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Recurrence Relations of the Multi-Indexed Orthogonal Polynomials : II

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Abstract

In a previous paper we presented $3+2M$ term recurrence relations with variable dependent coefficients for M-indexed orthogonal polynomials of Laguerre, Jacobi, Wilson and Askey-Wilson types. In this paper we present (conjectures of) the recurrence relations with constant coefficients for these multi-indexed orthogonal polynomials. The simplest recurrence relations have $3+2\ell$ terms, where $\ell\,(\geq M)$ is the degree of the lowest member of the multi-indexed orthogonal polynomials.

1 Introduction

Exactly solvable quantum mechanical systems in one dimension have seen remarkable developments in recent years and the central role is played by exceptional orthogonal polynomials [1]–[30](and the references therein). A set of polynomials $\{\mathcal{P}_n(\eta)|n\in\mathbb{Z}_{\geq 0}\}$ is called exceptional orthogonal polynomials, when the following conditions (i)–(iii) plus (iv) are satisfied; (i) they are orthogonal with respect to some inner product, (ii) there are missing degrees, i.e., $\{\deg\mathcal{P}_n|n\in\mathbb{Z}_{\geq 0}\}\subsetneqq\mathbb{Z}_{\geq 0}$, (iii) but they form a complete set, and (iv) they satisfy second order differential or difference equations. The constraints of Bochner's theorem and its generalizations [31, 32] are avoided by the condition (ii). We want to distinguish the following two cases; the set of missing degrees $\mathcal{I}=\mathbb{Z}_{\geq 0}\backslash\{\deg\mathcal{P}_n|n\in\mathbb{Z}_{\geq 0}\}$ is case (1): $\mathcal{I}=\{0,1,\ldots,\ell-1\}$, case (2) $\mathcal{I}\neq\{0,1,\ldots,\ell-1\}$, where ℓ is a positive integer. The situation of case (1) is called stable in [12]. The first example of the case (1) exceptional orthogonal polynomials, X_1 Laguerre and Jacobi, was found by Gómez-Ullate, Kamran and Milson [1], and its quantum mechanical formulation was given by Quesne [2]. Based on the quantum mechanical formulation (ordinary quantum mechanics (oQM), discrete quantum mechanics with pure imaginary shifts (idQM)[16]), Sasaki and the present author constructed X_ℓ polynomials and their generalizations, multi-indexed orthogonal polynomials [4, 5, 17, 23]. The multi-indexed orthogonal polynomials of Laguerre, Jacobi, Wilson and Askey-Wilson types, which are obtained by multi-step Darboux transformations [33, 34, 35, 3, 9] with virtual state wavefunctions as seed solutions [17, 23], correspond to the case (1). The exceptional orthogonal polynomials, which are obtained by multi-step Darboux transformations with eigenstate or pseudo virtual state wavefunctions as seed solutions [25, 26, 9, 29], correspond to the case (2). For those having purely discrete orthogonality weight functions, the number of orthogonal polynomials may be finite. The multi-indexed (q-)Racah polynomials [13, 21], which are constructed based on discrete quantum mechanics with real shifts (rdQM) [16], correspond to the case (1).

The ordinary orthogonal polynomials $\{P_n(\eta)|n\in\mathbb{Z}_{\geq 0}, \deg P_n=n\}$ satisfy the three term recurrence relations, $\eta P_n(\eta)=A_nP_{n+1}(\eta)+B_nP_n(\eta)+C_nP_{n-1}(\eta)$ (A_n,B_n,C_n : constants), and conversely the polynomials satisfying the three term recurrence relations are orthogonal polynomials (Favard's theorem [32]). Since the exceptional orthogonal polynomials are not ordinary orthogonal polynomials, they do not satisfy the three term recurrence relations. In a previous paper [27], we showed that M-indexed orthogonal polynomials $P_{\mathcal{D},n}(\eta)$ of Laguerre, Jacobi, Wilson and Askey-Wilson types satisfy $3+2M$ term recurrence relations ($\mathcal{D}=\{d_1,\ldots,d_M\}$),

$$R^{[M]}_{n,0}(\eta)P_{\mathcal{D},n}(\eta)=-\sum_{\substack{k=-M-1\\k\neq 0}}^{M+1}R^{[M]}_{n,k}(\eta)P_{\mathcal{D},n+k}(\eta), \tag{1.1}$$

where $R^{[M]}_{n,k}(\eta)$'s are polynomials of degree $M+1-|k|$ in η. In contrast to the three term recurrence relations, the coefficients of (1.1) are not constants. The three term recurrence relations are used to study bispectral properties or dual polynomials [32, 36], in which the constant coefficients of the recurrence relations are important. To study bispectral properties, recurrence relations with constant coefficients are desired,

$$X(\eta)P_{\mathcal{D},n}(\eta)=\sum_{k=-L}^{L}r^{X,\mathcal{D}}_{n,k}P_{\mathcal{D},n+k}(\eta)\quad(\forall n\in\mathbb{Z}_{\geq 0}), \tag{1.2}$$

where $r^{X,\mathcal{D}}_{n,k}$'s are constants and $X(\eta)$ is some polynomial of degree L in η. Such recurrence relations for $M=1$ case were first given by Sasaki, Tsujimoto and Zhedanov [11]. They found $1+4\ell$ term recurrence relations. Recently Miki and Tsujimoto found different recurrence relations with $3+2\ell$ terms [37]. Their choices of $X(\eta)$ are $\Xi_\ell(\eta)^2$ and $\int_0^\eta \Xi_\ell(y)dy$, respectively. Durán studied recurrence relations with constant coefficients for several exceptional orthogonal polynomials including exceptional Laguerre polynomials by using duality [38]. The exceptional Laguerre polynomials in [38] correspond to eigenstates and type I virtual states deletion, and our multi-indexed polynomials correspond to type I and II virtual states deletion (see § 4).

In this paper we present infinitely many (conjectures of) recurrence relations with constant coefficients for M-indexed orthogonal polynomials of Laguerre, Jacobi, Wilson and Askey-Wilson types, namely we present infinitely many polynomials $X(\eta)$ leading to (1.2). The minimal degree of $X(\eta)$ is (conjectured as) $\ell_{\mathcal{D}}+1$, where $\ell_{\mathcal{D}}$ is the degree of the lowest member multi-indexed orthogonal polynomial $P_{\mathcal{D},0}(\eta)$, and this gives $3+2\ell_{\mathcal{D}}$ term recurrence relations with constant coefficients.

This paper is organized as follows. In section 2 we recapitulate some fundamental formulas of the multi-indexed orthogonal polynomials and present a method deriving recurrence relations with constant coefficients. Section 3 is the main part of the paper. After discussing a necessary condition for $X(\eta)$, we present (conjectures of) recurrence relations with constant coefficients for the multi-indexed orthogonal polynomials of Laguerre, Jacobi, Wilson and Askey-Wilson types, Conjecture 1 and Conjecture 2. The final section is for a summary and comments. Some useful formulas of the multi-indexed orthogonal polynomials of Laguerre, Jacobi, Wilson and Askey-Wilson types are listed in Appendix A. Some examples are presented in Appendix B.

2 Method

In this section we explain an idea for deriving the recurrence relations with constant coefficients. We follow the notation of [27]. The virtual state wavefunction $\tilde{\phi}(x)$ is characterized by the degree v and the type t (I or II), like $\tilde{\phi}^{\mathrm{t}}_{\mathrm{v}}(x)$. For simplicity, we suppress type t in many places.

The fundamental formulas of the multi-indexed orthogonal polynomials of Laguerre, Jacobi, Wilson and Askey-Wilson types are found in [27]. Among them we recall that

$$\hat{\mathcal{A}}_{d_1\ldots d_s}\phi_{d_1\ldots d_{s-1}\,n}(x)=\phi_{d_1\ldots d_s\,n}(x),\quad \hat{\mathcal{A}}^{\dagger}_{d_1\ldots d_s}\phi_{d_1\ldots d_s\,n}(x)=(\mathcal{E}_n-\tilde{\mathcal{E}}_{d_s})\phi_{d_1\ldots d_{s-1}\,n}(x),\qquad(2.1)$$

$$\phi_{d_1\ldots d_s\,n}(x)=\Psi_{d_1\ldots d_s}(x)P_{d_1\ldots d_s,n}\bigl(\eta(x)\bigr)\ \ (n\in\mathbb{Z}_{\geq 0}),\quad P_{d_1\ldots d_s,n}(\eta)\stackrel{\text{def}}{=}0\ \ (n<0), \tag{2.2}$$

$$\deg P_{d_1\ldots d_s,n}(\eta)=\ell_{d_1\ldots d_s}+n,\quad \deg\Xi_{d_1\ldots d_s}(\eta)=\ell_{d_1\ldots d_s},\quad d_j>0,$$

$$\ell_{d_1\ldots d_s}=\sum_{j=1}^{s}d_j-\tfrac{1}{2}s(s-1)+2s_{\text{I}}s_{\text{II}},\quad s_{\text{t}}=\#\{d_j|d_j\colon\text{type t},\,j=1,\ldots,s\}\ (\text{t}=\text{I},\text{II}), \tag{2.3}$$

$$(\phi_{d_1\ldots d_s\,n},\phi_{d_1\ldots d_s\,m})=(\Psi_{d_1\ldots d_s}^2P_{d_1\ldots d_s,n},P_{d_1\ldots d_s,m})=h_{d_1\ldots d_s,n}\delta_{nm},$$

$$h_{d_1\ldots d_s,n}=\prod_{j=1}^{s}(\mathcal{E}_n-\tilde{\mathcal{E}}_{d_j})\cdot h_n. \tag{2.4}$$

The relations (2.1) are rewritten by using the step forward ($\hat{\mathcal{F}}$) and backward ($\hat{\mathcal{B}}$) shift operators as

$$\hat{\mathcal{F}}_{d_1\ldots d_s}P_{d_1\ldots d_{s-1},n}(\eta)=P_{d_1\ldots d_s,n}(\eta),\quad \hat{\mathcal{B}}_{d_1\ldots d_s}P_{d_1\ldots d_s,n}(\eta)=(\mathcal{E}_n-\tilde{\mathcal{E}}_{d_s})P_{d_1\ldots d_{s-1},n}(\eta). \tag{2.5}$$

Here $\hat{\mathcal{F}}_{d_1\ldots d_s}$ and $\hat{\mathcal{B}}_{d_1\ldots d_s}$ are defined by

$$\hat{\mathcal{F}}_{d_1\ldots d_s}\stackrel{\text{def}}{=}\Psi_{d_1\ldots d_s}(x)^{-1}\circ\hat{\mathcal{A}}_{d_1\ldots d_s}\circ\Psi_{d_1\ldots d_{s-1}}(x),\quad \hat{\mathcal{B}}_{d_1\ldots d_s}\stackrel{\text{def}}{=}\Psi_{d_1\ldots d_{s-1}}(x)^{-1}\circ\hat{\mathcal{A}}_{d_1\ldots d_s}^{\dagger}\circ\Psi_{d_1\ldots d_s}(x), \tag{2.6}$$

and their explicit forms are given in (A.1)–(A.2) and (A.8)–(A.9). This gives Rodrigues type formula, $P_{d_1\ldots d_s,n}(\eta)=\hat{\mathcal{F}}_{d_1\ldots d_s}\cdots\hat{\mathcal{F}}_{d_1d_2}\hat{\mathcal{F}}_{d_1}P_n(\eta)$. These formulas were not presented explicitly in our previous papers. We remark that, from their explicit forms, $\hat{\mathcal{F}}_{d_1\ldots d_s}$ and $\hat{\mathcal{B}}_{d_1\ldots d_s}$ map rational functions of η to rational functions of η. For an appropriate parameter range (for example, see [17, 23, 39]), the Hamiltonians $\mathcal{H}_{d_1\ldots d_s}$ are non-singular and their eigenfunctions $\{\phi_{d_1\ldots d_s\,n}(x)\}_{n=0}^{\infty}$ form a complete set of the Hilbert space. For any polynomial $X(\eta)$ in η, the function $X\bigl(\eta(x)\bigr)\phi_{d_1\ldots d_s\,n}(x)$ belongs to the Hilbert space.

The Hamiltonian $\mathcal{H}_{d_1\ldots d_s}$ does not depend on the order of d_j's. On the other hand, the multi-indexed orthogonal polynomial $P_{d_1\ldots d_s,n}(\eta)$ changes the sign under a permutation of d_j's, $P_{d_{\sigma_1}\ldots d_{\sigma_s},n}(\eta)=\text{sgn}\bigl(\begin{smallmatrix}1&\ldots&s\\ \sigma_1&\ldots&\sigma_s\end{smallmatrix}\bigr)P_{d_1\ldots d_s,n}(\eta)$. The denominator polynomial $\Xi_{d_1\ldots d_s}(\eta)$ also changes the sign, $\Xi_{d_{\sigma_1}\ldots d_{\sigma_s}}(\eta)=\text{sgn}\bigl(\begin{smallmatrix}1&\ldots&s\\ \sigma_1&\ldots&\sigma_s\end{smallmatrix}\bigr)\Xi_{d_1\ldots d_s}(\eta)$. We write $\mathcal{H}_{d_1\ldots d_M}$, $\phi_{d_1\ldots d_M\,n}(x)$, $P_{d_1\ldots d_M,n}(\eta)$, $\Xi_{d_1\ldots d_M}(\eta)$, $h_{d_1\ldots d_M,n}$, etc. as $\mathcal{H}_{\mathcal{D}}$, $\phi_{\mathcal{D}\,n}(x)$, $P_{\mathcal{D},n}(\eta)$, $\Xi_{\mathcal{D}}(\eta)$, $h_{\mathcal{D},n}$, etc., respectively ($\mathcal{D}=\{d_1,\ldots,d_M\}$).[1]

First we note the following property of orthogonal polynomials.

[1] In the Appendix of [27], we assumed the 'standard order' $\mathcal{D}=\{d_1^{\text{I}},\ldots,d_{M_{\text{I}}}^{\text{I}},d_1^{\text{II}},\ldots,d_{M_{\text{II}}}^{\text{II}}\}$ for simplicity. We do not assume it in this paper.

Lemma 1 *Let us assume for a certain polynomial $X(\eta)$ of degree L in η that*

$$X(\eta)P_{\mathcal{D},n}(\eta)=\sum_{k=-n}^{L}r_{n,k}^{X,\mathcal{D}}P_{\mathcal{D},n+k}(\eta)\quad(\forall n\in\mathbb{Z}_{\geq 0}).\tag{2.7}$$

Here $r_{n,k}^{X,\mathcal{D}}$'s are constants. The sum $\sum\limits_{k=-n}^{L}$ can be replaced by $\sum\limits_{k=-L}^{L}$.

<u>Proof</u> Multiplying by $\Psi_{\mathcal{D}}(x)$ to (2.7), we have

$$X(\eta)\phi_{\mathcal{D},n}(x)=\sum_{k=-n}^{L}r_{n,k}^{X,\mathcal{D}}\phi_{\mathcal{D}\,n+k}(x).$$

By using (2.4) we have

$$\begin{aligned}(\phi_{\mathcal{D}\,m},X\phi_{\mathcal{D}\,n})&=\sum_{k=-n}^{L}r_{n,k}^{X,\mathcal{D}}h_{\mathcal{D},m}\delta_{m,n+k}=\theta(m\leq n+L)\,r_{n,m-n}^{X,\mathcal{D}}h_{\mathcal{D},m}\\&=(X\phi_{\mathcal{D}\,m},\phi_{\mathcal{D}\,n})=\sum_{k=-m}^{L}r_{m,k}^{X,\mathcal{D}}h_{\mathcal{D},n}\delta_{n,m+k}=\theta(m\geq n-L)\,r_{m,n-m}^{X,\mathcal{D}}h_{\mathcal{D},n},\end{aligned}\tag{2.8}$$

where $\theta(P)$ is a step function for a proposition P, $\theta(P)=1$ (P : true), 0 (P : false). This means $(\phi_{\mathcal{D}\,m},X\phi_{\mathcal{D}\,n})=0$ unless $n-L\leq m\leq n+L$. Namely $r_{n,k}^{X,\mathcal{D}}=0$ unless $-L\leq k\leq L$.

□

Remark Although the inner product formulas used in the proof are valid only for 'real' $X(\eta)$ ($X^*=X$) and an appropriate parameter range such that the Hamiltonian is non-singular, the final result, which represents the polynomial equations, is valid for any parameter values and complex $X(\eta)$.

Next we explain a method to obtain recurrence relations with constant coefficients. Let $X(\eta)$ be a polynomial of degree L in η. Since $X(\eta)\phi_{\mathcal{D}\,n}(x)$ belongs to the Hilbert space and $\{\phi_{\mathcal{D}\,n}(x)\}_{n=0}^{\infty}$ is a complete set, we have the expansion

$$X(\eta)\phi_{\mathcal{D}\,n}(x)=\sum_{k=-n}^{\infty}r_{n,k}^{X,\mathcal{D}}\phi_{\mathcal{D}\,n+k}(x),\tag{2.9}$$

where $r_{n,k}^{X,\mathcal{D}}$'s are constants. By using this and (2.4), we obtain

$$(\phi_{\mathcal{D}\,m},X\phi_{\mathcal{D}\,n})=\sum_{k=-n}^{\infty}r_{n,k}^{X,\mathcal{D}}h_{\mathcal{D},m}\delta_{m,n+k}=r_{n,m-n}^{X,\mathcal{D}}h_{\mathcal{D},m}.\tag{2.10}$$

On the other hand, by using (2.1)–(2.2) and (2.6), we obtain

$$
\begin{aligned}
&(\phi_{\mathcal{D}\,m}, X\phi_{\mathcal{D}\,n})\\
&= (\hat{\mathcal{A}}_{d_1\ldots d_M}\cdots\hat{\mathcal{A}}_{d_1d_2}\hat{\mathcal{A}}_{d_1}\phi_m, X\phi_{\mathcal{D}\,n})\\
&= \bigl(\phi_m, \hat{\mathcal{A}}_{d_1}^{\dagger}\hat{\mathcal{A}}_{d_1d_2}^{\dagger}\cdots\hat{\mathcal{A}}_{d_1\ldots d_M}^{\dagger}(X\phi_{\mathcal{D}\,n})\bigr)\\
&= \bigl(\phi_0P_m, (\phi_0\hat{\mathcal{B}}_{d_1}\Psi_{d_1}^{-1})(\Psi_{d_1}\hat{\mathcal{B}}_{d_1d_2}\Psi_{d_1d_2}^{-1})\cdots(\Psi_{d_1\ldots d_{M-1}}\hat{\mathcal{B}}_{d_1\ldots d_M}\Psi_{d_1\ldots d_M}^{-1})(\Psi_{d_1\ldots d_M}XP_{\mathcal{D},n})\bigr)\\
&= \bigl(\phi_0P_m, \phi_0\hat{\mathcal{B}}_{d_1}\hat{\mathcal{B}}_{d_1d_2}\cdots\hat{\mathcal{B}}_{d_1\ldots d_M}(XP_{\mathcal{D},n})\bigr)\\
&= \bigl(\phi_0^2P_m, \hat{\mathcal{B}}_{d_1}\hat{\mathcal{B}}_{d_1d_2}\cdots\hat{\mathcal{B}}_{d_1\ldots d_M}(XP_{\mathcal{D},n})\bigr).
\end{aligned}
\tag{2.11}
$$

From the property of $\hat{\mathcal{B}}_{d_1\ldots d_s}$, the function $\hat{\mathcal{B}}_{d_1}\hat{\mathcal{B}}_{d_1d_2}\cdots\hat{\mathcal{B}}_{d_1\ldots d_M}(XP_{\mathcal{D},n})$ is a rational function of η. When it is not a polynomial in η, we have infinitely many m such that $(\phi_{\mathcal{D}\,m}, X\phi_{\mathcal{D}\,n}) \neq 0$, namely r.h.s. of (2.9) is an infinite sum. Let us consider the case that $\hat{\mathcal{B}}_{d_1}\hat{\mathcal{B}}_{d_1d_2}\cdots\hat{\mathcal{B}}_{d_1\ldots d_M}$ $(XP_{\mathcal{D},n})$ is a polynomial of degree $n+L'$ in η. Since any polynomial in η can be expanded in $P_n(\eta)$, we have

$$
\hat{\mathcal{B}}_{d_1}\hat{\mathcal{B}}_{d_1d_2}\cdots\hat{\mathcal{B}}_{d_1\ldots d_M}(XP_{\mathcal{D},n}) = \sum_{k=-n}^{L'} r_{n,k}^{(0)\,X,\mathcal{D}}P_{n+k}(\eta),
\tag{2.12}
$$

where $r_{n,k}^{(0)\,X,\mathcal{D}}$'s are constants. Substituting this to (2.11), we obtain

$$
(\phi_{\mathcal{D}\,m}, X\phi_{\mathcal{D}\,n}) = \sum_{k=-n}^{L'} r_{n,k}^{(0)\,X,\mathcal{D}}h_m\delta_{m\,n+k}, = \theta(m \le L'+n)\, r_{n,m-n}^{(0)X,\mathcal{D}}h_m.
\tag{2.13}
$$

Eqs.(2.10) and (2.13) imply

$$
r_{n,k}^{X,\mathcal{D}} = 0 \quad (k > L'), \qquad r_{n,k}^{X,\mathcal{D}}h_{\mathcal{D},n+k} = r_{n,k}^{(0)\,X,\mathcal{D}}h_{n+k} \quad (-n \le k \le L').
\tag{2.14}
$$

Thus we obtain

$$
X(\eta)\phi_{\mathcal{D}\,n}(x) = \sum_{k=-n}^{L'} r_{n,k}^{X,\mathcal{D}}\phi_{\mathcal{D}\,n+k}(x),
\tag{2.15}
$$

namely,

$$
X(\eta)P_{\mathcal{D},n}(\eta) = \sum_{k=-n}^{L'} r_{n,k}^{X,\mathcal{D}}P_{\mathcal{D},n+k}(\eta).
\tag{2.16}
$$

By comparing the degree of both sides, we have $L'=L$. By Lemma 1, the sum $\sum\limits_{k=-n}^{L}$ can be replaced by $\sum\limits_{k=-L}^{L}$.

We summarize this argument as the following proposition.

Proposition 1 *Let us assume for a certain polynomial $X(\eta)$ of degree L in η that the function $\hat{\mathcal{B}}_{d_1}\hat{\mathcal{B}}_{d_1d_2}\cdots\hat{\mathcal{B}}_{d_1\ldots d_M}(XP_{\mathcal{D},n})$ is a polynomial in η. Expand it as (2.12). We have $1+2L$ term recurrence relations with constant coefficients for $P_{\mathcal{D},n}(\eta)$:*

$$X(\eta)P_{\mathcal{D},n}(\eta)=\sum_{k=-L}^{L}r_{n,k}^{X,\mathcal{D}}P_{\mathcal{D},n+k}(\eta)\quad(\forall n\in\mathbb{Z}_{\geq 0}),\quad r_{n,k}^{X,\mathcal{D}}=\frac{r_{n,k}^{(0)\,X,\mathcal{D}}}{\prod_{j=1}^{M}(\mathcal{E}_{n+k}-\tilde{\mathcal{E}}_{d_j})}. \tag{2.17}$$

Remark 1 See Remark below Lemma 1.

Remark 2 Under the assumption of this proposition, the functions $\hat{\mathcal{B}}_{d_1\ldots d_j}\hat{\mathcal{B}}_{d_1\ldots d_jd_{j+1}}\cdots\hat{\mathcal{B}}_{d_1\ldots d_M}$ $(XP_{\mathcal{D},n})$ $(j=2,\ldots,M)$ are also polynomials in η.

Remark 3 The function $\hat{\mathcal{B}}_{d_1}\hat{\mathcal{B}}_{d_1d_2}\cdots\hat{\mathcal{B}}_{d_1\ldots d_M}(XP_{\mathcal{D},n})$ is rewritten as

$$\hat{\mathcal{B}}_{d_1}\hat{\mathcal{B}}_{d_1d_2}\cdots\hat{\mathcal{B}}_{d_1\ldots d_M}(XP_{\mathcal{D},n})=(\hat{\mathcal{B}}_{d_1}\hat{\mathcal{B}}_{d_1d_2}\cdots\hat{\mathcal{B}}_{d_1\ldots d_M}X\hat{\mathcal{F}}_{d_1\ldots d_M}\cdots\hat{\mathcal{F}}_{d_1d_2}\hat{\mathcal{F}}_{d_1})P_n. \tag{2.18}$$

This operator $\hat{\mathcal{B}}_{d_1}\cdots X\cdots\hat{\mathcal{F}}_{d_1}$ maps polynomials in η to rational functions of η. To find a proper polynomial $X(\eta)$ giving recurrence relations with constant coefficients is rephrased as follows; Find a polynomial $X(\eta)$ such that the operator $\hat{\mathcal{B}}_{d_1}\hat{\mathcal{B}}_{d_1d_2}\cdots\hat{\mathcal{B}}_{d_1\ldots d_M}X\hat{\mathcal{F}}_{d_1\ldots d_M}$ $\cdots\hat{\mathcal{F}}_{d_1d_2}\hat{\mathcal{F}}_{d_1}$ maps polynomials in η to polynomials in η.

3 Recurrence Relations with Constant Coefficients

In this section we present recurrence relations with constant coefficients (1.2) for the multi-indexed orthogonal polynomials of Laguerre, Jacobi, Wilson and Askey-Wilson types.

3.1 Multi-indexed Laguerre and Jacobi polynomials

In this subsection we discuss the recurrence relations with constant coefficients for the multi-indexed Laguerre and Jacobi polynomials. We note that the first order differential operator of the form $a(x)\frac{d}{dx}+b(x)$ $(a(x),b(x)$: functions of $x)$ acts on the product of two functions $f(x)$ and $g(z)$ as

$$\Bigl(a(x)\frac{d}{dx}+b(x)\Bigr)\bigl(f(x)g(x)\bigr)=f(x)\Bigl(a(x)\frac{d}{dx}+b(x)\Bigr)g(x)+a(x)\frac{df(x)}{dx}g(x). \tag{3.1}$$

First we consider a necessary condition for $X(\eta)$ giving recurrence relations with constant coefficients. Let us assume (2.7) for a polynomial $X(\eta)$ of degree L in η. Applying $\hat{\mathcal{B}}_{\mathcal{D}}=$

$\hat{\mathcal{B}}_{d_1\ldots d_M}$ (A.2) to (2.7), we have

$$
\begin{aligned}
&\hat{\mathcal{B}}_{\mathcal{D}}\big(X(\eta)P_{\mathcal{D},n}(\eta)\big)=\sum_{k=-n}^{L}r_{n,k}^{X,\mathcal{D}}\hat{\mathcal{B}}_{\mathcal{D}}P_{\mathcal{D},n+k}(\eta)=\sum_{k=-n}^{L}r_{n,k}^{X,\mathcal{D}}(\mathcal{E}_{n+k}-\tilde{\mathcal{E}}_{d_M})P_{d_1\ldots d_{M-1},n}(\eta)\\
&=X(\eta)\hat{\mathcal{B}}_{\mathcal{D}}P_{\mathcal{D},n}(\eta)-c_{\mathcal{F}}^2e_{\mathcal{D}}^{\hat{\mathcal{B}}}(\eta)\frac{\Xi_{d_1\ldots d_{M-1}}(\eta)}{\Xi_{\mathcal{D}}(\eta)}\frac{dX(\eta)}{d\eta}P_{\mathcal{D},n}(\eta)\\
&=(\mathcal{E}_n-\tilde{\mathcal{E}}_{d_M})X(\eta)P_{d_1\ldots d_{M-1},n}(\eta)-c_{\mathcal{F}}^2e_{\mathcal{D}}^{\hat{\mathcal{B}}}(\eta)\frac{\Xi_{d_1\ldots d_{M-1}}(\eta)}{\Xi_{\mathcal{D}}(\eta)}\frac{dX(\eta)}{d\eta}P_{\mathcal{D},n}(\eta),
\end{aligned}
\tag{3.2}
$$

where (3.1) and (2.5) are used. Since the expression in the first line is a polynomial in η, the expression in the last line should be so. The denominator polynomial $\Xi_{\mathcal{D}}(\eta)$ does not have common roots with $e_{\mathcal{D}}^{\hat{\mathcal{B}}}(\eta)$ and $P_{\mathcal{D},n}(\eta)$ for some n. Therefore, if $\Xi_{\mathcal{D}}(\eta)=\Xi_{d_1\ldots d_M}(\eta)$ does not have common roots with $\Xi_{d_1\ldots d_{M-1}}(\eta)$, the polynomial $\frac{dX(\eta)}{d\eta}$ should be divisible by $\Xi_{\mathcal{D}}(\eta)$.

We summarize this argument as follows.

Proposition 2 *Let $X(\eta)$ be a polynomial of degree L in η. Assume* (2.7) *and*

$$
\frac{dX(\eta)}{d\eta}=\Xi_{\mathcal{D}}(\eta)Y(\eta),\quad Y(\eta):\text{ a polynomial in }\eta. \tag{3.3}
$$

Then one action of $\hat{\mathcal{B}}_{\mathcal{D}}$ to the both sides of (2.7) *keeps the polynomiality intact.*

Remark If two polynomials in η, $\Xi_{\mathcal{D}}(\eta)=\Xi_{d_1\ldots d_M}(\eta)$ and $\Xi_{d_1\ldots d_{M-1}}(\eta)$, do not have common roots, the polynomial of degree L in η, $X(\eta)$, satisfying (2.7) should satisfy (3.3) for some polynomial $Y(\eta)$.

The overall normalization and the constant term of $X(\eta)$ are not important, because the change of the former induces that of the overall normalization of $r_{n,k}^{X,\mathcal{D}}$ and the shift of the latter induces that of $r_{n,0}^{X,\mathcal{D}}$. By taking the constant term of $X(\eta)$ as $X(0)=0$, the condition for the candidate of $X(\eta)$ (3.3) gives

$$
X(\eta)=\int_0^{\eta}\Xi_{\mathcal{D}}(y)Y(y)dy,\quad \deg X(\eta)=L=\ell_{\mathcal{D}}+\deg Y(\eta)+1. \tag{3.4}
$$

The minimal degree candidate of $X(\eta)$, which corresponds to $Y(\eta)=1$, is

$$
X_{\min}(\eta)=\int_0^{\eta}\Xi_{\mathcal{D}}(y)dy,\quad \deg X_{\min}(\eta)=\ell_{\mathcal{D}}+1. \tag{3.5}
$$

Based on these properties we present our main result. After one action of $\hat{\mathcal{B}}_{\mathcal{D}}$ to (2.7), further actions of $\hat{\mathcal{B}}_{d_1\ldots d_{M-1}},\hat{\mathcal{B}}_{d_1\ldots d_{M-2}},\ldots$ give more conditions for $X(\eta)$. However, it seems that these additional conditions are satisfied automatically by the original condition (3.3) and by the properties of $P_{d_1\ldots d_s,n}(\eta)$ and $\Xi_{d_1\ldots d_s}(\eta)$, for example, see the proof of $M=2$ case in Remark 5 below. We conjecture that this candidate $X(\eta)$ (3.4) actually gives recurrence relations with constant coefficients.

Conjecture 1 *For any polynomial $Y(\eta)$, we take $X(\eta)$ as* (3.4)*. Then the multi-indexed Laguerre and Jacobi polynomials $P_{\mathcal{D},n}(\eta)$ satisfy $1+2L$ term recurrence relations with constant coefficients* (1.2)*.*

Remark 1 If two polynomials in η, $\Xi_{\mathcal{D}}(\eta)=\Xi_{d_1\ldots d_M}(\eta)$ and $\Xi_{d_1\ldots d_{M-1}}(\eta)$, do not have common roots, this conjecture exhausts all possible $X(\eta)$ giving recurrence relations with constant coefficients, and the minimal degree choice $X(\eta)=X_{\min}(\eta)$ (3.5) gives $3+2\ell_{\mathcal{D}}$ term recurrence relations.

Remark 2 The minimal degree polynomial $X_{\min}(\eta)$ can be divisible by η. The degree of $\frac{X_{\min}(\eta)}{\eta}$ is $\ell_{\mathcal{D}}$, which is the lowest degree of $P_{\mathcal{D},n}(\eta)$. The recurrence relations (1.2), $\frac{X_{\min}(\eta)}{\eta}\times\eta P_{\mathcal{D},n}(\eta)=\sum\cdots$, can be regarded as a natural generalization of the three term recurrence relations of the ordinary orthogonal polynomial $P_n(\eta)$, $1\times\eta P_n(\eta)=A_nP_{n+1}(\eta)+B_nP_n(\eta)+C_nP_{n-1}(\eta)$.

Remark 3 Since $Y(\eta)$ is arbitrary, we obtain infinitely many recurrence relations. However not all of them are independent. For '$M=0$ case' (namely, ordinary orthogonal polynomials), it is trivial that recurrence relations obtained from arbitrary $Y(\eta)$ ($\deg Y\geq 1$) are derived by the three term recurrence relations.

Remark 4 For $M=1$ case ($\mathcal{D}=\{\ell\}$), the minimal degree choice $X(\eta)=X_{\min}(\eta)$, which gives $3+2\ell$ term recurrence relations, was given by Miki and Tsujimoto [37], and the choice $X(\eta)=\Xi_\ell(\eta)^2$, which corresponds to $Y(\eta)=2\Xi_\ell(\eta)$ and gives $1+4\ell$ term recurrence relations, was given by Sasaki, Tsujimoto and Zhedanov [11]. For general M, $Y(\eta)=2\partial_\eta\Xi_{\mathcal{D}}(\eta)p(\eta)+\Xi_{\mathcal{D}}(\eta)\partial_\eta p(\eta)$, where $p(\eta)$ is any polynomial in η, gives $X(\eta)=\Xi_{\mathcal{D}}(\eta)^2p(\eta)$.

Remark 5 Direct verification of this conjecture is rather straightforward for lower M and smaller d_j, n and $\deg Y$, by a computer algebra system, e.g. Mathematica. The coefficients $r_{n,k}^{X,\mathcal{D}}$ are explicitly obtained for small d_j and n. However, to obtain the closed expression of $r_{n,k}^{X,\mathcal{D}}$ for general n is not an easy task even for small d_j, and it is a different kind of problem. We present some examples in Appendix B.

Remark 6 For $M=1,2$ we can prove this conjecture. Since we have Proposition 1, it is sufficient to show that $\hat{\mathcal{B}}_{d_1}\hat{\mathcal{B}}_{d_1d_2}\cdots\hat{\mathcal{B}}_{d_1\ldots d_M}(XP_{\mathcal{D},n})$ is a polynomial in η.

$\underline{M=1}$ From (3.2) we have

$$\hat{\mathcal{B}}_{d_1}\big(X(\eta)P_{d_1,n}(\eta)\big)=(\mathcal{E}_n-\tilde{\mathcal{E}}_{d_1})X(\eta)P_n(\eta)-c_{\mathcal{F}}^2e_{d_1}^{\hat{\mathcal{B}}}(\eta)Y(\eta)P_{d_1,n}(\eta).$$

This is a polynomial in η. Thus $M=1$ case is proved. $\square$

$\underline{M=2}$ From (3.2) and (A.2) we have

$$\begin{aligned}
&\hat{\mathcal{B}}_{d_1}\hat{\mathcal{B}}_{d_1d_2}\big(X(\eta)P_{d_1d_2,n}(\eta)\big)\\
&=\hat{\mathcal{B}}_{d_1}\Big((\mathcal{E}_n-\tilde{\mathcal{E}}_{d_2})X(\eta)P_{d_1,n}(\eta)-c_{\mathcal{F}}^2e_{d_1d_2}^{\hat{\mathcal{B}}}(\eta)\Xi_{d_1}(\eta)Y(\eta)P_{d_1d_2,n}(\eta)\Big)\\
&=(\mathcal{E}_n-\tilde{\mathcal{E}}_{d_2})\Big((\mathcal{E}_n-\tilde{\mathcal{E}}_{d_1})X(\eta)P_n(\eta)-c_{\mathcal{F}}^2\frac{e_{d_1}^{\hat{\mathcal{B}}}(\eta)}{\Xi_{d_1}(\eta)}\Xi_{d_1d_2}(\eta)Y(\eta)P_{d_1,n}(\eta)\Big)\\
&\quad+c_{\mathcal{F}}^4e_{d_1}^{\hat{\mathcal{B}}}(\eta)\partial_\eta\big(e_{d_1d_2}^{\hat{\mathcal{B}}}(\eta)Y(\eta)P_{d_1d_2,n}(\eta)\big)+c_{\mathcal{F}}^4\frac{e_{d_1}^{\hat{\mathcal{B}}}(\eta)}{\Xi_{d_1}(\eta)}e_{d_1d_2}^{\hat{\mathcal{B}}}(\eta)\partial_\eta\Xi_{d_1}(\eta)Y(\eta)P_{d_1d_2,n}(\eta)\\
&\quad-c_{\mathcal{F}}^3\tilde{e}_{d_1}^{\hat{\mathcal{B}}}e_{d_1d_2}^{\hat{\mathcal{B}}}(\eta)Y(\eta)P_{d_1d_2,n}(\eta).
\end{aligned}$$

By using (A.6) with $s=2$, this becomes

$$\begin{aligned}
&=(\mathcal{E}_n-\tilde{\mathcal{E}}_{d_2})(\mathcal{E}_n-\tilde{\mathcal{E}}_{d_1})X(\eta)P_n(\eta)\\
&\quad+c_{\mathcal{F}}^4e_{d_1}^{\hat{\mathcal{B}}}(\eta)\partial_\eta\big(e_{d_1d_2}^{\hat{\mathcal{B}}}(\eta)Y(\eta)P_{d_1d_2,n}(\eta)\big)+c_{\mathcal{F}}^4e_{d_1}^{\hat{\mathcal{B}}}(\eta)e_{d_1d_2}^{\hat{\mathcal{B}}}(\eta)Y(\eta)\partial_\eta P_{d_1d_2,n}(\eta)\\
&\quad-c_{\mathcal{F}}^3\big(\tilde{e}_{d_1}^{\hat{\mathcal{B}}}e_{d_1d_2}^{\hat{\mathcal{B}}}(\eta)+e_{d_1}^{\hat{\mathcal{B}}}(\eta)\tilde{e}_{d_1d_2}^{\hat{\mathcal{B}}}\big)Y(\eta)P_{d_1d_2,n}(\eta).
\end{aligned}$$

This is a polynomial in η. Thus $M=2$ case is proved. $\square$

3.2 Multi-indexed Wilson and Askey-Wilson polynomials

In this subsection we discuss the recurrence relations with constant coefficients for the multi-indexed Wilson and Askey-Wilson polynomials. We restrict the parameters: $\{a_1^*,a_2^*\}=\{a_1,a_2\}$ (as a set) and $\{a_3^*,a_4^*\}=\{a_3,a_4\}$ (as a set).

The sinusoidal coordinate $\eta(x)$ is $\eta(x)=x^2$ for Wilson case and $\eta(x)=\cos x$ for Askey-Wilson case. They satisfy [40]

$$\frac{\eta(x-i\frac{\gamma}{2})^{n+1}-\eta(x+i\frac{\gamma}{2})^{n+1}}{\eta(x-i\frac{\gamma}{2})-\eta(x+i\frac{\gamma}{2})}=\sum_{k=0}^n g_n'^{(k)}\eta(x)^{n-k}\quad(n\in\mathbb{Z}_{\geq0}),\tag{3.6}$$

where $g_n'^{(k)}$ is given by [41]

$$\eta(x)=x^2:\quad g_n'^{(k)}=\frac{(-1)^k}{2^{2k+1}}\binom{2n+2}{2k+1},$$

$$\eta(x)=\cos x:\quad g_n'^{(k)}=\theta(k:\text{ even})\frac{(n+1)!}{2^k}\sum_{r=0}^{\frac{k}{2}}\binom{n-k+r}{r}\frac{(-1)^r[\![n-k+1+2r]\!]'}{(\frac{k}{2}-r)!\,(n-\frac{k}{2}+1+r)!},$$
$$[\![n]\!]'\stackrel{\text{def}}{=}\frac{e^{-\frac{\gamma}{2}n}-e^{\frac{\gamma}{2}n}}{e^{-\frac{\gamma}{2}}-e^{\frac{\gamma}{2}}}.\tag{3.7}$$

For a polynomial $p(\eta)$ in η, when it is regarded as a function of x, we denote it by adding a check,

$$\check{p}(x)\stackrel{\text{def}}{=}p\bigl(\eta(x)\bigr).\tag{3.8}$$

Since $\eta(x-i\frac{m}{2}\gamma)+\eta(x+i\frac{m}{2}\gamma)$ and $\eta(x-i\frac{m}{2}\gamma)\eta(x+i\frac{m}{2}\gamma)$ $(m\in\mathbb{Z})$ are expressed as polynomials in $\eta(x)$, any symmetric polynomial in $\eta(x-i\frac{m}{2}\gamma)$ and $\eta(x+i\frac{m}{2}\gamma)$ is expressed as a polynomial in $\eta(x)$ [40, 27]. For example, the followings are polynomials in $\eta(x)$ (p,p_1,p_2 : polynomials in η):

$$\begin{aligned}
&\check{p}(x-i\tfrac{\gamma}{2})+\check{p}(x+i\tfrac{\gamma}{2}),\quad \frac{\check{p}(x-i\frac{\gamma}{2})-\check{p}(x+i\frac{\gamma}{2})}{\eta(x-i\frac{\gamma}{2})-\eta(x+i\frac{\gamma}{2})},\\
&\check{p}(x-i\tfrac{\gamma}{2})\check{p}(x+i\tfrac{\gamma}{2}),\quad \frac{\check{p}_1(x-i\gamma)\check{p}_2(x-i\frac{\gamma}{2})-\check{p}_1(x+i\gamma)\check{p}_2(x+i\frac{\gamma}{2})}{\eta(x-i\frac{\gamma}{2})-\eta(x+i\frac{\gamma}{2})},\\
&\eta(x)=x^2:\quad x\check{p}_1(x+i\tfrac{\gamma}{2})\check{p}_2(x-i\tfrac{\gamma}{2})+x\check{p}_1(x-i\tfrac{\gamma}{2})\check{p}_2(x+i\tfrac{\gamma}{2}),\\
&\eta(x)=\cos x:\quad e^{\pm ix}\check{p}_1(x+i\tfrac{\gamma}{2})\check{p}_2(x-i\tfrac{\gamma}{2})+e^{\mp ix}\check{p}_1(x-i\tfrac{\gamma}{2})\check{p}_2(x+i\tfrac{\gamma}{2}).
\end{aligned}\tag{3.9}$$

For a polynomial $p(\eta)$ in η, let us define a polynomial in η, $I[p](\eta)$, as follows:

$$p(\eta)=\sum_{k=0}^{n}a_k\eta^k\mapsto I[p](\eta)=\sum_{k=0}^{n+1}b_k\eta^k,\tag{3.10}$$

where b_k's are defined by

$$b_{k+1}=\frac{1}{g_k'^{(0)}}\Bigl(a_k-\sum_{j=k+1}^{n}g_j'^{(j-k)}b_{j+1}\Bigr)\quad(k=n,n-1,\ldots,1,0),\quad b_0=0.\tag{3.11}$$

The constant term of $I[p](\eta)$ is chosen to be zero. It is easy to show that this polynomial $I[p](\eta)=P(\eta)$ satisfies

$$\frac{\check{P}(x-i\frac{\gamma}{2})-\check{P}(x+i\frac{\gamma}{2})}{\eta(x-i\frac{\gamma}{2})-\eta(x+i\frac{\gamma}{2})}=\check{p}(x).\tag{3.12}$$

The operator of the form $a(x)e^{\frac{\gamma}{2}p}-b(x)e^{-\frac{\gamma}{2}p}$ ($a(x),b(x)$: functions of x) acts on the product of two functions $f(x)$ and $g(z)$ as

$$\bigl(a(x)e^{\frac{\gamma}{2}p}-b(x)e^{-\frac{\gamma}{2}p}\bigr)\bigl(f(x)g(x)\bigr)$$

$$
\begin{aligned}
&= a(x)f(x-i\tfrac{\gamma}{2})g(x-i\tfrac{\gamma}{2}) - b(x)f(x+i\tfrac{\gamma}{2})g(x+i\tfrac{\gamma}{2})\\
&= f^{(+)}(x)\big(a(x)g(x-i\tfrac{\gamma}{2}) - b(x)g(x+i\tfrac{\gamma}{2})\big) - if^{(-)}(x)\big(a(x)g(x-i\tfrac{\gamma}{2}) + b(x)g(x+i\tfrac{\gamma}{2})\big)\\
&= f^{(+)}(x)\big(a(x)e^{\frac{\gamma}{2}p} - b(x)e^{-\frac{\gamma}{2}p}\big)g(x) - if^{(-)}(x)\big(a(x)g(x-i\tfrac{\gamma}{2}) + b(x)g(x+i\tfrac{\gamma}{2})\big),
\end{aligned}
\tag{3.13}
$$

where $f^{(\pm)}(x)$ are defined by [27]

$$
f^{(+)}(x) \stackrel{\text{def}}{=} \frac{1}{2}\big(f(x-i\tfrac{\gamma}{2}) + f(x+i\tfrac{\gamma}{2})\big), \quad f^{(-)}(x) \stackrel{\text{def}}{=} \frac{i}{2}\big(f(x-i\tfrac{\gamma}{2}) - f(x+i\tfrac{\gamma}{2})\big). \tag{3.14}
$$

The auxiliary function $\varphi(x)$ is rewritten as

$$
\varphi(x) = ic_{\varphi}\big(\eta(x-i\tfrac{\gamma}{2}) - \eta(x-i\tfrac{\gamma}{2})\big), \quad c_{\varphi} = \begin{cases} 1 & : \mathrm{W} \\ (\sinh\frac{-\gamma}{2})^{-1} & : \mathrm{AW} \end{cases}. \tag{3.15}
$$

First we consider a necessary condition for $X(\eta)$ giving recurrence relations with constant coefficients. Let us assume (2.7) for a polynomial $X(\eta)$ of degree L in η. Applying $\hat{\mathcal{B}}_{\mathcal{D}} = \hat{\mathcal{B}}_{d_1\ldots d_M}$ (A.9) to (2.7), we have

$$
\begin{aligned}
&\hat{\mathcal{B}}_{\mathcal{D}}\big(\check{X}(x)\check{P}_{\mathcal{D},n}(x)\big) = \sum_{k=-n}^{L} r_{n,k}^{X,\mathcal{D}}\hat{\mathcal{B}}_{\mathcal{D}}\check{P}_{\mathcal{D},n+k}(x) = \sum_{k=-n}^{L} r_{n,k}^{X,\mathcal{D}}(\mathcal{E}_{n+k} - \tilde{\mathcal{E}}_{d_M})\check{P}_{d_1\ldots d_{M-1},n}(\eta)\\
&= \check{X}^{(+)}(x)\hat{\mathcal{B}}_{\mathcal{D}}\check{P}_{\mathcal{D},n}(x) - i\check{X}^{(-)}(x)\Big(\frac{ic_{\mathcal{D}}^{\hat{\mathcal{B}}}}{\varphi(x)\check{\Xi}_{\mathcal{D}}(x)}e_{\mathcal{D}}^{\hat{\mathcal{B}}}(x)\check{\Xi}_{d_1\ldots d_{M-1}}(x+i\tfrac{\gamma}{2})\check{P}_{\mathcal{D},n}(x-i\tfrac{\gamma}{2})\\
&\qquad\qquad + \frac{ic_{\mathcal{D}}^{\hat{\mathcal{B}}}}{\varphi(x)\check{\Xi}_{\mathcal{D}}(x)}e_{\mathcal{D}}^{\hat{\mathcal{B}}*}(x)\check{\Xi}_{d_1\ldots d_{M-1}}(x-i\tfrac{\gamma}{2})\check{P}_{\mathcal{D},n}(x+i\tfrac{\gamma}{2})\Big)\\
&= (\mathcal{E}_n - \tilde{\mathcal{E}}_{d_M})\check{X}^{(+)}(x)\check{P}_{d_1\ldots d_{M-1},n}(x)\\
&\quad + \frac{c_{\mathcal{D}}^{\hat{\mathcal{B}}}}{\check{\Xi}_{\mathcal{D}}(x)}\frac{\check{X}^{(-)}(x)}{\varphi(x)}\Big(e_{\mathcal{D}}^{\hat{\mathcal{B}}}(x)\check{\Xi}_{d_1\ldots d_{M-1}}(x+i\tfrac{\gamma}{2})\check{P}_{\mathcal{D},n}(x-i\tfrac{\gamma}{2})\\
&\qquad\qquad + e_{\mathcal{D}}^{\hat{\mathcal{B}}*}(x)\check{\Xi}_{d_1\ldots d_{M-1}}(x-i\tfrac{\gamma}{2})\check{P}_{\mathcal{D},n}(x+i\tfrac{\gamma}{2})\Big),
\end{aligned}
\tag{3.16}
$$

where (3.13) and (2.5) are used. Since the expression in the first line is a polynomial in η, the expression in the last line should be so. By (3.9) and (A.12), $\frac{\check{X}^{(-)}(x)}{\varphi(x)} = \frac{\check{X}(x-i\frac{\gamma}{2}) - \check{X}(x+i\frac{\gamma}{2})}{2c_{\varphi}(\eta(x-i\frac{\gamma}{2}) - \eta(x+i\frac{\gamma}{2}))}$ and $e_{\mathcal{D}}^{\hat{\mathcal{B}}}(x)\check{\Xi}_{d_1\ldots d_{M-1}}(x+i\frac{\gamma}{2})\check{P}_{\mathcal{D},n}(x-i\frac{\gamma}{2}) + e_{\mathcal{D}}^{\hat{\mathcal{B}}*}(x)\check{\Xi}_{d_1\ldots d_{M-1}}(x-i\frac{\gamma}{2})\check{P}_{\mathcal{D},n}(x+i\frac{\gamma}{2})$ are polynomials in $\eta(x)$. If the latter is not divisible by $\check{\Xi}_{\mathcal{D}}(x)$, the former should be divisible by $\check{\Xi}_{\mathcal{D}}(x)$.

We summarize this argument as follows.

Proposition 3 *Let $X(\eta)$ be a polynomial of degree L in η. Assume* (2.7) *and*

$$
\frac{\check{X}(x-i\frac{\gamma}{2}) - \check{X}(x+i\frac{\gamma}{2})}{\eta(x-i\frac{\gamma}{2}) - \eta(x+i\frac{\gamma}{2})} = \check{\Xi}_{\mathcal{D}}(x)\check{Y}(x), \quad Y(\eta):\ \text{a polynomial in } \eta. \tag{3.17}
$$

Then one action of $\hat{\mathcal{B}}_{\mathcal{D}}$ to the both sides of (2.7) *keeps the polynomiality intact.*

Remark If two polynomials in η, $\check{\Xi}_{\mathcal{D}}(x)=\check{\Xi}_{d_1\ldots d_M}(x)$ and $e^{\hat{\mathcal{B}}}_{\mathcal{D}}(x)\check{\Xi}_{d_1\ldots d_{M-1}}(x+i\frac{\gamma}{2})\check{P}_{\mathcal{D},n}(x-i\frac{\gamma}{2})+e^{\hat{\mathcal{B}}*}_{\mathcal{D}}(x)\check{\Xi}_{d_1\ldots d_{M-1}}(x-i\frac{\gamma}{2})\check{P}_{\mathcal{D},n}(x+i\frac{\gamma}{2})$, have no common roots for some n, the polynomial of degree L in η, $X(\eta)$, satisfying (2.7) should satisfy (3.17) for some polynomial $Y(\eta)$.

By taking the constant term of $X(\eta)$ as $X(0)=0$, the condition for the candidate of $X(\eta)$ (3.17) gives

$$X(\eta)=I[\Xi_{\mathcal{D}}Y](\eta),\quad \deg X(\eta)=L=\ell_{\mathcal{D}}+\deg Y(\eta)+1. \tag{3.18}$$

The minimal degree candidate of $X(\eta)$, which corresponds to $Y(\eta)=1$, is

$$X_{\min}(\eta)=I[\Xi_{\mathcal{D}}](\eta),\quad \deg X_{\min}(\eta)=\ell_{\mathcal{D}}+1. \tag{3.19}$$

Based on these properties we present our another main result. Like as § 3.1, we conjecture that this candidate $X(\eta)$ (3.18) actually gives recurrence relations with constant coefficients.

Conjecture 2 *For any polynomial $Y(\eta)$, we take $X(\eta)$ as* (3.18)*. Then the multi-indexed Wilson and Askey-Wilson polynomials $P_{\mathcal{D},n}(\eta)$ satisfy $1+2L$ term recurrence relations with constant coefficients* (1.2).

Remark 1 If two polynomials in η, $\check{\Xi}_{\mathcal{D}}(x)=\check{\Xi}_{d_1\ldots d_M}(x)$ and $e^{\hat{\mathcal{B}}}_{\mathcal{D}}(x)\check{\Xi}_{d_1\ldots d_{M-1}}(x+i\frac{\gamma}{2})\check{P}_{\mathcal{D},n}(x-i\frac{\gamma}{2})+e^{\hat{\mathcal{B}}*}_{\mathcal{D}}(x)\check{\Xi}_{d_1\ldots d_{M-1}}(x-i\frac{\gamma}{2})\check{P}_{\mathcal{D},n}(x+i\frac{\gamma}{2})$, have no common roots for some n, this conjecture exhausts all possible $X(\eta)$ giving recurrence relations with constant coefficients, and the minimal degree choice $X(\eta)=X_{\min}(\eta)$ (3.19) gives $3+2\ell_{\mathcal{D}}$ term recurrence relations.
Remark 2 See Remark 2 and 3 below Conjecture 1.
Remark 3 By (3.9), for any polynomial $p(\eta)$, $\check{Y}(x)=\frac{\check{\Xi}_{\mathcal{D}}(x-i\gamma)\check{p}(x-i\frac{\gamma}{2})-\check{\Xi}_{\mathcal{D}}(x+i\gamma)\check{p}(x+i\frac{\gamma}{2})}{\eta(x-i\frac{\gamma}{2})-\eta(x+i\frac{\gamma}{2})}$ is also a polynomial in η. This $Y(\eta)$ gives $\check{X}(x)=\check{\Xi}_{\mathcal{D}}(x-i\frac{\gamma}{2})\check{\Xi}_{\mathcal{D}}(x+i\frac{\gamma}{2})\check{p}(x)$, which corresponds to Remark 4 below Conjecture 1.
Remark 4 See Remark 5 below Conjecture 1.
Remark 5 For $M=1$ we can prove this conjecture. Since we have Proposition 1, it is sufficient to show that $\hat{\mathcal{B}}_{d_1}\hat{\mathcal{B}}_{d_1d_2}\cdots\hat{\mathcal{B}}_{d_1\ldots d_M}(XP_{\mathcal{D},n})$ is a polynomial in η.
$\underline{M=1}$ From (3.16) we have

$$\begin{aligned}&\hat{\mathcal{B}}_{d_1}\big(\check{X}(x)\check{P}_{d_1,n}(x)\big)\\&=(\mathcal{E}_n-\tilde{\mathcal{E}}_{d_1})\check{X}^{(+)}(x)\check{P}_n(x)+\frac{c^{\hat{\mathcal{B}}}_{d_1}}{2c_{\varphi}}\check{Y}(x)\big(e^{\hat{\mathcal{B}}}_{d_1}(x)\check{P}_{d_1,n}(x-i\tfrac{\gamma}{2})+e^{\hat{\mathcal{B}}*}_{d_1}(x)\check{P}_{d_1,n}(x+i\tfrac{\gamma}{2})\big).\end{aligned}$$

This is a polynomial in η. Thus $M=1$ case is proved. □

4 Summary and Comments

In addition to $3+2M$ term recurrence relations with variable dependent coefficients presented in [27], we have presented (conjectures of) the recurrence relations with constant coefficients for the multi-indexed orthogonal polynomials of Laguerre, Jacobi, Wilson and Askey-Wilson types, Conjecture 1 and Conjecture 2. Since $Y(\eta)$ is arbitrary, we obtain infinitely many recurrence relations. However not all of them are independent. The most important one is the minimal degree one $X_{\min}(\eta)$ (3.5) or (3.19), which gives $3+2\ell_{\mathcal{D}}$ term recurrence relations. Here $\ell_{\mathcal{D}}$ ($\geq M$) is the degree of the lowest member polynomial $P_{\mathcal{D},0}(\eta)$. For this case, the coefficients $r_{n,k}^{X_{\min},\mathcal{D}}$ may have nice forms. Both derivations given in [27] and present paper are based on multi-step Darboux transformations. Although we have discussed Laguerre, Jacobi, Wilson and Askey-Wilson cases, the method is applicable to the multi-indexed (q-)Racah polynomials (which correspond to the case (1) $\mathcal{I}=\{0,1,\ldots,\ell-1\}$) and various case (2) polynomials. The results in [11] and [37] (type I and II) correspond to special cases of our results. In [37], type III case is also studied, which corresponds to case (2). In [38], exceptional Charlier, Meixner, Hermite and Laguerre polynomials are studied and some recurrence relations, which have minimal order (minimalness is stated as a conjecture), are proved by a different method from our paper. The exceptional Laguerre polynomials in [38] are labeled by the two sets, $F_1=\{f_1,\ldots,f_{k_1}\}$ (labels of eigenstates) and $F_2=\{f'_1,\ldots,f'_{k_2}\}$ (labels of type I virtual states). For $F_1=\emptyset$, they correspond to special cases of our multi-indexed Laguerre polynomials with $\mathcal{D}=\{d^{\mathrm{I}}_1,\ldots,d^{\mathrm{I}}_M\}=F_2$ (no type II), for which existence of (minimal degree) recurrence relations are proved [38]. For $F_1\neq\emptyset$, they correspond to case (2). For general case (2), the referee suggests that the minimal degree is equal to the number of missing degrees. It is an interesting problem to show this suggestion. We hope that Conjecture 1 and 2 will be proved in the near future.

Multi-indexed orthogonal polynomials of Laguerre, Jacobi, Wilson and Askey-Wilson types are labeled by an index set $\mathcal{D}$ but different index sets may give the same multi-indexed orthogonal polynomials, $P_{\mathcal{D},n}(\eta;\boldsymbol{\lambda})\propto P_{\mathcal{D}',n}(\eta;\boldsymbol{\lambda}')$ [30]. For example, $\mathcal{D}_1=\{1^{\mathrm{II}},3^{\mathrm{II}},4^{\mathrm{II}},5^{\mathrm{II}},8^{\mathrm{II}}\}$ with $\boldsymbol{\lambda}$, $\mathcal{D}_2=\{1^{\mathrm{I}},2^{\mathrm{I}},6^{\mathrm{I}},8^{\mathrm{I}}\}$ with $\boldsymbol{\lambda}-9\tilde{\boldsymbol{\delta}}_{\mathrm{I}}$ and $\mathcal{D}_3=\{3^{\mathrm{I}},5^{\mathrm{I}},2^{\mathrm{II}}\}$ with $\boldsymbol{\lambda}-6\tilde{\boldsymbol{\delta}}_{\mathrm{I}}$ give the same multi-indexed orthogonal polynomials, $P_{\mathcal{D}_1,n}(\eta;\boldsymbol{\lambda})\propto P_{\mathcal{D}_2,n}(\eta;\boldsymbol{\lambda}-9\tilde{\boldsymbol{\delta}}_{\mathrm{I}})\propto P_{\mathcal{D}_3,n}(\eta;\boldsymbol{\lambda}-6\tilde{\boldsymbol{\delta}}_{\mathrm{I}})$. The $3+2M$ term recurrence relations given in [27] states that these polynomials $P_{\mathcal{D}_1,n}(\eta;\boldsymbol{\lambda})$, $P_{\mathcal{D}_2,n}(\eta;\boldsymbol{\lambda}-9\tilde{\boldsymbol{\delta}}_{\mathrm{I}})$ and $P_{\mathcal{D}_3,n}(\eta;\boldsymbol{\lambda}-6\tilde{\boldsymbol{\delta}}_{\mathrm{I}})$ satisfy 13, 11 and 9 term recurrence relations with variable dependent coefficients, respectively. But the above equivalence implies that all of them satisfy 9 term recurrence relations with variable dependent coefficients. On the other hand, the degrees of lowest members $P_{\mathcal{D},0}(\eta)$ are $\ell_{\mathcal{D}_1}=\ell_{\mathcal{D}_2}=\ell_{\mathcal{D}_3}=11$, and for each case the minimal degree polynomial $X_{\min}(\eta)$ gives 25 term recurrence relations with constant coefficients. The above equivalence, which gives $\Xi_{\mathcal{D}_1}(\eta;\boldsymbol{\lambda})\propto\Xi_{\mathcal{D}_2}(\eta;\boldsymbol{\lambda}-9\tilde{\boldsymbol{\delta}}_{\text{I}})\propto\Xi_{\mathcal{D}_3}(\eta;\boldsymbol{\lambda}-6\tilde{\boldsymbol{\delta}}_{\text{I}})$, implies that these three 25 term recurrence relations are essentially same.

The $3+2M$ term recurrence relations with variable dependent coefficients can be used to calculate the multi-indexed orthogonal polynomials effectively and it needs $M+1$ initial data $P_{\mathcal{D},n}(\eta)$ $(n=0,1,\ldots,M)$ [27]. The simplest recurrence relations with constant coefficients corresponding to $X_{\min}(\eta)$ has $3+2\ell_{\mathcal{D}}$ terms and it needs $\ell_{\mathcal{D}}+1$ initial data. The difference of M and $\ell_{\mathcal{D}}$, $\ell_{\mathcal{D}}-M=\sum_{j=1}^{M}(d_j-1)+2M_{\text{I}}M_{\text{II}}$, becomes large for large d_j. In order to calculate the multi-indexed orthogonal polynomials by using recurrence relations, the $3+2M$ term recurrence relations with variable dependent coefficients are useful. On the other hand, in order to study bispectral properties etc., recurrence relations with constant coefficients are needed.

We hope that the recurrence relations with constants coefficients obtained in this paper will be used as a starting point for theoretical developments of various problems involving bispectrality, generalizations of the Jacobi matrix, spectral theory, etc.

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A Some Formulas

The notation and fundamental formulas of the multi-indexed orthogonal polynomials of Laguerre, Jacobi, Wilson and Askey-Wilson types are found in [27]. See also [23]. In this appendix we present other basic formulas. See footnote in § 2. We write parameter ($\boldsymbol{\lambda}$) dependence explicitly.

A.1 Multi-indexed Laguerre and Jacobi polynomials

Explicit forms of the operators (2.6) are

$$\hat{\mathcal{F}}_{d_1\ldots d_s}(\boldsymbol{\lambda}) = c_{\mathcal{F}}^{-1}\frac{\Xi_{d_1\ldots d_s}(\eta;\boldsymbol{\lambda})}{\Xi_{d_1\ldots d_{s-1}}(\eta;\boldsymbol{\lambda})}\Big(c_{\mathcal{F}}e^{\hat{\mathcal{F}}}_{d_1\ldots d_s}(\eta)\Big(\frac{d}{d\eta}-\frac{\partial_\eta\Xi_{d_1\ldots d_s}(\eta;\boldsymbol{\lambda})}{\Xi_{d_1\ldots d_s}(\eta;\boldsymbol{\lambda})}\Big)+\tilde{e}^{\hat{\mathcal{F}}}_{d_1\ldots d_s}(\boldsymbol{\lambda})\Big), \tag{A.1}$$

$$\hat{\mathcal{B}}_{d_1\ldots d_s}(\boldsymbol{\lambda}) = c_{\mathcal{F}}\frac{\Xi_{d_1\ldots d_{s-1}}(\eta;\boldsymbol{\lambda})}{\Xi_{d_1\ldots d_s}(\eta;\boldsymbol{\lambda})}\Big(c_{\mathcal{F}}e^{\hat{\mathcal{B}}}_{d_1\ldots d_s}(\eta)\Big(-\frac{d}{d\eta}+\frac{\partial_\eta\Xi_{d_1\ldots d_{s-1}}(\eta;\boldsymbol{\lambda})}{\Xi_{d_1\ldots d_{s-1}}(\eta;\boldsymbol{\lambda})}\Big)+\tilde{e}^{\hat{\mathcal{B}}}_{d_1\ldots d_s}(\boldsymbol{\lambda})\Big), \tag{A.2}$$

where $e^{\hat{\mathcal{F}}}_{d_1\ldots d_s}(\eta)$, $e^{\hat{\mathcal{B}}}_{d_1\ldots d_s}(\eta)$, $\tilde{e}^{\hat{\mathcal{F}}}_{d_1\ldots d_s}(\boldsymbol{\lambda})$ and $\tilde{e}^{\hat{\mathcal{B}}}_{d_1\ldots d_s}(\boldsymbol{\lambda})$ are given by

$$\begin{aligned}
\mathrm{L}:\quad & e^{\hat{\mathcal{F}}}_{d_1\ldots d_s}(\eta)=\begin{cases}1 & : d_s=d^{\mathrm{I}}_s\\ \eta & : d_s=d^{\mathrm{II}}_s\end{cases},\quad e^{\hat{\mathcal{B}}}_{d_1\ldots d_s}(\eta)=\begin{cases}\eta & : d_s=d^{\mathrm{I}}_s\\ 1 & : d_s=d^{\mathrm{II}}_s\end{cases},\\
& \tilde{e}^{\hat{\mathcal{F}}}_{d_1\ldots d_s}(\boldsymbol{\lambda})=\begin{cases}-2 & : d_s=d^{\mathrm{I}}_s\\ 2(g+s_{\mathrm{I}}-s_{\mathrm{II}})+1 & : d_s=d^{\mathrm{II}}_s\end{cases},\\
& \tilde{e}^{\hat{\mathcal{B}}}_{d_1\ldots d_s}(\boldsymbol{\lambda})=\begin{cases}-2(g+s_{\mathrm{I}}-s_{\mathrm{II}})+1 & : d_s=d^{\mathrm{I}}_s\\ 2 & : d_s=d^{\mathrm{II}}_s\end{cases},
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\mathrm{J}:\quad & e^{\hat{\mathcal{F}}}_{d_1\ldots d_s}(\eta)=\begin{cases}\frac{1+\eta}{2} & : d_s=d^{\mathrm{I}}_s\\ \frac{1-\eta}{2} & : d_s=d^{\mathrm{II}}_s\end{cases},\quad e^{\hat{\mathcal{B}}}_{d_1\ldots d_s}(\eta)=\begin{cases}\frac{1-\eta}{2} & : d_s=d^{\mathrm{I}}_s\\ \frac{1+\eta}{2} & : d_s=d^{\mathrm{II}}_s\end{cases},\\
& \tilde{e}^{\hat{\mathcal{F}}}_{d_1\ldots d_s}(\boldsymbol{\lambda})=\begin{cases}-2(h+s_{\mathrm{II}}-s_{\mathrm{I}})-1 & : d_s=d^{\mathrm{I}}_s\\ 2(g+s_{\mathrm{I}}-s_{\mathrm{II}})+1 & : d_s=d^{\mathrm{II}}_s\end{cases},\\
& \tilde{e}^{\hat{\mathcal{B}}}_{d_1\ldots d_s}(\boldsymbol{\lambda})=\begin{cases}-2(g+s_{\mathrm{I}}-s_{\mathrm{II}})+1 & : d_s=d^{\mathrm{I}}_s\\ 2(h+s_{\mathrm{II}}-s_{\mathrm{I}})-1 & : d_s=d^{\mathrm{II}}_s\end{cases}.
\end{aligned} \tag{A.4}$$

By (A.1)–(A.2), eqs. (2.5) are

$$\begin{aligned}
&P_{d_1\ldots d_s,n}(\eta;\boldsymbol{\lambda})\\
&=\frac{1}{\Xi_{d_1\ldots d_{s-1}}(\eta;\boldsymbol{\lambda})}\Big(e^{\hat{\mathcal{F}}}_{d_1\ldots d_s}(\eta)\big(\Xi_{d_1\ldots d_s}(\eta;\boldsymbol{\lambda})\partial_\eta P_{d_1\ldots d_{s-1},n}(\eta;\boldsymbol{\lambda})-\partial_\eta\Xi_{d_1\ldots d_s}(\eta;\boldsymbol{\lambda})P_{d_1\ldots d_{s-1},n}(\eta;\boldsymbol{\lambda})\big)\\
&\qquad+c_{\mathcal{F}}^{-1}\tilde{e}^{\hat{\mathcal{F}}}_{d_1\ldots d_s}(\boldsymbol{\lambda})\Xi_{d_1\ldots d_s}(\eta;\boldsymbol{\lambda})P_{d_1\ldots d_{s-1},n}(\eta;\boldsymbol{\lambda})\Big),
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
&\big(\mathcal{E}_n(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{d_s}(\boldsymbol{\lambda})\big)P_{d_1\ldots d_{s-1},n}(\eta;\boldsymbol{\lambda})\\
&=\frac{c_{\mathcal{F}}^2}{\Xi_{d_1\ldots d_s}(\eta;\boldsymbol{\lambda})}\Big(e^{\hat{\mathcal{B}}}_{d_1\ldots d_s}(\eta)\big(-\Xi_{d_1\ldots d_{s-1}}(\eta;\boldsymbol{\lambda})\partial_\eta P_{d_1\ldots d_s,n}(\eta;\boldsymbol{\lambda})+\partial_\eta\Xi_{d_1\ldots d_{s-1}}(\eta;\boldsymbol{\lambda})P_{d_1\ldots d_s,n}(\eta;\boldsymbol{\lambda})\big)\\
&\qquad+c_{\mathcal{F}}^{-1}\tilde{e}^{\hat{\mathcal{B}}}_{d_1\ldots d_s}(\boldsymbol{\lambda})\Xi_{d_1\ldots d_{s-1}}(\eta;\boldsymbol{\lambda})P_{d_1\ldots d_s,n}(\eta;\boldsymbol{\lambda})\Big).
\end{aligned} \tag{A.6}$$

A.2 Multi-indexed Wilson and Askey-Wilson polynomials

We restrict the parameters: $\{a_1^*, a_2^*\} = \{a_1, a_2\}$ (as a set) and $\{a_3^*, a_4^*\} = \{a_3, a_4\}$ (as a set).

Explicit form of the potential function $\hat{V}_{d_1\ldots d_s}(x;\boldsymbol{\lambda})$ is

$$\hat{V}_{d_1\ldots d_s}(x;\boldsymbol{\lambda}) = \frac{\check{\Xi}_{d_1\ldots d_{s-1}}(x+i\frac{\gamma}{2};\boldsymbol{\lambda})}{\check{\Xi}_{d_1\ldots d_{s-1}}(x-i\frac{\gamma}{2};\boldsymbol{\lambda})}\frac{\check{\Xi}_{d_1\ldots d_s}(x-i\gamma;\boldsymbol{\lambda})}{\check{\Xi}_{d_1\ldots d_s}(x;\boldsymbol{\lambda})} \times \begin{cases} \kappa^{s_{\text{I}}-s_{\text{II}}-1}\alpha^{\text{I}}(\boldsymbol{\lambda})V\big(x;\mathfrak{t}^{\text{I}}(\boldsymbol{\lambda}^{[s_{\text{I}}-1,s_{\text{II}}]})\big) & : d_s = d_s^{\text{I}} \\ \kappa^{s_{\text{II}}-s_{\text{I}}-1}\alpha^{\text{II}}(\boldsymbol{\lambda})V\big(x;\mathfrak{t}^{\text{II}}(\boldsymbol{\lambda}^{[s_{\text{I}},s_{\text{II}}-1]})\big) & : d_s = d_s^{\text{II}} \end{cases}. \tag{A.7}$$

Explicit forms of the operators (2.6) are

$$\hat{\mathcal{F}}_{d_1\ldots d_s}(\boldsymbol{\lambda}) = \frac{ic^{\hat{\mathcal{F}}}_{d_1\ldots d_s}(\boldsymbol{\lambda})}{\varphi(x)\check{\Xi}_{d_1\ldots d_{s-1}}(x;\boldsymbol{\lambda})}\Big(e^{\hat{\mathcal{F}}}_{d_1\ldots d_s}(x;\boldsymbol{\lambda})\check{\Xi}_{d_1\ldots d_s}(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda})e^{\frac{\gamma}{2}p} - e^{\hat{\mathcal{F}}*}_{d_1\ldots d_s}(x;\boldsymbol{\lambda})\check{\Xi}_{d_1\ldots d_s}(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda})e^{-\frac{\gamma}{2}p}\Big), \tag{A.8}$$

$$\hat{\mathcal{B}}_{d_1\ldots d_s}(\boldsymbol{\lambda}) = \frac{ic^{\hat{\mathcal{B}}}_{d_1\ldots d_s}(\boldsymbol{\lambda})}{\varphi(x)\check{\Xi}_{d_1\ldots d_s}(x;\boldsymbol{\lambda})}\Big(e^{\hat{\mathcal{B}}}_{d_1\ldots d_s}(x;\boldsymbol{\lambda})\check{\Xi}_{d_1\ldots d_{s-1}}(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda})e^{\frac{\gamma}{2}p} - e^{\hat{\mathcal{B}}*}_{d_1\ldots d_s}(x;\boldsymbol{\lambda})\check{\Xi}_{d_1\ldots d_{s-1}}(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda})e^{-\frac{\gamma}{2}p}\Big), \tag{A.9}$$

where $c^{\hat{\mathcal{F}}}_{d_1\ldots d_s}(\boldsymbol{\lambda})$, $c^{\hat{\mathcal{B}}}_{d_1\ldots d_s}(\boldsymbol{\lambda})$, $e^{\hat{\mathcal{F}}}_{d_1\ldots d_s}(x;\boldsymbol{\lambda})$ and $e^{\hat{\mathcal{B}}}_{d_1\ldots d_s}(x;\boldsymbol{\lambda})$ are given by

$$c^{\hat{\mathcal{F}}}_{d_1\ldots d_s}(\boldsymbol{\lambda})^{-1} = c^{\hat{\mathcal{B}}}_{d_1\ldots d_s}(\boldsymbol{\lambda}) = \begin{cases} \kappa^{\frac{s-1}{2}+s_{\text{II}}}\alpha^{\text{I}}(\boldsymbol{\lambda})^{\frac{1}{2}} & : d_s = d_s^{\text{I}} \\ \kappa^{\frac{s-1}{2}+s_{\text{I}}}\alpha^{\text{II}}(\boldsymbol{\lambda})^{\frac{1}{2}} & : d_s = d_s^{\text{II}} \end{cases}, \tag{A.10}$$

$$e^{\hat{\mathcal{F}}}_{d_1\ldots d_s}(x;\boldsymbol{\lambda}) = \begin{cases} v_1(x;\boldsymbol{\lambda}^{[s_{\text{I}},s_{\text{II}}]}) & : d_s = d_s^{\text{I}} \\ v_2(x;\boldsymbol{\lambda}^{[s_{\text{I}},s_{\text{II}}]}) & : d_s = d_s^{\text{II}} \end{cases}, \tag{A.11}$$

$$e^{\hat{\mathcal{B}}}_{d_1\ldots d_s}(x;\boldsymbol{\lambda}) = \begin{cases} v_2(x;\boldsymbol{\lambda}^{[s_{\text{I}}-1,s_{\text{II}}]}) & : d_s = d_s^{\text{I}} \\ v_1(x;\boldsymbol{\lambda}^{[s_{\text{I}},s_{\text{II}}-1]}) & : d_s = d_s^{\text{II}} \end{cases}. \tag{A.12}$$

Here $v_1(x;\boldsymbol{\lambda})$ and $v_2(x;\boldsymbol{\lambda})$ are given in [23]:

$$v_1(x;\boldsymbol{\lambda}) = \begin{cases} \prod_{j=1}^2(a_j+ix) & : \text{W} \\ e^{-ix}\prod_{j=1}^2(1-a_je^{ix}) & : \text{AW} \end{cases}, \quad v_2(x;\boldsymbol{\lambda}) = \begin{cases} \prod_{j=3}^4(a_j+ix) & : \text{W} \\ e^{-ix}\prod_{j=3}^4(1-a_je^{ix}) & : \text{AW} \end{cases}. \tag{A.13}$$

Note that $v_1^*(x;\boldsymbol{\lambda}) = v_1(-x;\boldsymbol{\lambda})$ and $v_2^*(x;\boldsymbol{\lambda}) = v_2(-x;\boldsymbol{\lambda})$. By (A.8)–(A.9), eqs. (2.5) are

$$\check{P}_{d_1\ldots d_s,n}(x;\boldsymbol{\lambda}) = \frac{ic^{\hat{\mathcal{F}}}_{d_1\ldots d_s}(\boldsymbol{\lambda})}{\varphi(x)\check{\Xi}_{d_1\ldots d_{s-1}}(x;\boldsymbol{\lambda})}\Big(e^{\hat{\mathcal{F}}}_{d_1\ldots d_s}(x;\boldsymbol{\lambda})\check{\Xi}_{d_1\ldots d_s}(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda})P_{d_1\ldots d_{s-1},n}(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda}) - e^{\hat{\mathcal{F}}*}_{d_1\ldots d_s}(x;\boldsymbol{\lambda})\check{\Xi}_{d_1\ldots d_s}(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda})P_{d_1\ldots d_{s-1},n}(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda})\Big), \tag{A.14}$$

$$
\begin{aligned}
&\big(\mathcal{E}_n(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{d_s}(\boldsymbol{\lambda})\big)P_{d_1\ldots d_{s-1},n}(x;\boldsymbol{\lambda})\\
&\quad=\frac{ic^{\hat{\mathcal{B}}}_{d_1\ldots d_s}(\boldsymbol{\lambda})}{\varphi(x)\check{\Xi}_{d_1\ldots d_s}(x;\boldsymbol{\lambda})}\Big(e^{\hat{\mathcal{B}}}_{d_1\ldots d_s}(x;\boldsymbol{\lambda})\check{\Xi}_{d_1\ldots d_{s-1}}(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda})P_{d_1\ldots d_s,n}(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda})\\
&\qquad\qquad\qquad\qquad\quad-e^{\hat{\mathcal{B}}*}_{d_1\ldots d_s}(x;\boldsymbol{\lambda})\check{\Xi}_{d_1\ldots d_{s-1}}(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda})P_{d_1\ldots d_s,n}(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda})\Big).
\end{aligned}
\tag{A.15}
$$

B Some Examples

For illustration, we present some examples of the coefficients $r^{X,\mathcal{D}}_{n,k}$ of the recurrence relations (1.2) for $X(\eta)=X_{\text{min}}(\eta)$ and small d_j.

B.1 Multi-indexed Laguerre polynomials

Ex.1 $\mathcal{D}=\{1^{\text{I}}\}$: 5-term recurrence relations

$$
\begin{aligned}
&X(\eta)=X_{\text{min}}(\eta)=\tfrac{1}{2}\eta(\eta+2g+1),\\
&r^{X,\mathcal{D}}_{n,2}=\tfrac{1}{2}(n+1)(n+2),\quad r^{X,\mathcal{D}}_{n,-2}=\tfrac{1}{8}(2g+2n-3)(2g+2n+3),\\
&r^{X,\mathcal{D}}_{n,1}=-(n+1)(2g+2n+3),\quad r^{X,\mathcal{D}}_{n,-1}=-\tfrac{1}{2}(2g+2n-1)(2g+2n+3),\\
&r^{X,\mathcal{D}}_{n,0}=\tfrac{1}{8}\big(24n^2+4(10g+11)n+(2g+1)(6g+13)\big),
\end{aligned}
\tag{B.1}
$$

Ex.2 $\mathcal{D}=\{1^{\text{I}},2^{\text{I}}\}$: 7-term recurrence relations

$$
\begin{aligned}
&X(\eta)=X_{\text{min}}(\eta)=\tfrac{1}{24}\eta\big(4\eta^2+6(2g+1)\eta+3(2g+1)(2g+3)\big),\\
&r^{X,\mathcal{D}}_{n,3}=-\tfrac{1}{6}(n+1)_3\,,\quad r^{X,\mathcal{D}}_{n,-3}=-\tfrac{1}{12}(2g+2n-5)(g+n+\tfrac{3}{2})_2\,,\\
&r^{X,\mathcal{D}}_{n,2}=\tfrac{1}{2}(n+1)_2(2g+2n+5),\quad r^{X,\mathcal{D}}_{n,-2}=\tfrac{1}{2}(2g+2n-3)(g+n+\tfrac{3}{2})_2\,,\\
&r^{X,\mathcal{D}}_{n,1}=-\tfrac{1}{4}(n+1)(2g+2n+3)(4g+5n+12),\\
&r^{X,\mathcal{D}}_{n,-1}=-\tfrac{1}{8}(2g+2n-1)(2g+2n+5)(4g+5n+7),\\
&r^{X,\mathcal{D}}_{n,0}=\tfrac{1}{48}\big(160n^3+96(4g+7)n^2+8(36g^2+132g+97)n+(2g+1)(2g+5)(14g+45)\big),
\end{aligned}
\tag{B.2}
$$

Ex.3 $\mathcal{D}=\{1^{\text{I}},1^{\text{II}}\}$: 9-term recurrence relations

$$
\begin{aligned}
&X(\eta)=X_{\text{min}}(\eta)=\tfrac{1}{8}\eta\big(2\eta^3+4(2g-1)\eta^2+3(2g-3)(2g+1)\eta+(2g-3)(2g-1)(2g+1)\big),\\
&r^{X,\mathcal{D}}_{n,4}=\frac{(n+1)_4(2g+2n-3)}{4(2g+2n+5)},\quad r^{X,\mathcal{D}}_{n,-4}=\tfrac{1}{16}(2g+2n-7)(g+n-\tfrac{3}{2})_2(2g+2n+3),\\
&r^{X,\mathcal{D}}_{n,3}=-(n+1)_3(2g+2n-3),\quad r^{X,\mathcal{D}}_{n,-3}=-(g+n-\tfrac{5}{2})_3(2g+2n+3),\\
&r^{X,\mathcal{D}}_{n,2}=\frac{(n+1)_2(2g+2n-3)}{4(2g+2n+1)}\big(28n^2+2(26g+29)n+3(2g+1)(4g+7)\big),
\end{aligned}
$$

$$
\begin{aligned}
r^{X,\mathcal{D}}_{n,-2}&=\tfrac{1}{16}(2g+2n-3)(2g+2n+3)\big(28n^2+2(26g-27)n+24g^2-50g+17\big),\\
r^{X,\mathcal{D}}_{n,1}&=-\tfrac{1}{2}(n+1)(2g+2n-3)(2g+2n+3)(4g+7n+5),\\
r^{X,\mathcal{D}}_{n,-1}&=-(g+n-\tfrac{3}{2})_2(2g+2n+3)(4g+7n-2),\\
r^{X,\mathcal{D}}_{n,0}&=\tfrac{1}{64}\big(1120n^4+160(22g+3)n^3+8(492g^2+168g-299)n^2\\
&\qquad+8(224g^3+156g^2-328g-135)n+(2g-3)(2g+1)(6g+5)(10g+19)\big).
\end{aligned}
\tag{B.3}
$$

Note that we have equivalences [30],

$$
P_{\{1^{\mathrm{I}},2^{\mathrm{I}}\},n}(\eta;g)=\frac{1}{g+n+\frac{1}{2}}P_{\{2^{\mathrm{II}}\},n}(\eta;g+3),
\tag{B.4}
$$

$$
P_{\{1^{\mathrm{I}},1^{\mathrm{II}}\},n}(\eta;g)=-3(g+n-\tfrac{3}{2})P_{\{1^{\mathrm{I}},3^{\mathrm{I}}\},n}(\eta;g-2).
\tag{B.5}
$$

Recurrence relations for $P_{\{1^{\mathrm{I}}\},n}(\eta;g)$ were given in [37, 38] and those for $P_{\{2^{\mathrm{II}}\},n}(\eta;g)$ were given in [37].

B.2 Multi-indexed Jacobi polynomials

We set $a=g+h$ and $b=g-h$.

Ex.1 $\mathcal{D}=\{1^{\mathrm{I}}\}$: 5-term recurrence relations

$$
\begin{aligned}
X(\eta)&=X_{\min}(\eta)=\tfrac{1}{4}\eta\big((b+2)\eta+2(a-1)\big),\\
r^{X,\mathcal{D}}_{n,2}&=\frac{(n+1)_2(b+2)(a+n)_2(2h+2n-3)}{(a+2n)_4(2h+2n+1)},\\
r^{X,\mathcal{D}}_{n,-2}&=\frac{(b+2)(2g+2n-3)(2g+2n+3)(h+n-\frac{3}{2})_2}{4(a+2n-3)_4},\\
r^{X,\mathcal{D}}_{n,1}&=\frac{(n+1)(a-1)(a+n)(2g+2n+3)(2h+2n-3)}{(a+2n-1)_3(a+2n+3)},\\
r^{X,\mathcal{D}}_{n,-1}&=\frac{(a-1)(2g+2n-1)(2g+2n+3)(h+n-\frac{3}{2})_2}{(a+2n-3)(a+2n-1)_3},\\
r^{X,\mathcal{D}}_{n,0}&=\frac{b+2}{4(a+2n-2)_2(a+2n+1)_2}\Big(-b(b+4)\big(2n(a+n)-(a-2)(a-1)\big)\\
&\qquad+(a+2n-1)(a+2n+1)\big(2n(a+n)-(a-2)(2a-1)\big)\Big),
\end{aligned}
\tag{B.6}
$$

Ex.2 $\mathcal{D}=\{1^{\mathrm{I}},2^{\mathrm{I}}\}$: 7-term recurrence relations

$$
\begin{aligned}
X(\eta)&=X_{\min}(\eta)=\tfrac{1}{48}(b+4)\eta\big((b+2)(b+3)\eta^2+3(b+3)(a-1)\eta+3(a^2-2a+b+3)\big),\\
r^{X,\mathcal{D}}_{n,3}&=\frac{(n+1)_3(b+2)_3(a+n)_3(h+n-\frac{5}{2})_2}{6(a+2n)_6(h+n+\frac{1}{2})_2},
\end{aligned}
$$

$$
\begin{aligned}
r_{n,-3}^{X,\mathcal{D}} &= \frac{(b+2)_3(2g+2n-5)(g+n+\frac{3}{2})_2(h+n-\frac{5}{2})_3}{12(a+2n-5)_6},\\
r_{n,2}^{X,\mathcal{D}} &= \frac{(n+1)_2(b+3)_2(a-1)(a+n)_2(2g+2n+5)(h+n-\frac{5}{2})_2}{(a+2n-1)_5(a+2n+5)(2h+2n+1)},\\
r_{n,-2}^{X,\mathcal{D}} &= \frac{(b+3)_2(a-1)(2g+2n-3)(g+n+\frac{3}{2})_2(h+n-\frac{5}{2})_3}{2(a+2n-5)(a+2n-3)_5},\\
r_{n,1}^{X,\mathcal{D}} &= \frac{(b+4)(a+n)(2g+2n+3)(2h+2n-5)}{8(a+2n-2)_4(a+2n+3)_2}\\
&\quad\times\Big(b(b+9)(n+1)\big(n(n+a+1)-(a-2)_2\big)\\
&\qquad+(n+1)\big(2(9-4a+2a^2)n(n+a+1)+(a-3)_3(a+6)\big)\Big),\qquad\qquad\text{(B.7)}\\
r_{n,-1}^{X,\mathcal{D}} &= \frac{(b+4)(2g+2n-1)(2g+2n+5)(h+n-\frac{3}{2})_2}{8(a+2n-4)_2(a+2n-1)_4}\\
&\quad\times\Big(b(b+9)\big(n(n+a-1)-(a-1)^2-1\big)\\
&\qquad+2n(2a^2-4a+9)(n+a-1)+(a-1)^4-23(a-1)^2-14\Big),\\
r_{n,0}^{X,\mathcal{D}} &= \frac{(b+4)(a-1)}{48(a+2n-3)_3(a+2n+1)_3}\Big(b^4(b+17)\big(6n(n+a)-(a-2)(a-3)\big)\\
&\qquad-b^3\big(48n^3(n+2a)+48(a^2+a-14)n^2+48a(a-14)n\\
&\qquad\qquad-(a-2)(a-3)(3a^2+3a-104)\big)\\
&\quad-2b^2\big(264n^3(n+2a)+6(45a^2+42a-181)n^2+6a(a^2+42a-181)n\\
&\qquad\qquad-(a-2)(a-3)(15a^2+12a-137)\big)\\
&\qquad+3b\big(32n^5(n+3a)+16(6a^2+3a-43)n^4+32a(a^2+3a-43)n^3\\
&\qquad\qquad-2(3a^4-36a^3+358a^2+316a-625)n^2\\
&\qquad\qquad-2a(3a^4-12a^3+14a^2+316a-625)n\\
&\qquad\qquad-(a-2)(a-3)(a^4+2a^3-32a^2-14a+99))\\
&\qquad+3(a+2n-1)(a+2n+1)\big(24n^3(n+2a)+2(7a^2+22a-111)n^2\\
&\qquad\qquad\quad-2a(5a^2-22a+111)n-(a-2)(a-3)(4a^2+9a-33)\big)\Big),
\end{aligned}
$$

<u>Ex.3</u> $\mathcal{D}=\{1^{\mathrm{I}},1^{\mathrm{II}}\}$: 9-term recurrence relations

$$
\begin{aligned}
X(\eta)=X_{\min}(\eta)&=-\frac{1}{64}\eta\big((b-2)b(b+2)\eta^3+4b^2(a-1)\eta^2+6b(a-1)^2\eta\\
&\qquad\qquad+4(a-3)(a-1)(a+1)\big),\\
r_{n,4}^{X,\mathcal{D}} &= -\frac{(n+1)_4(b-2)b(b+2)(a+n)_4(2g+2n-3)(2h+2n-3)}{4(a+2n)_8(2g+2n+5)(2h+2n+5)},
\end{aligned}
$$

$$
\begin{aligned}
r^{X,\mathcal{D}}_{n,-4} &= -\frac{(b-2)b(b+2)}{64(a+2n-7)_8}(2g+2n-7)(g+n-\tfrac{3}{2})_2(2g+2n+3)\\
&\qquad \times (2h+2n-7)(h+n-\tfrac{3}{2})_2(2h+2n+3),\\
r^{X,\mathcal{D}}_{n,3} &= -\frac{(n+1)_3 b^2(a-1)(a+n)_3(2g+2n-3)(2h+2n-3)}{2(a+2n-1)_7(a+2n+7)},\\
r^{X,\mathcal{D}}_{n,-3} &= -\frac{b^2(a-1)(g+n-\frac{5}{2})_3(2g+2n+3)(h+n-\frac{5}{2})_3(2h+2n+3)}{2(a+2n-7)(a+2n-5)_7},\\
r^{X,\mathcal{D}}_{n,2} &= \frac{(n+1)_2 b(a+n)_2(2g+2n-3)(2h+2n-3)}{8(a+2n-2)_6(a+2n+5)_2(2g+2n+1)(2h+2n+1)}\\
&\quad \times \Big(b^4\big(2n(n+a+2)-3(a-1)(a-2)\big)\\
&\qquad - b^2\big(8n^3(n+2a+4)-2(7a^2-50a-15)n^2-2(a+2)(11a^2-34a+1)n\\
&\qquad\qquad -3(a-1)(a-2)(2a^2+9a+11)\big)\\
&\qquad -(a+2n-1)(a+2n+5)\big(4(3a^2-6a+1)n(n+a+2)\\
&\qquad\qquad\qquad\qquad +3(a-2)(a+1)^2(a+2)\big)\Big),\\
r^{X,\mathcal{D}}_{n,-2} &= \frac{b(2g+2n-3)(2g+2n+3)(2h+2n-3)(2h+2n+3)}{128(a+2n-6)_2(a+2n-3)_6}\\
&\quad \times \Big(b^4\big(2n(n+a-2)-3a^2+5a-6\big)\\
&\qquad - b^2\big(8n^4+16(a-2)n^3-2(7a^2-2a-15)n^2-2(a-2)(11a^2-18a+1)n\\
&\qquad\qquad -6a^4+35a^3-68a^2+49a-66\big)\\
&\qquad -(2n+a+1)(2n+a-5)\big(4(3a^2-6a+1)n(n+a-2)\\
&\qquad\qquad\qquad\qquad +(a-3)(3a^3-9a^2+12a+4)\big)\Big), \qquad\qquad (B.8)\\
r^{X,\mathcal{D}}_{n,1} &= -\frac{(n+1)(a-1)(a+n)(2g+2n-3)(2g+2n+3)(2h+2n-3)(2h+2n+3)}{8(a+2n-3)_5(a+2n+3)_3}\\
&\quad \times \Big(b^2\big(3n(n+a+1)-(a-2)(a-3)\big)+(a+1)(a-3)(a+2n-2)(a+2n+4)\Big),\\
r^{X,\mathcal{D}}_{n,-1} &= -\frac{(a-1)(g+n-\frac{3}{2})_2(2g+2n+3)(h+n-\frac{3}{2})_2(2h+2n+3)}{2(a+2n-5)_3(a+2n-1)_5}\\
&\quad \times \Big(b^2\big(3n(n+a-1)-a^2+2a-6\big)+(a-3)(a+1)(a+2n-4)(a+2n+2)\Big),\\
r^{X,\mathcal{D}}_{n,0} &= -\frac{b}{64(a+2n-4)_4(a+2n+1)_4}\\
&\quad \times \Big(b^6\big(6n^3(n+2a)-6(a^2-5a+5)n^2-6a(2a^2-5a+5)n+(a-4)_4\big)\\
&\qquad -2b^4\big(24n^5(n+3a)+6(a^2+28a-9)n^4-12a(9a^2-28a+9)n^3\\
&\qquad\qquad -2(38a^4-71a^3-17a^2+5a+117)n^2-2a(5a^4+13a^3-44a^2+5a+117)n
\end{aligned}
$$

$$
\begin{aligned}
&\quad + (a-4)_4(2a^2+3a+11)\big)\\
&+ b^2\big(96n^7(n+4a) + 48(a^2+26a-15)n^6 - 48a(25a^2-78a+45)n^5\\
&\quad - 6(279a^4-616a^3+98a^2+376a-417)n^4\\
&\quad - 12a(75a^4-96a^3-202a^2+376a-417)n^3\\
&\quad - 2(87a^6+153a^5-1139a^4+1262a^3-931a^2-1031a+2775)n^2\\
&\quad + 2a(6a^6-129a^5+353a^4-134a^3-320a^2+1031a-2775)n\\
&\quad + (a-4)_4(6a^4+18a^3+37a^2+114a+153)\big)\\
&+ 2(a+2n-3)(a+2n+3)\big(48(2a^2-4a+1)n^5(n+3a)\\
&\qquad + 4(76a^4-124a^3-73a^2+124a-39)n^4\\
&\qquad + 8a(16a^4-4a^3-103a^2+124a-39)n^3\\
&\qquad + 2(3a^6+78a^5-274a^4+142a^3+385a^2-544a-114)n^2\\
&\qquad - 2a(5a^6-38a^5+56a^4+106a^3-463a^2+544a+114)n\\
&\qquad - (a-4)(a-2)_2(a+1)_2(2a^3-3a^2-2a+21)\big)\Big).
\end{aligned}
$$

Note that we have equivalences [30],

$$
P_{\{1^{\mathrm{I}},2^{\mathrm{I}}\},n}(\eta;g,h) = -\frac{(g-h+4)(h+n-\frac{5}{2})_2}{4(g+n+\frac{1}{2})}P_{\{2^{\mathrm{II}}\},n}(\eta;g+3,h-3), \tag{B.9}
$$

$$
P_{\{1^{\mathrm{I}},1^{\mathrm{II}}\},n}(\eta;g,h) = \frac{3(g+n-\frac{3}{2})}{(g-h+1)(h+n+\frac{1}{2})}P_{\{1^{\mathrm{I}},3^{\mathrm{I}}\},n}(\eta;g-2,h+2). \tag{B.10}
$$

Recurrence relations for $P_{\{1^{\mathrm{I}}\},n}(\eta;g,g)$ and $P_{\{2^{\mathrm{II}}\},n}(\eta;g,g)$ were given in [37].

B.3 Multi-indexed Wilson polynomials

We set $b_1=a_1+a_2+a_3+a_4$, $\sigma_1=a_1+a_2$, $\sigma_2=a_1a_2$, $\sigma'_1=a_3+a_4$ and $\sigma'_2=a_3a_4$.

<u>Ex.1</u> $\mathcal{D}=\{1^{\mathrm{I}}\}$: 5-term recurrence relations

$$
\begin{aligned}
X(\eta) &= X_{\min}(\eta) = \tfrac{1}{4}\eta\big(2(\sigma_1-\sigma'_1-2)\eta + 4(\sigma_2\sigma'_1-\sigma_1\sigma'_2-\sigma_1\sigma'_1+2\sigma'_2)+\sigma_1+3\sigma'_1-2\big),\\
r^{X,\mathcal{D}}_{n,2} &= \frac{(\sigma_1-\sigma'_1-2)(b_1+n-1)_2(\sigma_1+n-2)}{2(b_1+2n-1)_4(\sigma_1+n)},\\
r^{X,\mathcal{D}}_{n,-2} &= \frac{n(n-1)(\sigma_1-\sigma'_1-2)}{2(b_1+2n-4)_4}(\sigma_1+n-2)_2(\sigma'_1+n-2)(\sigma'_1+n+1)\\
&\quad\times\prod_{i=1}^{2}\prod_{j=3}^{4}(a_i+a_j+n-2)_2\,,
\end{aligned}
$$

$$
\begin{aligned}
r^{X,\mathcal{D}}_{n,1} &= -\frac{2(b_1+n-1)(\sigma_1+n-2)(\sigma'_1+n+1)}{(b_1+2n-2)_3(b_1+2n+2)} \\
&\quad\times \big((\sigma_1-\sigma'_1-2)n(n+b_1)-(b_1-2)(\sigma'_1-\sigma_2+\sigma'_2+1)\big), \qquad \text{(B.11)}\\
r^{X,\mathcal{D}}_{n,-1} &= \frac{2n(\sigma_1+n-2)_2(\sigma'_1+n-1)(\sigma'_1+n+1)}{(b_1+2n-4)(b_1+2n-2)_3}\prod_{i=1}^{2}\prod_{j=3}^{4}(a_i+a_j+n-1) \\
&\quad\times \big((2-\sigma_1+\sigma'_1)n(n+b_1-2)+(\sigma_1-2)b_1-(\sigma_2-\sigma'_2)(b_1-2)\big),\\
r^{X,\mathcal{D}}_{n,0} &= \frac{1}{32(b_1+2n-3)_2(b_1+2n)_2}A.
\end{aligned}
$$

Here A is a polynomial of degree 8 in n, whose coefficients are polynomials in σ_1, σ_2, σ'_1 and σ'_2. Since A has a lengthy expression, we do not write down it here and put it on the web page [42].

B.4 Multi-indexed Askey-Wilson polynomials

We set $b_4=a_1a_2a_3a_4$, $\sigma_1=a_1+a_2$, $\sigma_2=a_1a_2$, $\sigma'_1=a_3+a_4$ and $\sigma'_2=a_3a_4$.

Ex.1 $\mathcal{D}=\{1^{\text{I}}\}$: 5-term recurrence relations

$$
\begin{aligned}
X(\eta) &= X_{\min}(\eta)=\frac{\eta}{(1+q)\sigma_1}\Big(2q^{\frac12}(\sigma_2-\sigma'_2q^2)\eta-(1+q)\big(\sigma_1(1-\sigma'_2)q+\sigma'_1(\sigma_2-q^2)\big)\Big),\\
r^{X,\mathcal{D}}_{n,2} &= \frac{q^{\frac32}(1-\sigma_2^{-1}\sigma'_2q^2)(b_4q^{n-1};q)_2(1-\sigma_2q^{n-2})}{2(1+q)(b_4q^{2n-1};q)_4(1-\sigma_2q^n)},\\
r^{X,\mathcal{D}}_{n,-2} &= \frac{(q^{n-1};q)_2(1-\sigma_2^{-1}\sigma'_2q^2)}{2(1+q)q^{\frac12}(b_4q^{2n-4};q)_4}(\sigma_2q^{n-2};q)_2(1-\sigma'_2q^{n-2})(1-\sigma'_2q^{n+1})\prod_{i=1}^{2}\prod_{j=3}^{4}(a_ia_jq^{n-2};q)_2,\\
r^{X,\mathcal{D}}_{n,1} &= -\frac{(1-b_4q^{n-1})(1-\sigma_2q^{n-2})(1-\sigma'_2q^{n+1})}{2q^{\frac12}\sigma_2(b_4q^{2n-2};q)_3(1-b_4q^{2n+2})}\\
&\quad\times\Big(q(b_4q^{2n}+1)\big(q\sigma_1(1-\sigma'_2)+\sigma'_1(\sigma_2-q^2)\big)\\
&\qquad-(1+q^2)q^n\big(\sigma_1\sigma'_2(\sigma_2-q^2)+\sigma'_1\sigma_2q(1-\sigma'_2)\big)\Big), \qquad \text{(B.12)}\\
r^{X,\mathcal{D}}_{n,-1} &= -\frac{(1-q^n)(\sigma_2q^{n-2};q)_2(1-\sigma'_2q^{n-1})(1-\sigma'_2q^{n+1})}{2q^{\frac52}\sigma_2(1-b_4q^{2n-4})(b_4q^{2n-2};q)_3}\prod_{i=1}^{2}\prod_{j=3}^{4}(1-a_ia_jq^{n-1})\\
&\quad\times\Big((b_4q^{2n}+q^2)\big(q\sigma_1(1-\sigma'_2)+\sigma'_1(\sigma_2-q^2)\big)\\
&\qquad-(1+q^2)q^n\big(\sigma_1\sigma'_2(\sigma_2-q^2)+\sigma'_1\sigma_2q(1-\sigma'_2)\big)\Big),\\
r^{X,\mathcal{D}}_{n,0} &= \frac{1}{2q^{\frac{11}{2}}\sigma_2(1+q)(b_4q^{2n-3};q)_2(b_4q^{2n};q)_2}A,
\end{aligned}
$$

Here A is a polynomial of degree 8 in q^n, whose coefficients are polynomials in σ_1, σ_2, σ'_1 and σ'_2. Since A has a lengthy expression, we put it on the web page [42].

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