - \widehat{Q}_{+1})S_1 = S_2(\widehat{Q}_{-2} - \widehat{Q}_{+2})S_2 = \tag{5.5}$$

$$= \frac{1}{2}\begin{pmatrix} 2\sqrt{\alpha\beta} & -\gamma_0 & & & & & 0 \\ -\gamma_0 & 2\sqrt{\alpha\beta} & -\gamma_1 & & & & \\ & -\gamma_1 & 2\sqrt{\alpha\beta} & -\gamma_2 & & & \\ & & -\gamma_2 & 2\sqrt{\alpha\beta} & -\gamma_3 & & \\ & & & -\gamma_3 & 2\sqrt{\alpha\beta} & -\gamma_4 & \\ & & & & -\gamma_4 & 2\sqrt{\alpha\beta} & \ddots \\ 0 & & & & & \ddots & \ddots \end{pmatrix}. \tag{5.6}$$

We set $F = J((\gamma_n)_{n=0}^{\infty}, 0)$. Then $S_1(\widehat{Q}_{-1} - \widehat{Q}_{+1})S_1 = 2\sqrt{\alpha\beta} - F$. Since S_1 is self-adjoint and invertible, we have

$$(\widehat{Q}_{-1} - \widehat{Q}_{+1}) \geq (2\sqrt{\alpha\beta} - \|F\|)S_1^{-2} \geq (2\sqrt{\alpha\beta} - \|F\|)\min\{\sqrt{\alpha/\beta}, \sqrt{\beta/\alpha}\}.$$

Similarly we have $(\widehat{Q}_{-2} - \widehat{Q}_{+2}) \geq (2\sqrt{\alpha\beta} - \|F\|)\min\{\sqrt{\alpha/\beta}, \sqrt{\beta/\alpha}\}$. Hence it is sufficient to prove $\|F\| < 2(1 + 1/1600000000)$. Let $v = (v_n)_{n=0}^{\infty} \in \ell^2$. Then we have

$$\begin{aligned} |\,(v, Fv)\,| &= \left|\sum_{n=0}^{\infty}(\overline{v_n}\gamma_n v_{n+1} + v_n\gamma_n\overline{v_{n+1}})\right| \\ &\leq 2\sum_{n=0}^{\infty}|v_n|\gamma_n|v_{n+1}| \leq \sum_{n=0}^{\infty}\left(a_n|v_n|^2 + \frac{\gamma_n^2}{a_n}|v_{n+1}|^2\right) \end{aligned}$$

for any $a_n > 0$. So it follows that

$$|\,(v, Fv)\,| \leq a_0|v_0|^2 + \sum_{n=1}^{\infty}(a_n + \frac{\gamma_{n-1}^2}{a_{n-1}})|v_n|^2. \tag{5.7}$$

We split (5.7) as

$$|(v,Fv)| \le a_0|v_0|^2 + \sum_{n=1}^{N_0}(a_n + \frac{\gamma_{n-1}^2}{a_{n-1}})|v_n|^2 + \sum_{n=N_0+1}^{\infty}(a_n + \frac{\gamma_{n-1}^2}{a_{n-1}})|v_n|^2 \tag{5.8}$$

for some N_0. We recursively define a_n by

$$a_0 = 2, \quad a_n = 2 - \frac{\gamma_{n-1}^2}{a_{n-1}} \ (n = 1,2,3,\cdots,N_0), \quad a_n = 1 \ (n \ge N_0+1). \tag{5.9}$$

We can compute the numerical value of a_n from (5.9), e.g. $a_1 = 1.464\cdots$, $a_2 = 1.305\cdots$, $a_3 = 1.228\cdots$. We take $N_0 = 10000$. Then one can easily check that $a_n > 0$ for all $n < N_0$ and $a_{N_0} > 1$. Hence the inequality (5.8) is valid for $N_0 = 10000$ and we have

$$\begin{aligned} |(v,Fv)| &\le 2|v_0|^2 + 2\sum_{n=1}^{N_0}|v_n|^2 + \sum_{n=N_0+1}^{\infty}(a_n + \frac{\gamma_{n-1}^2}{a_{n-1}})|v_n|^2 \\ &< 2|v_0|^2 + 2\sum_{n=1}^{N_0}|v_n|^2 + \sum_{n=N_0+1}^{\infty}(1+\gamma_{n-1}^2)|v_n|^2. \end{aligned}$$

where we used (5.9). On the other hand, we have $\gamma_{n-1}^2 = 1 + \frac{1}{(2n+\sqrt{4n^2-1})^2}$. In particular γ_{n-1} is monotonously decreasing. Therefore we have

$$|(v,Fv)| \le (1+\gamma_{N_0}^2)\sum_{n=0}^{\infty}|v_n|^2, \tag{5.10}$$

which implies that $\|F\| \le 1 + \gamma_{N_0}^2$. Note that

$$\gamma_{N_0}^2 < \gamma_{N_0-1}^2 < 1 + \frac{1}{(4N_0)^2} = 1 + \frac{1}{1600000000}. \tag{5.11}$$

Therefore $\|F\| < 2(1 + 1/1600000000)$. □

The map $(\alpha,\beta) \mapsto \lambda_{\sigma\mathrm{p}}(n) = \lambda_{\sigma\mathrm{p}}(n,\alpha,\beta)$ is an eigenvalue-curve. It is immediate to see the corollary below by Theorem 5.1. .

Corollary 5.2. *Let*

$$D = \{(\alpha,\beta) \in \mathbb{R}\times\mathbb{R} | \alpha > 0, \beta > 0, \alpha \ne \beta, \sqrt{\alpha\beta} > 1 + \frac{1}{1600000000}\}.$$

Fix $\mathrm{p} = 1,2$. *Then two eigenvalue-curves* $\lambda_{-\mathrm{p}}(n)$ *and* $\lambda_{+\mathrm{p}}(n)$ *have no crossing in the region* D *for all* n.

6. Numerical results

For finite sequences $a = (a_0, \cdots, a_{N-1})$ and $b = (b_0, \cdots, b_N)$, we define the $(N+1)$-dimensional Jacobi matrix, $J(a, b)$, by

$$J(a,b) = \begin{pmatrix} b_0 & a_0 & & & 0 \\ a_0 & b_1 & a_1 & & \\ & a_1 & b_2 & \ddots & \\ & & \ddots & \ddots & a_{N-1} \\ 0 & & & a_{N-1} & b_N \end{pmatrix}. \tag{6.1}$$

For $\sigma = \pm$ and $\mathrm{p} = 1, 2$, we set $a_\sigma^N = (a_\sigma(n))_{n=0}^{N-1}$ and $b_{\sigma\mathrm{p}}^N = (b_{\sigma\mathrm{p}}(n))_{n=0}^{N}$. Define a finite Jacobi matrix by $\widehat{Q}_{\sigma\mathrm{p}}(N) = \frac{1}{2} J(a_\sigma^N, b_{\sigma\mathrm{p}}^N)$. We set

$$\Lambda_{+1}(N) = \frac{1}{2}(\alpha\beta - 1) \times \begin{cases} \min\{\alpha^{-1}(2N + \frac{3}{2}), \beta^{-1}(2N + \frac{7}{2})\} & \text{if } N \text{ is even} \\ \min\{\beta^{-1}(2N + \frac{3}{2}), \alpha^{-1}(2N + \frac{7}{2})\} & \text{if } N \text{ is odd} \end{cases} \tag{6.2}$$

$$\Lambda_{+2}(N) = \Lambda_{+1}(N)\Big|_{(\alpha,\beta)\to(\beta,\alpha)} \tag{6.3}$$

$$\Lambda_{-1}(N) = \frac{1}{2}(\alpha\beta - 1) \begin{cases} \min\{\alpha^{-1}(2N + \frac{5}{2}), \beta^{-1}(2N + \frac{9}{2})\} & \text{if } N \text{ is even} \\ \min\{\beta^{-1}(2N + \frac{5}{2}), \alpha^{-1}(2N + \frac{9}{2})\} & \text{if } N \text{ is odd} \end{cases} \tag{6.4}$$

$$\Lambda_{-2}(N) = \Lambda_{-1}(N)\Big|_{(\alpha,\beta)\to(\beta,\alpha)} \tag{6.5}$$

and

$$\delta_{\pm,1}(N) = \begin{cases} \frac{1}{2}\alpha|a_\pm(N)| & \text{if } N \text{ is even} \\ \frac{1}{2}\beta|a_\pm(N)| & \text{if } N \text{ is odd,} \end{cases} \qquad \delta_{\pm,2}(N) = \delta_{\pm,1}(N)\Big|_{(\alpha,\beta)\to(\beta,\alpha)}. \tag{6.6}$$

Since $\alpha\beta > 1$, one has $\Lambda_{\sigma\mathrm{p}}(N) = O(N) \to +\infty$ $(N \to +\infty)$. Let p_n be the orthogonal projection onto $e_n = (\delta_{n,j})_{j=0}^\infty \in \ell^2$. For a self-adjoint operator T, $\mu_n(T)$, $n = 1, 2, \cdots$, denotes the n-th eigenvalue of T counting multiplicity. For $n = 0, 1, \cdots, N$, we set

$$\begin{aligned} \lambda_{\sigma\mathrm{p},N}(n) &= \mu_n(\widehat{Q}_{\sigma\mathrm{p}}(N)), \\ \lambda_{\sigma\mathrm{p},N}^{\mathrm{upper}}(n) &= \mu_n(\widehat{Q}_{\sigma\mathrm{p}}(N) + \delta_{\sigma\mathrm{p}}(N)p_N), \\ \lambda_{\sigma\mathrm{p},N}^{\mathrm{lower}}(n) &= \mu_n(\widehat{Q}_{\sigma\mathrm{p}}(N) - \delta_{\sigma\mathrm{p}}(N)p_N). \end{aligned}$$

The eigenvalues of $\widehat{Q}_{\sigma\mathrm{p}}$ can be approximated by the eigenvalues of the $(N+1)$-dimensional matrix $\widehat{Q}_{\sigma\mathrm{p}}(N)$ in the following sense.

Theorem 6.1. *Fix $N \in \mathbb{N}$, $\sigma = \pm$ and $\mathrm{p} = 1, 2$. Let $n \in \mathbb{N}$ be a number such that*

$$\lambda_{\sigma\mathrm{p},N}^{\mathrm{upper}}(n) \le \Lambda_{\sigma\mathrm{p}}(N). \tag{6.7}$$

Then it follows that

$$\lambda_{\sigma\mathrm{p},N}^{\mathrm{lower}}(n) \le \lambda_{\sigma\mathrm{p}}(n) \le \lambda_{\sigma\mathrm{p},N}^{\mathrm{upper}}(n) \tag{6.8}$$

In particular, the error is estimated as $|\lambda_{\sigma\mathrm{p}}(n) - \lambda_{\sigma\mathrm{p},N}(n))| \le \lambda_{\sigma\mathrm{p},N}^{\mathrm{upper}}(n) - \lambda_{\sigma\mathrm{p},N}^{\mathrm{lower}}(n)$.

We give an example below:

Example 6.2. We set $\mathcal{Q}_\pm = \widehat{Q}_{+1}(N) \pm \delta_{+1}(N)p_N$. We apply Theorem 6.1 to the case $\alpha = 1$, $\beta = 2$ and $N = 10$. Then $\Lambda_{+1}(N) = 5.875$ and

$$\begin{aligned}
&\lambda^{\rm upper}_{+1,N}(0) = 0.366917859 \pm 0.000000001, && \lambda^{\rm lower}_{+1,N}(0) = 0.366917862 \pm 0.000000001,\\
&\lambda^{\rm upper}_{+1,N}(1) = 2.432911 \pm 0.000001, && \lambda^{\rm lower}_{+1,N}(1) = 2.432920 \pm 0.000001,\\
&\lambda^{\rm upper}_{+1,N}(2) = 4.7145 \pm 0.0001, && \lambda^{\rm lower}_{+1,N}(2) = 4.7164 \pm 0.0001\\
&\lambda^{\rm upper}_{+1,N}(3) = 6.2717 \pm 0.0001, && \lambda^{\rm lower}_{+1,N}(3) = 6.2789 \pm 0.0001.
\end{aligned}$$

Since $\lambda^{\rm upper}_{+1,N}(2) \le \Lambda_{+1}(N) = 5.875$, by Theorem 6.1 we have numerical bounds:

$$\begin{aligned}
0.36691785 \le &\lambda_{\sigma{\rm p}}(0) \le 0.36691786,\\
2.43291 \le &\lambda_{\sigma{\rm p}}(1) \le 2.43292,\\
4.714 \le &\lambda_{\sigma{\rm p}}(2) \le 4.717.
\end{aligned}$$

This example does not include the bound on $\lambda_{\sigma{\rm p}}(3)$, since the condition (6.7) is not valid for $n = 3$.

Proof of Theorem 6.1: We prove the theorem only for the case of $\sigma = +$ and ${\rm p} = 1$. The other cases can be similarly proven. For $u, v \in \ell^2$, we define the operator $u \odot v : \ell^2 \to \ell^2$ by $(u \odot v)\Phi = (v, \Phi)\, u$, for $\Phi \in \ell^2$. Then operator $\widehat{Q}_{+1}$ can be expressed as

$$\widehat{Q}_{+1} = \widehat{Q}_{+1}(N) \oplus 0 + \sum_{n=N+1}^{\infty} b_{+1}(n)p_n + \sum_{n=N}^{\infty} a_+(n)(e_n \odot e_{n+1} + e_{n+1} \odot e_n).$$

We can show that $u \odot v + v \odot u \le \epsilon u \odot u + \epsilon^{-1} v \odot v$ for all $\epsilon > 0$. By using this inequality, we have

$$\begin{aligned}
&\sum_{n=N}^{\infty} a_+(n)(e_n \odot e_{n+1} + e_{n+1} \odot e_n) \le \sum_{n=N}^{\infty} |a_+(n)|(\epsilon_n e_n \odot e_n + \epsilon_n^{-1} e_{n+1} \odot e_{n+1})\\
&= |a_+(N)|\epsilon_N p_N + \sum_{n=N+1}^{\infty} (\epsilon_n |a_+(n)| + \epsilon_{n-1}^{-1} |a_+(n-1)|) p_n
\end{aligned}$$

for all $\epsilon_n > 0$. We take $\epsilon_{2n+1} = \beta$ and $\epsilon_{2n} = \alpha$ for even N, and $\epsilon_{2n+1} = \alpha$ and $\epsilon_{2n} = \beta$ for odd N. Note that $|a_+(N)|\epsilon_N = \delta_{+1}(N)$. First we consider the case of even N.

Then, we have

$$\begin{aligned}
&\sum_{n=N+1}^{\infty}(\epsilon_n|a_+(n)|+\epsilon_{n-1}^{-1}|a_+(n-1)|)p_n \\
&=\sum_{n=0}^{\infty}(\epsilon_{N+n+1}|a_+(N+n+1)|+\epsilon_{N+n}^{-1}|a_+(N+n)|)p_{N+n+1} \\
&=\sum_{n=0}^{\infty}(\epsilon_{N+2n+1}|a_+(N+2n+1)|+\epsilon_{N+2n}^{-1}|a_+(N+2n)|)p_{N+2n+1} \\
&\quad+\sum_{n=0}^{\infty}(\epsilon_{N+2n+2}|a_+(N+2n+2)|+\epsilon_{N+2n+1}^{-1}|a_+(N+2n+1)|)p_{N+2n+2} \\
&=\sum_{n=0}^{\infty}(\beta|a_+(N+2n+1)|+\alpha^{-1}|a_+(N+2n)|)p_{N+2n+1} \\
&\quad+\sum_{n=0}^{\infty}(\alpha|a_+(N+2n+2)|+\beta^{-1}|a_+(N+2n+1)|)p_{N+2n+2}.
\end{aligned}$$

Since $|a_+(n)|\le 2n+\frac{3}{2}$, we have

$$\begin{aligned}
&\sum_{n=N+1}^{\infty}(\epsilon_n|a_+(n)|+\epsilon_{n-1}^{-1}|a_+(n-1)|)p_n \\
&\le\sum_{n=0}^{\infty}(\beta(2N+4n+2+\frac{3}{2})+\alpha^{-1}(2N+4n+\frac{3}{2}))p_{N+2n+1} \\
&\quad+\sum_{n=0}^{\infty}(\alpha(2N+4n+4+\frac{3}{2})+\beta^{-1}(2N+4n+2+\frac{3}{2}))p_{N+2n+2}.
\end{aligned}$$

By the definition of $b_{+1}(n)$, we have

$$\begin{aligned}
\widehat{Q}_{+1}\ge\ &\widehat{Q}_{+1}(N)\oplus 0-\delta_{+1}(N)p_N \\
&+\frac{1}{2}\sum_{n=0}^{\infty}\left(\beta(4N+8n+5)-\beta(2N+4n+\frac{7}{2})-\alpha^{-1}(2N+4n+\frac{3}{2})\right)p_{N+2n+1} \\
&+\frac{1}{2}\sum_{n=0}^{\infty}\left(\alpha(4N+8n+9)-\alpha(2N+4n+\frac{11}{2})-\beta^{-1}(2N+4n+\frac{7}{2})\right)p_{N+2n+2} \\
\ge\ &\widehat{Q}_{+1}(N)\oplus 0-\delta_{+1}(N)p_N+\frac{1}{2}\sum_{n=0}^{\infty}(\beta-\alpha^{-1})(2N+4n+\frac{3}{2})p_{N+2n+1} \\
&+\frac{1}{2}\sum_{n=0}^{\infty}(\alpha-\beta^{-1})(2N+4n+\frac{7}{2})p_{N+2n+2}.
\end{aligned}$$

Thus we have $\widehat{Q}_{+1} \geq (\widehat{Q}_{+1}(N) - \delta_{+1}(N)p_N) \oplus (\Lambda_{+1}(N))$. We can obtain the same inequality for odd N. In a similar way, we can furthermore obtain the upper bound $\widehat{Q}_{+1} \leq (\widehat{Q}_{+1}(N) + \delta_{+1}(N)p_N) \oplus R(N)$, where $R(N)$ is an operator such that $R(N) \geq \Lambda_{+1}(N)$. By the min-max principle, we have

$$\begin{aligned}\mu_n((\widehat{Q}_{+1}(N) - \delta_{+1}(N)p_N) \oplus \Lambda_{+1}(N)) &\leq \mu_n(\widehat{Q}_{+1})\\ &\leq \mu_n((\widehat{Q}_{+1}(N) + \delta_{+1}(N)p_N) \oplus R(N)).\end{aligned}$$

Suppose that $\mu_n(\widehat{Q}_{+1}(N) + \delta_{+1}(N)p_N) \leq \Lambda_{+1}(N)$. Then

$$\begin{aligned}\mu_n((\widehat{Q}_{+1}(N) - \delta_{+1}(N)p_N) \oplus \Lambda_{+1}(N)) &= \mu_n(\widehat{Q}_{+1}(N) - \delta_{+1}(N)p_N),\\ \mu_n((\widehat{Q}_{+1}(N) + \delta_{+1}(N)p_N) \oplus R(N)) &= \mu_n(\widehat{Q}_{+1}(N) + \delta_{+1}(N)p_N).\end{aligned}$$

This proves (6.8). □

7. Concluding remarks

We can extend non-commutative harmonic oscillators to an infinite dimensional version. Let $\mathscr{F} = \oplus_{n=0}^{\infty} L^2_{\mathrm{sym}}(\mathbb{R}^n)$ be the boson Fock space, where $L^2_{\mathrm{sym}}(\mathbb{R}^n)$, $n \geq 1$, denotes the set of symmetric square integrable functions, and $L^2(\mathbb{R}^0) = \mathbb{C}$. Let $a(f)$ and $a^*(f)$, $f \in L^2(\mathbb{R})$, be the annihilation operator and the creation operator, respectively, which satisfy canonical commutation relations $[a(f), a^*(g)] = (\bar{f}, g)$, $[a(f), a(g)] = 0 = [a^*(f), a^*(g)]$, and adjoint relation $(a(f))^* = a^*(\bar{f})$. Let $d\Gamma(\omega) = \int \omega(k)a^*(k)a(k)dk$ be the second quantization of a real-valued multiplication ω. The scalar field is defined by $\phi(f) = \frac{1}{\sqrt{2}}(a^*(f) + a(\bar{f}))$ and its momentum conjugate by $\pi(f) = \frac{i}{\sqrt{2}}(a^*(f) - a(\bar{f}))$. Thus we define the self-adjoint operator

$$H = A \otimes d\Gamma(\omega) + J \otimes \left(i\phi(f)\pi(f) + \frac{1}{2}\|f\|^2\right)$$

on $\mathbb{C}^2 \otimes \mathscr{F}$. The spectrum of H is not purely discrete. It is interesting to consider the existence of a ground state of H and to estimate its multiplicity.

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