Stabilization of a planar slow-fast system at a non-hyperbolic point

Jardón Kojakhmetov, Hildeberto; Scherpen, Jacquelien M.A.

Published in:
Proceedings of the 22nd International Symposium on the Mathematical Theory of Networks and Systems

22nd International Symposium on
Mathematical Theory of Networks and Systems
July 11-15, 2016. Minneapolis, MN, USA

Stabilization of a planar slow-fast system at a non-hyperbolic point

H. Jardón-Kojakhmetov[1] and Jacquelien M.A. Scherpen[1]

***Abstract*— In this document we study the stabilization problem of a planar slow-fast system at a non-hyperbolic point. At these type of points, the classical theory of singular perturbations is not applicable and new techniques need to be introduced in order to design a controller that stabilizes such a point. We show that using geometric desingularization (also known as blow up), it is possible to design, in a simple way, controllers that stabilize non-hyperbolic equilibrium points of slow-fast systems. Our results are exemplified on the van der Pol oscillator.**

I. INTRODUCTION

In this document we study the stabilization of a planar slow-fast system at a non-hyperbolic point of its critical manifold. By a slow-fast system (SFS), we mean a singularly perturbed ordinary differential equation of the form

$$\begin{aligned} \dot{x} &= f(x,z,\varepsilon) \\ \varepsilon \dot{z} &= g(x,z,\varepsilon), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}$, $z \in \mathbb{R}$, and f and g are assumed to be C^∞. The parameter $\varepsilon > 0$ is assumed to be small, i.e., $\varepsilon \ll 1$. Note that by this assumption z evolves much faster than x and therefore we refer to z, resp. x, as the fast, resp. slow, variable. For $\varepsilon > 0$ we can define a new time parameter τ by $\tau = {}^t/_\varepsilon$. With this new time (1) is rewritten as

$$\begin{aligned} x' &= \varepsilon f(x,z,\varepsilon) \\ z' &= g(x,z,\varepsilon), \end{aligned} \tag{2}$$

where now the prime denotes the derivative with respect to the scaled time parameter τ. Note that for $\varepsilon > 0$ and $f \neq 0$, the systems (1) and (2) are equivalent. In the limit $\varepsilon \to 0$ we have that (1) and (2) become

$$\begin{aligned} \dot{x} &= f(x,z,0) \\ 0 &= g(x,z,0), \end{aligned} \tag{3}$$

and

$$\begin{aligned} x' &= 0 \\ z' &= g(x,z,0), \end{aligned} \tag{4}$$

respectively. The system given by (3) is known as *Differential Algebraic Equation (DAE)* (or also Constrained Differential Equation (CDE) [24]) while (4) is called *the layer equation* [28]. Associated to these two systems, the following important set is defined.

Definition 1: The critical manifold is defined by

$$S = \{(x,z) \in \mathbb{R}^m \times \mathbb{R}^n \,|\, g(x,z,0) = 0\}.$$

[1]H. Jardón-Kojakhmetov and Jacquelien M.A. Scherpen are with the Engineering and Technology Institute (ENTEG), University of Groningen, Nijenborgh 4, 9747 AG Groningen, The Netherlands. {h.jardon.kojakhmetov, j.m.a.scherpen}@rug.nl

Remark 1:

- In e.g. [1] it is proved that for generic maps $g(x,z,0)$, S is indeed a smooth m-dimensional manifold.
- The critical manifold S serves as the phase space of the DAE (3) and as the set of equilibrium points of the layer equation (4).

Associated to the layer equation we now recall the definition of normal hyperbolicity.

Definition 2: Let X_ε be an ε-parameter family of smooth vector fields given by (2). Denote by S the set of equilibrium points of X_0. The manifold S is called *normally hyperbolic* if each point of S is a hyperbolic equilibrium point of X_0.

Remark 2: A hyperbolic point is also known as a singularity of index-1 in the field of DAEs [6], [20].

In the context of SFSs, the importance of normal hyperbolicity is due to [5], [7], see also [11], [12]. Briefly put, if $S_0 \subset S$ is a compact, normally hyperbolic subset of S, then there exists a manifold S_ε (the slow manifold) which is invariant under the flow of X_ε. Moreover, S_ε is diffeomorphic to S_0, lies within distance of order $O(\varepsilon)$ from S_0; and the flow of (3) restricted to S_0 provides a first approximation of the flow of X_ε along S_ε. However, these conclusions are not valid around non-normally hyperbolic points of S, and the analysis of the corresponding dynamics is much more complicated compared to the classical situation, see e.g. [4], [8], [17].

In the context of control theory, a lot of attention has been given to problems of the form

$$\begin{aligned} \dot{x} &= f(x,z,\varepsilon) + u(x,z,\varepsilon) \\ \varepsilon \dot{z} &= g(x,z,\varepsilon) + v(x,z,\varepsilon), \end{aligned}$$

where u and v are control functions and where the associated critical manifold S, of the open-loop system, is normally hyperbolic, see for example [13], [14]. Normal hyperbolicity has been the key ingredient in order to design simplified controllers in the slow and fast time scales, some examples are given in [3], [15], [21], [23], [29]. Less attention has been given to the situation where S is not normally hyperbolic, especially in the nonlinear case. At non-hyperbolic points, the dynamics of a certain system may change drastically via jumps. This behavior is interesting as it is present in many phenomena [2], [8], [16], [30], [22], [25], [26], [27], however it is difficult to analyze.

In this document we investigate the stabilization problem of a SFS with two novel features: 1) The stabilization problem is developed at a non-hyperbolic point; in other words, we do not make the classical assumption that (1) satisfies $\frac{\partial g}{\partial z}(0) \neq 0$. In this sense we give the first steps towards an extension of the theory of singular perturbations in control systems. 2) The critical manifold S (see Definition 1) is left invariant. In practical terms, this means that the controller to be designed does not modify the overall behavior of the system, like rapid transitions between stable states or the dynamics along normally hyperbolic parts of the critical manifold, see Section V.

II. Setting of the problem

In the rest of this document we study the stabilization problem at the origin of the planar SFS

$$\begin{aligned} x' &= \varepsilon(Ax + Bz + u(x,z,\varepsilon)) \\ z' &= -(z^2 + x). \end{aligned} \tag{5}$$

where $A \in \mathbb{R}$ and $B \in \mathbb{R}$. The motivation behind studying (5) is that it is one of the simplest systems to have a non-hyperbolic point (at the origin) but yet it has linear slow dynamics. Note the absence of control signal in the equation of z'. The associated critical manifold is given by $S = \left\{(x,z) \in \mathbb{R}^2 \,|\, x = z^2\right\}$. To avoid working with an ε-family of vector fields as (5), it is customary [4], [19], [18] to incorporate the trivial equation $\varepsilon' = 0$ and then consider the three-dimensional vector field

$$X : \begin{cases} x' &= \varepsilon(Ax + Bz + u(x,z,\varepsilon)) \\ z' &= -(z^2 + x) \\ \varepsilon' &= 0. \end{cases} \tag{6}$$

Note that the origin is a *nilpotent singularity* of (6).

Remark 3:

- Any compact subset $S_0 \subset S$ around the origin is not normally hyperbolic.
- The control problem (6) has the important characteristic of leaving the critical manifold S *invariant*. Note a linear feedback $v = -z$ could be proposed so that the closed loop system is of the form

$$\begin{aligned} \dot{x} &= Ax + Bz + u(x,z,\varepsilon) \\ \varepsilon\dot{z} &= -(z^2 - x) - z. \end{aligned}$$

  In this way the 'closed-loop critical manifold' would be normally hyperbolic in a compact neighborhood of the origin. Hence, classical techniques could be used to design a controller u. However in such a case the topological properties of the critical manifold are lost. More precisely, a jump at the origin (due to non-hyperbolicity) would disappear due to the action of the controller. Therefore, we emphasize that a novelty of our approach is to propose a controller that does not change S.
- The main goal of our contribution is to extend the theory of singular perturbations for control systems to non-hyperbolic points. An important ingredient in this process is the geometric desingularization technique, see Section III-B.

III. Preliminaries

In this section we present the open loop dynamics of the problem of interest and point-out the main properties of the geometric desingularization technique.

A. The open-loop dynamics

First of all, note that the slow manifold S is a parabola as depicted in Figure 1.

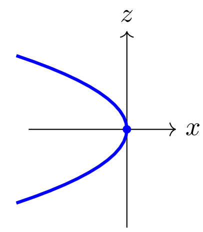


Fig. 1: The critical manifold $S = \left\{(x,z) \in \mathbb{R}^2 \,|\, x = -z^2\right\}$. The origin is also called the fold point.

The corresponding DAE and layer equations related to (6), for $u = 0$, are given by

$$\begin{aligned} \dot{x} &= Ax + Bz \\ 0 &= -(z^2 + x), \end{aligned} \tag{7}$$

and

$$\begin{aligned} x' &= 0 \\ z' &= -(z^2 + x), \end{aligned} \tag{8}$$

respectively.

Remark 4: Our analysis is of local nature. Therefore we assume that A and B are suitably chosen constants such that in a sufficiently large neighborhood U of the origin, the fold point $(x,y) = (0,0)$ is the only singularity of the vector field $x' = Ax + Bz$.

By a simple analysis it can be shown that the local phase portraits (in a small neighborhood of the origin) of (7) and (8) are as depicted in Figure 2.

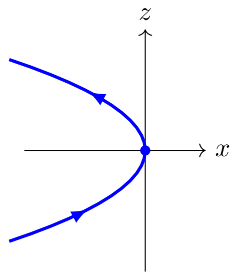
B. Geometric desingularization

In order to design the controller u of (6) we propose to use the *geometric desingularization* or blow up method. This technique was introduced in the context of SFSs in [4] (see also [18]). However, to the authors' best knowledge, geometric desingularization has not been used to design controllers of singularly perturbed control systems around non-hyperbolic points before.

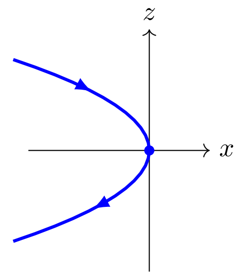
Briefly speaking, geometric desingularization is a well suited change of coordinates under which the non-hyperbolic singularity (the fold point) of (6) is simplified. By this we mean that after the coordinate transformation, the new singularities of the induced vector field are hyperbolic or semi-hyperbolic. Such a change of coordinates is of the form

$$x = r^{\alpha_1}\bar{x}, \; z = r^{\alpha_2}\bar{z}, \; \varepsilon = r^{\alpha_3}\bar{\varepsilon}, \tag{9}$$

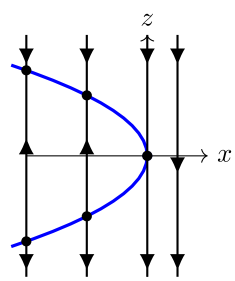
where $(\bar{x},\bar{z},\bar{\varepsilon}) \in \mathbb{S}^2$ and $r \in [0,\infty)$, and where $\alpha_1, \alpha_2, \alpha_3$ are suitable positive integers depending on the vector field. Since we have assumed that $\varepsilon > 0$, we may also assume that $\bar{\varepsilon} \in [0,\infty)$. Let $\Phi : \mathbb{S}^2 \times [0,\infty) \to \mathbb{R}^3$ denote the blow up map (9). Note that Φ maps the the sphere $\mathbb{S}^2 \times \{0\}$ to the origin of $\mathbb{R}^3$. Moreover, the map Φ induces a vector field $\tilde{X}$ defined by $\Phi_*\tilde{X} = X$ (where X is given by (6)). It



(a) Phase portrait of (7) for $B < 0$



(b) Phase portrait of (7) for $B > 0$



(c) Phase portrait of (8).

Fig. 2: Phase portraits of (7) and (8) in a sufficiently small neighborhood of the origin.

may happen that $\tilde{X}$ is degenerate along $\mathbb{S}^2 \times \{0\}$ in which case one defines a new vector field $\bar{X}$ by $\bar{X} = {}^1\!/\!_{r^m}\tilde{X}$ for a suitable integer m such that $\bar{X}$ is not degenerate at $\mathbb{S}^2 \times \{0\}$. In this way, the dynamics of $\tilde{X}$ and $\bar{X}$ are equivalent outside $\mathbb{S}^2 \times \{0\}$ and thus it is equally useful to study $\bar{X}$. One then obtains a complete description of the dynamics of X around the origin by studying $\bar{X}$ around $\mathbb{S}^2 \times [0, r_0)$ for $r_0 > 0$.

When studying SFSs of dimensions greater than 2 it is more convenient to use *charts* [2], [4], [8], [17], [18]. A chart is a parametrization of distinct hemispheres of $\mathbb{S}^2 \times [0, r_0)$. More precisely in our particular problem, the charts are defined by

$$K_{\pm\bar{x}} = \{\bar{x} = \pm 1\}\,,\ K_{\pm\bar{z}} = \{\bar{z} = \pm 1\}\,,$$
$$K_{\bar{\varepsilon}} = \{\bar{\varepsilon} = 1\}\,.$$

We show in the following section that a controller designed for the blown up vector field $\bar{X}$ induces a controller for X. Moreover, the closed-loop characteristics of $\bar{X}$ are carried over X.

IV. Controller design via geometric desingularization

For the specific problem given by (6), the blow up map reads as

$$x = r^2\bar{x},\ z = r\bar{z},\ \varepsilon = r^3\bar{\varepsilon}. \tag{10}$$

Next, the most important chart to consider is $K_{\bar{\varepsilon}}$ since in this chart we desingularize the singular behavior induced by the parameter ε. Moreover, the dynamics in $K_{\bar{\varepsilon}}$ are equivalent to the dynamics of (10) in a small neighborhood U_ε of the origin of size $O(\varepsilon^{2/3}) \times O(\varepsilon^{1/3})$.

Remark 5: The analysis of the remaining charts ($K_{\pm\bar{x}}$ and $K_{\pm\bar{z}}$) is non-trivial and may provide insightful information on the dynamics of (6) near the origin.

A. Analysis in the chart $K_{\bar{\varepsilon}}$

In this chart the blow up map is given by

$$x = r^2\bar{x},\ z = r\bar{z}, \varepsilon = r^3. \tag{11}$$

The corresponding blown up vector field $\bar{X}$ reads as

$$\bar{X}: \begin{cases} r' &= 0 \\ \bar{x}' &= Ar^2\bar{x} + Br\bar{z} + \bar{u}(\bar{x}, \bar{z}, r) \\ \bar{z}' &= -(\bar{z}^2 + \bar{x}), \end{cases} \tag{12}$$

which is obtained after rescaling time by a factor of r and where the prime denotes time derivative with respect to this re-scaled time. Furthermore, $\bar{u}$ denotes the transformation of u under the blow up map (11) that is $\bar{u}(\bar{x}, \bar{z}, r) = u(r^2\bar{x}, r\bar{z}, r^3)$.

Theorem 1: Consider the 'blown up' control problem (12). Let the controller $\bar{u}$ be given by $\bar{u} = -Ar^2\bar{x} - Br\bar{z} + \alpha\bar{x} + \beta\bar{z}$ with $\alpha < 0,\ \beta > 0$. Then, the origin is a locally asymptotically stable equilibrium point of the closed-loop system.

Proof: The closed loop dynamics of (12) given by the controller $\bar{u} = -Ar^2\bar{x} - Br\bar{z} + \alpha\bar{x} + \beta\bar{z}$ are uniform in r and read as

$$\bar{X}_{cl}: \begin{cases} r' &= 0 \\ \bar{x}' &= \alpha\bar{x} + \beta\bar{z} \\ \bar{z}' &= -(\bar{z}^2 + \bar{x}). \end{cases} \tag{13}$$

It is easy to verify that the eigenvalues of the Jacobian $D\bar{X}_{cl}(0)$ are $\lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}$. It follows from classical stability arguments that $\alpha < 0,\ \beta > 0$ are necessary and sufficient conditions to make the origin locally asymptotically stable. ■

The controller designed in Theorem 1 provides necessary and sufficient conditions for local asymptotic stability of the origin. For didactic purposes let us choose α and β in such a way that the origin has a pair of complex-conjugated stable eigenvalues[1]. Thus, let us choose $\alpha < -K < 0$, with $K > 0$, and $\alpha^2 - 4\beta < -Q < 0$ with $Q > 0$. Next, note that the closed-loop system has another equilibrium point $p' = \left(-\left(\frac{\beta}{\alpha}\right)^2, \frac{\beta}{\alpha}\right)$. We want to place this secondary equilibrium point sufficiently away from the origin and therefore let us further choose $\beta > -\alpha$, compare with Remark 4. The phase portrait of (13) is shown in Figure 3.

B. Region of attraction

It is interesting to see the qualitative properties of the region of attraction of the origin in the closed loop system (13). For this we study the local properties of the equilibrium point $p' = \left(-\left(\frac{\beta}{\alpha}\right)^2, \frac{\beta}{\alpha}\right)$.

[1]The case where the origin has a pair of purely real stable eigenvalues is completely similar to the one presented here.

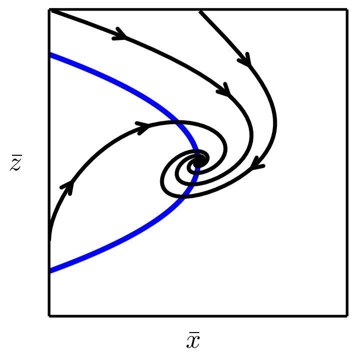


Fig. 3: Phase portrait of (13) for $\alpha = -1$, $\beta = 2$.

Proposition 1: The equilibrium point p' is a saddle point with eigenvalues $\lambda_{1,2} = \frac{-\rho \pm \sqrt{\rho^2+4\beta}}{2}$, where $\rho = \frac{2\beta}{\alpha} - \alpha$. The stable ($E^s$) and unstable ($E^u$) eigenspaces are given by

$$E^s = \left\{ \begin{bmatrix} v^- \\ 1 \end{bmatrix} \right\}, \qquad E^u = \left\{ \begin{bmatrix} v^+ \\ 1 \end{bmatrix} \right\},$$

where $v^{\pm} = -\frac{\alpha}{2} - \frac{\beta}{\alpha} \mp \frac{\sqrt{\frac{4\beta^2}{\alpha^2}+\alpha^2}}{2}$. Moreover we have $0 < v^+ < v^-$.

Proof: The result follows from standard linear analysis at the equilibrium point p' and the assumption that $\frac{\beta}{\alpha} < -1$. ■

It follows from Proposition 1 that there exist 1-dimensional stable ($W^s(p')$) and unstable ($W^u(p')$) invariant manifolds intersecting at p'.

Let $\bar{S}$ denote the manifold $\bar{S} = \left\{ \bar{z}^2 + \bar{x} = 0 \right\}$. We have that $W^s(p')$ intersects transversally $\bar{S}$ as shown by the following Lemma.

Lemma 1: Let $s > 0$ denote the slope of the tangent line of the manifold $\bar{S}$ at p'. Then $\frac{1}{v^-} < s$.

Proof: First, it is straightforward to show that the slope s is given by $s = -\frac{1}{2}\frac{\alpha}{\beta}$. On the other hand, the slope of $W^s(p')$ at p' is $\frac{1}{v^-}$. Next, recall that $\frac{\beta}{\alpha} < -1$ and note that

$$\begin{aligned} v^- &= \underbrace{-\frac{\alpha}{2} - \frac{\beta}{\alpha}}_{>0} + \frac{\sqrt{\frac{4\beta^2}{\alpha^2} + \alpha^2}}{2} \\ &= \sqrt{\frac{\alpha^2}{4} + \frac{\beta^2}{\alpha^2} + \beta} + \sqrt{\frac{\alpha^2}{4} + \frac{\beta^2}{\alpha^2}} > -2\frac{\beta}{\alpha} > 1. \end{aligned}$$

The proof is concluded by noting that $\frac{1}{v^-} < -\frac{1}{2}\frac{\alpha}{\beta} = s$. ■

From the results of this section it follows that the region of attraction of the origin is bounded by the stable manifold $W^s(p')$ as shown in Figure 4.

C. The induced controller

From the blow up map (11), it follows that the corresponding controller u obtained from $\bar{u}$ is $u = \bar{u} \circ \Phi^{-1}$. Therefore, due to Theorem 1, the induced controller in coordinates (x, z, ε) is given by

$$u = -Ax - Bz + \alpha\varepsilon^{-2/3}x + \beta\varepsilon^{-1/3}z.$$

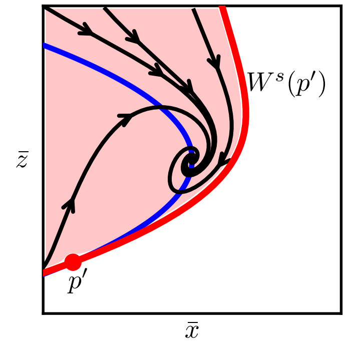


Fig. 4: Region of attraction of the closed-loop system $\bar{X}_{cl}$.

The rational powers in the controller are important and depend on the blow up map (11). Note that εu is well defined in the limit $\varepsilon \to 0$. Moreover, the closed loop system (6) reads as

$$X : \begin{cases} x' &= \alpha\varepsilon^{1/3}x + \beta\varepsilon^{2/3}z \\ z' &= -(z^2 + x) \\ \varepsilon' &= 0. \end{cases} \tag{14}$$

The corresponding phase portrait of (14) is shown in Figure 5

Remark 6 (On Lyapunov functions): Even though we used the direct Lyapunov Method to design the controller, recall that Lyapunov functions are invariant under change of coordinates. In fact, let X be a smooth vector field on a manifold M and $\Phi : N \to M$ a blow up map. Let $\bar{X}$ be the induced blown up vector field on N defined by $\Phi_*\bar{X} = X$ (up to equivalence). Let W be a Lyapunov function for the vector field $\bar{X}$. Let $V = W \circ \Phi^{-1}$. Let $\bar{\zeta}$ and ζ be local coordinates on the manifolds N and M respectively. By definition, the Lyapunov function W satisfies

- $W(\bar{\zeta}^*) = 0$
- $W(\bar{\zeta}) > 0, \ \forall \bar{\zeta} \in \bar{U} \setminus \{0\}$
- $W'(\bar{\zeta}) \leq 0, \ \forall \bar{\zeta} \in \bar{U}$

where $\bar{U}$ is some neighborhood of $\bar{\zeta}^* = \Phi^{-1}(0)$. It follows that

- $V(0) = W \circ \Phi^{-1}(0) = 0$

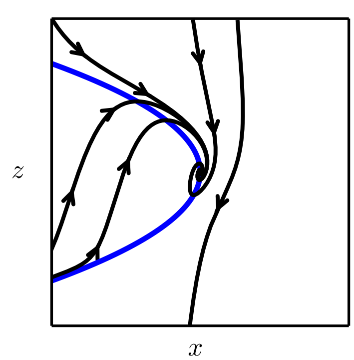


Fig. 5: Phase portrait of (14) for $\alpha = -1$, $\beta = 2$, $\varepsilon = 0.05$.

- $V(\zeta) = W \circ \Phi^{-1}(\zeta) > 0,\ \forall \zeta \in U \backslash \{0\}$
- $V'(\zeta) = \frac{d}{dt}\left(W \circ \Phi^{-1}(\zeta)\right) \leq 0,\ \forall \zeta \in U,$

where U is a neighborhood of $0 \in M$ defined by $U = \Phi(\bar{U})$. The last equality is true since the blow up map restricted to U has positive definite Jacobian. Note that the same conclusion holds for asymptotic stability, i.e., for $W'(\bar{\zeta}) < 0$.

D. *The induced region of attraction*

Let us denote by $\bar{U}$ the region of attraction found in Section IV-B, see Figure 4. Following the arguments of Section IV-C we have that $\bar{U}$ is also mapped (via the blow-up map (11)) to a region U of attraction in the original coordinates (x, z), that is $U = \Phi(\bar{U})$. This induced region depends on ε and has a well defined limit as $\varepsilon \to 0$. Just as in Section IV-B, it is bounded by the stable manifold of the induced equilibrium point $p = \Phi(p')$. The corresponding region of attraction and its limit as $\varepsilon \to 0$ are shown in Figure 6.

Remark 7: The regions of attraction $\bar{U}$ and U are topologically equivalent. Moreover, they are diffeomorphic for $\varepsilon > 0$. The difference on their shape is due to the dependence of U on ε.

V. Application: Trigger control of the van der Pol Oscillator

Let us consider the van der Pol oscillator given by

$$\begin{aligned} \dot{x} &= z - a + u \\ \varepsilon \dot{z} &= -(z^3 - z + x), \end{aligned} \tag{15}$$

where $a \in \mathbb{R}$ is a parameter that defines the position of the equilibrium point of the slow dynamics. For simplicity let $a = 0$, in this way there is no equilibrium point along the stable branch of the slow manifold $S = \left\{z^3 - z + x = 0\right\}$. In turn, there exists a unique stable limit cycle as shown in Figure 7.

By using geometric desingularization we want to design a controller that stabilizes one of the fold points, in particular

$$p = (x^*, z^*) = \left(\left(\frac{4}{27}\right)^{1/2}, \left(\frac{1}{3}\right)^{1/2}\right).$$

Moreover, we shall provide a trigger signal that, together with the controller, decides when the system oscillates.

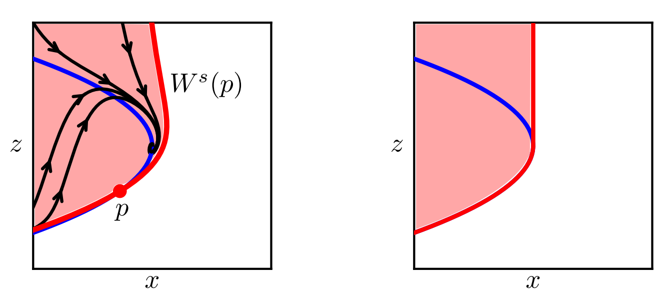


Fig. 6: Left: Region of attraction U of the closed-loop system (14). The point p is given by $p = \Phi(p')$, compare with Figure 4. Right: limit of the region of attraction as $\varepsilon \to 0$

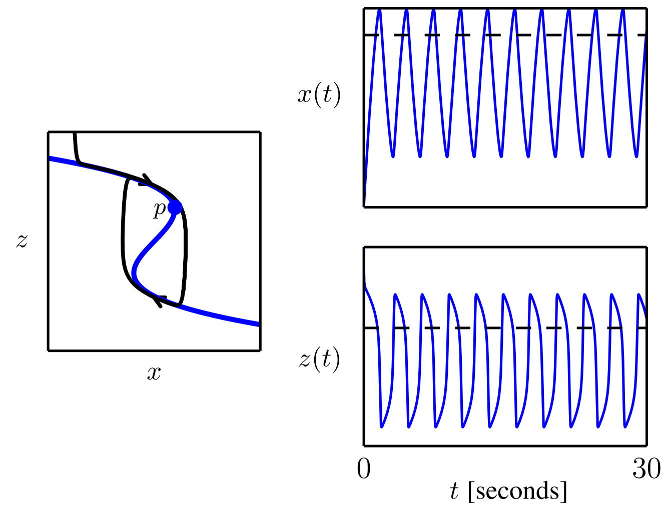


Fig. 7: Left: Phase portrait of the open-loop dynamics of (15). Right: Signals $x(t)$ and $z(t)$, the dashed line represents the values of the fold point p.

Proposition 2: Consider the van der Pol oscillator (15). The controller

$$u = -z + \alpha\varepsilon^{-2/3}\left(x - (4/27)^{1/2}\right) + \beta\varepsilon^{-1/3}\left(z - (1/3)^{1/2}\right),$$

with $\alpha < 0$ and $\beta > 0$ makes the fold point p locally asymptotically stable.

Proof: The proof follows from the exposition of Section IV, so let us provide only a sketch. The proof can be divided in three steps: 1) Move the origin to the singular point $p = (x^*, z^*)$; in this way, the local system is of the form studied above. 2) Design the controller following Section IV. 3) Return to the original coordinates. ∎

In Figure (8) we show a simulation of Proposition 2. The controller u is applied at certain intervals to allow the trajectories reach the equilibrium point. After a while the controller is turned off to allow a rapid transition to the lower stable branch of the critical manifold. Then the trajectories converge again to the fold point p.

VI. Conclusions and final remarks

In this document we have studied the stabilization problem of a planar SFS at a non-hyperbolic point. We have applied the technique called *geometric desingularization*. Several advantages are carried from this method:

- The control problem of a SFS at a non-hyperbolic point (where classical techniques do not apply) is translated to the control problem of a non linear vector field via geometric desingularization.
- The local stability properties of the blown up system are equivalent to those of the original (slow-fast) system.
- Although we have studied the planar case, it is evident from our analysis that the results are immediately applicable to slow-fast control systems of the form

$$X_\varepsilon : \begin{cases} x' &= \varepsilon(L(x,z) + u(x,z,\varepsilon)) \\ z' &= -(z^2 + x_1), \end{cases}$$

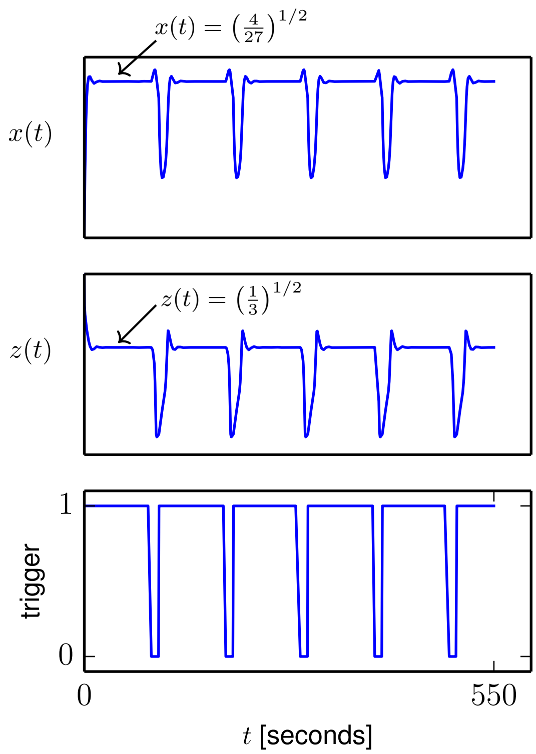


Fig. 8: Top and middle: the corresponding $x(t)$ and $z(t)$ signals of (15). Bottom: the trigger signal for the controller u. Note that when the controller is active, the trajectories $(x(t), z(t))$ converge to the non-hyperbolic equilibrium point p. However, when the controller is off, we allow a fast transition towards the lower branch of S.

where $x \in \mathbb{R}^m$, $z \in \mathbb{R}$, $L : \mathbb{R}^{m+1} \to \mathbb{R}^m$ is *a linear map* and $u : \mathbb{R}^{m+2} \to \mathbb{R}^m$.

- The controller design is not only valid at non-hyperbolic points but also within arbitrarily small neighborhoods of such points.
- Remark 6 suggests that Lyapunov based controllers are also applicable to SFS even at non-hyperbolic points, see also [10].
- The analysis of more general slow-fast control systems near non-hyperbolic points is an ongoing research topic. A general treatment and results shall appear in [9].

References

[1] John. M. Boardman. Singularities of differentiable maps. *Publications Mathématiques de l'IHÉS*, 33, pages 21–57.

[2] Henk W. Broer, Tasso J. Kaper, and Martin Krupa. Geometric Desingularization of a Cusp Singularity in SlowâĂŞFast Systems with Applications to Zeeman's Examples. *Journal of Dynamics and Differential Equations*, 25(4), pages 925–958, aug 2013.

[3] Alessandro De Luca. Flexible Robots. In John Baillieul and Tariq Samad, editors, *Encyclopedia of Systems and Control*. Springer-Verlag London, 2014.

[4] Freddy Dumortier and Robert H. Roussarie. *Canard Cycles and Center Manifolds*, volume 121. American Mathematical Society, 1996.

[5] N. Fenichel. Geometric singular perturbation theory. *JDE*, pages 53–98, 1979.

[6] C W Gear. Differential-Algebraic Equations. In Edward J Haug, editor, *Computer Aided Analysis and Optimization of Mechanical System Dynamics*, volume 9 of *NATO ASI Series*, pages 323–334. Springer Berlin Heidelberg, 1984.

[7] M W Hirsch, C C Pugh, and M Shub. Invariant manifolds. *Bulletin of the American Mathematical Society*, 76(5), pages 1015–1019, 1970.

[8] H. Jardón-Kojakhmetov. *Classification of constrained differential equations embedded in the theory of slow-fast systems*. PhD Thesis, University of Groningen, 2015.

[9] H. Jardón-Kojakhmetov and Jacquelien M. A. Scherpen. Stabilization of slow-fast systems at non-hyperbolic points. *In preparation*, 2016.

[10] H. Jardón-Kojakhmetov, Jacquelien M. A. Scherpen and D. del Puerto-Flores. Nonlinear adaptive stabilization of a planar slow-fast system at a non-hyperbolic point. *submitted to CDC*, 2016.

[11] C.K.R.T Jones. Geometric singular perturbation theory. In *Dynamical Systems*, LNM 1609, pages 44–120. Springer-Verlag, 1995.

[12] Tasso J. Kaper. An Introduction to Geometric Methods and Dynamical Systems Theory for Singular Peturbation Problems. In *Symposia in Applied Mathematics*, volume 56, pages 85–131. AMS, 1999.

[13] Petar V. Kokotovic. Applications of Singular Perturbation Techniques to Control Problems. *SIAM Review*, 26(4), pages 501–550, 1984.

[14] Peter V. Kokotovic, John O'Reilly, and Hassan K. Khalil. *Singular Perturbation Methods in Control: Analysis and Design*. Academic Press, Inc., Orlando, FL, USA, 1986.

[15] P.V. Kokotovic, R.E. O'Malley, and P. Sannuti. Singular perturbations and order reduction in control theory: An overview. *Automatica*, 12(2), pages 123–132, 1976.

[16] Ilona Kosiuk and Peter Szmolyan. Scaling in Singular Perturbation Problems: Blowing Up a Relaxation Oscillator. *{SIAM} J. Applied Dynamical Systems*, 10(4), pages 1307–1343, 2011.

[17] M. Krupa and P. Szmolyan. Extending geometric singular perturbation theory to non hyperbolic points: fold and canard points in two dimensions. *SIAM J. Math. Anal.*, 33, pages 286–314, 2001.

[18] M. Krupa and P. Szmolyan. Geometric analysis of the singularly perturbed planar fold. In *Multiple-Time.Scale Dynamical Systems*, LNM 1609, pages 89–116. Springer-Verlag, 2001.

[19] Martin Krupa and Martin Wechselberger. Local analysis near a folded saddle-node singularity. *Journal of Differential Equations*, 248(12), pages 2841–2888, 2010.

[20] Werner C Rheinboldt. Differential-Algebraic Systems as Differential Equations on Manifolds. *Mathematics of Computation*, 43(168), pages 473–482, oct 1984.

[21] V.R. Saksena, J. O'Reilly, and P.V. Kokotovic. Singular perturbations and time-scale methods in control theory: Survey 1976–1983. *Automatica*, 20(3), pages 273–293, 1984.

[22] S. Sastry and C. Desoer. Jump behavior of circuits and systems. *IEEE Transactions on Circuits and Systems*, 28(12), pages 1109–1124, dec 1981.

[23] M. W. Spong. Modeling and Control of Elastic Joint Robots. *Journal of Dynamic Systems, Measurement, and Control*, 109(4):310, 1987.

[24] F Takens. Constrained Equations: a Study of Implicit Differential Equations and their Discontinuous Solutions. In *Structural Stability, the Theory of Catastrophes, and Applications in the Sciences*, LNM 525, pages 134–234. Springer-Verlag, 1976.

[25] Tina Thiessen, Martin Gutschke, Philipp Blanke, Wolfgang Mathis, and Franz-erich Wolter. Differential Geometric Methods for Jump Effects in Nonlinear Circuits. Number 1, pages 1–4, 2012.

[26] Tina Thiessen and Wolfgang Mathis. Geometrical interpretation of jump phenomena in nonlinear dynamical circuits. In *Proceedings of the Joint INDS'11 & ISTET'11*, pages 1–5. IEEE, 2011.

[27] Tina Thiessen, Michael Popp, Christoph Zorn, and Wolfgang Mathis. Generalization of the jump postulate and Brayton-Moser's mixed potential for the analysis of RTD circuits. *International Journal of Circuit Theory and Applications*, 2015.

[28] Ferdinand Verhulst. *Methods and Applications of Singular Perturbations*, volume 50 of *Texts in Applied Mathematics*. Springer, 2005.

[29] Valery D. Yurkevich. A unified approach to two-time scale control systems design: a tutorial. In *2nd IASTED Int. Multi-Conference Automation, Control and Applications*, pages 314–319, 2005.

[30] E C Zeeman. Differential equations for the heart beat and nerve impulse. In *Towards a theoretical biology*, volume 4, pages 8–67. Edinburgh University Press, 1975.