Long-term development of how students interpret a model; Complementarity of contexts and mathematics

Vos, Pauline; Roorda, Gerrit

Published in:
Mathematical Modelling and Applications

Publication date:
2017

Long-term development of how students interpret a model; Complementarity of contexts and mathematics

Pauline Vos
University of Agder, Norway

Gerrit Roorda
University of Groningen, the Netherlands

Abstract

When students engage in rich mathematical modelling tasks, they have to handle real-world contexts and mathematics in chorus. This is not easy. In this chapter, contexts and mathematics are perceived as complementary, which means they can be integrated. Based on four types of approaches to modelling tasks (ambivalent, reality bound, mathematics bound, or integrating), we used task-based interviews to study the development of students' approaches while the students moved from grade 11 to 12. Our participants were ten Dutch students. We found that their approaches initially were either ambivalent, reality bound or mathematics bound. In subsequent interviews the preference was maintained, and in the end the approaches of four students were integrating. Both a reality bound and a mathematics bound preference could lead to a more advanced integrating approach.

1 Introduction

In mathematical modelling students have to deal with real-world contexts on the one hand, and mathematics on the other hand. The variety of prompts within a task activates students' knowledge of the context, or their knowledge of mathematics, or both. As a result, students' thinking and acting will be very dynamic and diverse.

Borromeo Ferri (2010) studied patterns in students' approaches to modelling problems, finding that students followed their own *modelling routes*. Borromeo Ferri related students' modelling routes to their learning styles, which revealed an underlying preference to task approaches. Busse (2011) also studied patterns in students' approaches to modelling tasks. He found four different types of approaches of how students dealt with the real-world context within a modelling task. Students' approaches could be: ambivalent, reality bound, mathematics bound, or integrating. For example, an approach was considered *reality bound*, if only extra-mathematical concepts and methods were applied. An approach was considered *mathematics bound* if the real-world context was treated as a mere decoration and the task was solved exclusively by mathematical methods.

These four types of approaches that Busse identified are *ideal types*. Ideal types are intellectual constructions emerging from interpretative research, whereby categories are developed to describe and analyse phenomena in reality (Bikner-Ahsbahs 2014). In his study, Busse determined a hierarchy between the ideal types, with *ambivalent* at the lowest cognitive level, *reality bound* and *mathematics bound* at an intermediate level, and *integrating* at the highest level. There is not a hierarchy between reality bound and mathematics bound, see Fig 1. Both Busse (2011) and Borromeo Ferri (2010) found that patterns in problem solving could differ between students and between tasks. Therefore, the four ideal types are neither attributes of a student nor of a task, but they are a characterization of how a particular student deals with a particular task.

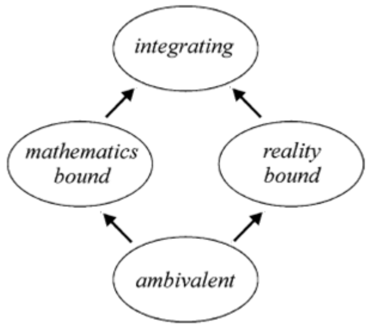


Fig. 1.1 Ideal types of dealing with a real-world context within a modelling task (Busse 2011)

Our study takes a longitudinal perspective on modelling. Instead of researching students at just one moment in their educational career, we were interested in their growth, or lack thereof. The deeper aim of this research is to obtain a better insight into how students deal with real-life contexts and mathematics, what blockages and opportunities to occur when students move from contexts to mathematics and back, and how students develop modelling competencies. To study this, we assumed that Busse's ideal types are a characterization of how a particular student deals with a particular task at a particular moment in time. By keeping task and students as constants, and having time as independent variable, we had as research question: how do students' problem solving approaches when characterised by Busse's ideal types develop over time?

In mathematical modelling tasks, the dynamics of dealing with real-life contexts and mathematics occurs in particular during the phase of mathematising and the phase of interpreting. The study presented here only deals with the activity of interpreting.

2 Theoretical background

Pollak (1979) conceptualized how mathematical modelling is an activity that takes place in two disjoint spheres: in mathematics and 'the rest of the world'. With 'the rest of the world' he meant all outside mathematics including nature, society, everyday life and other scientific disciplines. Other authors followed this description (e.g. Blum 2002). However, this distinction can be challenged, because mathematics can be found scattered within nature, society, everyday life and other scientific disciplines. So, it may not always be possible to clearly distinguish between the different spheres. Also, if in modelling we move between the two spheres, where are we when we are in a transition between the two? Below, we discuss the nature of this distinction.

In this chapter we will speak of contexts instead of 'the rest of the world'. By contexts, we mean the real-life situations described in mathematical modelling tasks. A context can be more or less close to reality, and this context may be recognized and understood by students in different ways.

Pollak's (1979) original terminology suggests a dichotomy of contexts and mathematics, that is: contexts and mathematics are mutually exclusive and cannot overlap. This dichotomy is confirmed by Busse's (2011) findings, in which some students were more mathematics bound, while others were more reality bound. However, the higher achieving students were able to integrate mathematics and contexts. This observation is confirmed by Vos and Roorda (2007), who used the term *reconciliation* of mathematics and context for a similar case, in which one of the smarter students manages to see the context through the mathematics and vice versa. Thus, a distinction between mathematics and contexts requires the option that they can be integrated.

In this chapter, we take contexts and mathematics as being complementary. Complementarity is a notion with origins in the work by Niels Bohr, who worked on a dilemma in physics, needing to integrate two conceptions of light: one as a particle and the other one as a wave. The two notions offer different ways of understanding light, they are not mutually exclusive, and they can support each other. As such, complementarity differs from notions such as dichotomy or duality. In the educational setting of mathematical modelling, complementarity of mathematics and contexts means that the two are different, but that they can be integrated and then strengthen each other. This fits Busse's (2011) ideal types, in which the highest cognitive level is termed *integrating*.

3 Methods

We carried out a longitudinal multiple case study with a detailed analysis of work by individual students (Yin 2003). While the students moved from grade 11 to grade 12, we administered three task-based interviews (Goldin 2000) at successive moments. In each interview we used several tasks which were not shown to the students beforehand. The tasks were rotated between interviews, and not all tasks were used in all interviews. The study described in this chapter was part of a larger study (Roorda 2012, Roorda et al. 2015). It is based on one task, which deals with derivatives and interpreting these within a context. More details about the task are described below.

The first interview was held in the 3rd month in grade 11; a few weeks after the mathematics teacher had introduced derivatives. Interview 2 was held six months later, and Interview 3 was held a year later. Between the first and the final interview, derivatives were a recurring topic in mathematics lessons and for some students in their elective subjects (physics or economics) as well. We observed that the curriculum in between interviews focused primarily on calculations, and did not contain interpretation tasks such as the one used in the interview. To enable comparison across interviews, exactly the same task was used, as small changes in a task can yield large differences in students' approaches. The time interval of six months was considered sufficient to limit inter-interview effects.

3.1. The task

We adapted a task from Kaiser-Messmer (1986), which is set in the context of cars, petrol consumption and the distance driven. Central is a function $V(a)$ for the volume of petrol (in litres) that depends on the travelled distance a (in km). The word for distance in Dutch is *afstand*, hence a is used for this variable. The task is rich in resources: there are different representations (graph, table) and students can address different aspects of the derivative:

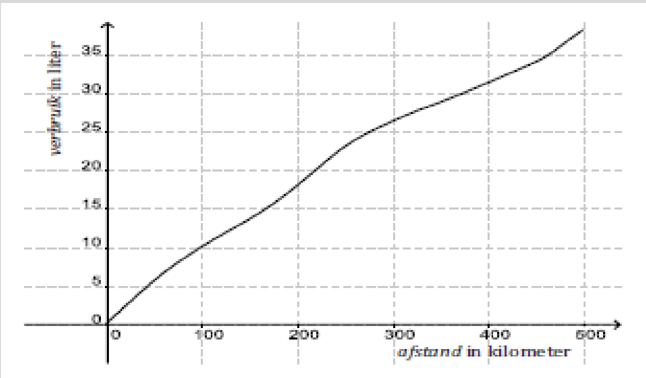
the average rate of change on an interval (with data from the table), the rate of change in a point, a tangent, slope, limits, and so forth. Also, students can reason about the real-life context: the average petrol consumption over a distance of h kilometre.

Petrol

In a car a measuring system was installed, which measures the petrol consumption of the car every 10 km. During a trip of 500 km the measurements were written down. In the table you see some of the measurements during this trip. The travelled distance is a (in km) and the petrol consumption is V (in litre).

a (km)	10	20	30	50	100	200	300	400	500
V (litre)	1,3	2,7	4,0	6,4	10,3	18,3	26,6	31,2	39,7

The measurement points were plotted into a graph by drawing a smooth line through the points.



What is the meaning of $\dfrac{V(a+h)-V(a)}{h}$ in this situation? (h is a value, which you can choose)

Differences to the original task from Kaiser-Messmer (1986) are as follows. (1) To make the task more realistic, we added details to the context by describing a system for measuring the petrol consumption. (2) We added a table to increase variety in representations. (3) We removed a second question about the interpretation of the limit for $h \to 0$ of the same difference quotient, because this would give a cue about h possibly being small. This would hinder us from observing students' spontaneous reflections about limits.

The *Petrol task* has a number of specific features. (1) Formula $V(a)$ is not given as a symbolical expression with variable a, from which volume V can be calculated. (2) The task is about interpretation, and not about standard mathematical activities such as calculating or solving. (3) One can give an interpretation of the difference quotient without knowledge of the derivative. (4) The task context can be regarded as realistic (recognizable, possibly existing in real life) but inauthentic (there is no evidence of a really existing car with a measuring system). (5) The difference quotient has h as additional variable (or parameter) to V and a, therefore three symbols need to be considered, while the table and the graph suggest only two dimensions.

3.2. Participants, interview protocol and data analysis

We selected ten pre-university students (6 boys, 4 girls), who took mathematics at an advanced level. The mathematics teacher had indicated one student as weak, four as average and five as good. In our study weak students are underrepresented because we looked for students who most likely would move up from grade 11 to grade 12 without delay. The study was carried out at two schools to reduce inter-student communication about the tasks between interviews. The students' pseudonyms are: Andy, Bob, Casper, Dorien and Elly from School I, and Karin, Maaike, Nico, Otto and Piet from School II.

The interview started by asking the student to solve the task. During the solving the interviewer did not interfere. If a student thought for over a minute, he or she was asked for an explication. To enhance the reasoning and interpretation process, the interviewer would ask students about the effect of the size of h in the formula. This hint could offer students the opportunity to reason about a limit. The interviewer would **not** use words that directed towards mathematical concepts, such as 'derivative', 'differentiation', 'rate of change', 'tangent' or 'slope'. By avoiding these words, we did not lead students to more mathematics than the task already did. In case a student would reason completely in terms of the situation (cars, petrol consumption, distance travelled), an additional question was, whether the student had seen the formula before.

Both author independently analysed the transcripts of the interviews and the written answers to the task, thereafter reaching agreement on labelling students' problem solving approaches using Busse's ideal types. We identified utterances as being more reality bound, when a student spoke about average consumption. We identified utterances as being more mathematics bound, when a student spoke about aspects of the derivative, such as rates of change, slope, decreasing difference intervals. Additionally, we coded students' expressions on a simple scale: accurate and clear – somewhat accurate or clear – unclear.

4 Results

Below we report on four students and their approaches to the *Petrol task* in the three subsequent interviews. We selected these because of their illuminating differences. The approaches of the six others are reported in detail in Roorda (2012). At the end of this paragraph, we synthesize the findings over all ten students.

4.1. The case of Nico

In the first interview Nico started by saying: *"So, the steeper the line goes, the more his petrol consumption per kilometer is."* This was a correct interpretation of the graph, but not of the difference quotient. He interpreted $V(a+h)$ as multiplication $Va+Vh$. He remarked that he had no idea about the meaning of h. When prompted by the interviewer for a meaning of the formula, he said: *"It is the average consumption of the car, of course, what else would you want to calculate?"* but he did not link this correct statement to the formula.

In Interview 2 Nico started by thinking that $V(a)$ is a multiplication, but then corrected himself spontaneously and recognized that $V(a)$ is the petrol consumption after a kilometer, and rewrote the formula into $V(a)+V(h)-V(a)/h$, then $V(h)/h$ and then wrote: *V with 1 unit h on average*. He explained this as the consumption after one kilometer. After being prompted to further explain, he took numbers: at 100 km the consumption is 10 litre. The value 10/100 is 0,1 litre per kilometer, and according to Nico, this was the average consumption. When the interviewer asked about the effect of the size of *h* in the formula, Nico reasoned that it does not matter, because *h*/*h* is equal to 1.

In Interview 3 Nico used the table to calculate 39,7/500 and 1,3/10 (these numbers are *V*(500)/500 and *V*(10)/10) and said that the consumption is not constant, *"otherwise the graph would be straight"*. He went on to interpret the difference quotient as: the consumption at *h* divided by *h*. Thereafter, he said that it was about a route: *"It is the extra distance h that one travels, and that divided by h (…), so h is the consumption per kilometer h. So the formula means what the consumption is in kilometers h on a certain kilometer* [points at different points in the graph] *on that route. Approximately I think."* He then wrote: *the consumption per kilometer during distance h.*

We interpreted Nico's utterances in all interviews as being reality bound, because he mainly talked in terms of consumption and distances. We interpreted his explanation in Interview 3 as being reality bound, and quite clear and correct.

4.2. The case of Elly

In Interview 1 Elly wondered what *h* could be: *"I don't understand at all what my h is."* She inserted numbers by taking *a*=10 and *h*=4 and said: *"It will become 10 + 4 – 10 divided by 4, but what this means, no idea."* She clearly could not interpret the function notation. In Interview 2 Elly said: *"I don't understand what this h is, and why you can choose it."* She used numbers from the table and wrote: 1,3(10+10) – 1,3(10) / 10. She obtained 1,3 and said: *"I get a number I already had."* Again, she could not interpret the function notation. In the final interview, Interviews 3, she changed the *h* in the formula into an *x* and said: *"Then I will not think all the time that h is the height or something."* She wrote 1,3 (10+3) – 1,3 (10) / 10 and said: *"I don't get what they want with this formula.... what it means, and for what you can use it. No idea."*

In all interviews Elly interpreted the notation *V*(*a*+*h*) as multiplication *Va* + *Vh*. Not once did she relate the formula to a rate of change, nor to an average petrol consumption. In all interviews we considered her as mathematics bound, unclear and inaccurate.

4.3. The case of Bob

In Interview 1 Bob took *a*=40 and said: *"Here you could have the consumption 40 and here the consumption 40 plus a certain value."* He then said that the formula was about the average consumption in liters per kilometer.

In Interview 2 Bob took the petrol consumption at distances 200 and 300 and said: *"It is the petrol consumption between two points of the distance travelled.... how much he used while driving those 100 km"*. He said that the formula is like $V_{end} - V_{start}$ divided by the travelled distance: *"Yes, in fact this is the average consumption per km."*

In Interview 3 Bob first interpreted the formula as *V*(*h*)/*h*, but changed this because already *a* kilometer has been travelled. He drew a line with points 0, *a* and *a*+*h* and indicated that it is the consumption between *a* and *a*+*h*: "*It is the consumption per* *kilometer within this piece.*" When prompted to explain the role of *h*, he said: *"I think it often is 1, then you will have the consumption on one moment, that is more precise (..) for example you take a=400 then you will know how much he uses from 400 to 401, that is approximately the consumption on 400. That has something of a limit from mathematics in it, then you can make h smaller like 0,001 or something."*

In all interviews Bob's approach to the task was reality bound, as he used terms such as average consumption per kilometer, and liters per kilometer. From the first interview onwards, he interpreted the formula as a difference of consumption between two points, and from the second interview onwards this difference was divided by the distance. In Interview 3 he related the formula to limits, which we interpreted as – somewhat – integrating.

4.4. The case of Dorien

In Interview 1 Dorien recognized the formula: *"We did this in the chapter on derivatives (..) with adding small values, first 0,3 and then 0,03 and then you came closer every time."* She thought the formula was about used liters of petrol, but she could not explain this.

In Interview 2 Dorien said: *"With this formula I had to calculate the slope, and later also the derivative. This formula was used for the proof for another, faster formula, and then we had to use the other one, and not this one anymore."* She explained that the formula has to do with limits, by saying: *"I recognize it from how the formula is built, that h was first larger, and then you could make it smaller and then you reached a limit, and that was a number that you never reached, that was the slope in one point."* She also said that the formula is *"how much litre is used per km"*, explaining: *"If you take for example 300 and 400, then you will know the slope, and that is how many liters is used per kilometer"* and she drew Fig. 1.2.

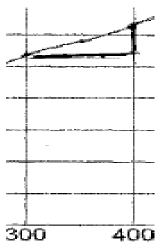


Fig. 1.2 Dorien's illustration of a slope into the graph in Interview 2

In Interview 3 Dorien first said that the formula is about limits and that she is a little allergic to them. She learnt them before they did the derivative. She explained that the formula is a $\Delta y/\Delta x$. She also explained it as a derivative, which can calculate how many liters are used per kilometer. It is *"some kind of speed of petrol consumption in fact, in liters per kilometer."* She also connected the formula to gradients, and explained the limiting process: *"If you take h smaller and smaller, then h becomes nearly zero. That is called a limit, and it became more precise. I know exactly that it was on that page, it was the first paragraph."*

In the first interview Dorien’s approach was mathematics bound, and she could not explain the formula well within the context. From the second interview onwards her approach was integrating, explaining the formula both mathematically and in its context.

4.5. Synthesis of results

Table 1.1 gives an overview of students’ approaches to the *Petrol task* in the three subsequent interviews. The first two students, Andy and Nico (see paragraph 4.1), maintained a reality bound approach throughout all interviews, and their statements became more accurate and clear. The next four students, Elly (see paragraph 4.2), Maaike, Casper and Piet, maintained a mathematics bound approach throughout all interviews. From these, Casper and Piet became more accurate and clear. The next two students, Karin and Bob (see paragraph 4.3), started with reality bound approaches, and these became integrating. The final two students, Otto and Dorien (see paragraph 4.4) started with mathematics bound approaches and these became more integrating in Interview 3.

Table 1.1 Results of students’ approaches being reality bound or mathematics bound

	Andy			Nico			Elly			Maaike			Casper			Piet			Karin			Bob			Otto			Dorien		
Interview #	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
Reality bound	o	+	*	o	o	*													o		*	o	*	*	o		o		*	*
Mathematics bound							o	o	o	+	+		+	+	*	+	*	*			o			+	o	+	*	o	+	*

Note. * accurate and clear + somewhat accurate or clear o unclear

Table 1.1 shows that in the first interview all students’ approaches are either mathematics bound or reality bound, with the exception of Otto (not reported here): his approach is ambivalent. In the subsequent interviews, the students maintain their preference and their statements become more accurate and clear. In the final interview four students have – somewhat – integrating approaches.

5 Conclusion and discussion

Our study was guided by the research question: how do students’ problem solving approaches when characterised by Busse’s ideal types (ambivalent, reality bound, mathematics bound, or integrating) develop over time? Our results show that the approaches to the *Petrol task* can be associated to all four ideal types, and that students’ approaches can change from one ideal type to another. In the course of a year, while the students followed the same curriculum for learning about derivatives, the development of approaches followed different paths. Not one student had a mathematics bound approach in one interview and reality bound in a subsequent interview, or vice versa. All students’ approaches were first either reality bound, mathematics bound, or ambivalent. An integrating approach could be observed with students, who earlier had a mathematics bound, or a reality bound approach. This confirms Busse’s hierarchy, in which integrating has a higher cognitive level than both mathematics bound and reality bound approaches (see Fig. 1.1). An integrating approach was independent of the initial preference.

We cannot confirm Busse's hierarchy with ambivalent approaches at the lowest level. The weakest student in our study, Elly, had a mathematics bound preference, albeit unclear and inaccurate. She took $V(a+h)$ as multiplication in all interviews, and this inability to recognize a function notation probably hindered her progress in learning about derivatives. This may explain the absence of growth in her approaches to the *Petrol task.*

Also, we see that in the first and second interview, not one approach is integrating. We see students grow: their vocabulary becomes more accurate, they become more flexible in using different representations, and their confidence grows. After the introduction of derivatives, it takes the best students, Bob and Dorien, a year to reach the integrating level. This confirms that it is not easy for students to integrate contexts and mathematics in modelling tasks, and that learning to integrate these takes time: at least a year.

Busse's (2011) ideal types proved extremely useful to analyze students' different approaches to tasks, and how their preferences develop. Also, the ideal types can assist teachers to analyze students' approaches, and develop instructional methods to encourage the uptake of complementary approaches. The framework shows that contexts and mathematics are not disjoint spheres, but that students can integrate these.

References

Bikner-Ahsbahs, A. (2015). Empirically grounded building of ideal type. In A. Bikner-Ahsbahs, Chr. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education* (pp. 137-154). Dordrecht, The Netherlands: Springer.

Blum, W. (2002). ICMI Study 14: Applications and modelling in mathematics education – Discussion document. *Educational Studies in Mathematics, 51*(1-2), 149-171.

Borromeo Ferri, R. (2010). On the influence of mathematical thinking styles on learners' modeling behavior. *Journal für Mathematik-Didaktik*, *31*(1), 99-118.

Busse, A. (2011). Upper secondary students' handling of real-world contexts. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G.A. Stillman (Eds.), *Trends in teaching and learning of mathematical modelling* (pp. 37-46). Dordrecht, the Netherlands: Springer.

Goldin, G.A. (2000). A scientific perspective on structured task-based interviews in mathematics education research. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517- 545). Mahwah, NJ: Lawrence Erlbaum.

Kaiser-Messmer, G. (1986). Modelling in calculus instruction - empirical research towards an appropriate introduction of concepts. In J. Berry et.al. (Eds.), *Mathematical modelling methodology, models and Micros* (pp. 36-47). Chichester: Ellis Horwood.

Pollak, H. O. (1979). The Interaction between mathematics and other school subjects. In UNESCO (Ed.), *New trends in mathematics teaching* (pp. 232-248). Paris: UNESCO.

Roorda, G. (2012). *Ontwikkeling in verandering; ontwikkeling van wiskundige bekwaamheid van leerlingen met betrekking tot het concept afgeleide [Development of students' mathematical proficiency with respect to the concept of derivative]* (PhD thesis). University of Groningen, The Netherlands.

Roorda, G., Vos, P., & Goedhart, M.J. (2015). An actor-oriented transfer perspective on high school students' development of the use of procedures to solve problems on 'rate of change'. *International Journal of Mathematics Education in Science and Technology, 13*(4), 863-889.

Vos, P., & Roorda, G. (2007). Interpreting velocity and stopping distance; complementarity, context and mathematics. In *Proceedings of the the 5th Congress of the European Society for Research in Mathematics Education (CERME-5).* Nicosia, Cyprus: University of Cyprus.

Yin, R.K. (2003). *Case study research: Design and methods*. Thousand Oaks, CA: Sage Publications.