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# Dichotomic differential inequalities and multi-agent coordination

Anton V. Proskurnikov and Ming Cao

**Abstract**—Distributed algorithms of multi-agent coordination have attracted substantial attention from the research communities. The most investigated are Laplacian-type dynamics over time-varying weighted graphs, whose applications include, but are not limited to, the problems of consensus, opinion dynamics, aggregation and containment control, target surrounding and distributed optimization. While the algorithms solving these problems are similar, for their analysis different mathematical techniques have been used. In this paper, we propose a novel approach, allowing to prove the stability of many Laplacian-type algorithms, arising in multi-agent coordination problems, in a unified elegant way. The key idea of this approach is to consider an associated *linear differential inequality* with the Laplacian matrix, satisfied by some bounded outputs of the agents (e.g. the distances to the desired set in aggregation and containment control problems). Although such inequalities have many unbounded solutions, under natural connectivity conditions all their *bounded* solutions converge (and even reach consensus), entailing the convergence of the original protocol. The differential inequality thus admits only convergent but not “oscillatory” *bounded* solutions. This property, referred to as the *dichotomy*, has been long studied in the theory of differential equations. We show that a number of recent results from multi-agent control can be derived from the dichotomy criteria for Laplacian differential inequalities, developed in this paper, discarding also some technical restrictions.

## I. INTRODUCTION

Problems of distributed multi-agent coordination, achieved via local interaction among the agents, have attracted substantial attention from the research community. Besides giving insight into many natural phenomena, multi-agent control has found numerous applications in mobile robotics, sensor networks, social network analysis and distributed optimization; some relevant applications and main historical milestones can be found in [1]–[3] and references therein.

Problems of *consensus* and *synchronization*, where the agents aim to agree on some quantity of interest or synchronize some outputs, are studied now up to a certain exhaustiveness. The simplest consensus algorithms date back to the iterative procedures of decision making [4]. A group of agents, modeled by the continuous-time single integrators and applying such a protocol, obeys the linear system

$$\dot{x}(t) = -L(t)x(t),$$

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where  $x(t)$  stands for the vector of agents’ states  $x_i(t) \in \mathbb{R}$  and  $L(t)$  is the *Laplacian matrix* [1], [2] of their interaction graph (in general, directed and weighted). This protocol and its discrete-time counterpart has been thoroughly studied by using Lyapunov methods [5]–[7] and matrix analysis [1], [2], [8], paying special attention to the effects of time-varying interaction topologies. For bidirectional and “cut-balanced” graphs necessary and sufficient consensus criteria have been obtained [9]–[11]. At the same time, for general directed graph convergence to consensus is proved under the sufficient (but not necessary) assumption of “uniform quasi-strong connectivity (UQSC)” [5]–[7].

In this paper, we are interested in consensus properties of differential *inequalities*, also employing the graph Laplacian

$$\dot{x}(t) \leq -L(t)x(t) \quad \text{or} \quad \dot{x}(t) \geq -L(t)x(t).$$

where the inequality is satisfied elementwise. It may seem surprising that one-sided inequalities can imply consensus; however, this can be proved for any *bounded* solution under modest connectivity assumptions. Hence any solution of the inequality either grows unbounded or converges to a finite limit. This property, referred to as the system’s *dichotomy*, has been thoroughly studied in the theory of differential equations [12], being intimately related with *absolute stability* of nonlinear control systems [13].

As will be shown, the dichotomy property of the Laplacian differential inequalities allows to prove convergence of many distributed protocols, similar in structure to consensus dynamics, however, providing more complicated behaviors of the agents. Among them are the models of opinion polarization [11], [14], containment control [3] and target aggregation [15] algorithms, protocols over complex-weighted graphs for target surrounding [16], [17] and some recent algorithms on distributed optimization [18]. Dichotomy criteria, elaborated in this paper, give a unified and elegant way to derive these mentioned results, whose independent proofs are different and non-trivial, relaxing or discarding also some technical assumptions such as e.g. the existence of a positive dwell-time between consecutive switches of the matrix  $L(t)$ .

## II. PRELIMINARIES

In this section, we recall some concepts and results important for the future considerations. Henceforth  $\underline{N}$  stands for the set  $\{1, \dots, N\}$  and  $\mathbf{1}_N \triangleq (1, 1, \dots, 1)^\top \in \mathbb{R}^N$ . Given two vectors  $x, y \in \mathbb{R}^N$ , we write  $x \leq y$  (respectively  $x < y$ ) if  $x_i \leq y_i \forall i$  (respectively,  $x_i < y_i$ ). Given a vector  $x \in \mathbb{R}^N$ ,  $|x| \triangleq \sqrt{x^\top x}$  denotes its Euclidean norm. The sign of a real number  $x \in \mathbb{R}$  is denoted by  $\text{sgn } x \in \{\pm 1, 0\}$ . Given a complex number  $z \in \mathbb{C}$ ,  $\bar{z}$  stands for its conjugate.

### A. Weighted graphs and Laplacian matrices

A *weighted* (directed) graph is a triple  $G = (\mathcal{V}, E, A)$ , where  $\mathcal{V} = \{v_1, \dots, v_N\}$  stands for the set of *nodes*,  $E \subset \mathcal{V} \times \mathcal{V}$  is a set of (directed) *arcs* and  $A = (a_{jk}) \in \mathbb{R}^{N \times N}$  stands for a non-negative *adjacency matrix*, whose positive entries  $a_{jk} > 0$  correspond to arcs  $(v_k, v_j) \in E$ . We assume that the set of nodes  $\mathcal{V}$  is fixed and hence the graph  $G = G[A]$  is uniquely defined by its adjacency matrix.

Given a weighted graph  $G[A]$ , its *Laplacian matrix* is

$$L[A] = (l_{ij})_{i,j=1}^N, \quad l_{ij} \triangleq \begin{cases} -a_{ij}, & i \neq j \\ \sum_{j \neq i} a_{ij}, & i = j. \end{cases} \quad (1)$$

A *walk* connecting nodes  $v$  and  $v'$  is a sequence of nodes  $v_{i_0} := v, v_{i_1}, \dots, v_{i_{n-1}}, v_{i_n} := v'$  ( $n \geq 1$ ) such that  $(v_{i_{k-1}}, v_{i_k}) \in E$  for  $k \in 1 : n$ . A walk where  $v_{i_0} = v_{i_n}$  is referred to as a *cycle*. A node is called a *root* if a walk from it to any other node exists; a graph is *strongly connected* (SC) if any of its nodes is a root and *quasi-strongly connected* (QSC) or *rooted* if at least one root exists. The QSC condition is equivalent to the existence of a (directed) *spanning tree*.

We also use a concept of  $\delta$ -connectivity. Given an adjacency matrix  $A = (a_{ij})$  and  $\delta > 0$ , define its “truncation”  $A^\delta = (a_{ij}^\delta)$  as follows:  $a_{ij}^\delta = a_{ij}$  if  $a_{ij} \geq \delta$  and otherwise  $a_{ij}^\delta = 0$ . The graph  $G[A]$  is strongly (quasi-strongly)  $\delta$ -connected if its subgraph  $G[A^\delta]$ , obtained by removing “light” arcs, is SC (respectively, QSC).

Consider now a *time-varying* adjacency matrix  $A(t) = (a_{ij}(t))$ , defined for  $t \geq 0$ . The graph  $G \left[ \int_{t_0}^{t_1} A(s) ds \right]$  can be treated as the infinite union of the graphs  $G[A(t)]$  over the interval  $[t_0; t_1]$ . We call the time-varying graph  $G[A(\cdot)]$  *uniformly strongly connected* (USC) if a period  $T > 0$  and a threshold  $\delta > 0$  exist such that any union of the graphs  $G \left[ \int_t^{t+T} A(s) ds \right]$  (where  $t \geq 0$ ) over the period  $T$  is strongly  $\delta$ -connected. The *uniform quasi-strong connectivity* (UQSC) is defined similarly:  $G \left[ \int_t^{t+T} A(s) ds \right]$  is quasi-strongly  $\delta$ -connected for any  $t \geq 0$ . Notice that these definitions do not assume the piecewise-continuity of the matrix  $A(t)$ , whose entries  $a_{ij}(t) \geq 0$  can be arbitrary Lebesgue measurable functions. A USC graph is a special case of *infinitely strongly connected* (ISC) graphs. The ISC condition (referred to also as the *essential connectivity* [10], [11]) requires the graph  $G_\infty = (\mathcal{V}, E_\infty)$ , where  $E_\infty = \{(i, j) \in \mathcal{V} : \int_0^\infty a_{ij}(t) dt = \infty\}$ , to be strongly connected.

Following [9], we call a time-varying graph (uniformly) *cut-balanced*, if a constant  $K \geq 1$  exists such that

$$K^{-1} \sum_{i \in \mathcal{V}_1, j \in \mathcal{V}_2} a_{ij}(t) \leq \sum_{i \in \mathcal{V}_1, j \in \mathcal{V}_2} a_{ji}(t) \leq K \sum_{i \in \mathcal{V}_1, j \in \mathcal{V}_2} a_{ij}(t)$$

for any  $t \geq 0$  and any partition  $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ , where  $\mathcal{V}_1, \mathcal{V}_2 \neq \emptyset$  and  $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ . Special cases of cut-balanced graphs are weight-balanced graphs ( $\sum_{k=1}^N a_{jk} = \sum_{k=1}^N a_{kj} \forall j$ ) and undirected graphs (which means that  $A = A^T$ ).

### B. Consensus protocols and their convergence

The graph Laplacians are closely related to dynamical systems over graphs, whose mathematical models are known as *consensus* (or agreement) protocols [1]–[3]. Given a time-varying graph  $G(t) = (\mathcal{V}, E(t), A(t))$ , the associated consensus dynamics is given by

$$\dot{x}(t) = -L[A(t)]x(t) \in \mathbb{R}^N, \quad t \geq 0. \quad (2)$$

The system (2) obeys the dynamics of  $N$  independent agents  $\dot{x}_i(t) = u_i(t)$ , whose aim is to “agree” on some quantity of interest, where agreement means that

$$\exists x_i^0 \triangleq \lim_{t \rightarrow \infty} x_i(t) \in \mathbb{R} \quad \text{and} \quad x_1^0 = \dots = x_N^0. \quad (3)$$

Seeking for such an agreement, the agents apply a protocol

$$u_i(t) = \sum_{k=1}^N a_{jk}(t)(x_k(t) - x_j(t)), \quad (4)$$

which, obviously, corresponds to the closed-loop system (2). It can be noticed that any point  $x^0 = c\mathbf{1}_N$  with  $c \in \mathbb{R}$  is an equilibrium point of (2). The extensive studies on consensus protocols have mainly focused on the problem whether any solution of (2) converges to one of these equilibria.

Henceforth we confine ourselves to *bounded* (almost everywhere) adjacency matrices, adopting the following.

*Assumption 1:* There exists a constant  $M > 0$  such that  $a_{ij}(t) \leq M \forall i, j$  for almost all  $t \geq 0$ .

The standard criterion for consensus (3) is as follows.

*Lemma 1:* Suppose the graph  $G[A(t)]$  is UQSC or cut-balanced and ISC. Then any solution of (2) reaches consensus (3) and  $x_i^0 = \xi^T x(0)$ , where  $\xi \neq 0$  is some constant vector. In the UQSC graph case the convergence is exponential:  $|x_i(t) - x_i^0| \leq C e^{-\alpha t} x(0)$  for some  $C, \alpha > 0$ .

The case of cut-balanced ISC graph is proved in [9] (alternative proofs can be found in [10], [11]). The case of UQSC graph is usually examined under the assumption of piecewise-continuous Laplacian with positive dwell time between consecutive switches [1], [6], [7]. The proof in the general situation is beyond the scope of this paper. However, we will use Lemma 1 only in the special case, where the graphs  $G \left[ \int_t^{t+T} A(s) ds \right]$  are not only rooted, but have a *common* root, e.g. all agents follow a dedicated “leader”. In this special case Lemma 1 follows from [5, Theorem 1].

### III. DIFFERENTIAL LAPLACIAN INEQUALITIES

In this paper, we examine dynamics similar to (2), replacing the differential equation with the *differential inequality*

$$\dot{y}(t) \leq -L[A(t)]y(t), \quad y(t) \in \mathbb{R}^N, \quad t \geq 0. \quad (5)$$

Unlike the differential equations (2), the solution of (5) is not uniquely defined by the initial condition  $y(0)$ , and hence (5) cannot be used as a control algorithm. However, as will be shown in the subsequent examples, the convergence of many distributed algorithms for multi-agent coordination boils down to analysis of the inequalities (5).

Besides consensus equilibria  $y(t) = c\mathbf{1}_N$  (with  $c \in \mathbb{R}$ ), inequalities (5) have many other solutions. For instance,

choose  $y(t) = y^0 - tc\mathbb{1}_N$  where  $y^0 \in \mathbb{R}^N$  and  $c > 0$  is sufficiently large, then  $\dot{y}(t) = -c\mathbb{1}_N \leq -L(t)y^0$  when  $c > 0$  is large due to Assumption 1. Similarly, choosing  $y^0 < 0$  and  $\alpha > 0$  sufficiently large, the function  $y(t) = y^0 e^{\alpha t}$  is a solution of (5) since  $\alpha y^0 \leq -Ly^0$ . Notice, however, that in these examples the solutions are *unbounded*.

In this paper we are primarily interested in the properties of *bounded* solutions of (5). In fact, the solution boundedness is equivalent to the existence of a finite *lower bound*, since the upper bound is guaranteed by the following lemma.

*Lemma 2:* For any solution  $y(t)$  of (5), one has  $y_i(t) \leq \max_j y_j(0)$  for any  $t \geq 0$  and  $i \in \mathcal{V}$ .

We are going to show that under some connectivity assumptions, any *bounded* solution converges to a consensus equilibrium. In particular, any solution is either *convergent* to an equilibrium or *unbounded*. For differential equations this property is referred to as *dichotomy* [12], [13].

*Definition 1:* We say the system of differential inequalities (5) is *dichotomic*, if any of its solutions  $y(t)$  is either unbounded or converges to a finite limit  $y^0 \triangleq \lim_{t \rightarrow \infty} y(t)$ . A dichotomic system (5) is called *consensus dichotomic*, if any limit  $y^0$  is a consensus equilibrium  $y^0 = c^0 \mathbb{1}_N$ ,  $c^0 \in \mathbb{R}$ .

The problem we address in this paper is to disclose conditions, which guarantee consensus dichotomy of the system (5). Since any solution of (2) satisfies (5), consensus dichotomy automatically implies consensus in the network (2). The inverse, however, is not always the case. As will be shown, the UQSC condition in general, is *not sufficient* for consensus dichotomy, whereas a natural sufficient condition is the uniform *strong* connectivity (USC). For cut-balanced graphs, however, is necessary and sufficient for both consensus dichotomy in (5) and consensus in (2).

*Remark 1:* Instead of (5), the reverse inequality

$$\dot{y}(t) \geq -L[A(t)]y(t), \quad y(t) \in \mathbb{R}^N, t \geq 0 \quad (6)$$

can be considered. The solutions of (6) are bounded from below; any bounded solution  $y(t)$  of (6) corresponds to a bounded solution  $(-y(t))$  of (5) and vice versa, hence the inequality (6) is dichotomic (consensus dichotomic) if and only if the corresponding condition holds for (5).

#### IV. DICHOTOMY CRITERIA

In this section we formulate general dichotomy criteria for the Laplacian inequalities (5). We start with the case of *cut-balance* graphs, where necessary and sufficient condition for the consensus dichotomy is given by the following.

*Theorem 1:* Let the graph  $G[A(\cdot)]$  be cut-balanced. Then for any bounded solution of (5) and  $j, k \in \underline{N}$  one has  $a_{jk}|y_k - y_j| \in L_1[0; \infty]$  and  $\dot{y}_j \in L_1[0; \infty]$ , and thus the inequality (5) is dichotomic. It is *consensus dichotomic* if and only if the graph  $G[A(\cdot)]$  is ISC.

In the case of a general time-varying graph the USC condition is sufficient for consensus dichotomy.

*Theorem 2:* Let the graph  $G[A(\cdot)]$  be USC. Then the inequality (5) is consensus dichotomic. Moreover, for *unbounded* solutions  $y(t)$  one has  $\max_i y_i(t) \xrightarrow{t \rightarrow \infty} -\infty$ ,

whereas for any *bounded* solution the function  $\Delta(t) \triangleq -L[A(t)]y(t) - \dot{y}(t) \geq 0$  satisfies the following condition

$$\int_t^{t+T_0} \Delta(s) ds \xrightarrow{t \rightarrow \infty} 0 \quad \forall T_0 \geq 0. \quad (7)$$

As will be shown below (Section V), the USC condition here cannot be relaxed to UQSC. Whereas UQSC guarantees consensus in (2), the inequalities (5) can have *periodic* solutions. At the same time, UQSC guarantees a “weaker” dichotomy condition for graphs with special structures.

*Definition 2:* We say that a UQSC graph  $G(t) = (\mathcal{V}, E(t), A(t))$  has a *principal root*  $v \in \mathcal{V}$  if for any  $t \geq 0$  the node  $v$  has no incoming arcs  $\{j : (j, v) \in E(t)\} = \emptyset$ .

Since the definition of the UQSC requires the graphs  $G \left[ \int_t^{t+T} A(s) ds \right]$  to be QSC, the principal root  $v \in \mathcal{V}$  (if exists) is (the only) common root of these graphs. Dealing with distributed protocols (4), the corresponding agent (whose state remains constant) is usually referred to as the (static) *leader*; in presence of such a leader consensus (3) means that the states of all other agents converge to the leader’s state:  $x_j(t) \rightarrow x_v$  as  $t \rightarrow \infty$ .

*Theorem 3:* Let the graph  $G[A(t)]$  be UQSC with a principal root  $v$ . Then the following “relaxed” dichotomy condition holds for (5): either  $\liminf_{t \rightarrow \infty} \min_i y_i(t) < y_v(0)$  or  $y_i(t) \xrightarrow{t \rightarrow \infty} y_v(0) \forall i$ , and the convergence is exponential.

Notice that (5) *does not* imply that the leader’s state  $y_v(t)$  is static, entailing only that  $\dot{y}_v(t) \leq 0$ .

Proofs of the main results will be given in Section VI. In the next section we demonstrate their applications in several problems of multi-agent coordination and opinion dynamics.

#### V. EXAMPLES AND APPLICATIONS

In this section we demonstrate that the results, formulated in Section IV, allow to obtain a number of recent results in a unified way, relaxing also some assumptions.

##### A. Opinion dynamics with antagonistic interactions

Social agents usually fail to reach consensus, but rather exhibit *clustering* [19] of opinions or other types of persistent disagreement. Whereas protocols, leading to consensus, have been thoroughly studied up to certain exhaustiveness, it remain a tough problem to obtain a realistic model of opinion dynamics, “complex” enough to include the possibilities of both consensus and disagreement and yet sufficiently “simple” to admit rigorous analysis. The most studied in literature are *bounded confidence* models [20], where the agents ignore opinions outside their confidence intervals, and hence the interaction graph may loose its connectivity. Another reason for disagreement and clustering in social networks is the agents heterogeneity, caused e.g. by their “stubbornness” [21]: some agents are “attached” to their initial opinions and take them into account on each iteration of opinion change (such agents are also called “informed” [19]).

A completely different type of opinion dynamics was suggested in [14], dealing with protocols over *signed* graphs

$$\dot{x}_i(t) = \sum_{j=1}^N |w_{ij}(t)| (x_j(t) \operatorname{sgn} w_{ij}(t) - x_i(t)) \in \mathbb{R}, \quad \forall i. \quad (8)$$

Here  $W = (w_{ij})$  is a signed ‘‘adjacency’’ matrix that may be time-varying, the coupling term  $|w_{ij}|(x_j \operatorname{sgn} w_{ij} - x_i) = w_{ij}x_j - |w_{ij}|x_i$  infinitesimally drives  $x_i$  to  $x_j$  when  $w_{ij} > 0$  and to  $-x_j$  if  $w_{ij} < 0$ . Thus arcs of positive weight correspond to ‘‘collaboration’’ or ‘‘trust’’ among the agents, whereas negative weights stand for their ‘‘competition’’ or ‘‘distrust’’. Under assumptions of strongly connected static graph ( $w_{ij}(t) = \operatorname{const}$ ) the *structural balance* [14] of positive and negative arcs implies *polarization* of opinions, whereas structural imbalance entails *exponential stability* of the system, where all the opinions converge to 0 independent of the initial condition. In both situations *modulus consensus* is established, where opinions agree in modulus but their signs can differ. In the recent papers [11], [22] the results from [14] are extended to time-varying case.

To formulate a rigorous criterion of ‘‘modulus consensus’’, assume that  $a_{ij}(t) = |w_{ij}(t)|$  are bounded and let  $y_i(t) \triangleq |x_i(t)|$ . As was noticed in [11], denoting  $\varepsilon_{ij}(t) \triangleq \operatorname{sgn} w_{ij}(t) \operatorname{sgn} x_j(t) \operatorname{sgn} x_i(t)$ , (8) implies that

$$\dot{y}_i(t) = \sum_{j=1}^N a_{ij}(t)(\varepsilon_{ij}(t)y_j(t) - y_i(t)). \quad (9)$$

By noticing that  $\varepsilon_{ij}(t)y_j(t) \leq y_j(t)$ , one easily shows that  $y(t)$  is a solution of (5). On the other hand, by definition  $y(t) \geq 0$ . Applying Theorem 1 and 2, we arrive at the following criterion of modulus consensus, obtained in [11].

*Theorem 4:* Let the graph  $G[A(\cdot)]$  be either USC or cut-balanced and ISC. Then the protocol (8) establishes *modulus consensus*: the common limit  $x_* \triangleq \lim_{t \rightarrow \infty} |x_i(t)| \geq 0$  exists.

It should be noticed that in general modulus consensus does not exclude the ‘‘degenerate’’ case where the system (8) is globally asymptotically stable. In the case of cut-balanced graph and static graph it is possible to refine the results of Theorem 4 and obtain necessary and sufficient conditions for stability and non-degenerate opinion polarization [11], [22].

*Remark 2:* As demonstrated by [11, Example 2], the assumptions of Theorem 4 cannot be relaxed to the UQSC condition. Namely, there exist periodic UQSC graphs  $G[A(t)]$  leading to non-constant periodic solutions  $x(t)$ . Hence the UQSC condition *does not* provide the dichotomy of (5).

### B. Networks with complex-valued Laplacians

Further extension of (8) was proposed in [17] and deals with complex-valued agents  $\hat{x}_i = u_i \in \mathbb{C}$  and the protocol

$$u_i(t) = \sum_{j=1}^N (w_{ij}(t)x_j(t) - |w_{ij}(t)|x_i(t)) \in \mathbb{C}, \quad \forall i. \quad (10)$$

Here  $w_{ij}(t) \in \mathbb{C}$  are *complex* weights; the model (8) corresponds to a special case where  $w_{ij}(t) \in \mathbb{R}$ . A special case of this model was also addressed in [16] (Section III-C).

Let  $a_{ij}(t) \triangleq |w_{ij}(t)| \geq 0$ . Notice that for any  $z_1, z_2 \in \mathbb{C}$  one has  $\bar{z}_1 z_2 = |z_1||z_2| \cos \varphi$ , where  $\varphi = \angle(z_1, z_2)$  is the angle between  $z_1, z_2$ ; for technical reasons it is convenient to put  $\varphi = \pi/2$  when  $|z_1| = 0$  or  $|z_2| = 0$ . It now can be easily shown that functions  $y_j(t) = |x_j(t)|$  obey (5). Let

$\theta_{jk} = \angle(w_{jk}x_k, x_j)$ . Since  $y_i(t)$  is locally Lipschitz,  $\dot{y}(t)$  exists for almost all  $t \geq 0$ . Assuming that  $y_i(t) > 0$ , one has

$$\dot{y}_j = \operatorname{Re} \frac{\bar{x}_j \dot{x}_j}{y_j} = \sum_{k=1}^N a_{jk}(y_k \cos \theta_{jk} - y_j). \quad (11)$$

Notice now that if  $y_i(t) = 0$  at some  $t > 0$ , then  $t$  is the global minimum point and thus  $\dot{y}_i(t) = 0$ . In both cases it is obvious that (5) is valid, and we arrive at the following.

*Theorem 5:* Theorem 4 retains its validity, replacing the Altafini protocol (8) with its complex counterpart (10).

A natural question arises whether the limit modulus  $x_* = \lim_{t \rightarrow \infty} |x_i(t)|$  can be positive, which situation is called in [17] *complex consensus*. In the case of *constant* graph  $W(t) \equiv W$  this is equivalent to the existence of a spanning tree in  $G[A]$  and the ‘‘essential nonnegativity’’ of the matrix  $W$ . The latter property, referred in [16] to as *consistency*, implies that the product of weights  $w_{jk}$  over any cycle is a positive real number (in the case of real signed weights this condition is referred to as the *structural balance* [11], [14]). Equivalently, there exist complex numbers  $p_1, \dots, p_N$  such that  $|p_i| = 1$  and  $w_{jk} = a_{jk}p_k \bar{p}_j$  (the construction of such numbers is discussed in [16, Section III-B]).

Assuming now that the matrix  $W(t)$  is consistent for any  $t \geq 0$  and the numbers  $p_j(t) \equiv p_j$  are *constant*, the system (10) converges to a circular formation [16], [17].

*Theorem 6:* Assume that  $w_{ij}(t) = a_{ij}(t)p_j \bar{p}_i$ , where  $a_{ij}(t) \geq 0$ ,  $p_i$  are constant and  $|p_i| = 1 \forall i$ . If the graph  $G[A(t)]$  is UQSC or cut-balanced and ISC, then a vector  $\eta \in \mathbb{C}^N \setminus \{0\}$  exists such that  $x_i(t) \rightarrow \bar{p}_i \eta^\top x(0)$ .

The proof is immediate from Lemma 1 since  $z_i = p_i x_i$ , obviously, obey the usual consensus protocol (4) (formally Lemma 1 provides consensus for scalar agents, but one can apply it independently to the real and imaginary parts).

### C. Problem of a target set surrounding

Further extension of the models (8),(10) was suggested in [16] (see also [23] for some important extensions and detailed proofs). A convex compact set  $X \subset \mathbb{C}$  is fixed, which is known by all of agents. Furthermore, the agents are able to calculate their *projections*  $P_X(x_i(t))$ , where  $x_i(t) \in \mathbb{C}$  stands for the state of the  $i$ th agent and  $P_X$  is the projection operator. A well-known property of the projector  $P_X$ , defining it uniquely and valid in any dimension, is that  $\langle x - P_X(x), y - P_X(x) \rangle \leq 0$  for any  $y \in X$  and hence  $\langle x - P_X(x), x - y \rangle \geq |x - P_X(x)|^2$ . Here  $\langle \cdot, \cdot \rangle$  stands for the scalar product  $\langle z_1, z_2 \rangle = \operatorname{Re} \bar{z}_1 z_2$ . The main concern of the paper [16] is the convergence of the distributed protocol

$$\dot{x}_j(t) = \sum_{k=1}^N a_{jk}(t)(w_{jk}x_k^p(t) - x_j^p(t)), \quad (12)$$

where  $x_j^p \triangleq x_j - P_X(x_j)$ ,  $a_{jk}(t) \in \{0; 1\}$  is an adjacency matrix, defining a time-varying interaction topology, and  $W = (w_{jk}) \in \mathbb{C}^{N \times N}$  is a *constant* matrix,  $|w_{jk}| \in \{0; 1\}$ . In the case of  $X = \{0\}$  the protocol (12) is a special case of (10). In general, the whole set  $X$  consists of equilibria

points, but there are other equilibria, corresponding to agents' arrangements, equidistant from the set  $X: |x_1^p| = \dots = |x_N^p|$ .

As was proved in [16], under the USC condition (and positive dwell time) the protocol (12) always renders the agents equidistant from  $X$ . This result is a special case of Theorem 2, since, as was shown in [16], the distances  $y_j(t) = |x_j^p(t)|$  satisfy (11) with the only difference that  $\bar{x}_j \hat{x}_j$  in the second expression has to be replaced with  $\bar{x}_j^p \hat{x}_j$  and  $\theta_{jk}(t)$  denotes the angle between  $w_{jk} x_k^p(t)$  and  $x_j^p(t)$ . In particular,  $y_j(t)$  obey (5), which allows to apply Theorem 1 or Theorem 2. Furthermore, these theorems guarantee that

$$\int_t^{t+T_0} a_{jk}(t) y_k(t) (1 - \cos \theta_{jk}(t)) \xrightarrow{t \rightarrow \infty} 0 \quad \forall T_0 \geq 0 \quad \forall j, k,$$

which can be transformed, assuming the graph is USC, into the so-called *target surrounding* condition [16]

$$|w_{jk} x_k^p(t) - x_j^p(t)|^2 \xrightarrow{t \rightarrow \infty} 0. \quad (13)$$

The alternative proof of (13) under additional assumptions of piecewise-continuous matrix  $A(t)$  and positive dwell-time between its consecutive switches is available in [16], [23].

The condition (13) implies that  $x_i(t) \rightarrow X$  unless the matrix  $W$  is "consistent" [16], that is, the product of  $w_{ij}$  over any cycle in the graph  $G[W]$  equals to 1. In the consistent case nonzero entries are decomposed as  $w_{ij} = p_i \bar{p}_j$ , where  $|p_i| = 1 \forall i$ , and the convergence  $w_{jk} x_k^p(t) - x_j^p(t) \rightarrow 0$  can be treated as *surrounding* of the target set  $X$ : the agents are equidistant from the set  $X$  and angles between  $x_j^p(t)$  converge to some fixed values [16].

*Theorem 7:* Suppose the graph  $G[A(t)]$  is either USC or cut-balanced and ISC. Then the protocol (8) deploys the agents' at equal distances from the set  $X$ : there exists a limit  $d_* \triangleq \lim_{t \rightarrow \infty} |x_i^p(t)| \geq 0$ , which is independent of  $i \in \mathcal{V}$ . If the graph is USC, then the set is surrounded (13).

Some conditions providing non-trivial set surrounding ( $d_* > 0$ ) can be found in [16]. Notice that Theorem 7 remains valid for arbitrary bounded adjacency matrix  $A(t)$ , does not require the dwell time positivity and partially extends the result from [16] to cut-balanced ISC graphs.

#### D. Containment control with multiple leaders

In the previous example all of the agents are aware of the target set position, so they could gather in it without distributed control. In this subsection we consider the so-called *containment control* problem [3], where the desired set is pointed out by multiple static leaders, and the others have to converge into the convex hull, spanned by their states.

Hence, besides the  $N$  agents indexed 1 through  $N$  we consider  $L$  leaders  $\hat{x}_i(t) = 0$ , where  $i = N+1, \dots, N+L$ . The distributed protocol for the "followers", obeying the model  $\dot{x}_j = u_j$ , is similar to (4) but involves some extra terms, attracting the agents towards the desired convex hull

$$u_j(t) = \sum_{k=1}^{N+L} a_{jk}(t) (x_k(t) - x_j(t)) \in \mathbb{R}^d, \quad j = 1, \dots, N. \quad (14)$$

Let  $X$  be the convex hull, spanned by the leaders' states, following the notation from the previous section, we put  $x_i^p = x_i - P_X(x_i)$ . We are going to find conditions providing that  $y_j(t) = |x_j^p(t)| \rightarrow 0$  for any  $j = 1, \dots, N$ . As was noticed, for each  $y \in X$  one has  $\langle x - y, x^p \rangle \geq |x^p|^2$ . In particular,  $\langle x_k - x_j, x_j^p \rangle \leq -|x_j^p|^2$  for any  $j \leq N$  and  $k > N$ . Furthermore, for  $k \leq N$  we have  $\langle x_k - x_j, x_j^p \rangle \leq \langle x_k^p - x_j^p, x_j^p \rangle$ , as shown in [15], [18]. Since  $\frac{d}{dt} y_j^2(t) = \langle \dot{x}_j^p, \hat{x}_j \rangle$  (see [15]), the "outputs"  $y_j(t)$  obey the inequalities

$$\dot{y}_j(t) \leq \sum_{k=1}^N a_{jk}(t) (y_k(t) - y_j(t)) - a_{j0}(t) y_j(t). \quad (15)$$

Here  $a_{j0} \triangleq \sum_{k=N+1}^{N+L} a_{jk}(t)$  for  $j = 1, \dots, N$  and we put  $a_{0j} \triangleq 0$  for any  $j$ . Considering an extended vector  $\hat{y}(t) = (0, y_1(t), \dots, y_N(t))$  and extended adjacency matrix  $\hat{A}(t) = (a_{ij}(t))_{i,j=0}^N$ , (15) shapes into

$$\frac{d}{dt} \hat{y}(t) \leq -L[\hat{A}(t)] \hat{y}(t). \quad (16)$$

Since  $y_j(t) \geq y_0(0)$ , Theorem 3 now implies the following.

*Theorem 8:* Suppose that  $G[\hat{A}(t)]$  is a UQSC graph (obviously, 0 is its principal root). Then all the agents' states exponentially converge to the set  $X$ .

Theorem 8 was proved in [3] (Theorem 5.3) under stronger condition of the "united spanning tree" existence and piecewise constant graph; result from [3] also does not guarantee exponential convergence. In the case of cut-balance graphs conditions can be further relaxed. Applying Theorem 1 to the vector  $y(t)$ , (15), one proves that the limits  $y_i^0 = \lim_{t \rightarrow \infty} y_j(t)$  exist and coincide  $y_1^0 = \dots = y_N^0 = y^0$ , and, moreover,  $\int_0^\infty a_{j0}(t) y_j(t) < \infty$ . This implies that  $y^0 = 0$  unless  $\int_0^\infty a_{i0}(t) dt < \infty \forall i \in \mathcal{V}$ , which yields in the following.

*Theorem 9:* Suppose that  $G[A(t)]$  is a cut-balanced ISC graph and  $\int_0^\infty a_{j0}(t) dt = \infty$  for at least one  $j = 1, \dots, N$ . Then all the agents' states converge to the set  $X$ .

Similarly, our results allow to prove in a unified manner the convergence of protocols for target aggregation control [15] and distributed optimization algorithms from [18].

## VI. PROOFS OF THEOREMS 1-3

In this section we give outlines of the proofs of the main results. The proof of Lemma 2 is standard and omitted due to space limitations. We start with Theorem 3, being fully independent of the other criteria, and then introduce some techniques, enabling us to cope with the other results.

#### A. Proof of Theorem 3

Consider the consensus protocol (2) and let  $\Phi(t; s)$  stands for the Cauchy evolutionary matrix of the linear system (2), which is known to be stochastic [1]. Furthermore, since  $v \in \mathcal{V}$  is a principal root, consensus means that  $x_i(t) \rightarrow x_v$  independent of the initial condition, which means that  $\lim_{t \rightarrow \infty} \Phi(t; 0)$  is a matrix, whose entries are zeros except for

the  $v$ th column, equal to  $\mathbf{1}_N$ . If  $y(t)$  is a solution of (5) and  $y_i(t) \geq y_v(0) \forall i \in \mathcal{V}$ , then  $\dot{y}(t) \leq -L(t)y(t)$  leads to

$$y_v(0)\mathbf{1}_N \leq y(t) \leq \Phi(t; 0)y(0) \xrightarrow[t \rightarrow \infty]{} y_v(0)\mathbf{1}_N,$$

where the convergence is exponential by Lemma 1. ■

### B. Proof of Theorem 1

We introduce an ordering  $\xi_1(t) \leq \dots \leq \xi_N(t)$  of the values  $y_1(t), \dots, y_N(t)$ ; let  $\xi_k(t) = y_{i_k(t)}(t)$ . Although  $i_k(t)$  are defined non-uniquely, one can always choose them  $i_k(t)$  Lebesgue measurable [11]. Moreover,  $\xi_k(t)$  are absolutely continuous and  $\dot{\xi}_k(t) = \dot{y}_{i_k(t)}(t)$  for almost all  $t \geq 0$ .

For technical reasons it is easier to consider the inverted inequality (6). Let  $b_{jk}(t) \triangleq a_{i_j(t)i_k(t)}(t)$ . It can be shown that  $G[B(\cdot)]$  is a cut-balanced graph. Since  $\dot{\xi}_k(t) = \dot{y}_{i_k(t)}(t)$ , one has  $\xi_j(t) \geq \xi_j(0) + \int_0^t b_{jk}(t)(x_k(t) - x_j(t))$ . Retracing now the proof of Theorem 1 in [9], one shows that if  $\xi_i(t)$  are bounded from above, then  $\sum_{j,k=1}^N \int_0^\infty b_{jk}(t)|\xi_k(t) - \xi_j(t)|dt = \sum_{j,k=1}^N \int_0^\infty a_{jk}(t)|y_k(t) - y_j(t)|dt < \infty$  and  $\dot{y}_j \in L_1[0; \infty]$ , i.e.  $y_j(t)$  converge to finite limits and  $y_j(t) \rightarrow y_j^0$ , which limits coincide under the ISC condition. Since ISC condition is necessary [9], [10] for consensus under protocol (2), it is necessary for the consensus dichotomy. ■

### C. Proof of Theorem 2

The proof is based on the following technical lemma, proved similarly to Lemmas 14 and 15 in [11].

*Lemma 3:* Let  $y(t)$  obey (5) and  $y_* = \max_i y_i(t_0)$ . For any  $T > 0$  there exists  $\theta = \theta(T) \in [0; 1]$  such that

$$y_i(t) \leq \theta y_i(t_0) + (1 - \theta)y_*, \quad \forall i \in \mathcal{V}, \forall t \in [t_0; t_0 + T].$$

Furthermore, for any  $\delta > 0$  there exists  $\theta_0 = \theta_0(T, \delta) \in [0; 1]$  such that if  $\int_{t_0}^{t_0+T} a_{jk}(t)dt \geq \delta$  for some  $j$  and  $k$ , then

$$y_j(t_0 + T) \leq \theta_0 y_k(t_0) + (1 - \theta_0)y_*.$$

*Proof of Theorem 2* is now similar to the proof of Theorem 2 in [11]. We again use the ordering  $\xi_1 \leq \dots \leq \xi_N$ . If  $\xi_N(t)$  is bounded from below then the limit  $\xi_* \triangleq \lim_{t \rightarrow \infty} \xi_N(t) > -\infty$  exists. We are going to show via induction on  $j$  that  $\xi_* = \lim_{t \rightarrow \infty} \xi_j(t)$ . For  $j = N$  this claim is obvious. Suppose the claim is proved for  $j = s + 1, \dots, N$ . To prove it for  $j = s$ , assume on the contrary that  $\underline{\lim} \xi_s(t) < \xi_*$ , that is, there exist  $q < \xi_*$  and sequence  $t_n \uparrow \infty$  such that  $\xi_s(t_n) \leq q$ . Consider two sets  $I_n = \{i_1(t_n), \dots, i_s(t_n)\}$  and  $J_n = \{i_{s+1}(t_n), \dots, i_N(t_n)\}$ . Let  $T, \delta$  be the constants from the definition of USC, then  $j_n \in J_n, k_n \in I_n$  exist such that  $\int_{t_n}^{t_n+T} a_{j_n k_n}(t) \geq \delta$ . Using Lemma 3, we prove that

$$y_i(t_n + T) \leq \theta_1 q + (1 - \theta_1)\xi_N(t_n), \quad \forall i \in I_n \cap \{j_n\},$$

where  $\theta_1 \triangleq \max(\theta_0, \theta)$ . By definition this implies that  $\xi_{s+1}(t_n + T) \leq \theta_1 q + (1 - \theta_1)\xi_N(t_n)$ . Passing to the limit as  $n \rightarrow \infty$ , we arrive at the contradiction with the induction hypothesis  $\xi_* = \lim_{t \rightarrow \infty} \xi_{s+1}(t)$ . This proves our claim for  $j = s$  and finishes the proof of the consensus dichotomy. The convergence of  $y_i(t)$  to finite limits also implies (7). ■

## VII. CONCLUSIONS

In this paper, we study properties of linear differential inequalities with time-varying Laplacian matrices. An important property of such inequalities, valid under mild connectivity assumptions, is their *dichotomy*: any bounded solution converges to a consensus equilibrium point. The dichotomy criteria allow to prove convergence of many Laplacian-type protocols for multi-agent coordination in a unified way.

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