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Networks of conforming or nonconforming individuals tend to reach satisfactory decisions

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Binary decisions of agents coupled in networks can often be classified into two types: “coordination,” where an agent takes an action if enough neighbors are using that action, as in the spread of social norms, innovations, and viral epidemics, and “anticoordination,” where too many neighbors taking a particular action causes an agent to take the opposite action, as in traffic congestion, crowd dispersion, and division of labor. Both of these cases can be modeled using linear-threshold-based dynamics, and a fundamental question is whether the individuals in such networks are likely to reach decisions with which they are satisfied. We show that, in the coordination case, and perhaps more surprisingly, also in the anticoordination case, the agents will indeed always tend to reach satisfactory decisions, that is, the network will almost surely reach an equilibrium state. This holds for every network topology and every distribution of thresholds, for both asynchronous and partially synchronous decision-making updates. These results reveal that irregular network topology, population heterogeneity, and partial synchrony are not sufficient to cause cycles or nonconvergence in linear-threshold dynamics; rather, other factors such as imitation or the coexistence of coordinating and anticoordinating agents must play a role.

evolutionary game theory | network games | best-response dynamics | linear-threshold model | equilibrium convergence

The sharing of resources, division of labor, and dispersion of crowds represent a few examples of collective behaviors that can emerge on networks of interacting agents when the adoption of a particular action by too many individuals deters others from adopting that action. On the other hand, when individuals’ taking an action makes it more likely that others will adopt that action, behaviors such as the spread of social norms, technological innovations, and viral infections can occur. In either case, various factors are thought to contribute to the emergence of such collective behaviors, including network topology, interaction payoffs, strategy update rules, and population diversity (1–3). In particular, when agents can take one of a finite number of states at a given time, whether by voluntary decisions or involuntary reactions, the fundamental questions include (i) whether the agents will eventually settle on a fixed state/action, (ii) which distribution of actions the network will converge to, and (iii) how long it will take for the network to converge. Although most of the literature focuses on the second and especially third questions, particularly for homogeneous populations where individuals share the same utility function (4–7), the first question is perhaps more fundamental because it deals with convergence at the agent level. Undoubtedly, this action settling is crucial to a wide range of decision-making populations, from collective nest selection by ants (8) to stabilization of financial markets (9). Moreover, without establishing such convergence, answers to questions ii and iii cannot be used to design policies at the local or neighborhood level (10), which may help to improve cost efficiency by targeting influential nodes in a complex network.

“Best-response dynamics” is one of the most widely used models to study the types of problems mentioned above; this is due in part to its rational nature as well as its broad applicability. Indeed, recent experimental studies have shown that humans often use myopic best responses when making decisions in social

networks (11). The idea behind best-response dynamics is simple: in a network of interacting agents, each agent takes the action that results in the best cumulative outcome against its neighbors. This framework is frequently considered in a game-theoretic context (12–15), but its impact is further broadened by the fact that when there are two available actions for each agent, it is equivalent to a “linear-threshold model” (16). This comes with an alternative yet also intuitive interpretation: when a sufficient fraction of my neighbors are taking action *A*, I will also take action *A*. The reverse is also possible: if too many of my neighbors are taking action *A*, I will switch to action *B*. We call agents of the former type “coordinating” and the latter, “anticoordinating,” according to the standard coordination and anticoordination games, which are also sometimes referred to as games of strategic complements and strategic substitutes, respectively (17). These switches may be considered voluntary as in changing an opinion, preference, or habit, or involuntary, as getting infected by virus or defaulting on a loan. In general, the thresholds of the agents need not be the same, and this is equivalent to agents having asymmetric payoffs in the corresponding matrix game. Depending on the application, this can represent differences in the willingness to follow a crowd, susceptibility to infection, perceived value of interactions, and many other individual characteristics (18). Variation in thresholds can thus be thought of as heterogeneity of a population.

The linear-threshold model was first motivated and discussed by Mark Granovetter (18) in the context of fully connected networks, with an example of how the incitement of riots in crowds depends critically on the distribution of individual thresholds that will cause each person to join the riot. Since then, a rich literature has developed around questions related to stability and convergence of best-response dynamics in matrix

Significance

Many real-life decisions where one out of two actions must be chosen can be modeled on networks consisting of individuals who are either coordinating, that is, take an action only if sufficient neighbors are also doing so, or anticoordinating, that is, take an action only if too many neighbors are doing the opposite. It is not yet known whether such networks tend to reach a state where every individual is satisfied with his decision. We show that indeed any network of coordinating, and any network of anticoordinating individuals always reaches a satisfactory state, regardless of how they are connected, how different their preferences are, and how many simultaneous decisions are made over time.

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games on well-mixed populations (19, 20). Convergence of homogeneous well-mixed populations under more general non-matrix games has been established in ref. 21 in the context of congestion games, using potential functions. The convergence properties of networks with arbitrary structure has also been investigated under various conditions, particularly in the homogeneous (symmetric) case (22–24). Recently in ref. 17, a combination of mean-field approximations and simulations were used to show that synchronous best-response dynamics in symmetric coordination or antcoordination games tend to always converge to Nash equilibria. However, in the case of asymmetric coordination games, it was shown in ref. 25 that synchronous best-response dynamics may not converge to a single action but rather to a cycle of alternating actions. Symmetry thus seems to be a significant factor in the convergence of the threshold model, but we will show that asynchrony perhaps plays an even more important role, because regardless of the symmetry, the network always converges in the asynchronous case. In addition to making the convergence more likely, compared with synchronous models, asynchronous dynamics can provide a more realistic model of the timeline over which independent agents make decisions and receive information, and they are particularly suitable when the payoff dynamics can be thought of as fast compared with the update dynamics. For example, decisions on matters such as which product to buy, which political party to vote for, or which traffic route to take may be the culmination of many individual interactions. Nevertheless, our results do not require that only one agent can update at a time; in fact, we show that even small asynchronous perturbations to fully synchronous dynamics lead to equilibrium convergence.

In this paper, we show that every network consisting of anti-coordinating agents with asynchronous best-response dynamics will eventually reach an equilibrium state, even if each agent has a different threshold. Moreover, we show that the same result holds for networks of coordinating agents. As a corollary, we establish the existence of pure-strategy Nash equilibria in both cases for arbitrary networks and arbitrary payoffs. On the question of convergence time, we show that, in such networks, the total number of strategy switches is no greater than six times the number of edges in the network. In the case of partial synchrony, when a random number of agents can update simultaneously, we show that the network still almost surely reaches an equilibrium. It follows that irregular network topology, population heterogeneity, and synchrony of decisions between two or more agents (as long as random asynchronous updating is not completely excluded) are not sufficient to cause nonconvergence or cycles in best-response dynamics; rather, possible causes include the occasional use of non-best-response strategies, randomization, or a mixture of coordinating and antcoordinating agents. Indeed, we provide a small example demonstrating the possibility of cycles in networks containing both coordinating and antcoordinating agents.

Asynchronous Best-Response Dynamics

Consider an undirected network $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ where the nodes $\mathcal{V} = \{1, \dots, n\}$ correspond to agents and each edge in the set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represents a 2-player game between neighboring agents. Each agent $i \in \mathcal{V}$ chooses pure strategies from a binary set $\mathcal{S} := \{A, B\}$ and receives a payoff upon completion of game according to the matrix:

$$\begin{matrix} & \begin{matrix} A & B \end{matrix} \\ \begin{matrix} A \\ B \end{matrix} & \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \quad a_i, b_i, c_i, d_i \in \mathbb{R}. \end{matrix}$$

The dynamics take place over a sequence of discrete time $k = 0, 1, 2, \dots$. Let $x_i(k) \in \mathcal{S}$ denote the strategy of agent i at time

k , and denote the number of neighbors of agent i playing A and B at time k by $n_i^A(k)$ and $n_i^B(k)$, respectively. When there is no ambiguity, we may sometimes omit the time k for compactness of notation. The total payoffs to each agent i at time k are accumulated over all neighbors, and are therefore equal to $a_i n_i^A(k) + b_i n_i^B(k)$ when $x_i(k) = A$, or $c_i n_i^A(k) + d_i n_i^B(k)$ when $x_i(k) = B$.

In asynchronous (myopic) best-response dynamics, one agent at a time becomes active and chooses a single action to play against all neighbors. The active agent at time k updates at time $k + 1$ to the strategy that achieves the highest total payoff, that is, is the best response, against the strategies of its neighbors at time k :

$$x_i(k+1) = \begin{cases} A, & \text{if } a_i n_i^A + b_i n_i^B > c_i n_i^A + d_i n_i^B \\ B, & \text{if } a_i n_i^A + b_i n_i^B < c_i n_i^A + d_i n_i^B \\ z_i, & \text{if } a_i n_i^A + b_i n_i^B = c_i n_i^A + d_i n_i^B \end{cases}$$

In the case that strategies A and B result in equal payoffs, both strategies are best responses and we use the notation z_i , which is defined to be A , B , or $x_i(k)$, to allow for several possibilities for this equality case. Note that we do not require all agents to have the same z_i . That is, when both A and B are best responses, some agents may choose A , others may choose B , and others may keep their current strategy; however, the agents cannot change their choice of z_i over time.

It is convenient to rewrite these dynamics in terms of the number of neighbors playing each strategy. Let deg_i denote the total number of neighbors of agent i . We can simplify the conditions above by using the fact that $n_i^B = \text{deg}_i - n_i^A$ and rearranging terms:

$$\begin{aligned} a_i n_i^A + b_i (\text{deg}_i - n_i^A) &> c_i n_i^A + d_i (\text{deg}_i - n_i^A) \\ n_i^A (a_i - c_i + d_i - b_i) &> \text{deg}_i (d_i - b_i) \\ \delta_i n_i^A &> \gamma_i \text{deg}_i, \end{aligned}$$

where $\delta_i := a_i - c_i + d_i - b_i$ and $\gamma_i := d_i - b_i$. The cases of “ $<$ ” and “ $=$ ” can be handled similarly. Consider the case when $\delta_i \neq 0$, and let $\tau_i := \gamma_i / \delta_i$ denote a “threshold” for agent i . Depending on the sign of δ_i , we have two possible types of best-response update rules. If $\delta_i > 0$, the update rule is given by the following:

$$x_i(k+1) = \begin{cases} A & \text{if } n_i^A(k) > \tau_i \text{deg}_i \\ B & \text{if } n_i^A(k) < \tau_i \text{deg}_i \\ z_i & \text{if } n_i^A(k) = \tau_i \text{deg}_i \end{cases} \quad [1]$$

These agents switch to strategy A if a sufficient fraction of neighbors are using that strategy, and likewise for strategy B . On the other hand, for $\delta_i < 0$, if a sufficient fraction of neighbors are playing A , the agent will switch to B , and vice versa. This update rule is given by the following:

$$x_i(k+1) = \begin{cases} A & \text{if } n_i^A(k) < \tau_i \text{deg}_i \\ B & \text{if } n_i^A(k) > \tau_i \text{deg}_i \\ z_i & \text{if } n_i^A(k) = \tau_i \text{deg}_i \end{cases} \quad [2]$$

In the case that $\tau_i \notin [0, 1]$, it is straightforward to show that there exists a strictly dominant strategy, and the update rule Eq. 1 or 2 is equivalent to one in which $\tau_i \in \{0, 1\}$ and $z_i \in \{A, B\}$. The same holds for when $\delta_i = 0$. Agents for which $\delta_i \geq 0$ are called coordinating and can be modeled by Eq. 1. Agents for which $\delta_i \leq 0$ are called antcoordinating and can be modeled by Eq. 2. Therefore, every agent can be described as a coordinating or an antcoordinating agent (or both).

Let $\Gamma := (\mathbb{G}, \tau, \{+, -, \pm\})$ denote a “network game,” which consists of the network \mathbb{G} , a vector of agent thresholds $\tau = (\tau_1, \dots, \tau_n)^\top$, and one of $+$, $-$, or \pm , corresponding to the cases of all coordinating, all antcoordinating, or a mixture of both types of agents, respectively. The dynamics in Eqs. 1 and 2 are in the form of the standard linear-threshold model (18). An equilibrium state in the threshold model is a state in which the number

of A -neighbors of each agent does not violate the threshold that would cause them to change strategies. For example, in a network of anticonordinating agents in which $z_i = B$ for all i , this means that for each agent $i \in \mathcal{V}$, $x_i(k) = A$ implies $n_i^A(k) < \tau_i \deg_i$ and $x_i(k) = B$ implies $n_i^A(k) \geq \tau_i \deg_i$. Note that this notion of equilibrium is equivalent to a pure-strategy Nash equilibrium in the corresponding network game.

Convergence Results

We investigate the equilibrium convergence properties of the agent-based threshold models in Eqs. 1 and 2 that we defined in the previous section.

Before providing the main results, we precisely define the nature of the asynchronous dynamics. We require only that at any given time step, each agent is guaranteed to be active at some finite future time. Let i^k denote the agent who is active at time k and let $(i^k)_{k=0}^\infty$ denote a sequence of active agents. We say that such a sequence is “persistent” if for every agent $j \in \mathcal{V}$ and every time $k \geq 0$, there exists some finite later time $k' > k$ at which agent j is again active ($i^{k'} = j$).

Assumption 1. Every activation sequence driving the dynamics in Eq. 1 or Eq. 2 is persistent.

Remark 1. In stochastic settings, Assumption 1 holds almost surely whenever agents activate infinitely many times with probability one, for example, if each agent activates at a rate determined by a Poisson process.

We divide the convergence analysis into two main parts corresponding to the cases of anticonordinating and coordinating agents. In what follows, we use $\mathbf{1}$ to denote the n -dimensional vector containing all ones. We refer the reader to *SI Appendix* for the detailed proofs of the results.

All Agents Are Anticonordinating

Theorem 1. Every network of anticonordinating agents who update asynchronously under Assumption 1 will reach an equilibrium state in finite time.

The sketch of the proof is as follows. We begin by showing that an arbitrary network game $\Gamma = (\mathbb{G}, (1/2)\mathbf{1}, -)$ consisting of anticonordinating agents in which the threshold of each agent is $1/2$ will reach an equilibrium in finite time. Then we extend the result to a heterogeneous-threshold network game $\Gamma = (\mathbb{G}, \tau, -)$ by constructing a homogeneous-threshold “augmented” network game $\hat{\Gamma} = (\hat{\mathbb{G}}, (1/2)\mathbf{1}, -)$ that is dynamically equivalent to Γ . We complete the proof by showing that if the augmented network game $\hat{\Gamma}$ reaches an equilibrium, then Γ does as well. The following lemma establishes convergence of the homogeneous-threshold network game $(\mathbb{G}, (1/2)\mathbf{1}, -)$.

Lemma 1. Every network of anticonordinating agents who update asynchronously under Assumption 1, with $\tau_i = 1/2$ for each agent $i \in \mathcal{V}$, will reach an equilibrium state in finite time.

The proof of the lemma revolves around the following potential function:

$$\Phi_i(k) = \begin{cases} n_i^A(k) - \hat{n}_i^A & \text{if } x_i(k) = A \\ \hat{n}_i^A + 1 - n_i^A(k) & \text{if } x_i(k) = B \end{cases}$$

where \hat{n}_i^A denotes the maximum number of A -neighbors of agent i that will not cause agent i to switch to B when playing A . The proof follows from the fact that the function is lower bounded and decreases every time an agent in the network switches strategies.

To motivate our approach for extending this result to an arbitrary distribution of thresholds, consider for example an agent i with 4 neighbors whose threshold is $1/3$. When playing A , this agent can tolerate up to 1 A -neighbor (Fig. 1A), but 2 or more

will cause a switch to B . Similarly, when playing B , the agent needs at least 2 A -neighbors to remain playing B , whereas 1 or fewer will cause a switch to A . Now consider an agent \hat{i} whose threshold is $1/2$ but who has one additional neighbor who always plays A , for a total of 5 neighbors, as shown in Fig. 1B. When playing A , this agent can tolerate up to 2 A -neighbors before switching to B , and as a B -agent needs at least 3 A -neighbors, whereas 2 or fewer will cause a switch to A . Notice, however, that with respect to the original 4 neighbors, the dynamics of agents i and \hat{i} are indistinguishable. It turns out that whenever $\tau_i < 1/2$, by adding a sufficient number of fixed A -neighbors, we can always construct a dual node \hat{i} with threshold $1/2$ whose dynamics are equivalent to the dynamics of the original node i . Moreover, if $\tau_i > 1/2$, we can achieve the same result by adding fixed B -neighbors. To ensure that the added nodes do not change strategies, we simply add two opposite strategy neighbors to these nodes (Fig. 1C). It is then straightforward to show that the strategies of all added nodes remain constant. We now formalize this argument for arbitrary networks of anticonordinating agents, using some techniques similar to those that have already proven useful in studying the convergence of synchronous networks (25).

We define the “augmented network game” $\hat{\Gamma} := (\hat{\mathbb{G}}, (1/2)\mathbf{1}, -)$ based on Γ as follows. Let $\hat{\mathbb{G}} = (\hat{\mathcal{V}}, \hat{\mathcal{E}})$. Define a “ V -block” as a triplet of nodes $\{v_1, v_2, v_3\} \subseteq \hat{\mathcal{V}}$ along with the edges $\{\{v_1, v_2\}, \{v_1, v_3\}\} \subseteq \hat{\mathcal{E}}$. For each agent $i \in \mathcal{V}$, we introduce a “dual agent” $\hat{i} \in \hat{\mathcal{V}}$ with the same initial strategy, that is, $x_{\hat{i}}(0) = x_i(0)$, and with $z_{\hat{i}} = z_i$. Corresponding to each dual agent \hat{i} , there are m_i number of V -blocks in $\hat{\mathbb{G}}$ such that the v_1 -node of each block is connected to \hat{i} , with m_i being defined as follows: if $\tau_i = 1/2$, then $m_i = 0$; otherwise, m_i depends on which one of the following three conditions on τ_i holds:

$$m_i = \begin{cases} |(1 - 2\tau_i)\deg_i| & \tau_i \deg_i \in \mathbb{Z} \\ |\deg_i - r - 1| & \exists r \in 2\mathbb{Z} : \frac{r}{2} < \tau_i \deg_i < \frac{r+1}{2} \\ |\deg_i - r| & \exists r \in 2\mathbb{Z} + 1 : \frac{r}{2} \leq \tau_i \deg_i < \frac{r+1}{2} \end{cases},$$

where $2\mathbb{Z}$ and $2\mathbb{Z} + 1$ denote the set of even and odd numbers, respectively. If $\tau_i < 1/2$, then the initial strategies of each V -block connected to the dual agent \hat{i} are $x_{v_1}(0) = A$ and $x_{v_2}(0) = x_{v_3}(0) = B$; and if $\tau_i > 1/2$, then $x_{v_1}(0) = B$ and $x_{v_2}(0) = x_{v_3}(0) = A$. In total, $\hat{\mathcal{V}}$ has $n + \sum_{i=1}^n m_i$ agents, the thresholds of all of which are set to $1/2$. For $\hat{\mathcal{E}}$, in addition to the edges for the V -blocks, there is an edge between any two dual agents \hat{i} and \hat{j} in $\hat{\mathcal{V}}$, if and only if there is an edge between i and j in \mathcal{V} .

Next, we show that if whenever an agent in \mathbb{G} activates, its dual in $\hat{\mathbb{G}}$ also activates (while neglecting the time steps that a V -block

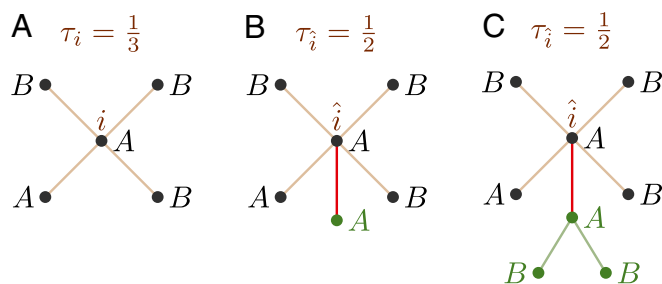


Fig. 1. (A and B) Agents i and \hat{i} have thresholds $\tau_i = 1/3$ and $\tau_{\hat{i}} = 1/2$, but because an extra A -neighbor has been added to \hat{i} , they are dynamically equivalent. (C) To ensure that the added node does not switch, two B neighbors are added who also will not switch, forming a V -block.

agent is active), then the dynamics of each node in \mathbb{G} are the same as the dynamics of its dual node in $\hat{\mathbb{G}}$ (again while neglecting the time steps that a V -block agent is active). Then *Theorem 1* can be proven accordingly.

As a result of *Theorem 1*, we have the following corollary, which to the best of our knowledge has not been shown in the literature to date.

Corollary 1. *Every network of anticonordinating agents admits a pure-strategy Nash equilibrium.*

All Agents Are Coordinating

Theorem 2. *Every network of coordinating agents who update asynchronously under Assumption 1 will reach an equilibrium state in finite time.*

The proof of *Theorem 2* follows similar steps as the anti-coordinating case. The key difference is that the potential function becomes the following:

$$\Phi_i(k) = \begin{cases} \tilde{n}_i^A - n_i^A(k) & \text{if } x_i(k) = A \\ n_i^A(k) - \tilde{n}_i^A + 1 & \text{if } x_i(k) = B \end{cases}, \quad [3]$$

where \tilde{n}_i^A is defined as the minimum number of A -neighbors required for an A -playing agent to continue playing A . The maximum number of A -neighbors that a B agent can tolerate before switching to A is then given by $\tilde{n}_i^A - 1$. We refer the reader to *SI Appendix* for a complete proof of the theorem.

Theorem 2 also leads to a corollary on existence of equilibria for networks of coordinating agents, but in this case we confirm an already known result (e.g., ref. 26).

Corollary 2. *Every network of coordinating agents admits a pure-strategy Nash equilibrium.*

The following corollary follows directly from *Theorem 1* and *Theorem 2*, and the fact that a network of homogeneous agents can always be described as all coordinating or all anticonordinating.

Corollary 3. *Every network of homogenous agents who update asynchronously under Assumption 1 will reach an equilibrium state in finite time.*

Coordinating and Anticoordinating Agents Coexist. Given the previous two results, the natural question arises of what we can say about convergence when both coordinating and anticonordinating agents are present in a network. Although there may be particular configurations that converge, we can demonstrate that this will not hold in general with a simple example on a network consisting of only two agents connected to each other, namely a path of length one, that is, $\mathcal{V} := \{1, 2\}$, $\mathcal{E} := \{(1, 2), (2, 1)\}$. Suppose that agent 1 is coordinating, agent 2 is anticonordinating, and their thresholds are both equal to $1/2$. Given their respective update rules in Eqs. 1 and 2, agent 1 will always switch to match agent 2 while agent 2 will always switch to oppose the strategy of agent 1. In other words, the following transitions will occur with probability one: $(A, A) \rightarrow (A, B) \rightarrow (B, B) \rightarrow (B, A) \rightarrow (A, A)$, resulting in a neverending cycle. This formulation is equivalent to a repeated version of the well-known matching pennies game. From this, we can conclude that convergence is not a trivial result to be expected in all cases, and can only be guaranteed when all agents are either coordinating or anticonordinating.

Convergence Time. The following corollary follows from the fact that the potential function $\Phi(k)$ is bounded from above and below and decreases by at least one every time an agent switches strategies.

Corollary 4. *Every network of all coordinating or all anticonordinating agents will reach an equilibrium state after no more than $6|\mathcal{E}|$ agent switches.*

This implies that agents cannot switch an arbitrary number of times before the network reaches equilibrium. It follows that when agent activation times are independent and identically distributed, the upper bound on the expected time to reach equilibrium is linear in the number of edges in the network.

Synchronous and Partially Synchronous Updating

So far, any network of all coordinating or all anticonordinating agents is shown to reach an equilibrium state, as long as the agents update asynchronously. However, the importance of asynchronous updating to the convergence results remains an open problem. In this section, we show that, although full synchrony may not always result in convergence, the results indeed still hold for partial synchrony, in which a random number of agents update at each time step.

Synchronous Updating. We show that networks in which updates are fully synchronous may never reach an equilibrium state, by presenting counterexamples with only two agents.

First, suppose that both agents are anticonordinating and start from the strategy vector (A, A) . The agents update synchronously, that is, at each time step, both agents activate and update their strategies according to the update rule in Eq. 2. Therefore, the dynamics will be deterministic, and the following transitions will occur on the strategies of the agents: $(A, A) \rightarrow (B, B) \rightarrow (A, A)$, resulting in a cycle of length 2.

Now suppose that both agents are coordinating and they start from the strategy vector (A, B) . Following the update rule in Eq. 1, the following transitions take place under the synchronous updating: $(A, B) \rightarrow (B, A) \rightarrow (A, B)$, again resulting in a cycle of length 2.

The above examples prove that equilibrium convergence is no longer guaranteed if the agents update in full synchrony. However, it is known that any network game governed by synchronous best-response dynamics will reach a cycle of length at most 2, even when both coordinating and anticonordinating agents coexist in the network (27).

Partially Synchronous Updating. To understand what happens in the case of partially synchronous updates, we need to relax the assumption that only one agent can update at a given time. We must therefore decouple the activation sequence from the discrete-time dynamics, and consider the activations to occur in continuous time. Let $\{t^j\}_{j=1}^\infty$ denote a sequence of times and $\{\tilde{i}^j\}_{j=1}^\infty$, a sequence of agents such that agent \tilde{i}^j activates at time t^j . Fig. 2 shows the relationship of the discrete time steps with the agent activation sequence.

It is now possible that multiple agents update between consecutive time steps. In particular, all agents who are active in the time interval $[k, k + 1)$ update at time $k + 1$ based on the state of the network at time k . Let \mathcal{A}_k denote the set of all agents who are

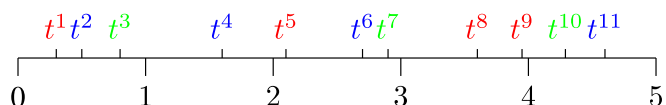


Fig. 2. In the partially synchronous update model, multiple agents can activate in a single discrete time step. All agents who are active in the absolute time interval $[k, k + 1)$ update at time $k + 1$ based on the information at time k . The colors red, blue, and green represent three different agents.

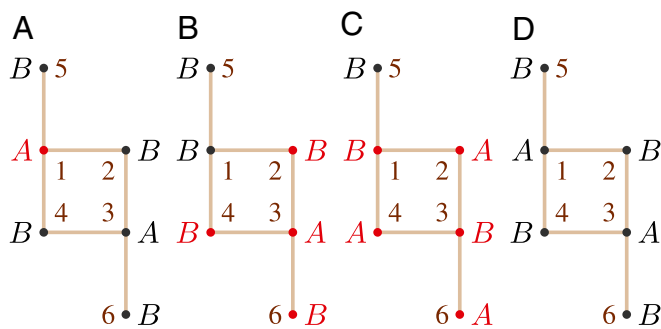


Fig. 3. (A–D) Strategy evolution in the network game $\Gamma = (\mathbb{G}, (1/2)\mathbf{1}, +)$ under the activation subsequence $\alpha_1 = (\{1\}, \{2,3,4,6\}, \{1,2,3,4,6\})$. Red circles indicate those nodes that are activated at the corresponding time step.

active during the time interval $[k, k+1)$. We can express the partially synchronous dynamics as follows:

$$x_i(k+1) = \begin{cases} x_i^{br}(k) & \text{if } i \in \mathcal{A}_k \\ x_i(k) & \text{otherwise} \end{cases}$$

where $x_i^{br}(k)$ denotes the best response of agent i to the strategies of its neighbors at time k . To provide a general framework for independent stochastic activation sequences, we make the following assumption.

Assumption 2. The interactivation times for each agent are drawn from mutually independent probability distributions with support on $\mathbb{R}_{\geq 0}$.

One standard model that satisfies Assumption 2 is to use exponential distributions with mean $1/\lambda$. This represents the case in which each agent updates according to a Poisson clock with rate λ , and the expected number of agents updating in one unit of time is λn .

From the previous sections, we know that the best-response dynamics do not necessarily converge to an equilibrium when the updating is synchronous, yet do converge when the updating is asynchronous. Therefore, the natural question arising is the following: how much asynchrony do the partially synchronous best-response dynamics need for convergence? It turns out that, even under the relatively mild assumption on the partially synchronous updates in Assumption 2, we still have convergence to an equilibrium state almost surely.

Theorem 3. Every network of all coordinating or all anticonducting agents who update with partially synchronous dynamics that satisfy Assumption 2 almost surely reaches an equilibrium state in finite time.

To prove the theorem, we model the network game as a Markov chain and show that it is absorbing (28). We refer the reader to *SI Appendix* for a detailed proof.

Note that one technically equivalent but perhaps less practical modeling of the partially synchronous updates would be to simply assume that at each time step $k=0,1,\dots$, a random number of random agents activate simultaneously and then update at $k+1$. In other words, there is a fixed probability that a particular group of agents activate simultaneously at every time step. Then Theorem 3 still holds because the probability that each agent activates asynchronously is bounded below by a positive constant. From a broader point of view, Theorem 3 holds whenever random asynchronous activations are not completely excluded from the partially synchronous dynamics. Convergence, however, may not be achieved

for particular nonrandom activation sequences, as discussed in the following.

Zero-Probability Nonconvergence. We now show that Theorem 3 holds only almost surely. In other words, there exist activation sequences generated under Assumption 2 that do not result in equilibrium convergence; however, the probability of such sequences happening is zero. Consider the network game $\Gamma = (\mathbb{G}, (1/2)\mathbf{1}, +)$ where \mathbb{G} and the initial strategies $x(0)$ are as in Fig. 3A. First, we show that if any single agent activates when the strategy state equals $x(0)$, then there exists a finite sequence of agent activations that return the state to $x(0)$. Consider the case when agent 1 activates exclusively at $k=0$ (Fig. 3A), and hence, switches to B at $k=1$. Then if agents 2, 3, 4, 6 activate at $k=1$ and agents 1, 2, 3, 4, 6 at $k=2$, the strategy state returns to $x(0)$ at $k=3$, that is, $x(3) = x(0)$; the process is shown in Fig. 3B–D. Denote the corresponding activation subsequence by $\alpha_1 = (\{1\}, \{2,3,4,6\}, \{1,2,3,4,6\})$. Similarly, as shown in Fig. 4, there exists an activation sequence that returns the state of the system to $x(0)$ when agent 2 activates at $k=0$. The corresponding activation subsequence would be $\alpha_2 = (\{2\}, \{1,3,4,5,6\}, \{1,2,3,4,5,6\})$. Moreover, due to the symmetric distribution of the strategies $x(0)$ in the network, the same can be shown for when agents 3, 4, or 6 are activated at $k=0$ (which are similar to agents 1, 2, and 5, respectively). Denote the corresponding activation subsequences by α_3, α_4 , and α_6 . Now consider the event X made by α_1 and α_2 as follows:

$$X = (x_i)_{i=1}^{\infty}, \quad x_i \in \{\alpha_1, \alpha_2\}.$$

Any activation sequence in X can be generated under Assumption 2 and has the property that all agents activate exclusively infinitely many times. However, no sequence in X results in convergence of the network game Γ . However, this does not contradict Theorem 3 because $\mathbb{P}[X] = 0$, under Assumption 2.

Concluding Remarks

We have shown that arbitrary networks consisting of all coordinating or all anticonducting agents who update with asynchronous best responses will reach equilibrium in finite time. Moreover, when updates are partially synchronous, we have shown that the network still almost surely reaches an equilibrium under mild conditions on the independence and randomness of agent updates. For the case of anticonducting agents, these results have important implications in social contexts where individuals prefer an action only if a small enough portion of neighbors are using that action, for example, deciding which route to take to avoid traffic congestion, volunteering for a dangerous but important public service position, contributing money or time toward a crowd-sourced project, etc. For coordinating agents, the results apply to social contexts where each agents prefer an action only if a sufficient

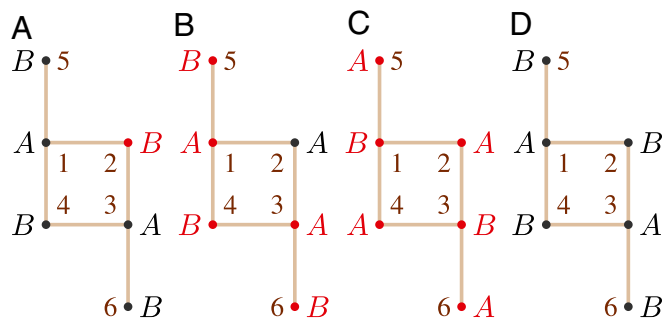


Fig. 4. (A–D) Strategy evolution in the network game $\Gamma = (\mathbb{G}, (1/2)\mathbf{1}, +)$ under the activation subsequence $\alpha_2 = (\{2\}, \{1,3,4,5,6\}, \{1,2,3,4,5,6\})$.

number of neighbors are using that action, for example, the spread of social behaviors, technological innovations, viral infections. Our results suggest that, in both cases, no matter how different the individuals are, which neighbors affect their decisions, or how many simultaneous decisions are made, everyone will tend to settle on a particular action with which they are satisfied. This means that the presence of cycles or nonconvergence must result from other factors such as imitation or other unmodeled effects

in the update dynamics, or a mixture of coordinating and anti-coordinating agents. These results also open the door to characterizing the equilibria and investigating possibilities for payoff-based incentive control of the network.

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**Supplementary Material for
Networks of Conforming or Non-Conforming Individuals Tend
to Reach Satisfactory Decisions**

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This document contains the proofs for the three theorems and associated lemmas and corollaries in the paper Agree or Agree-to-Disagree: Networks of Conforming or Non-Conforming Individuals Tend to Reach Satisfactory Decisions, in the Proceedings of the National Academy of Sciences.

I. PROOF OF THEOREM 1

Lemma 1. *Every network of anti-coordinating agents who update asynchronously under Assumption 1, with $\tau_i = \frac{1}{2}$ for each agent $i \in \mathcal{V}$, will reach an equilibrium state in finite time.*

Proof: Let \hat{n}_i^A denote the maximum number of A -neighbors of agent i that will not cause agent i to switch to B when playing A . Define the function $\Phi(k) = \sum_{i=1}^n \Phi_i(k)$, where

$$\Phi_i(k) = \begin{cases} n_i^A(k) - \hat{n}_i^A & \text{if } x_i(k) = A \\ \hat{n}_i^A + 1 - n_i^A(k) & \text{if } x_i(k) = B \end{cases}$$

$\Phi(k)$ is clearly lower bounded by $\Phi(k) \geq -\sum_{i=1}^n \deg_i$ for all k . Consider a time step k , and let i denote the index of the active agent at that time. One of the following three cases must happen:

- 1) Agent i does not switch strategies at time $k + 1$. This implies $\Phi(k + 1) = \Phi(k)$.
- 2) Agent i switches from A to B at time $k + 1$. This implies $n_i^A(k) \geq \hat{n}_i^A + 1$. Then, since $n_i^A(k) = n_i^A(k + 1)$, we have

$$\begin{aligned} \Phi_i(k + 1) - \Phi_i(k) &= \hat{n}_i^A + 1 - n_i^A(k) - n_i^A(k) + \hat{n}_i^A \\ &= 2(\hat{n}_i^A - n_i^A(k)) + 1 \leq -1. \end{aligned} \quad (3)$$

Moreover, for each $j \in \mathcal{N}_i$, if $x_j(k) = A$, it holds that

$$\begin{aligned} \Phi_j(k + 1) - \Phi_j(k) &= n_j^A(k + 1) - \hat{n}_j^A - n_j^A(k) + \hat{n}_j^A \\ &= -1, \end{aligned} \quad (4)$$

and if $x_j(k) = B$, it holds that

$$\Phi_j(k + 1) - \Phi_j(k) = -n_j^A(k + 1) + n_j^A(k) = 1. \quad (5)$$

According to (2), the fact that agent i switches from A to B at time $k + 1$ implies $n_i^A(k) \geq \frac{1}{2} \deg_i$, regardless of how z_i is defined. Hence, by combining (3), (4), and (5), we have

$$\begin{aligned} \Phi(k + 1) - \Phi(k) &= \sum_{j \in \mathcal{N}_i \cup \{i\}} \Phi_j(k + 1) - \Phi_j(k) \\ &= \underbrace{\Phi_i(k + 1) - \Phi_i(k)}_{\leq -1} - \underbrace{n_i^A(k) + (\deg_i - n_i^A(k))}_{\leq 0} \\ &\leq -1. \end{aligned} \quad (6)$$

- 3) Agent i switches from B to A at time $k + 1$. This implies $n_i^A(k) \leq \hat{n}_i^A$. Hence,

$$\Phi_i(k + 1) - \Phi_i(k) = 2(n_i^A(k) - \hat{n}_i^A) - 1 \leq -1. \quad (7)$$

Moreover, for each $j \in \mathcal{N}_i$, if $x_j(k) = A$, it holds that

$$\Phi_j(k + 1) - \Phi_j(k) = n_j^A(k + 1) - n_j^A(k) = 1, \quad (8)$$

and if $x_j(k) = B$, it holds that

$$\Phi_j(k + 1) - \Phi_j(k) = -n_j^A(k + 1) + n_j^A(k) = -1. \quad (9)$$

According to (2), the fact that agent i switches from B to A at time $k + 1$ implies $n_i^A(k) \leq \frac{1}{2} \deg_i$, regardless of how z_i is defined. Hence, by combining (7), (8), and (9), we have

$$\begin{aligned} & \Phi(k+1) - \Phi(k) \\ &= \underbrace{\Phi_i(k+1) - \Phi_i(k)}_{\leq -1} + \underbrace{n_i^A(k) - (\deg_i - n_i^A(k))}_{\leq 0} \\ &\leq -1. \end{aligned}$$

By summarizing the above three cases, we have that

$$\Phi(k+1) \leq \Phi(k) \quad \forall k \geq 0. \quad (10)$$

Moreover, we have shown that every time an agent switches strategies, the function $\Phi(k)$ decreases by at least one. The case where all thresholds are equal to $\frac{1}{2}$ is thus a generalized ordinal potential game, by the definition given in [1]. However, as shown in [2], this does not necessarily imply convergence to an equilibrium. Hence, we complete the proof by contradiction.

Assume that the network does not reach an equilibrium in finite time. Hence, at every time step $k = 0, 1, \dots$, there exists an agent i^k whose strategy violates its threshold. Denote by \tilde{k} the first time after k at which agent i^k is active. The existence of \tilde{k} is guaranteed by Assumption 1. At time \tilde{k} , agent i^k 's threshold either remains violated, implying the agent will switch strategies at time $\tilde{k} + 1$, or is no longer violated, implying that some neighbors have changed their strategies during the time interval $[\tilde{k} + 1, \tilde{k}]$. Hence, at least one switch occurs in each interval $\mathcal{I}^k = [k + 1, \tilde{k} + 1]$. Now consider the sequence of intervals $\mathcal{I}^{k_1}, \mathcal{I}^{k_2}, \dots$ where the indices $k_j, j = 1, 2, \dots$, are such that $k_{j+1} > \tilde{k}_j + 1$. This sequence is infinite, and the intersection of each pair of intervals from the sequence is empty. Therefore, an infinite number of switches occur in the network over time. Namely, there exists an infinite time sequence $(\kappa^j)_{j=1}^\infty, \kappa^j \in \mathcal{I}^{k_j}$, such that an agent switches strategies at each κ^j . Hence, either Case 2 or 3 occurs at each κ^j , resulting in $\Phi(\kappa^j + 1) \leq \Phi(\kappa^j) - 1$. Hence, in view of (10),

$$\Phi(k) \leq \Phi(\kappa^j) - 1 \quad \forall k \geq \kappa^j + 1. \quad (11)$$

Since (11) holds for all $j = 1, 2, \dots$, we get that

$$\Phi(k) \leq \Phi(\kappa^1) - j \quad \forall k \geq \kappa^j + 1 \quad \forall j \in \mathbb{N}.$$

The above inequality implies that Φ is not lower bounded, which is a contradiction. Hence, the proof is complete. \blacksquare

We define the *augmented network game* $\hat{\Gamma} := (\hat{\mathbb{G}}, \frac{1}{2}\mathbf{1}, -)$ based on Γ as follows. Let $\hat{\mathbb{G}} = (\hat{\mathcal{V}}, \hat{\mathcal{E}})$. Define a *V-block* as a triplet of nodes $\{v_1, v_2, v_3\} \subseteq \hat{\mathcal{V}}$ along with the edges $\{\{v_1, v_2\}, \{v_1, v_3\}\} \subseteq \hat{\mathcal{E}}$. For each agent $i \in \mathcal{V}$, we introduce a *dual agent* $\hat{i} \in \hat{\mathcal{V}}$ with the same initial strategy, i.e., $x_{\hat{i}}(0) = x_i(0)$, and with $z_{\hat{i}} = z_i$. Corresponding to each dual agent \hat{i} , there are m_i number of *V-blocks* in $\hat{\mathbb{G}}$ such that the v_1 -node of each block is connected to \hat{i} , with m_i being defined as follows: if $\tau_i = \frac{1}{2}$, then $m_i = 0$; otherwise, m_i depends on which one of the following three conditions on τ_i holds:

$$m_i = \begin{cases} |(1 - 2\tau_i) \deg_i| & \tau_i \deg_i \in \mathbb{Z} \\ |\deg_i - r - 1| & \exists r \in 2\mathbb{Z} : \frac{r}{2} < \tau_i \deg_i < \frac{r+1}{2} \\ |\deg_i - r| & \exists r \in 2\mathbb{Z} + 1 : \frac{r}{2} \leq \tau_i \deg_i < \frac{r+1}{2} \end{cases} \quad (12)$$

where $2\mathbb{Z}$ and $2\mathbb{Z} + 1$ denote the set of even and odd numbers, respectively. If $\tau_i < \frac{1}{2}$, then the initial strategies of each *V-block* connected to the dual agent \hat{i} are $x_{v_1}(0) = A$ and $x_{v_2}(0) = x_{v_3}(0) = B$; and if $\tau_i > \frac{1}{2}$, then $x_{v_1}(0) = B$ and $x_{v_2}(0) = x_{v_3}(0) = A$. In total, $\hat{\mathcal{V}}$ has $n + \sum_{i=1}^n m_i$ agents, the thresholds of all of which are set to $\frac{1}{2}$. For $\hat{\mathcal{E}}$, in addition to the edges for the *V-blocks*, there is an edge between any two dual agents \hat{i} and \hat{j} in $\hat{\mathcal{V}}$, if and only if there is an edge between i and j in \mathcal{V} .

Lemma 2. *The strategy of each V-block agent is fixed over time.*

Proof: We prove by contradiction. Assume there exists some time when the strategy of one of the *V-block* agents changes. Let r denote the first time this happens, and denote the *V-block* agent who changes her strategy by

i . If i is a v_1 -node and $x_i(0) = A$, then $x_i(r-1) = A$ and $x_i(r) = B$. Agent i has two neighbors in the V -block who each play strategy B at time $r-1$. Since $\deg_i = 3$, we have

$$\left. \begin{array}{l} n_i^A(r-1) \leq 1 \\ \tau_i \deg_i = \frac{3}{2} \end{array} \right\} \Rightarrow n_i^A(r-1) < \tau_i \deg_i \Rightarrow x_i(r) = A,$$

which is a contradiction. If $x_i(0) = B$, then agent i has two A -neighbors in the V -block and $x_i(r) = A$. It follows that

$$\left. \begin{array}{l} n_i^A(r-1) \geq 2 \\ \tau_i \deg_i = \frac{3}{2} \end{array} \right\} \Rightarrow n_i^A(r-1) > \tau_i \deg_i \Rightarrow x_i(r) = B,$$

again a contradiction. Now if i is either a v_2 or a v_3 node, and $x_i(0) = A$, then $x_i(r-1) = A$, and i has only one neighbor, v_1 , whose strategy is B at time $r-1$. Hence,

$$\left. \begin{array}{l} n_i^A(r-1) = 0 \\ \tau_i \deg_i = \frac{1}{2} \end{array} \right\} \Rightarrow n_i^A(r-1) < \tau_i \deg_i \Rightarrow x_i(r) = A,$$

a contradiction. If on the other hand, $x_i(0) = B$, then

$$\left. \begin{array}{l} n_i^A(r-1) = 1 \\ \tau_i \deg_i = \frac{1}{2} \end{array} \right\} \Rightarrow n_i^A(r-1) > \tau_i \deg_i \Rightarrow x_i(r) = B,$$

which is a contradiction and completes the proof. ■

The following lemma takes the first step towards establishing equivalence of the dynamics between the original and augmented network games. In particular, we need to show that the thresholds of $\frac{1}{2}$ in the augmented network will be satisfied (respectively violated) whenever the thresholds of the corresponding agents in the original network are satisfied (respectively violated).

Lemma 3. *Let n_i^A denote an instance of $n_i^A(k)$ for some agent i . If $\tau_i < \frac{1}{2}$, then*

$$\text{sign}(n_i^A - \tau_i \deg_i) = \text{sign}\left(n_i^A + m_i - \frac{1}{2}(\deg_i + m_i)\right), \quad (13)$$

and if $\tau_i > \frac{1}{2}$, then

$$\text{sign}(n_i^A - \tau_i \deg_i) = \text{sign}\left(n_i^A - \frac{1}{2}(\deg_i + m_i)\right). \quad (14)$$

Proof: First consider the situation when $\tau_i < \frac{1}{2}$. In general, one of the following cases takes place:

1) $\tau_i \deg_i \in \mathbb{Z}$: If $n_i^A = \tau_i \deg_i$, using (12) we have

$$n_i^A = \frac{1}{2}(\deg_i - m_i) \Rightarrow n_i^A + m_i = \frac{1}{2}(\deg_i + m_i).$$

The cases $n_i^A < \tau_i \deg_i$ and $n_i^A > \tau_i \deg_i$ can be shown using the same approach, which verifies (13) for this case.

2) $\exists r \in 2\mathbb{Z} : \frac{r}{2} \leq \tau_i \deg_i < \frac{r+1}{2}$: Here, $\tau_i \deg_i \notin \mathbb{Z}$ implies that $n_i^A \neq \tau_i \deg_i$, so we need only to check the inequality cases. Using (12), $n_i^A > \tau_i \deg_i$ implies

$$\left. \begin{array}{l} n_i^A > \frac{r}{2} \\ \frac{r}{2}, n_i^A \in \mathbb{Z} \end{array} \right\} \Rightarrow n_i^A \geq \frac{r}{2} + 1 = \frac{1}{2}(\deg_i - m_i) + \frac{1}{2} \\ \Rightarrow n_i^A + m_i > \frac{1}{2}(\deg_i + m_i),$$

and $n_i^A < \tau_i \deg_i$ implies

$$\begin{aligned} n_i^A &< \frac{r+1}{2} = \frac{1}{2}(\deg_i - m_i) \\ \Leftrightarrow n_i^A + m_i &< \frac{1}{2}(\deg_i + m_i). \end{aligned}$$

Hence, (13) is confirmed for this case.

3) $\exists r \in 2\mathbb{Z} + 1 : \frac{r}{2} \leq \tau_i \deg_i < \frac{r+1}{2}$: Again, $\tau_i \deg_i \notin \mathbb{Z}$ implies that $n_i^A \neq \tau_i \deg_i$. Then $n_i^A > \tau_i \deg_i$ implies

$$n_i^A > \frac{r}{2} = \frac{1}{2}(\deg_i - m_i) \Leftrightarrow n_i^A + m_i > \frac{1}{2}(\deg_i + m_i),$$

and $n_i^A < \tau_i \deg_i$ implies

$$\left. \begin{aligned} n_i^A &< \frac{r+1}{2} \\ n_i^A &\in \mathbb{Z}, r \in 2\mathbb{Z} + 1 \end{aligned} \right\} \Rightarrow n_i^A \leq \frac{r-1}{2}$$

$$\Leftrightarrow n_i^A \leq \frac{1}{2}(\deg_i - m_i) - \frac{1}{2}$$

$$\Rightarrow n_i^A + m_i < \frac{1}{2}(\deg_i + m_i).$$

Hence, (13) holds for this case and for all $\tau_i < \frac{1}{2}$.

If $\tau_i > \frac{1}{2}$, then one of the following occurs:

1) $\tau_i \deg_i \in \mathbb{Z}$: If $n_i^A = \tau_i \deg_i$, then (12) implies $n_i^A = \frac{1}{2}(\deg_i + m_i)$. The cases $n_i^A < \tau_i \deg_i$ and $n_i^A > \tau_i \deg_i$ can be shown using the same approach, which verifies (14) for this case.

2) $\exists r \in 2\mathbb{Z} : \frac{r}{2} \leq \tau_i \deg_i < \frac{r+1}{2}$: First, we know that $n_i^A \neq \tau_i \deg_i$. Using (12), $n_i^A > \tau_i \deg_i$ implies

$$n_i^A \geq \frac{r}{2} + 1 = \frac{1}{2}(\deg_i + m_i) + \frac{1}{2} > \frac{1}{2}(\deg_i + m_i),$$

and $n_i^A < \tau_i \deg_i$ implies

$$n_i^A < \frac{r+1}{2} = \frac{1}{2}(\deg_i + m_i) \Leftrightarrow n_i^A < \frac{1}{2}(\deg_i + m_i).$$

Hence, (14) holds for this case.

3) $\exists r \in 2\mathbb{Z} + 1 : \frac{r}{2} \leq \tau_i \deg_i < \frac{r+1}{2}$: Once again, we know that $n_i^A \neq \tau_i \deg_i$. Using (12), $n_i^A > \tau_i \deg_i$ implies

$$n_i^A > \frac{r}{2} = \frac{1}{2}(\deg_i + m_i),$$

and $n_i^A < \tau_i \deg_i$ implies

$$n_i^A \leq \frac{r-1}{2} = \frac{1}{2}(\deg_i + m_i) - \frac{1}{2} < \frac{1}{2}(\deg_i + m_i).$$

Hence, (14) holds for this case and for all $\tau_i > \frac{1}{2}$, which completes the proof. \blacksquare

Next, we show in Lemma 4 that if whenever an agent in \mathbb{G} activates, its dual in $\hat{\mathbb{G}}$ also activates (while neglecting the time steps that a V -block agent is active), then the dynamics of each node in \mathbb{G} are the same as the dynamics of its dual node in $\hat{\mathbb{G}}$ (again while neglecting the time steps that a V -block agent is active).

Consider the network \mathbb{G} and let $i_{\mathbb{G}}^k$ denote the active agent at time k . Correspondingly, denote by $(i_{\mathbb{G}}^k)_{k=0}^{\infty}$, the sequence of active agents in \mathbb{G} . Similarly define $(i_{\hat{\mathbb{G}}}^k)_{k=0}^{\infty}$ as the sequence of active agents in $\hat{\mathbb{G}}$. Consider $(i_{\hat{\mathbb{G}}}^k)_{k=0}^{\infty}$ and exclude those agents $i_{\hat{\mathbb{G}}}^k$ that belong to one of the V -blocks, to get the subsequence $(i_{\hat{\mathbb{G}}}^{h_k})_{k=0}^{\infty}$. Denote the sequence of superscripts of $(i_{\hat{\mathbb{G}}}^{h_k})_{k=0}^{\infty}$ by $(h_k)_{k=0}^{\infty}$ which corresponds to the times at which the non- V -block agents in $\hat{\mathbb{G}}$ are active.

Lemma 4. *If $(i_{\mathbb{G}}^k)_{k=0}^{\infty} = (i_{\hat{\mathbb{G}}}^{h_k})_{k=0}^{\infty}$, then for $k = 0, 1, \dots$, it holds that*

$$x_i(k) = x_i(h_k) \quad \forall i \in \mathcal{V}. \quad (15)$$

where $\hat{i} \in \hat{\mathcal{V}}$ is the dual of agent i .

Proof: The proof is done via induction on k . By the definition of $\hat{\mathbb{G}}$, (15) holds for $k = 0$. Assume that (15) holds for $k = r \in \mathbb{Z}_{\geq 0}$.

Consider agent $i_{\hat{\mathbb{G}}}^r$ and its dual $\hat{i}_{\hat{\mathbb{G}}}^r$ whose threshold and degree are $\frac{1}{2}$ and $\deg_i + m_i$, respectively. Agent $i_{\hat{\mathbb{G}}}^r$ updates at time $k = r + 1$, and agent $\hat{i}_{\hat{\mathbb{G}}}^r$ updates at $k_{\hat{\mathbb{G}}} = h_r + 1$ where $k_{\hat{\mathbb{G}}}$ denotes the time in the augmented network game $\hat{\Gamma}$. If $\tau_i = \frac{1}{2}$, then both agents have the same threshold and number of A -neighbors. Hence, they update to the same strategy at the next time step. If $\tau_i < \frac{1}{2}$, then in view of Lemma 2 and since (15) holds for $k = r$, $\hat{i}_{\hat{\mathbb{G}}}^r$ has $n_{i_r^A}(r) + m_i$ A -neighbors. Therefore, according to (13) in Lemma 3, $\hat{i}_{\hat{\mathbb{G}}}^r$ updates to the same strategy that $i_{\hat{\mathbb{G}}}^r$ does. On the other hand, if $\tau_i > \frac{1}{2}$, then $\hat{i}_{\hat{\mathbb{G}}}^r$ has $n_{i_r^A}(r)$ A -neighbors. Hence, according to (14) in Lemma 3, $\hat{i}_{\hat{\mathbb{G}}}^r$ updates to the same strategy that $i_{\hat{\mathbb{G}}}^r$ does. Therefore, in all cases, agent $\hat{i}_{\hat{\mathbb{G}}}^r$ updates to the same strategy that agent $i_{\hat{\mathbb{G}}}^r$ does. That is,

$$x_{i_{\hat{\mathbb{G}}}^r}(r+1) = x_{\hat{i}_{\hat{\mathbb{G}}}^r}(h_r+1). \quad (16)$$

On the other hand, since no other agent has become active at times h_r or r ,

$$x_i(r+1) = x_i(r) \quad \forall i \in \mathcal{V} - \{i_{\hat{\mathbb{G}}}^r\}, \quad (17)$$

$$x_{\hat{i}}(h_r+1) = x_{\hat{i}}(h_r) \quad \forall \hat{i} \in \mathcal{V} - \{\hat{i}_{\hat{\mathbb{G}}}^r\}. \quad (18)$$

Due to the induction statement for $k = r$, it holds that $x_i(r) = x_{\hat{i}}(h_r)$ for all $i \in \mathcal{V} - \{i_{\hat{\mathbb{G}}}^r\}$. Hence, (17) and (18) result in

$$x_i(r+1) = x_{\hat{i}}(h_r+1) \quad \forall i \in \mathcal{V} - \{i_{\hat{\mathbb{G}}}^r\}.$$

Therefore, according to (16),

$$x_i(r+1) = x_{\hat{i}}(h_r+1) \quad \forall i \in \mathcal{V}. \quad (19)$$

Now since at each of the time steps $h_r+1, h_r+2, \dots, h_{r+1}-1$, the active agent is a V -block agent whose strategy remains fixed by Lemma 2, (19) results in

$$x_i(r+1) = x_{\hat{i}}(h_{r+1}) \quad \forall i \in \mathcal{V}.$$

Hence, (15) holds for $k = r + 1$, which completes the proof by induction. \blacksquare

The remaining step in proving Theorem 1 is to show that agents with arbitrary thresholds will indeed reach an equilibrium state in finite time.

Proof of Theorem 1: Towards a proof by contradiction, suppose that the original network game never converges, i.e., there exists an agent $j \in \mathcal{V}$ such that

$$\forall k^*, (\exists k > k^* : x_j(k) \neq x_j(k^*)).$$

Construct the sequence of active agents $(i_{\hat{\mathbb{G}}}^k)_{k=0}^{\infty}$ by inserting an agent \hat{i} uniformly at random from the set of augmented nodes $\hat{\mathcal{V}} - \mathcal{V}$ after every n elements of the sequence $(i_{\hat{\mathbb{G}}}^k)_{k=0}^{\infty}$. This is clearly a persistent activation sequence on the network $\hat{\mathbb{G}}$. By Lemma 1, we know that

$$\exists \tilde{k}^* : (\forall k > \tilde{k}^*, x_{\hat{i}}(k) = x_{\hat{i}}(\tilde{k}^*)). \quad (20)$$

On the other hand, by eliminating the V -block agents in $(i_{\hat{\mathbb{G}}}^k)_{k=0}^{\infty}$, we arrive at $(i_{\hat{\mathbb{G}}}^k)_{k=0}^{\infty}$. Hence, in view of Lemma 4, (20) implies that

$$\exists k^* : (\forall k > k^*, x_j(k) = x_j(k^*)),$$

which contradicts our initial statement. Therefore, $x(k)$ will reach an equilibrium in finite time. \blacksquare

II. PROOF OF THEOREM 2

The proof of Theorem 2 follows similar steps as the anti-coordinating case. The key difference is that the potential function becomes

$$\Phi_i(k) = \begin{cases} \tilde{n}_i^A - n_i^A(k) & \text{if } x_i(k) = A \\ n_i^A(k) - \tilde{n}_i^A + 1 & \text{if } x_i(k) = B \end{cases}, \quad (21)$$

where \check{n}_i^A is defined as the minimum number of A -neighbors required for an A -playing agent to continue playing A . The maximum number of A -neighbors that a B agent can tolerate before switching to A is then given by $\check{n}_i^A - 1$. As shown in the following lemma, this function also decreases by at least 1 with every change of strategy for the network game $\Gamma := (\mathbb{G}, \frac{1}{2}\mathbf{1}, +)$.

Lemma 5. *Every network of coordinating agents who update asynchronously under Assumption 1, with $\tau_i = \frac{1}{2}$ for each agent $i \in \mathcal{V}$, will reach an equilibrium state in finite time.*

Proof: Consider the function $\Phi(k) = \sum_{i=1}^n \Phi_i(k)$, where Φ_i is defined in (21). Clearly $\Phi(k)$ is lower bounded by $\Phi(k) \geq -\sum_{i=1}^n \deg_i$ for all k . Consider a time step k , and let i denote the active agent at that time. One of the following three cases must happen:

1) Agent i does not switch strategies at time $k + 1$. This implies $\Phi(k + 1) = \Phi(k)$.

2) Agent i switches from A to B at time $k + 1$. This implies $n_i^A(k) \leq \check{n}_i^A - 1$. Hence, since $n_i^A(k) = n_i^A(k + 1)$, we have

$$\begin{aligned} \Phi_i(k + 1) - \Phi_i(k) &= n_i^A(k) - \check{n}_i^A + 1 - \check{n}_i^A + n_i^A(k) \\ &= 2(n_i^A(k) - \check{n}_i^A) + 1 \leq -1. \end{aligned} \quad (22)$$

Moreover, for each $j \in \mathcal{N}_i$, if $x_j(k) = A$, it holds that

$$\begin{aligned} \Phi_j(k + 1) - \Phi_j(k) &= n_i^A - n_i^A(k + 1) - \check{n}_i^A + n_i^A(k) \\ &= 1, \end{aligned} \quad (23)$$

and if $x_j(k) = B$, it holds that

$$\Phi_j(k + 1) - \Phi_j(k) = n_i^A(k + 1) - n_i^A(k) = -1. \quad (24)$$

According to (1), the fact that agent i switches from A to B at time $k + 1$ implies $n_i^A(k) \leq \frac{1}{2} \deg_i$, regardless of how z_i is defined. Hence, by combining (22), (23), and (24), we have

$$\begin{aligned} \Phi(k + 1) - \Phi(k) &= \underbrace{\Phi_i(k + 1) - \Phi_i(k)}_{\leq -1} + \underbrace{n_i^A(k) - (\deg_i - n_i^A(k))}_{\leq 0} \\ &\leq -1. \end{aligned} \quad (25)$$

3) Agent i switches from B to A at time $k + 1$. This implies $n_i^A(k) \geq \check{n}_i^A$. Hence,

$$\Phi_i(k + 1) - \Phi_i(k) = 2(\check{n}_i^A - n_i^A(k)) - 1 \leq -1. \quad (26)$$

Moreover, for each $j \in \mathcal{N}_i$, if $x_j(k) = A$, it holds that

$$\Phi_j(k + 1) - \Phi_j(k) = -n_j^A(k + 1) + n_j^A(k) = -1, \quad (27)$$

and if $x_j(k) = B$, it holds that

$$\Phi_j(k + 1) - \Phi_j(k) = n_j^A(k + 1) - n_j^A(k) = 1. \quad (28)$$

According to (2), the fact that agent i switches from B to A at time $k + 1$ implies $n_i^A(k) \geq \frac{1}{2} \deg_i$, regardless of how z_i is defined. Hence, by combining (26), (27), and (28), we have

$$\Phi(k + 1) - \Phi(k) \quad (29)$$

$$= \underbrace{\Phi_i(k + 1) - \Phi_i(k) - n_i^A(k)}_{\leq -1} + \underbrace{(\deg_i - n_i^A(k))}_{\leq 0} \quad (30)$$

$$\leq -1. \quad (31)$$

By summarizing the above three cases, we have that

$$\Phi(k + 1) \leq \Phi(k) \quad \forall k \geq 0. \quad (32)$$

Moreover, we have shown that every time an agent switches strategies, the function $\Phi(k)$ decreases by at least one. The rest of the proof follows in the same way as that of Lemma 1. ■

By following the same process of constructing the network augmentation for anti-coordinating agents, we are able to extend the result of Lemma 5 to a network game with arbitrary thresholds. We define the augmented (coordinating) network game $\hat{\Gamma} := (\hat{\mathbb{G}}, \frac{1}{2}\mathbf{1}, +)$ based on the (coordinating) network game Γ , in the same way we defined the augmented network game for anti-coordinating agents, but with the following difference: If $\tau_i < \frac{1}{2}$, then the initial strategies of each V -block connected to the dual agent \hat{i} are $x_{v_1}(0) = x_{v_2}(0) = x_{v_3}(0) = A$, and if $\tau_i > \frac{1}{2}$, then $x_{v_1}(0) = x_{v_2}(0) = x_{v_3}(0) = B$. Similar to Lemma 2, the following lemma guarantees the invariance of the strategies of the V -block agents.

Lemma 6. *The strategy of each (coordinating) V -block agent is fixed over time.*

Proof: The proof is done via contradiction. Assume there exists some time when the strategy of one of the V -block agents changes. Let r denote the first time this happens, and denote the V -block agent who changes her strategy by i . If i is a v_1 -node and $x_i(0) = A$, then $x_i(r-1) = A$ and $x_i(r) = B$. Agent i has two neighbors in the V -block who each play strategy A at time $r-1$. Since $\deg_i = 3$, we have

$$\left. \begin{array}{l} n_i^A(r-1) \geq 2 \\ \tau_i \deg_i = \frac{3}{2} \end{array} \right\} \Rightarrow n_i^A(r-1) > \tau_i \deg_i \Rightarrow x_i(r) = A,$$

which is a contradiction. If $x_i(0) = B$, then agent i has two B -neighbors in the V -block and $x_i(r) = A$. It follows that

$$\left. \begin{array}{l} n_i^A(r-1) \leq 1 \\ \tau_i \deg_i = \frac{3}{2} \end{array} \right\} \Rightarrow n_i^A(r-1) < \tau_i \deg_i \Rightarrow x_i(r) = B,$$

again a contradiction. Now if i is either a v_2 or a v_3 node, and $x_i(0) = A$, then $x_i(r-1) = A$, and i has only one neighbor, v_1 , whose strategy is A at time $r-1$. Hence,

$$\left. \begin{array}{l} n_i^A(r-1) = 1 \\ \tau_i \deg_i = \frac{1}{2} \end{array} \right\} \Rightarrow n_i^A(r-1) > \tau_i \deg_i \Rightarrow x_i(r) = A,$$

a contradiction. If on the other hand, $x_i(0) = B$, then

$$\left. \begin{array}{l} n_i^A(r-1) = 0 \\ \tau_i \deg_i = \frac{1}{2} \end{array} \right\} \Rightarrow n_i^A(r-1) < \tau_i \deg_i \Rightarrow x_i(r) = B,$$

which is a contradiction and completes the proof. ■

Next, since Lemma 3 is independent of the type of agents, i.e., coordinating or anti-coordinating, it can be used here as well. Moreover, because of Lemma 6, the result of Lemma 4 can be readily extended to a network of coordinating agents. With these lemmas in hand, and with the help of Lemma 5, the proof of Theorem 2 can be done in the same way as that of Theorem 1.

III. CONVERGENCE TIME: PROOF OF COROLLARY 3

Corollary 3. *Every network of all coordinating or all anti-coordinating agents will reach an equilibrium state after no more than $6|\mathcal{E}|$ agent switches.*

Proof: To compute the maximum number of times any agent switches strategies before such a network reaches an equilibrium, we consider the augmented network game $\hat{\Gamma}$, which will undergo the same sequence of agent switches as the original network game Γ , provided that the dual agents in $\hat{\mathcal{V}}$ activate in the same order as the corresponding agents in \mathcal{V} . From (6) in the proof of Lemma 1 and (25) in the proof of Lemma 5, we know that whenever an agent $i \in \hat{\mathcal{V}}$ switches strategies, $\Phi(k+1) - \Phi(k) \leq -1$. Otherwise, $\Phi(k)$ remains constant. It follows that the total number of agent switches in $\hat{\Gamma}$ is bounded from above by $\Phi(0) - \Phi(k^*)$, where k^* is the time at which the network reaches an equilibrium. To obtain such a bound, we start by decomposing the augmented network into

three disjoint sets of agents such that $\hat{\mathcal{V}} = \hat{\mathcal{V}}_0 \cup \hat{\mathcal{V}}_1 \cup \hat{\mathcal{V}}_{23}$, where $\hat{\mathcal{V}}_0$ denotes the dual agents corresponding to the original agents \mathcal{V} , $\hat{\mathcal{V}}_1$ denotes the set of v_1 agents in the V -blocks, and $\hat{\mathcal{V}}_{23}$ denotes the set of v_2 and v_3 agents in the V -blocks (we refer the reader to the proof of Theorem 1 for definitions of the augmented network). We can now expand the expression for the upper bound as follows:

$$\Phi(0) - \Phi(k^*) = \sum_{i \in \hat{\mathcal{V}}} \Phi_i(0) - \Phi_i(k^*) = \sum_{i \in \hat{\mathcal{V}}_0} \Phi_i(0) - \Phi_i(k^*) + \sum_{i \in \hat{\mathcal{V}}_1} \Phi_i(0) - \Phi_i(k^*) + \sum_{i \in \hat{\mathcal{V}}_{23}} \Phi_i(0) - \Phi_i(k^*). \quad (33)$$

Since the V -block agents never change strategies (by Lemmas 2 and 6), $\Phi_i(k)$ is constant for all agents in $\hat{\mathcal{V}}_{23}$. The final term in (33) is therefore equal to zero. The agents in $\hat{\mathcal{V}}_1$ each have one neighbor in $\hat{\mathcal{V}}_0$ who might change strategies (the other two neighbors are in $\hat{\mathcal{V}}_{23}$ and remain fixed). Since $n_i^A(k)$ can change by at most one for these agents, it follows that the maximum change in $\Phi_i(k)$ for such an agent is one. Therefore, we have

$$\sum_{i \in \hat{\mathcal{V}}_1} \Phi_i(0) - \Phi_i(k^*) \leq \sum_{i \in \hat{\mathcal{V}}_1} 1 = |\hat{\mathcal{V}}_1| = \sum_{i \in \mathcal{V}} m_i < \sum_{i \in \mathcal{V}} \deg_i, \quad (34)$$

since the size of the set $\hat{\mathcal{V}}_1$ is simply the total number of V -blocks (m_i for each agent), and $m_i < \deg_i$ due to (12). Next, we consider the set $\hat{\mathcal{V}}_0$ of dual agents. For a network of anti-coordinating agents at time zero, we have for each $\hat{i} \in \hat{\mathcal{V}}_0$

$$\Phi_{\hat{i}}(0) = \begin{cases} n_{\hat{i}}^A(0) - \hat{n}_{\hat{i}}^A \leq \deg_{\hat{i}} - \frac{1}{2} \deg_{\hat{i}} + 1 & \text{if } x_{\hat{i}}(0) = A \\ \hat{n}_{\hat{i}}^A + 1 - n_{\hat{i}}^A(0) \leq \frac{1}{2} \deg_{\hat{i}} + 1 - 0 & \text{if } x_{\hat{i}}(0) = B \end{cases},$$

where we used the facts that $\tau_{\hat{i}} \deg_{\hat{i}} - 1 \leq \hat{n}_{\hat{i}}^A \leq \tau_{\hat{i}} \deg_{\hat{i}}$ and that the thresholds in the augmented network $\tau_{\hat{i}}$ are all equal to $\frac{1}{2}$. Similarly, for a network of coordinating agents, we have

$$\Phi_{\hat{i}}(0) = \begin{cases} \check{n}_{\hat{i}}^A - n_{\hat{i}}^A(0) \leq \frac{1}{2} \deg_{\hat{i}} + 1 - 0 & \text{if } x_{\hat{i}}(0) = A \\ n_{\hat{i}}^A(0) - \check{n}_{\hat{i}}^A + 1 \leq \deg_{\hat{i}} - \frac{1}{2} \deg_{\hat{i}} + 1 & \text{if } x_{\hat{i}}(0) = B \end{cases},$$

since it holds that $\tau_{\hat{i}} \deg_{\hat{i}} \leq \check{n}_{\hat{i}}^A \leq \tau_{\hat{i}} \deg_{\hat{i}} + 1$. The result is the following upper bound:

$$\Phi_{\hat{i}}(0) \leq \frac{1}{2} \deg_{\hat{i}} + 1 \text{ for all } \hat{i} \in \hat{\mathcal{V}}_0. \quad (35)$$

For a network of anti-coordinating agents at equilibrium (at time k^*), we have

$$\Phi_{\hat{i}}(k^*) = \begin{cases} n_{\hat{i}}^A(k^*) - \hat{n}_{\hat{i}}^A \geq 0 - \frac{1}{2} \deg_{\hat{i}} & \text{if } x_{\hat{i}}(k^*) = A \\ \hat{n}_{\hat{i}}^A + 1 - n_{\hat{i}}^A(k^*) \geq \frac{1}{2} \deg_{\hat{i}} - \deg_{\hat{i}} & \text{if } x_{\hat{i}}(k^*) = B \end{cases}.$$

Similarly, for a network of coordinating agents, we have

$$\Phi_{\hat{i}}(k^*) = \begin{cases} \check{n}_{\hat{i}}^A - n_{\hat{i}}^A(k^*) \geq \frac{1}{2} \deg_{\hat{i}} - \deg_{\hat{i}} & \text{if } x_{\hat{i}}(k^*) = A \\ n_{\hat{i}}^A(k^*) - \check{n}_{\hat{i}}^A + 1 \geq 0 - \frac{1}{2} \deg_{\hat{i}} & \text{if } x_{\hat{i}}(k^*) = B \end{cases}.$$

This yields the following lower bound:

$$\Phi_{\hat{i}}(k^*) \geq -\frac{1}{2} \deg_{\hat{i}} \text{ for all } \hat{i} \in \hat{\mathcal{V}}_0. \quad (36)$$

Using (35) and (36), we can bound the change in potential for the dual agents as follows:

$$\sum_{\hat{i} \in \hat{\mathcal{V}}_0} \Phi_{\hat{i}}(0) - \Phi_{\hat{i}}(k^*) \leq \sum_{\hat{i} \in \hat{\mathcal{V}}_0} (\deg_{\hat{i}} + 1).$$

For each dual agent $\hat{i} \in \hat{\mathcal{V}}_0$, let i denote the corresponding original agent in \mathcal{V} . Since $\deg_{\hat{i}} = \deg_i + m_i$ and $m_i < \deg_i$ due to (12), it holds that $\deg_{\hat{i}} \leq 2 \deg_i - 1$. It follows that

$$\sum_{\hat{i} \in \hat{\mathcal{V}}_0} \Phi_{\hat{i}}(0) - \Phi_{\hat{i}}(k^*) \leq 2 \sum_{i \in \mathcal{V}} \deg_i. \quad (37)$$

Substituting (34) and (37) into (33) results in

$$\Phi(0) - \Phi(k^*) \leq 3 \sum_{i \in \mathcal{V}} \deg_i = 6|\mathcal{E}|.$$

Finally, Lemma 4 implies that the sequence of agent switches between an original and augmented network are equivalent, as long as the dual agents activate in the same sequence as the agents in the original network. This completes the proof. ■

IV. PROOF OF THEOREM 3

Proof of Theorem 3: Since the updates to $x(k+1)$ depend only on the state $x(k)$, and since agent activations do not depend on time, the network game can be modeled as a Markov chain with dimension 2^n . The state transition probabilities depend on the probabilities that each of the sets \mathcal{A}_k will occur, along with the corresponding update dynamics. To prove almost sure convergence of the network game, it suffices to show that this Markov chain is *absorbing*, which requires satisfying two conditions [3, Definition 11.1, p416]. The first condition is that there exists at least one absorbing state. Absorbing states are equivalent to Nash equilibria of the network game, whose existence we have established in Corollaries 2 and 1. The second condition is that there exists a path in the Markov chain from every non-absorbing state to an absorbing state. Theorems 2 and 1 established the existence of such paths, which consist of finite sequences of asynchronous updates. It follows from Assumption 2, i.e., agent updates are independent and have support on $\mathbb{R}_{\geq 0}$, that the probabilities of each agent being the only active agent in a given time step are strictly positive (they can be computed from the probability distributions for the inter-activation times of each agent). Therefore, both conditions are met, and the Markov chain is absorbing, which implies that the corresponding network game will almost surely reach an equilibrium state in finite time [3, Theorem 11.3, p417]. ■

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