A discrete linearizability test based on multiscale analysis

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Abstract
In this paper we consider the classification of dispersive linearizable partial difference equations defined on a quad-graph by the multiple scale reduction around their harmonic solution. We show that the A_1, A_2 and A_3 linearizability conditions restrain the number of the parameters which enter into the equation. A subclass of the equations which pass the A_3 C-integrability conditions can be linearized by a Möbius transformation.

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1. Introduction

Calogero in 1991 [2] introduced the notion of S and C integrable equations to characterize those nonlinear partial differential equations (PDEs) which are solvable through an inverse scattering transform or linearizable through a change of variables. Using the multiscale reductive technique he was able to show that the nonlinear Schrödinger equation (NLSE)

$$\mathrm{i}\partial_t u = K_2[u] = \partial_{xx} u + \rho_2 |u|^2 u, \qquad u = u(x,t) \tag{1}$$

appears as a universal equation governing the evolution of slowly varying packets of quasi-monochromatic waves in a weakly nonlinear media featuring dispersion. The necessary conditions for the S-integrability is that ρ_2 is real. If, however, the equation is linearizable then ρ_2 must be null, the equation has to be linear or linearizable, as the Eckhaus equation [4, 20].

By going to a higher order in the expansion, the multiscale techniques have been used to find new S-integrable PDEs and to prove the integrability of new nonlinear equations

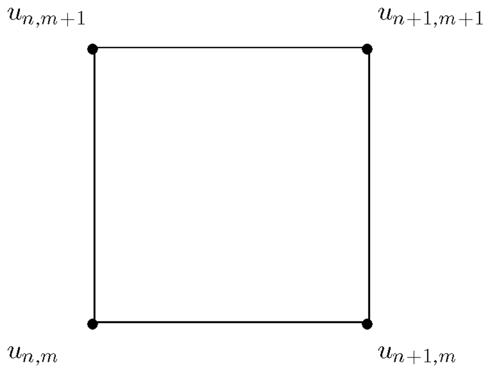

Figure 1. The $\mathbb{Z}^2$ square-lattice where the equation $\mathcal{Q} = 0$ is defined.

[7, 8, 13]. Probably the most important example of such nonlinear PDE is the Degasperis–Procesi equation [6]. The application of this test to the case of linearizable equations has not been done yet. However, Calogero and collaborators constructed many interesting linearizable nonlinear evolution equations by applying complicated transformations to linear equations [3, 5].

In the case of discrete equations it has been shown [1, 9–11, 16–19, 24] that a similar situation is also true. One can present the equivalent of the Calogero–Eckhaus theorem stating that a nonlinear dispersive PDE will not be S-integrable if its multiscale expansion on analytic functions will not give rise to an integrable NLSE at the lowest order. Moreover, it was shown on examples that a nonlinear PDE will be C-integrable if its multiscale expansion on analytic functions will give rise to a linear PDE [23].

As was shown in [10, 11, 17], the introduction of multiple scales on a lattice reduces the given discrete equation either to a local nonlinear PDE, by imposing a slow-varying condition, or to a PDE of infinite order when dealing with analytic functions. Here we choose the second alternative because, as shown in [10, 11], only in this case are the integrability conditions of the discrete equation preserved.

In this paper we provide necessary conditions for the linearizability of a class of real difference equations in the variable $u : \mathbb{Z}^2 \to \mathbb{R}$ and its three nearest neighbors defined on a $\mathbb{Z}^2$ square-lattice (see figure 1)

$$\mathcal{Q}(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}; \beta_1, \beta_2, \ldots) = 0, \tag{2}$$

where β_i's are the real parameters. Linearizability conditions will be determined through the multiple scale perturbative development.

We will suppose that equation (2) is linear affine in every variable, and our classification will be carried out up to the simultaneous Möbius transformations

$$u_{n,m} \mapsto (Au_{n,m} + B)\big/(Cu_{n,m} + D). \tag{3}$$

In section 2, we will briefly discuss the multiple scale expansion for equation (2) and present the integrability conditions which ensure that the given equation is a C-integrable equation of j order, with $j = 1, 2, 3$, i.e. it is such that asymptotically satisfies the C-integrability conditions up to third order. In section 3, we apply them to the equation (2) and present a sequence of theorems which give the conditions on the constants β_i's under which the system is A_1, A_2 or A_3 integrable. At the end we present some conclusive remarks.

2. Expansion of real dispersive partial difference equations

We now briefly illustrate all the ingredients of the reductive perturbative technique necessary to treat difference equations, as presented in [9, 10]. We will consider here the multiscale expansion in the case of analytic functions so as to preserve the linearizability of the given equation.

Introducing multiple lattice scales, under some obvious hypotheses on the $\mathcal{C}^{(\infty)}$ property of the function $u_{n,m}$ and on the radius of convergence of its Taylor expansion for all the n and m shifts involved in the difference equation (2), we can write a series representation of the shifted values of $u_{n,m}$ around the point (n, m). Choosing

$$\begin{aligned} n_i &= \varepsilon_{n_i} n, \qquad & \varepsilon_{n_i} &\doteq N_i \varepsilon^i, \qquad & 1 \leqslant i \leqslant K_n, \\ m_j &= \varepsilon_{m_j} m, \qquad & \varepsilon_{m_j} &\doteq M_j \varepsilon^j, \qquad & 1 \leqslant j \leqslant K_m, \end{aligned}$$

where the various constants N_i, M_j and ε are all real numbers and we will assume $K_n = 1$ and $K_m = K$ (eventually $K = +\infty$), the total shift operators T_n, T_m can be rewritten in terms of partial shift operators $\mathcal{T}_n$, $\mathcal{T}_m$ as

$$T_n = \mathcal{T}_n \mathcal{T}_{n_1}^{(\varepsilon_{n_1})} = \mathcal{T}_n \sum_{j=0}^{+\infty} \varepsilon^j \mathcal{A}_n^{(j)}, \qquad \mathcal{A}_n^{(j)} \doteq \frac{N_1^j}{j!} \partial_{n_1}^j, \tag{4a}$$

$$T_m = \mathcal{T}_m \prod_{j=1}^{K} \mathcal{T}_{m_j}^{(\varepsilon_{m_j})} = \mathcal{T}_m \sum_{j=0}^{+\infty} \varepsilon^j \mathcal{A}_m^{(j)}, \tag{4b}$$

$$T_n T_m = \mathcal{T}_n \mathcal{T}_m \mathcal{T}_{n_1}^{(\varepsilon_{n_1})} \prod_{j=1}^{K} \mathcal{T}_{m_j}^{(\varepsilon_{m_j})} = \mathcal{T}_n \mathcal{T}_m \sum_{j=0}^{+\infty} \varepsilon^j \mathcal{A}_{n,m}^{(j)}, \tag{4c}$$

and in terms of differential operators $\mathcal{A}_m^{(j)}$, $\mathcal{A}_{n,m}^{(j)}$. These operators are given by appropriate combinations of $\frac{M_k^j}{j!} \partial_{m_k}^j$ (explicit expressions and formulae can be found in [9], see [22] for more details). We can assume for the function $u_{n,m} = u(n, m, n_1, \{m_j\}_{j=1}^{K}, \varepsilon)$ a double expansion in harmonics and in the perturbative parameter ε

$$u_{n,m} = \sum_{\gamma=1}^{+\infty} \sum_{\theta=-\gamma}^{\gamma} \varepsilon^\gamma u_\gamma^{(\theta)}(n_1, m_j, j \geqslant 1)\, \mathrm{e}^{\mathrm{i}\theta(\kappa h n - \omega\sigma m)}, \tag{5}$$

with $u_\gamma^{(-\theta)}(n_1, m_j, j \geqslant 1) = \bar{u}_\gamma^{(\theta)}(n_1, m_j, j \geqslant 1)$ in order to ensure the reality of $u_{n,m}$. Then, inserting the explicit expressions (4) of the shift operators in terms of the derivatives with respect to the slow variables, equation (2) turns out to be a PDE of infinite order. Moreover, we require the functions $u_\gamma^{(\theta)}$ to satisfy the asymptotic conditions $\lim_{n_1 \to \pm\infty} u_\gamma^{(\theta)} = 0$, $\forall \gamma$ and θ, and the index γ chosen $\geqslant 1$ in order to let any nonlinear part of equation (2) to enter as a perturbation in the multiscale expansion.

2.1. The orders beyond the Schrödinger equation and the C-integrability conditions

The multiple scale expansion of a nonlinear PDE on analytic functions will thus give rise to continuous PDEs. So a multiple scale linearizability test will require that an equation is C-integrable if its multiple scale expansion will go into the hierarchy of the Schrödinger equation. To show this we need to consider the orders beyond that at which one obtains for the harmonic $u_1^{(1)}$ the Schrödinger equation. References [7, 12, 14, 15] contain different approaches to the higher order multiscale expansion of S-integrable nonlinear PDE's.

Following [7] we can remove all secular terms from the reduced equations order by order and thus, in agreement with Degasperis and Procesi [8], we can state the following theorem proved by Procesi [21].

Theorem 1. *A nonlinear dispersive partial difference equation is C-integrable only if its multiscale expansion is given by a uniform asymptotic series such that*

(1) the amplitude $u_1^{(1)}$ evolves at the slow-times m_σ, $\sigma \geqslant 2$ according to the σth equation of the linear Schrödinger hierarchy:

$$\partial_{m_\sigma} u_1^{(1)} = (-i)^{(\sigma-1)} B_\sigma \partial_{n_1}^{\sigma} u_1^{(1)}, \qquad B_\sigma \doteq -\frac{1}{\sigma!}\frac{\mathrm{d}^\sigma \omega(\kappa)}{\mathrm{d}\kappa^\sigma}, \tag{6}$$

where B_σ are constants;

(2) the amplitudes of the higher perturbations of the first harmonic $u_j^{(1)}$, $j \geqslant 2$ evolve at the slow-times m_σ, $\sigma \geqslant 2$ according to certain linear, nonhomogeneous equations when taking into account the proper asymptotic boundary conditions:

$$\partial_{m_\sigma} u_j^{(1)} - (-i)^{(\sigma-1)} B_\sigma \partial_{n_1}^{\sigma} u_j^{(1)} \equiv M_\sigma u_j^{(1)} = f_\sigma(j), \tag{7}$$

$\forall\, j$, $\sigma \geqslant 2$, where $B_\sigma \partial_{n_1}^{\sigma} u_1^{(j)}$ is the σth flow in the linear Schrödinger hierarchy (6). All the other $u_j^{(\theta)}$, $\theta \geqslant 2$, are expressed in terms of differential monomials of $u_\rho^{(1)}$, $\rho \leqslant j$.

If the asymptotic expansion is, besides a necessary condition for integrability, also a sufficient condition, has been only conjectured in [21]. The proof of the theorem takes into account that for a C-integrable equation by definition we have a linearizing transformation. The next step is to assume that the function satisfying the linearized equation admits a similar perturbative expansion as in (5) and then to perform a multiscale analysis of the linearizing transformation.

In equation (7), $f_\sigma(j)$ is a nonhomogeneous *nonlinear* forcing term and B_σ, $\sigma \geqslant 1$, introduced in equation (6) are the complex constants. Equation (6) represents a hierarchy of *compatible* evolutions for the function $u_1^{(1)}$. It is obvious that the operators M_σ defined in equation (7) commute among themselves. Thus, once we fix the index $j \geqslant 2$ in the set of equations (7), their compatibilities imply the following conditions:

$$M_\sigma f_{\sigma'}(j) = M_{\sigma'} f_\sigma(j), \qquad \forall\, \sigma, \sigma' \geqslant 2, \tag{8}$$

where as $f_\sigma(j)$ and $f_{\sigma'}(j)$ are functions of the different perturbations of the fundamental harmonic up to degree $j-1$, the time derivatives ∂_{t_σ}, $\partial_{t_{\sigma'}}$ of those harmonics appearing respectively in M_σ and $M_{\sigma'}$ have to be eliminated using the evolution equations (6) and (7) up to the index $j-1$. The commutativity conditions (8) turn out to be *an integrability test*.

To construct the functions $f_\sigma^{(j)}$ we define homogeneous spaces of functions of $u_j^{(1)}$, its n_1-derivative and its complex conjugate.

Definition 2.1. *A differential monomial $\rho\big[u_j^{(1)}\big]$, $j \geqslant 1$ in the functions $u_j^{(1)}$, its complex conjugate and its n_1-derivatives is a monomial of 'gauge' 1 if it possesses the transformation property*

$$\rho\big[\tilde{u}_j^{(1)}\big] = \mathrm{e}^{\mathrm{i}\theta}\rho\big[u_j^{(1)}\big], \qquad \tilde{u}_j^{(1)} \doteq \mathrm{e}^{\mathrm{i}\theta} u_j^{(1)}.$$

Definition 2.2. *A finite dimensional vector space $\mathcal{P}_\nu$, $\nu \geqslant 2$ is the set of all differential polynomials in the functions $u_j^{(1)}$, $j \geqslant 1$, their complex conjugates and their n_1-derivatives of order ν in ε and gauge 1 such that*

$$order\big(\partial_{n_1}^{\mu} u_j^{(1)}\big) = order\big(\partial_{n_1}^{\mu} \bar{u}_j^{(1)}\big) = \mu + j, \qquad \mu \geqslant 0.$$

Definition 2.3. *$\mathcal{P}_\nu(\mu)$, $\mu \geqslant 1$ and $\nu \geqslant 2$ is the subspace of $\mathcal{P}_\nu$ whose elements are differential polynomials in the functions $u_j^{(1)}$s, their complex conjugates and their n_1-derivatives of order ν in ε and gauge 1 for $1 \leqslant j \leqslant \mu$.*

The basis monomials of the spaces $\mathcal{P}_\nu(\mu)$ can be found, for example, in [22].

The functions $f_\sigma(j)$ belong to a well defined-finite dimensional space $\mathcal{P}_\nu(\mu)$ and if relations (8) are satisfied up to the index k, $k \geqslant 2$, we say that our equation is asymptotically C-integrable of degree k or A_k C-integrable.

2.2. *C-integrability conditions for the Schrödinger hierarchy*

In this subsection we present the conditions which must be satisfied for the asymptotic C-integrability of order k or the A_k C-integrability conditions with $k = 1, 2, 3$. To simplify the notation, we will use for $u_j^{(1)}$ the concise form $u(j)$.

The A_1 C-integrability condition is given by the absence of the coefficient ρ_2 of the nonlinear term in the NLSE.

The A_2 C-integrability condition is obtained by choosing $j = 2$ in the compatibility conditions (8) with $\sigma = 2$ and $\sigma' = 3$:

$$M_2 f_3(j) = M_3 f_2(j), \tag{9}$$

where $M_2 = \partial_{m_2} + \mathrm{i}B_2\partial_{n_1}^2$ and $M_3 = \partial_{m_3} + B_3\partial_{n_1}^3$.

In this case $f_2(2) \in \mathcal{P}_4(1)$ and $f_3(2) \in \mathcal{P}_5(1)$ with $\dim(\mathcal{P}_4(1)) = 2$ and $\dim(\mathcal{P}_5(1)) = 5$, so that $f_2(2)$ and $f_3(2)$ will be respectively identified by two and five complex constants

$$f_2(2) \doteq a u_{n_1}(1)|u(1)|^2 + b\bar{u}_{n_1}(1)u(1)^2, \tag{10a}$$

$$\begin{aligned} f_3(2) \doteq{}& \alpha|u(1)|^4 u(1) + \beta|u_{n_1}(1)|^2 u(1) + \gamma u_{n_1}(1)^2\bar{u}(1) \\ &+\delta\bar{u}_{n_1n_1}(1)u(1)^2 + \epsilon|u(1)|^2 u_{n_1n_1}(1). \end{aligned} \tag{10b}$$

In this way, eliminating from equation (9) the derivatives of $u(1)$ with respect to the slow-times m_2 and m_3 using evolutions (6) with $\sigma = 2, 3$ and equating term by term, we obtain that the A_2 C-integrability conditions give no constraint for the two coefficients a and b. As a consequence one can say that the A_1 C-integrability condition $\rho_2 = 0$ automatically implies A_2 C-integrability, or that it indeed represents an A_2 C-integrability condition. The expression of $\alpha, \beta, \gamma, \delta, \epsilon$ in terms of a and b are

$$\alpha = 0, \qquad \beta = -\frac{3\mathrm{i}B_3 b}{B_2}, \qquad \gamma = -\frac{3\mathrm{i}B_3 a}{2B_2}, \qquad \delta = 0, \qquad \epsilon = \gamma, \tag{11}$$

where, for convenience, from now on we will write B for B_3.

The A_3 C-integrability conditions are derived in a similar way setting $j = 3$ in equation (9). In this case we have that $f_2(3) \in \mathcal{P}_5(2)$ and $f_3(3) \in \mathcal{P}_6(2)$ with $\dim(\mathcal{P}_5(2)) = 12$ and $\dim(\mathcal{P}_6(2)) = 26$, so that $f_2(3)$ and $f_3(3)$ will be respectively identified by 12 and 26 complex constants

$$\begin{aligned} f_2(3) \doteq{}& \tau_1|u(1)|^4u(1) + \tau_2|u_{n_1}(1)|^2u(1) + \tau_3|u(1)|^2u_{n_1n_1}(1) + \tau_4\bar{u}_{n_1n_1}(1)u(1)^2 + \tau_5u_{n_1}(1)^2\bar{u}(1) \\ &+ \tau_6u_{n_1}(2)|u(1)|^2 + \tau_7\bar{u}_{n_1}(2)u(1)^2 + \tau_8u(2)^2\bar{u}(1) + \tau_9|u(2)|^2u(1) \\ &+ \tau_{10}u(2)u_{n_1}(1)\bar{u}(1) + \tau_{11}u(2)\bar{u}_{n_1}(1)u(1) + \tau_{12}\bar{u}(2)u_{n_1}(1)u(1), \end{aligned} \tag{12a}$$

$$\begin{aligned} f_3(3) \doteq{}& \gamma_1|u(1)|^4u_{n_1}(1) + \gamma_2|u(1)|^2u(1)^2\bar{u}_{n_1}(1) + \gamma_3|u(1)|^2u_{n_1n_1n_1}(1) \\ &+ \gamma_4u(1)^2\bar{u}_{n_1n_1n_1}(1) + \gamma_5|u_{n_1}(1)|^2u_{n_1}(1) + \gamma_6\bar{u}_{n_1n_1}(1)u_{n_1}(1)u(1) \end{aligned}$$

$$
\begin{aligned}
&+\gamma_7 u_{n_1n_1}(1)\bar{u}_{n_1}(1)u(1)+\gamma_8 u_{n_1n_1}(1)u_{n_1}(1)\bar{u}(1)+\gamma_9|u(1)|^4u(2)\\
&+\gamma_{10}|u(1)|^2u(1)^2\bar{u}(2)+\gamma_{11}\bar{u}_{n_1}(1)u(2)^2+\gamma_{12}u_{n_1}(1)|u(2)|^2+\gamma_{13}|u_{n_1}(1)|^2u(2)\\
&+\gamma_{14}|u(2)|^2u(2)+\gamma_{15}u_{n_1}(1)^2\bar{u}(2)+\gamma_{16}|u(1)|^2u_{n_1n_1}(2)+\gamma_{17}u(1)^2\bar{u}_{n_1n_1}(2)\\
&+\gamma_{18}u(2)\bar{u}_{n_1n_1}(1)u(1)+\gamma_{19}u(2)u_{n_1n_1}(1)\bar{u}(1)+\gamma_{20}\bar{u}(2)u_{n_1n_1}(1)u(1)\\
&+\gamma_{21}u(2)u_{n_1}(2)\bar{u}(1)+\gamma_{22}\bar{u}(2)u_{n_1}(2)u(1)+\gamma_{23}u_{n_1i}(2)u_{n_1}(1)\bar{u}(1)\\
&+\gamma_{24}u_{n_1}(2)\bar{u}_{n_1}(1)u(1)+\gamma_{25}\bar{u}_{n_1i}(2)u_{n_1}(1)u(1)+\gamma_{26}\bar{u}_{n_1}(2)u(2)u(1). \qquad (12b)
\end{aligned}
$$

Let us eliminate from equation (9) with $j = 3$ the derivatives of $u(1)$ with respect to the slow-times m_2 and m_3 using evolutions (6) respectively with $\sigma = 2, 3$ and the same derivatives of $u(2)$ using evolutions (7) with $\sigma = 2, 3$. Equating the remaining terms, term by term, the A_3 C-integrability conditions turn out to be

$$
\begin{aligned}
&\tau_1 = -\frac{\mathrm{i}}{4B_2}\left[b\left(\tau_{11}-2\tau_6\right)+\bar{a}\tau_7\right], \qquad \bar{b}\tau_7 = \frac{1}{2}\left(b-a\right)\left(\tau_{11}+\tau_{10}-\tau_6\right)+\bar{a}\tau_7,\\
&a\tau_8 = b\tau_8 = 0, \qquad a\tau_9 = b\tau_9 = 0, \qquad \bar{a}\tau_{12} = a\left(\tau_{10}-\tau_{11}\right)+b\tau_6+\bar{a}\tau_7, \qquad (13)\\
&(\bar{b}-\bar{a})\tau_{12} = (b-a)\tau_{10}.
\end{aligned}
$$

Sometimes a and b turn out to be both real. In this case the conditions given in equations (13) become

$$
\begin{aligned}
&R_1 = \frac{1}{4B_2}\left[b\left(I_{11}-2I_6\right)+aI_7\right], \qquad I_1 = -\frac{1}{4B_2}\left[b\left(R_{11}-2R_6\right)+aR_7\right],\\
&(b-a)\left(R_{11}+R_{10}-R_6-2R_7\right) = 0, \qquad (b-a)\left(I_{11}+I_{10}-I_6-2I_7\right) = 0,\\
&(b-a)R_8 = 0, \qquad (b-a)\,I_8 = 0, \qquad (b-a)\,R_9 = 0, \qquad (b-a)\,I_9 = 0, \qquad (14)\\
&a(R_{12}+R_{11}-R_{10}-R_7) = bR_6, \qquad a(I_{12}+I_{11}-I_{10}-I_7) = bI_6,\\
&(b-a)(R_{12}-R_{10}) = 0, \qquad (b-a)(I_{12}-I_{10}) = 0,
\end{aligned}
$$

where $\tau_j = R_j + \mathrm{i}I_j$ for $j = 1, \ldots, 12$. The expressions of the γ_j as functions of the τ_i are

$$
\begin{aligned}
&\gamma_1 = \frac{3B_3}{4B_2^2}(a\tau_6 - 4\mathrm{i}B_2\tau_1 + \bar{b}\tau_{12}), \qquad \gamma_2 = \frac{3B_3}{4B_2^2}\left(b\tau_6+\bar{a}\tau_7\right),\\
&\gamma_3 = -\frac{3\mathrm{i}B_3\tau_3}{2B_2}, \qquad \gamma_4 = 0, \qquad \gamma_5 = -\frac{3\mathrm{i}B_3\tau_2}{2B_2},\\
&\gamma_6 = -\frac{3\mathrm{i}B_3\tau_4}{B_2}, \qquad \gamma_7 = \gamma_5, \qquad \gamma_8 = \gamma_3 - \frac{3\mathrm{i}B_3\tau_5}{B_2}, \qquad \gamma_9 = 0, \qquad \gamma_{10} = 0,\\
&\gamma_{11} = 0, \qquad \gamma_{12} = -\frac{3\mathrm{i}B_3\tau_9}{2B_2}, \qquad \gamma_{13} = -\frac{3\mathrm{i}B_3\tau_{11}}{2B_2}, \qquad \gamma_{14} = 0, \qquad \gamma_{15} = -\frac{3\mathrm{i}B_3\tau_{12}}{2B_2},\\
&\gamma_{16} = -\frac{3\mathrm{i}B_3\tau_6}{2B_2}, \qquad \gamma_{17} = \gamma_{18} = 0, \qquad \gamma_{19} = -\frac{3\mathrm{i}B_3\tau_{10}}{2B_2},\\
&\gamma_{20} = \gamma_{15}, \qquad \gamma_{21} = -\frac{3\mathrm{i}B_3\tau_8}{B_2},\\
&\gamma_{22} = \gamma_{12}, \qquad \gamma_{23} = \gamma_{16}+\gamma_{19}, \qquad \gamma_{24} = \gamma_{13}, \qquad \gamma_{25} = -\frac{3\mathrm{i}B_3\tau_7}{B_2}, \qquad \gamma_{26} = 0.
\end{aligned}
$$

The conditions given in equations (13) and (14) appear to be new. Their importance resides in the fact that a C-integrable equation must satisfy those conditions.

3. Dispersive affine-linear equations on the square lattice

In this section we derive the necessary conditions for the linearizability of a dispersive and homogeneous affine-linear equation defined on the square lattice and belonging to class (2). The most general multilinear equation of class (2) has at most quartic nonlinearity. Let us introduce into its linear part the solution $u_{n,m} = K^n \Omega^m$ where $K = \mathrm{e}^{\mathrm{i}\kappa}$ and $\Omega = \mathrm{e}^{-\mathrm{i}\omega(\kappa)}$. We get that this equation is dispersive if homogeneous and written as

$$\begin{aligned}\mathcal{Q}_{\pm} = {} & a_1(u_{n,m} \pm u_{n+1,m+1}) + a_2(u_{n+1,m} \pm u_{n,m+1}) \\ & + (\alpha_1 - \alpha_2)\, u_{n,m}u_{n+1,m} + (\alpha_1 + \alpha_2)\, u_{n,m+1}u_{n+1,m+1} \\ & + (\beta_1 - \beta_2)\, u_{n,m}u_{n,m+1} + (\beta_1 + \beta_2)\, u_{n+1,m}u_{n+1,m+1} \\ & + \gamma_1 u_{n,m}u_{n+1,m+1} + \gamma_2 u_{n+1,m}u_{n,m+1} \\ & + (\xi_1 - \xi_3)\, u_{n,m}u_{n+1,m}u_{n,m+1} + (\xi_1 + \xi_3)\, u_{n,m}u_{n+1,m}u_{n+1,m+1} \\ & + (\xi_2 - \xi_4)\, u_{n+1,m}u_{n,m+1}u_{n+1,m+1} + (\xi_2 + \xi_4)\, u_{n,m}u_{n,m+1}u_{n+1,m+1} \\ & + \zeta\, u_{n,m}u_{n+1,m}u_{n,m+1}u_{n+1,m+1} = 0, \end{aligned} \tag{15}$$

where $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, $|a_1| \neq |a_2|$, are the coefficients appearing in the linear part while $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \xi_1, \ldots, \xi_4, \zeta$ are eleven real parameters to be determined by using the multiscale procedure described in section 2. The linear dispersion relation is given by

$$\omega(\kappa) = \arctan\left[\frac{\left(a_1^2 - a_2^2\right)\sin\kappa}{\left(a_1^2 + a_2^2\right)\cos\kappa + 2a_1a_2}\right]. \tag{16}$$

Let us look for those transformations which leave the class of equations $\mathcal{Q}_{\pm}$ invariant. They will give the equivalence conditions for our classification. It is well known that polynomial equations are invariant under simultaneous Möbius transformations $u_{n,m} \mapsto (Au_{n,m} + B)/(Cu_{n,m} + D)$. However, our class of equations is homogeneous with a restriction on the coefficients of the linear part. After a Möbius transformation a constant term

$$a_0 = B^4\zeta + 2B^3D(\xi_1 + \xi_2) + B^2D^2[\gamma_1 + \gamma_2 + 2(\alpha_1 + \beta_1)] + 2BD^3(a_1 + a_2) \tag{17}$$

will appear, and thus, if we do not want to restrict the coefficients of the equation, we have to set $B = 0$ to have $a_0 = 0$. Under the Möbius transformation with $B = 0$ the coefficients of $\mathcal{Q}_{\pm}$ become

$$a_1 \mapsto D^3a_1, \qquad a_2 \mapsto D^3a_2, \qquad \alpha_1 \mapsto D^2[\alpha_1 + C(a_1 + a_2)], \qquad \alpha_2 \mapsto D^2\alpha_2, \tag{18}$$

$$\beta_1 \mapsto D^2[\beta_1 + C(a_1 + a_2)], \qquad \beta_2 \mapsto D^2\beta_2, \tag{19}$$

$$\gamma_1 \mapsto D^2(\gamma_1 + 2Ca_1), \qquad \gamma_2 \mapsto D^2(\gamma_2 + 2Ca_2), \tag{20}$$

$$\xi_1 \mapsto D\xi_1 + \tfrac{1}{2}CD[3C(a_1 + a_2) + \gamma_1 + \gamma_2 + 2(\alpha_1 - \alpha_2 + \beta_1)], \tag{21}$$

$$\xi_2 \mapsto D\xi_2 + \tfrac{1}{2}CD[3C(a_1 + a_2) + \gamma_1 + \gamma_2 + 2(\alpha_1 + \alpha_2 + \beta_1)], \tag{22}$$

$$\xi_3 \mapsto D\xi_3 + \tfrac{1}{2}CD[C(a_1 - a_2) + \gamma_1 - \gamma_2 + 2\beta_2], \tag{23}$$

$$\xi_4 \mapsto D\xi_4 + \tfrac{1}{2}CD[C(a_1 - a_2) + \gamma_1 - \gamma_2 - 2\beta_2], \tag{24}$$

$$\zeta \mapsto \zeta + C^2[2C(a_1 + a_2) + \gamma_1 + \gamma_2 + 2(\alpha_1 + \beta_1)] + 2C(\xi_1 + \xi_2). \tag{25}$$

So our equivalence transformation with respect to which we will be classifying the equation $\mathcal{Q}_{\pm}$ is a restricted Möbius transformation of the form $u_{n,m} \mapsto u_{n,m}/(Cu_{n,m} + D)$.

In this paper we limit ourselves to consider the case of $\mathcal{Q}_{+}$. The case $\mathcal{Q}_{-}$ will be dealt with in a subsequent publication.

Imposing that $\rho_2 = 0$ we get he following proposition.

Proposition 1. *The lowest order necessary conditions for the linearizability of equations $\mathcal{Q}_+$ give six different classes of equations characterized by different non-superimposed ranges of values of the coefficients of the equation $\mathcal{Q}_+$. They are*

- *case L1:*

$$\begin{cases} \alpha_2 = \beta_2 = 0, \quad \alpha_1 = \beta_1, \quad \gamma_1 + \gamma_2 = 2\beta_1, \\ \xi_1 = \xi_2 = \frac{(a_1+a_2)^2\gamma_1\gamma_2 + (3a_1-2a_2)a_2\gamma_1\beta_1 - a_1(2a_1-3a_2)\gamma_2\beta_1}{4a_1a_2(a_1+a_2)}, \\ \xi_3 = \xi_4 = \frac{-(a_1-a_2)(a_1+a_2)^2\gamma_1\gamma_2 - a_2\left(-a_1^2-5a_2a_1+2a_2^2\right)\gamma_1\beta_1 + a_1\left(2a_1^2-5a_2a_1-a_2^2\right)\gamma_2\beta_1}{4a_1a_2(a_1+a_2)^2}. \end{cases} \tag{26}$$

- *case L2:*

$$\begin{cases} \alpha_2 = \beta_2 = 0, \quad \alpha_1 = \beta_1, \quad (3a_1 - 2a_2)a_2^2\gamma_1 + a_1^2(2a_1 - 3a_2)\gamma_2 = 4a_1(a_1 - a_2)a_2\beta_1, \\ \xi_1 = \xi_2 = \frac{(a_1+a_2)\left(a_2^2\gamma_1^2 - a_1^2\gamma_2^2\right) + 2a_2\left(a_1^2 - 2a_2^2\right)\gamma_1\beta_1 + 2a_1\left(2a_1^2 - a_2^2\right)\gamma_2\beta_1 - 6a_1(a_1-a_2)a_2\beta_1^2}{4a_1a_2\left(a_1^2 - a_2^2\right)}, \\ \xi_3 = \xi_4 = \frac{2a_1a_2(a_1+a_2)(\gamma_1-\gamma_2)\beta_1 + (a_1-a_2)(a_2\gamma_1 - a_1\gamma_2)^2 + 2a_1a_2(a_2-a_1)\beta_1^2}{4a_1a_2(a_1+a_2)^2}. \end{cases} \tag{27}$$

- *case L3:*

$$\begin{cases} \alpha_2 = \beta_2 = 0, \qquad a_1 = 2a_2, \qquad \gamma_1 = \frac{2}{3}(\alpha_1 + \beta_1), \qquad \gamma_2 = \frac{1}{3}(\alpha_1 + \beta_1), \\ \xi_1 = \xi_2 = -\frac{-5\alpha_1\beta_1 + \alpha_1^2 + \beta_1^2}{6a_2}, \qquad \xi_3 = \xi_4 = -\frac{(\alpha_1 - 2\beta_1)(2\alpha_1 - \beta_1)}{18a_2}. \end{cases} \tag{28}$$

- *case L4:*

$$\begin{cases} \alpha_2 = \beta_2 = 0, \qquad 2a_1 = a_2, \qquad \gamma_1 = \frac{1}{3}(\alpha_1 + \beta_1), \qquad \gamma_2 = \frac{2}{3}(\alpha_1 + \beta_1), \\ \xi_1 = \xi_2 = -\frac{-5\alpha_1\beta_1 + \alpha_1^2 + \beta_1^2}{6a_1}, \qquad \xi_3 = \xi_4 = \frac{(\alpha_1 - 2\beta_1)(2\alpha_1 - \beta_1)}{18a_1}. \end{cases} \tag{29}$$

- *case L5:*

$$\begin{cases} \alpha_2 = \beta_2, \qquad \alpha_1 = \beta_1, \qquad 2a_1 = a_2, \qquad \gamma_1 = \frac{2\beta_1}{3}, \qquad \gamma_2 = \frac{4\beta_1}{3}, \\ \xi_1 = \frac{3\beta_1^2 - 2\beta_2\beta_1 + \beta_2^2}{6a_1}, \qquad \xi_2 = \frac{3\beta_1^2 + 2\beta_2\beta_1 + \beta_2^2}{6a_1}, \\ \xi_3 = -\frac{\beta_1^2 - 6\beta_2\beta_1 + 7\beta_2^2}{18a_1}, \qquad \xi_4 = -\frac{\beta_1^2 + 6\beta_2\beta_1 + 7\beta_2^2}{18a_1}. \end{cases} \tag{30}$$

- *case L6:*

$$\begin{cases} \alpha_2 = -\beta_2, \qquad \alpha_1 = \beta_1, \qquad a_1 = 2a_2, \qquad \gamma_1 = \frac{4\beta_1}{3}, \qquad \gamma_2 = \frac{2\beta_1}{3}, \\ \xi_1 = \frac{3\beta_1^2 + 2\beta_2\beta_1 + \beta_2^2}{6a_2} \qquad \xi_2 = \frac{3\beta_1^2 - 2\beta_2\beta_1 + \beta_2^2}{6a_2}, \\ \xi_3 = \frac{\beta_1^2 + 6\beta_2\beta_1 + 7\beta_2^2}{18a_2} \qquad \xi_4 = \frac{\beta_1^2 - 6\beta_2\beta_1 + 7\beta_2^2}{18a_2}. \end{cases} \tag{31}$$

The corresponding six subclasses of equations are invariant under restricted Möbius transformations of the form $u_n \mapsto u_n/(Cu_n + D)$.

Proof. Following the procedure described in section 2 we expand the fields appearing in equation $\mathcal{Q}_+$ using definitions (5) (4). The lowest order necessary conditions for the integrability of $\mathcal{Q}_+$ are obtained at the order ε^3 of the multiple scale expansion. At this order we get the m_2-evolution equation for the harmonic $u_1^{(1)}$, that is in general a NLSE of the form

$$\mathrm{i}\partial_{m_2}u_1^{(1)} - B_2\partial_{n_2}^2 u_1^{(1)} - \rho_2 u_1^{(1)}|u_1^{(1)}|^2 = 0, \qquad \text{where,} \qquad n_2 = n_1 - \frac{d\omega}{d\kappa}m_1, \tag{32}$$

where the coefficients B_2 and ρ_2 will depend on the parameters of the equation $\mathcal{Q}_+$ and on the wave parameters κ and ω, with ω expressed in terms of κ through the dispersion relation (16).

Let us give just a sketch of the construction of equation (32), omitting all intermediate formulae. At $\mathcal{O}(\varepsilon)$ we get

- for $\theta = 1$ a linear equation which is identically satisfied by the dispersion relation (16);
- for $\theta = 0$ a linear equation whose solution is $u_1^{(0)} = 0$.

At $\mathcal{O}(\varepsilon^2)$, taking into account the dispersion relation (16), we get

- for $\theta = 2$ an algebraic relation between $u_2^{(2)}$ and $u_1^{(1)}$;
- for $\theta = 1$ a linear wave equation for $u_1^{(1)}$, whose solution is given by $u_1^{(1)}(n_1, m_1, m_2) = u_1^{(1)}(n_2, m_2)$, with n_2 given by equation (32);
- for $\theta = 0$ an algebraic relation between $u_1^{(1)}$ and $u_2^{(0)}$.

Let us stress here that at $\mathcal{O}(\varepsilon^2)$ we find that all the harmonics will depend on the slow-variables n_1 and m_1 through n_2.

At $\mathcal{O}(\varepsilon^3)$, for $\theta = 1$, by using the previous results, one gets the NLSE (32) with

$$B_2 = -\frac{a_1a_2\left(a_1^2 - a_2^2\right)\sin\kappa}{\left(a_1^2 + a_2^2 + 2a_1a_2\cos\kappa\right)^2}, \qquad \rho_2 = \mathcal{R}_1 + \mathrm{i}\mathcal{R}_2,$$

where

$$\mathcal{R}_1 = \frac{\sin\kappa\left[\mathcal{R}_1^{(0)} + \mathcal{R}_1^{(1)}\cos\kappa + \mathcal{R}_1^{(2)}\cos^2\kappa + \mathcal{R}_1^{(3)}\cos^3\kappa + \mathcal{R}_1^{(4)}\cos^4\kappa\right]}{(a_1 + a_2)\left(a_1^2 + a_2^2 + 2a_1a_2\cos\kappa\right)^2[(a_1 - a_2)^2 + 2a_1a_2\cos\kappa(1 + \cos\kappa)]}, \tag{33}$$

$$\mathcal{R}_2 = \frac{\mathcal{R}_2^{(0)} + \mathcal{R}_2^{(1)}\cos\kappa + \mathcal{R}_2^{(2)}\cos^2\kappa + \mathcal{R}_2^{(3)}\cos^3\kappa + \mathcal{R}_2^{(4)}\cos^4\kappa + \mathcal{R}_2^{(5)}\cos^5\kappa}{(a_1 + a_2)\left(a_1^2 + a_2^2 + 2a_1a_2\cos\kappa\right)^2[(a_1 - a_2)^2 + 2a_1a_2\cos\kappa(1 + \cos\kappa)]}. \tag{34}$$

Here the coefficients $\mathcal{R}_1^{(i)}$, $0 \leqslant i \leqslant 4$ and $\mathcal{R}_2^{(i)}$, $0 \leqslant i \leqslant 5$ are the polynomials depending on the coefficients $a_1, a_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \xi_1, \ldots, \xi_4$ and their expressions are cumbersome, so that we omit them.

Note that B_2 is a real coefficient depending only on the parameters of the linear part of $\mathcal{Q}_+$, while ρ_2 is a complex one. Hence, the linearizability of the NLS equation (32) is equivalent to the request $\rho_2 = 0\ \forall\kappa$, that is,

$$\mathcal{R}_j^{(i)} = 0, \qquad 0 \leqslant i \leqslant 5, j = 1, 2. \tag{35}$$

Equation (35) is a nonlinear algebraic system of eleven equations in twelve unknowns, the coefficient ζ not appearing at this order of the multiple scale expansion. By solving it with the help of the computer algebra software Mathematica one gets solutions (26)–(31). These solutions are computed taking into account that $a_1, a_2 \in \mathbb{R}\setminus\{0\}$ with $|a_1| \neq |a_2|$. Let us point out that the equations have been solved using only rational algebra, avoiding the unreliability of computer algebra calculations when using irrational functions.

One first solves the six equations corresponding to $\mathcal{R}_2^{(i)} = 0$, $0 \leqslant i \leqslant 5$. Two of them can be solved for ξ_1 and ξ_3 in (rational) terms of the remaining ten coefficients. The resulting system of equations turns out to be ξ_j-independent and linear in the four variables α_1, β_1, γ_1 and γ_2. Therefore, we may write the remaining four equations as a matrix equation with coefficients nonlinearly depending on α_2, β_2, a_1 and a_2. The rank of the matrix is three. Solutions (26)–(31) are obtained by requiring that the matrix be of rank 3, 2, 1 and 0. In this way we get

- *Case 1*:

$$\begin{cases} \alpha_2 = \beta_2 = 0, \\ \xi_1 = \xi_2, \qquad \xi_3 = \xi_4. \end{cases} \tag{36}$$

- *Case 2*:

$$\begin{cases} \alpha_2 = \beta_2, \qquad \alpha_1 = \beta_1, \\ a_1 = 2a_2, \\ \gamma_1 = 2\gamma_2, \\ a_1(\xi_1 - \xi_2) = -a_1(\xi_3 - \xi_4) = -2\alpha_2\gamma_2. \end{cases} \tag{37}$$

- *Case 3*:

$$\begin{cases} \alpha_2 = -\beta_2, \quad \alpha_1 = \beta_1, \\ a_2 = 2a_1, \\ \gamma_2 = 2\gamma_1, \\ a_1(\xi_1 - \xi_2) = a_1(\xi_3 - \xi_4) = -\alpha_2\gamma_1. \end{cases} \tag{38}$$

- *Case 4*:

$$\begin{cases} \alpha_1 = \beta_1 = \frac{1}{2}(1 + a_1/a_2)\gamma_2, \\ a_2\gamma_1 = a_1\gamma_2, \\ a_1(\xi_1 - \xi_2) = -\alpha_2\gamma_1, \\ a_1(\xi_3 - \xi_4) = \beta_2\gamma_1. \end{cases} \tag{39}$$

- *Case 5*:

$$\begin{cases} (a_2 - a_1)\beta_2 = (a_2 + a_1)\alpha_2, \\ 2a_1a_2(a_1 - a_2)\alpha_1 = (a_1 + a_2)\left(\gamma_2 a_1^2 - \gamma_1 a_2^2\right), \\ 2a_1a_2\beta_1 = \gamma_1 a_2^2 + \gamma_2 a_1^2, \\ (a_2 - a_1)(\xi_1 - \xi_2) = (\gamma_1 - \gamma_2)\alpha_2, \\ (a_2 - a_1)^2(\xi_3 - \xi_4) = [\gamma_2(a_2 - 3a_1) - \gamma_1(a_1 - 3a_2)]\,\alpha_2. \end{cases} \tag{40}$$

- *Case 6*:

$$\begin{cases} (a_2 + a_1)\beta_2 = (a_2 - a_1)\alpha_2, \\ 2a_1a_2\alpha_1 = \gamma_1 a_2^2 + \gamma_2 a_1^2, \\ 2a_1a_2(a_1 - a_2)\beta_1 = (a_1 + a_2)\left(\gamma_2 a_1^2 - \gamma_1 a_2^2\right), \\ \left(a_2^2 - a_1^2\right)(\xi_1 - \xi_2) = [\gamma_1(a_1 - 3a_2) - \gamma_2(a_2 - 3a_1)]\alpha_2, \\ (a_1 + a_2)(\xi_3 - \xi_4) = (\gamma_2 - \gamma_1)\alpha_2. \end{cases} \tag{41}$$

Then we impose the remaining five equations $\mathcal{R}_1^{(i)} = 0$, $0 \leqslant i \leqslant 4$ to each of the six obtained solutions (36)–(41). By a straightforward computer aided computation we conclude that case 1 has four subcases, cases (26)–(29), that pass those conditions while case 4 has two subcases (30) and (31). All the other cases provide only subcases of the six solutions L1–L6. Cases 1 and 4 represent the necessary and sufficient conditions for the coefficients a

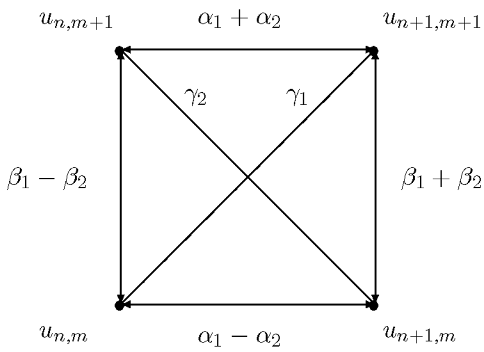

Figure 2. Representation of the quadratic nonlinearities of $\mathcal{Q}_{\pm}$.

and b in $(10a)$ to be real. A direct calculation proves invariance of the resulting equations with respect to the reduced Möbius transformation. Let us stress again that cases (26)–(31) are A_2 C-integrable. □

As a consequence of this proposition we can state the following obvious but important corollary.

Corollary 1. *If the coefficients* $a_1, a_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \xi_1, \ldots, \xi_4$ *of equations* $\mathcal{Q}_+$ *do not satisfy the conditions in equations (26)–(31) then* $\mathcal{Q}_+$ *is not linearizable.*

Note that the trivial linearizability condition $\alpha_1=\alpha_2=\beta_1=\beta_2=\gamma_1=\gamma_2=\xi_1=\xi_2=\xi_3=\xi_4=0$ is contained in equations (26)–(31) (cases L1 to L6).

A particularly interesting case is when we consider an equation with at most quadratic nonlinearity (see figure 2). In this case we get the following proposition.

Proposition 2. *If* $\xi_1=\xi_2=\xi_3=\xi_4=0$ *in equations* $\mathcal{Q}_+$, *then the lowest linearizability conditions are the following.*

- *Cases QL1a and QL1b:* $\alpha_2=\beta_2=0$, $\alpha_1=\beta_1$, $2a_1^2-7a_2a_1+2a_2^2=0 \Leftrightarrow a_1=\frac{1}{4}(7\pm\sqrt{33})a_2$, $\gamma_1=\left(1\mp\sqrt{\frac{3}{11}}\right)\beta_1, \gamma_2=\left(1\pm\sqrt{\frac{3}{11}}\right)\beta_1$.
- *Cases QL2a and QL2b:* $\alpha_2=\beta_2=0$, $\alpha_1=\beta_1$, $4a_1^4-12a_2a_1^3+9a_2^2a_1^2-12a_2^3a_1+4a_2^4=0 \Leftrightarrow a_1=\left(\frac{3}{4}+\frac{1}{\sqrt{2}}\pm\frac{1}{2}\sqrt{\frac{1}{4}+3\sqrt{2}}\right)a_2$, $\gamma_1=\mp2\sqrt{\frac{1}{41}(19+18\sqrt{2})}\beta_1$, $\gamma_2=\pm2\sqrt{\frac{1}{41}(19+18\sqrt{2})}\beta_1$.

Cases QL1a and QL1b are in case L1 and cases QL2a and QL2b are in case L2.

Let us consider now the higher order terms of the expansion for the cases L1–L6. Imposing the A_3 C-integrability conditions (14) on the real and imaginary parts of the coefficients τ_j, $j=1,\ldots,12$, defined in expression $(12a)$, one obtains the following proposition.

Proposition 3. *The most general* A_3 *C-integrable equation is represented by*

$$\begin{aligned} &\alpha_1=\beta_1=\frac{(a_1+a_2)\gamma_1}{2a_1}, \qquad \alpha_2=\beta_2=0, \qquad \gamma_2=\frac{a_2\gamma_1}{a_1}, \\ &\xi_1=\xi_2=\frac{3(a_1+a_2)\gamma_1^2}{8a_1^2}, \qquad \xi_3=\xi_4=\frac{(a_1-a_2)\gamma_1^2}{8a_1^2}. \end{aligned} \tag{42}$$

Case (42) is the intersection of cases (26) and (27). As a consequence of result (42) we have that an equation of the form Q_+ which satisfies the proposition 2, i.e. dispersive with at most quadratic nonlinearity, will never be C-integrable.

Moreover it is easy to prove the following theorem.

Theorem 2. *The equations Q_+ satisfying the A_3 C-integrability (42) can be linearized by a real Möbius transformation*

$$u_{n,m} = \frac{\alpha v_{n,m} + \beta}{\gamma v_{n,m} + \delta}$$

if and only if $\zeta = \frac{(a_1+a_2)\gamma_1^3}{4a_1^3}$. The coefficients of the linearizing transformation are

$$\beta = 0, \qquad \gamma = -\frac{\alpha\gamma_1}{2a_1},$$

while the resulting linearized equation is

$$v_{n,m} + v_{n+1,m+1} + \frac{a_2}{a_1}(v_{n+1,m} + v_{n,m+1}) = 0. \tag{43}$$

Equation (43) is the most general linear dispersive equation defined on the square. As the Möbius transformation used is restricted Möbius transformation, (43) can be assumed to be the canonical equation for this case.

4. Conclusions

In this work we presented the complete classification of the linerizable dispersive partial difference equations belonging to the Q_+ class using multiple scales expansions around a periodic discrete wave solution of the linearized equation, up to the fifth order in the perturbation expansion parameter. The resulting linearizable system depends on three free parameters, and requesting the resulting equation to explicitly linearize via a Möbius transformation reduces the free parameters to one. The proof of the linearizability of the equation Q_+ satisfying just the A_3 C-integrability conditions (42) is still an open problem. Maybe by going higher in the perturbation expansion we could be able to fix the parameter ζ according to the theorem in last section.

This calculation shows that the multiple scale expansion can be effectively used to classify discrete equations.

Work is in progress for the derivation of the integrable class of dispersive partial difference equations belonging to the Q_+ class. They are provided by a reduced set of equations with respect to the one considered in this case, as $\rho_2 \neq 0$ is just a real constant. Moreover, the integrability conditions are given by a larger number of nonlinear algebraic equations for the parameters entering into the equation and, as such, are much harder to solve.

Work is also in progress in the study of the Q_- case. In this case we get at the lowest order in the perturbative parameter a nonlinear system of PDEs relating the zeroth and the fundamental harmonic. The solution of this equation is the key ingredient in the classification of this class of equations which contains all dispersive equations belonging to the ABS classification of multilinear equations on the square.

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